Job Search and Savings: Wealth Effects and Duration Dependence

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Abstract

In this paper we study a risk averse worker's optimal savings and job-search behavior as she moves back and forth between employment and unemployment. Recent empirical findings suggest a negative association between a risk averse worker's search effort and wealth. We show that additively separable utility and bounded wealth imply that job search is negatively affected by wealth, and characterize the worker's consumption paths and wealth formation under these conditions. In general, all decisions will depend on the current level of wealth. Furthermore, given optimal search, savings still provide imperfect insurance against income fluctuations; precautionary savings are built up during employment spells and run down during unemployment spells but the consumption path is never going to be completely smooth over states. Finally, our results suggests that wealth introduces an element of positive duration dependence to the probability of leaving unemployment.

Keywords: Search, Consumption Smoothing, Duration Dependence.

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1 Introduction

In this paper, we study how a worker’s job search and savings decisions are affected by wealth. We set forth a general job search intensity model with savings and demonstrate that wealth can influence search effort both positively and negatively, and even non-monotonically. Recent empirical evidence suggests that search effort is negatively related to wealth.\(^1\) We give sufficient conditions for the search intensity to be monotonically decreasing in wealth and characterize the worker’s search and savings choices as well as wealth formation under these conditions.

Both the acts of job search and savings are undertaken in order to increase future consumption possibilities and both are part of a self-insurance effort: Savings allow the agent to maintain a smooth consumption path in the face of fluctuating income due to for instance the reoccurring event of job loss, while the choice of search effort acts to control the damage by shortening the length of the unemployment spell. Yet search and saving are rarely analyzed as interrelated choice problems. In the labor literature, the savings decision is typically moot because of risk neutrality of the worker or because wealth cannot be stored. Neither assumption is satisfactory in the context of a worker in a labor market with unemployment but has served as a useful simplification in the analysis of many aspects of the model. In the macro literature the savings problem is well understood but here the search decision is ignored as unemployed workers become re-employed at an exogenous rate.

Some important aspects of search models are not affected much by the savings decision but generally there are reasons to believe that savings and search behavior do influence each other. Search is likely to be affected by the level of savings; an unemployed worker with little wealth has less ability to smooth consumption and consequently has greater incentive to leave the state of unemployment. The savings decision might also be influenced by the costs of search; if search costs are low and unemployment therefore is expected to be brief, fewer savings are needed for the eventuality of unemployment. Thus, to understand the search and savings decisions better they need to be analyzed as a joint choice problem. Moreover, it is imperative in assessing the welfare implications of, for instance, unemployment insurance and employment protection that not only

\(^1\)See for example Algan, Chéron, Hairault, and Langot (2001), Lentz (2002), and Lentz and Tranaes (2003) in which unemployment spell duration is shown to be positively related to the worker’s wealth holdings. Direct evidence of the impact of wealth holdings on the search decision is given in Bloemen and Stancanelli (2001) and Alexopoulos and Gladden (2002) where wealth holdings are shown to be negatively related to the reservation wage and positively related to the search intensity choice in survey data.
search effort but also the worker’s ability to self-insure via savings is accounted for.

The job search model with savings does not lend itself easily to characterization which may account for the search literature’s reluctance to include the savings choice in the model. This paper provides an analytical characterization of the worker’s behavior in a sequential job search model with savings where the worker alternates back and forth between employment and unemployment and thus has incentives both to save and to search. In every period, the worker decides how much to consume and how much to save. The worker’s search choice when unemployed is modelled as a choice of search effort by which the worker can affect her probability of re-employment.

Disregarding the search costs, an unemployed worker with concave utility is less eager to move back into employment the more wealthy he or she is, which in isolation makes wealthier workers search less. The net effect is ambiguous, however, if the search costs are diminished as the worker becomes wealthier. Therefore, with specific assumptions on preferences and search costs one can generate almost any relationship between wealth and search effort including non-monotone ones. We illustrate this by simulating some selected cases. The case where search effort is negatively related to wealth is of particular interest because it is a plausible account for the empirical finding that wealthier workers experience longer unemployment durations.

The main contribution of our paper is to prove the following monotonicity result for the job-search and savings model described above: Under the assumptions of separability between consumption and search effort, concave utility from consumption, convex job search costs, and a technical assumption of the existence of a simple wealth lottery, the search effort depends negatively on wealth.

Under the assumption that employment is an absorbing state and that utility exhibits decreasing absolute risk aversion, a similar result is shown in Danforth (1979) where the reservation wage choice is shown to be positively related to wealth holdings. The assumption that employment is an absorbing state, that is, once employed the worker can never again lose her job is crucial to the line of reasoning in Danforth (1979) but limits the scope of the analysis: Unemployment is restricted to consist of only newly arrived workers. Once a worker is employed she has no precautionary motive for saving. Thus, the wealth that serves as insurance against unemployment in the model is never accumulated by the worker herself with the intension of self insurance. Also, the study of many labor market policy related issues such as unemployment insurance, employment protection
and the like makes little sense if the worker is at no risk of losing her job. The characterization of how the search behavior depends on wealth is considerably complicated by the assumption that jobs can be lost. We, nevertheless, insist that it is an important generalization. Both because it facilitates the links between search and savings decisions to be studied and because it constitutes a comprehensive framework for studying labor insurance issues.

Furthermore, allowing for income uncertainty in all states also makes for a more interesting consumption choice framework. Given the endogenous Markov income process and the borrowing constraint in our model, it is shown that savings do provide some insurance against income fluctuations but that it is not perfect. In the light of the results in Deaton (1991) this is expected. Deaton characterizes the consumption choice under borrowing constraints and exogenously given income processes. Our results extend the consumption characterization to a case of endogenous income processes.

Our work is related to the theoretical literature on duration dependence. Job search effort is shown to exhibit positive duration dependence in our model via its negative relationship with wealth. During spells of unemployment the worker will draw on savings to smooth consumption. Thus, as an unemployment spell progresses wealth is drawn down and the search intensity increases. Therefore, the model implies that the unemployment hazard rate is increasing in the duration of the spell.\(^2\) The literature on duration dependence of the unemployment hazard rate includes arguments in favor of both positive and negative duration dependence. Berkovitch (1990) suggests there is a stigma associated with long unemployment spells so that the hazard rate would show negative duration dependence. Others have associated negative duration dependence with loss of absolute or relative skills due to inactivity or separation from innovations. In Mortensen (1986) a simple liquidity constraint is built into a basic search model which generates a decreasing reservation wage as the unemployed worker moves closer and closer to the constraint. Thus implying positive duration dependence. It is worthwhile noting that the duration dependence in the job search model with savings is an endogenous feature whereas most other studies obtain duration dependence as a direct result of some sort of assumed duration dependence of exogenous parameters, be it offer

\(^2\)We assume in our analysis that the interest rate is less than the subjective rate at which the worker discounts future utility. The duration dependence can possibly be reversed in special cases where the interest rate is sufficiently greater than the subjective discount rate and when the worker holds sufficient amounts of wealth. In this case, the worker may want to increase savings even when unemployed (but at a lower rate than when employed). Thus, in this special case the sign of the duration dependence will potentially be wealth dependent.
arrival rates, unemployment benefits, etc.

Our work is also related to a number of more recent studies such as Acemoglu and Shimer (1999), Wang and Williamson (1999), and Gomes, Greenwood, and Rebelo (2001) all of which have moved beyond the expected income maximization or risk neutrality assumption. In a study of efficient unemployment insurance in a general equilibrium search model with savings, Acemoglu and Shimer (1999) consider the special case where utility is characterized by constant absolute risk aversion, marginal search costs are inversely related to consumption, and consumption is allowed to be negative.\(^3\) In a study of optimal unemployment insurance in a partial equilibrium setting, Flemming (1978) considers a similar household side of the model. In this special case, all wealth effects on the search decision are eliminated.\(^4\) Wang and Williamson (1999) allow for job separation in a model where individuals both search and save and by way of numerical methods they then characterize optimal unemployment insurance schemes. Finally, Gomes, Greenwood, and Rebelo (2001) use a job search model that includes savings and job separations to study labor market and business cycle regularities. In their model, a worker chooses a reservation wage given a constant search effort whereas in our model a worker chooses a level of search effort given a fixed wage opportunity. Still, their necessary conditions for a monotone relationship between wealth and the search decision are consistent with our model. However, Gomes, Greenwood, and Rebelo (2001) do not show that these necessary conditions are in fact satisfied. In the calibrated version of their model the conditions do hold, though.

The paper proceeds by presenting the model in Section 2. Section 3 demonstrates the broad range of possible wealth effects on the search decision in the general model and gives a simple intuition for the cause of the different effects. The main analysis of the negative wealth effect case is presented in Sections 4 and 5. Section 6 concludes the paper.

2 A Model of Job Search and Savings

Consider a worker who moves back and forth between employment and unemployment according to a two-state Markov process set in discrete time with time indexed by \(t\). The worker is risk averse in consumption and therefore wants to smooth consumption over the two states representing high

\(^3\) This refers to section V of Acemoglu and Shimer (1999).

\(^4\) Like in Danforth (1979) it is also assumed that employment is an absorbing state. But in this special case, the results extend to the framework where workers move in and out of employment.
and low income situations. The consumption smoothing is accomplished by use of capital markets
where the worker’s savings can be placed.

We assume that during unemployment the worker can affect the probability $s_t$ of moving back
into employment via the choice of search intensity. Assuming that there is a one-to-one relationship
between search intensity and the transition probability, $s_t$, we will simply let the worker choose
$s_t \in [0, 1]$ directly. Job separation is exogenous.

The worker derives utility from consumption and dis-utility from job search. It is assumed that
utility is separable over time with the period-by-period utility given by $v(c_t, s_t)$ where $c_t$ and $s_t$ are
consumption and search intensity in period $t$, respectively. In periods of employment, the search
choice is mute and $s = 0$. We assume throughout that $c_t \geq 0$ and that the worker discounts future
utility by the rate $\rho > 0$. Savings are assumed to carry a rate of return, $r < \rho$.

When employed, the worker receives wage $w$ and during periods of unemployment, the worker
gets compensation $b$. The compensation will be referred to as unemployment benefits but could
also include income from a secondary labor market, surplus from home production, etc.

The worker’s wealth at the beginning of period $t$, is $k_t$ and let $I_t \in \{b, w\}$ be the income in
period $t$. Thus, state variables at the beginning of period $t$ are $k_t$ and $I_{t-1}$, where $I_{t-1}$ tracks
the worker’s employment status in the previous period. The timing of events in each period is as
follows (see Figure 1 for the case of an unemployed worker): Given $(k_t, I_{t-1})$, the worker first has
a choice of putting her wealth into a lottery with immediate realization $k^*_t$. The lottery holds a
zero risk premium and will therefore only be entered into if the worker’s value function is convex
at $k_t$ and thus exhibits a risk loving attitude in wealth. Then given $k^*_t$, a search and separation
stage follows. If unemployed in the previous period, the worker decides on how much effort to put
into job search so as to affect the probability of moving into employment in the current period.
Immediately following the search effort choice the state of employment in period $t$ is realized and
along with it the period $t$ income, $I_t$. If employed in the previous period the worker does not search
and the search effort is trivially zero. Instead the worker is (exogenously) either separated from her
job or remains employed, which in this situation realizes $I_t$. Finally, based on $k^*_t$ and $I_t$, the worker
decides on how much to consume in the present period or equivalently, she decides how much to
save for the future which determines the wealth in the next period, $k_{t+1}$.

Unlike the job search model without savings one cannot in general rule out local convexities of
the worker’s value functions in the model with savings. Convexities of the value functions turn out to be highly problematic in terms of characterization of the worker’s search and savings choices.\footnote{Convexity of the value functions also imply rather perverse and counter-intuitive behavior on the part of the worker. For instance, consumption will be decreasing in wealth in this case.}

The lottery is introduced in order to mitigate this problem. It effectively smooths out any local convexities and ensures concavity of the value functions of the model with the lottery.\footnote{As such, the assumption of the existence of lotteries can be viewed as a sufficient condition for our results. However, simulations of our model suggest that it is not a necessary condition since all value functions turn out to be perfectly concave for a very wide range of model parameters in the model without the lottery. The issue of possible problems with convexities of value functions or concavities of cost functions in these types of models is noted in papers such as Gomes, Greenwood, and Rebelo (2001), Hopenhayn and Nicolini (1997) and Phelan and Townsend (1991).}

The lottery is modelled as yielding return \( k_l^t \leq k_t \) with probability \( \alpha \) and return \( k_h^t \geq k_t \) with probability \( 1 - \alpha \). The parameter \( \alpha \) is set so that the expected return of the lottery equals \( k_t \), that is \( \alpha = (k_h^t - k_l^t) / (k_h^t - k_l^t) \). The worker has access to a full menu of these lotteries in the sense of being able to choose \( k_l^t \) and \( k_h^t \) freely as long as the returns lie within the wealth bounds. Non-participation in the lottery is given by \( k_l^t = k_h^t = k_t \).

The following analysis will exclusively focus on the worker’s choice of savings and search intensity and we take no particular stand on the existence and determination of wages, the interest rate, and the lottery.

All in all, the worker is faced with the following problem,

\[
\max_{\{s_t, c_t, k_l^t, k_h^t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (1 + \rho)^{-t} v(c_t, s_t) \\
\text{s.t. : } k_{t+1} = (1 + r)k_t^* + I_t - c_t,
\]

where \( k_t^* = k_t, k_l^t, \) or \( k_h^t \) according to whether the worker chooses to participate in the period \( t \) lottery or not and what comes out of it. The income process is given by \( I_t \equiv w n_t + (1 - n_t)b \) where \( n_t \in \{0, 1\}; n_t = 0 \) implying that the worker is unemployed and \( n_t = 1 \) that the worker is employed. The random variable \( n_t \) follows a Markov process with transition function \( P(n_{t-1}, n_t) \).

\[\begin{array}{c|c|c|c|c}
\text{Beginning} & \text{Lottery} & \text{Lottery} & \text{Search} & \text{Savings} \\
\text{of period} & \text{is played} & \text{outcome is} & \text{outcome is} & \text{decision} \\
& k, I_{-1} & k_l^t, k_h^t & k^*(k_l^t, k_h^t) & s(k^*, I_{-1}) \\
& & & s(k^*, I_{-1}) & I(s^*, I_{-1}) \end{array}\]
\[ P(0,1) = s_t, \quad P(0,0) = 1 - s_t, \quad P(1,0) = \eta, \quad \text{and} \quad P(1,1) = 1 - \eta. \]

To make sure that our problem is well behaved we will assume the following concerning wealth.

**Assumption 1** The worker’s wealth at the beginning of period \( t \), \( k_t \), is assumed to be bounded both above and below, that is \( k_t \in [k, \bar{k}] \).

A lower bound *per se* can be justified as a borrowing limit imposed by the capital market. Aiyagari (1994) points out that a lower bound on wealth can also be motivated by requiring asymptotic present value budget balance (i.e. \( \lim_{t \to \infty} k_t/(1 + r)^t \geq 0 \)) combined with non-negative consumption. The upper bound is imposed in order to bound the problem and ensure existence of a solution. The bound \( \bar{k} \) will be set above the upper bound of the ergodic wealth distribution when possible. This can always be done as long as the upper bound on the ergodic wealth distribution is finite which it is for any \( r < \rho \) and in this case the assumption does not restrict the worker’s savings problem from above. Finally, when we write “for all \( k_t \)” below we mean for all \( k_t \in [k, \bar{k}] \).

The particular choice of timing of events in the model provides a convenient framework for the analytical results in the following sections. An alternative choice of timing is to have the worker make the search and consumption choices simultaneously. There is however no reason to suspect that the alternative timing will change the savings and search choices in any qualitative way. In fact, it is found in the numerical simulations in the next section that the characteristics of the search choices are identical across the two different timing assumptions.

Without further structure the model can support various different relationship between wealth and search effort, which we discuss in the next section.

### 3 Job Search and Wealth

Recent empirical findings suggest a negative association between a risk averse worker’s search effort and wealth. In this section we point out that while the finding is consistent with a theoretical analysis of a search and savings model, it is not the necessary outcome of such models and we present a counter example. The unemployed worker chooses search intensity so as to equate marginal gains to search with marginal search cost. We cast the discussion in terms of how these two measures are affected by changes in wealth.
In our model, wealth allows the worker a high level of consumption in periods with unemployment. The purpose of search, on the other hand, is to reduce the duration of such periods. But the wealthier the worker is, the less utility difference there is between the two states. Therefore, at higher wealth levels the worker has less incentive to reduce the duration of an unemployment spell and consequently, the less willing she is to suffer utility losses in order to increase the probability of transitioning back into employment.

Hence, in terms of characterizing how wealth influences search effort the key relationship is how search costs are affected by wealth - directly as well as indirectly. While the gains to search are decreasing in wealth, if search costs are also decreasing in wealth then the net wealth effect may be ambiguous. Generally, the cross derivative \( v''_{cs} (c, s) \) will be of particular interest in this respect. A specific example for which search costs are affected by wealth is the case where search costs are monetary: The search costs will be evaluated in terms of a consumption loss via the budget constraint. For a concave utility function, the marginal utility loss from extra search will decrease the higher the consumption level. Higher wealth implies higher consumption and consequently marginal search costs are decreasing in wealth in this case.

Figure 2 shows the choice of search intensity of an unemployed worker as a function of initial wealth for the model described in the previous section under two different utility functions; one where utility of consumption and dis-utility of search effort are separable and one where they are not.\(^7\) The search choice is also shown for an alternative model differing only in that the savings and search choices are made simultaneously and not sequentially.

In the separable-utility case, search costs are not affected by changes in wealth and the search choice is monotonically decreasing in wealth due to the lower gains to search at higher wealth levels. In the non-separable case, marginal search costs are decreasing in consumption. Since consumption

\[^7\text{The separable utility specification is given by:} v(c, s) = c^{5} - \frac{s}{1 - 10s},\]

and the non-separable utility function is given by:

\[v(c, s) = (c^{5} + (1 - 10s)^{5})^{5}.\]

The simulations were preformed under the following parameter specifications: The period length is weekly. The interest rate is set an annual rate of 4.5% and the subjective discount rate is set at an annual rate of 5.0%. The benefit level is normalized at 1 and the wage is set at 2, yielding a replacement rate of 50%. The lower bound on wealth is set at zero (implying no borrowing) and the upper bound is set so that it is above the upper bound of the ergodic wealth distribution. The job destruction rate is set so that the expected employment spell is 4.5 years.
Figure 2: The Unemployed Worker’s Search Intensity for Separable and Non-Separable Utility.

Note: Base model drawn in thick pen. The alternative timing model where the consumption and search decisions are made simultaneously is drawn in thin pen.

is an increasing function of wealth, marginal search costs decrease in wealth. For low wealth levels (and therefore low consumption levels), the wealth effect on search costs is strong and more than offsets the wealth effect on gains to search. Consequently, at low wealth levels the search intensity increases in wealth. At higher wealth levels, changes in wealth affect search costs less and the search intensity starts to decrease in wealth as the reduced gains to search eventually dominate the reduced costs. Thus, one finds a non-monotonic search intensity in wealth for this particular case.

Finally, it is seen that the choice of timing has very little effect on the choice of search intensity in this model.

In Flemming (1978) and section V of Acemoglu and Shimer (1999) the assumption is that search cost are monetary and that the basic utility function is characterized by constant absolute risk aversion.\(^8\) As with the non-separable example in Figure 2, the cross-derivative is positive,

\(^8\)Acemoglu and Shimer (1999) assume a directed search technology. Jobs differ with respect to their wage and the worker can choose which job to apply for. Higher wage jobs will have longer queues. Thus, like the reservation wage decision, a choice of a higher probability of moving into employment (a choice of a lower wage job) is associated with
that is $v''_{cs} (c, s) > 0$. Thus implying that as consumption increases, the marginal search costs are reduced. Therefore, while higher wealth implies fewer incentives to move back into employment it also increases consumption and consequently lowers the marginal search costs. It just so happens that in the constant absolute risk aversion case, the two effects exactly offset each other and eliminate wealth effects on the search decision altogether given that there is no lower bound on wealth. The liquidity effects associated with a lower wealth bound will make the search decision depend negatively on wealth even in this case. Thus, in order to completely eliminate wealth effects on the search choice it is necessary to allow negative consumption so as to eliminate the lower wealth bound.\footnote{As there is no restriction on how negative the asset position can be in Acemoglu and Shimer (1999), section V, the fixed UI benefits do not secure non-negative consumption.}

The above examples should make it clear that one can establish a broad range of relationships between wealth and the search decision depending on the assumptions made in relation to $v''_{cs} (c, s)$. The results in Danforth (1979) even suggest that one can establish monotonically increasing search intensity in wealth by assuming monetary search costs, no lower bound on wealth and a consumption utility function characterized by increasing absolute risk aversion.

As mentioned previously, empirical results suggest that the actual search intensity is decreasing in wealth. In the next section we provide sufficient conditions for the model to exhibit a negative relationship between search intensity and wealth.

### 4 Negative Wealth Effects on Job Search

In this section we will introduce some additional structure to the model of Section 2 and show that the model then exhibits negative wealth effects on the search decision. Specifically, we assume that utility is additive separable over not only time but also over consumption $c_t$ and search $s_t$, and that utility $u(\cdot)$ is strictly increasing and strictly concave in consumption $c_t$ and that search costs $e(\cdot)$ are strictly increasing and strictly convex in search effort $s_t$.

**Assumption 2** In any period $t$, the worker's utility from consumption and search is $v(c_t, s_t) = u(c_t) - e(s_t)$. Furthermore, $u'(\cdot) > 0$, $u''(\cdot) < 0$, $e'(\cdot) > 0$, and $e''(\cdot) > 0$, with $e(0) = 0$.

For convenience it is assumed that both $u(\cdot)$ and $e(\cdot)$ are differentiable. In order to fully specify the details and orders of moves, the model is expressed in terms of Bellman equations. Let $V_{\theta}(k)$ a future income loss.
and $U_g(k)$ be the value functions for the gambling stage of period $t$, given wealth $k_t$ for $I_{t-1} = w$ and $I_{t-1} = b$, respectively. Thus,

$$V_g(k_t) = \max_{k_h^t, k_l^t} \left[ \frac{k_h^t - k_l^t}{k_h^t - k_l^t} V\left(k_h^t\right) + \frac{k_l^t - k_h^t}{k_h^t - k_l^t} V\left(k_l^t\right) \right]$$

$$U_g(k_t) = \max_{k_h^t, k_l^t} \left[ \frac{k_h^t - k_l^t}{k_h^t - k_l^t} U\left(k_h^t\right) + \frac{k_l^t - k_h^t}{k_h^t - k_l^t} U\left(k_l^t\right) \right],$$

where $V$ and $U$ are the value functions at the search and separation stage given the employment status of the previous period:

$$V(k_t) = (1 - \eta) V_c(k_t) + \eta U_c(k_t)$$

$$U(k_t) = \max_{s \in [0,1]} \left[ -e(s) + s V_c(k_t) + (1 - s) U_c(k_t) \right],$$

where, finally, $V_c$ and $U_c$ are the value functions at the consumption and savings stage, which depend on the employment status of the current period, period $t$:

$$V_c(k_t) = \max_{k_{t+1} \in [\min(k_t(1+r)+w, k)]} \left[ u\left( (1 + r) k_t + w - k_{t+1} \right) + \frac{V_g(k_{t+1})}{1 + \rho} \right]$$

$$U_c(k_t) = \max_{k_{t+1} \in [\min(k_t(1+r)+b, k)]} \left[ u\left( (1 + r) k_t + b - k_{t+1} \right) + \frac{U_g(k_{t+1})}{1 + \rho} \right],$$

where by definition $(1 + r) k_t + I_t - k_{t+1} = c_t$. These six value functions represent the model and form the basis for the results we arrive at below.

Given the fact that the value functions are themselves time independent, we need only keep track of two periods at a time, and we suppress the index for the current period. For instance, $k$ and $k_{t+1}$ will be the workers wealth in two successive periods, the current and the next period.

To proceed, a little more notation is needed. Let $c(k; e)$ and $c(k; u)$ be the optimal choices of consumption given wealth $k$ under employment and unemployment, respectively. Let $s(k)$ be the optimal choice of search given wealth $k$ and let the optimal choices of next period’s wealth be defined by $k_{t+1}(k; e) \equiv k + w - c(k; e)$ and $k_{t+1}(k; u) \equiv k + b - c(k; u)$ again according to whether the worker is employed or unemployed in the current period. It is assumed the utility function is such that the only constraint that may be binding is the lower bound on wealth. The first order conditions associated with the optimal choices of consumption and search effort give us the
following characterizations:

\[ u'(c(k; e)) = \frac{V'(k+1(k; e))}{1 + \rho} + \lambda_V(k) \]  
\[ u'(c(k; u)) = \frac{U'(k+1(k; u))}{1 + \rho} + \lambda_U(k) \]  
\[ e'(s(k)) = V_c(k) - U_c(k), \]

where \( \lambda_V \) and \( \lambda_U \) are the non-negative Lagrange multipliers associated with the lower bound on wealth. By the envelope theorem it follows that:

\[ V_0'(k) = V'(k) \]  
\[ U_0'(k) = U'(k) , \quad \forall k^h \geq k^l. \]  

If the worker chooses not to participate in the lottery, that is \( k = k^l = k^h \), then (4) becomes:

\[ V_0'(k) = V'(k) \]  
\[ U_0'(k) = U'(k) , \quad \forall k^h = k^l = k. \]

Furthermore,

\[ V'(k) = (1 - \eta) V_c'(k) + \eta U_c'(k), \quad U'(k) = s(k) V_c'(k) + (1 - s(k)) U_c'(k), \]

\[ V_c'(k) = u'(c(k; e)) (1 + r), \quad \text{and} \quad U_c'(k) = u'(c(k; u)) (1 + r). \]

A main focus of the paper is to characterize the relationship between search intensity and wealth. To this end, (3) is differentiated with respect to \( k \) which yields,

\[ s'(k) = \frac{\partial s(k)}{\partial k} = \frac{V_c'(k) - U_c'(k)}{e''(s(k))} \geq 0 \quad \text{for} \ V_c'(k) - U_c'(k) \geq 0. \]

From (6) and (7) it follows that \( s'(k) \leq 0 \) globally if \( c(k; e) \geq c(k; u) \) for all \( k \), which is to say that search intensity increases when \( k \) falls. Furthermore, if \( k \) is decreasing over the unemployment spell, \( k_{+1}(k; u) \leq k \), this will yield positive duration dependence of the worker’s search intensity. In other words, the worker will search harder as a spell progresses. Notice that in this model search intensity of a risk neutral worker is constant over the duration of an unemployment spell.

### 4.1 Three Important Lemmas

Below it is shown that assumptions 1 and 2 are sufficient to establish that search effort is inversely related to wealth. To this end, we need to add a technical assumption to our description of the model.
above, namely that the period length is sufficiently small so that the transition probabilities between states will be small relative to unity. Specifically, it is assumed that $s(k) + \eta \leq 1$. Furthermore, we will be needing three lemmas. Lemma 1 establishes concavity of all value functions except $U(k)$, and Lemma 2 associates first derivatives across value functions. Both these lemmas are proven in the appendix.

**Lemma 1** The value functions $V_g(k)$ and $U_g(k)$ are concave, and $V(k)$, $U_c(k)$, and $V_c(k)$ are strictly concave.

**Lemma 2** $V_g'(k) = V'(k)$ and $U_g'(k) = U'(k^l(k)) = U'(k^h(k))$ for all $k$, where $(k^l(k), k^h(k))$ is an interior solution to the optimal lottery for a worker who was unemployed in the previous period.

The key to all of our results and the main difficulty with providing characterizations of savings and search behavior when individuals alternate between employment and unemployment is to establish that $V_c'(k) - U_c'(k) < 0$ for all $k$. Lemma 3 provides this result.

**Lemma 3** $V_c'(k) - U_c'(k) < 0$ for all $k$.

**Proof.** By concavity of $V(k)$ it follows that $V_g(k) = V(k)$. Thus it must be that:

\[
V_c(k) = \max_{k+i \in \Gamma_w(k)} \left[ u \left( (1 + r) k + w - k_{i+1} \right) + \frac{V(k_{i+1})}{1 + \rho} \right],
\]

\[
= \max_{k+i \in \Gamma_w(k)} \left[ u \left( (1 + r) k + w - k_{i+1} \right) + \frac{(1 - \eta) V_c(k_{i+1}) + \eta U_c(k_{i+1})}{1 + \rho} \right],
\]

where $\Gamma_i(k) = \left\{ \tilde{k} \in \mathbb{R} \mid \tilde{k} \leq \tilde{k} \leq \min \left[ (1 + r) k + i, \tilde{k} \right], i \in \{w, b\}$ . Similarly, $U_c(k)$ can be written in terms of $V_c$ and $U_c$. But in this case, one cannot disregard the lottery, so the expression is somewhat more complicated:

\[
U_c(k) = \max_{k+i \in \Gamma_h(k)} \left\{ u \left( (1 + r) k + b - k_{i+1} \right) + \frac{1}{1 + \rho} \max_{k^l, k^h} \left[ \frac{k^h - k_{i+1}}{k^h - k^l} \max_{s \in [0,1]} \left[ -e(s) + s V_c \left( k^l \right) + (1 - s) U_c(k^l) \right] \right] + \frac{k_{i+1} - k^l}{k^h - k^l} \max_{s \in [0,1]} \left[ -e(s) + s V_c \left( k^h \right) + (1 - s) U_c(k^h) \right] \right\}.
\]

Let $S$ be the set of all bounded, continuous functions. Then, (8) and (9) together define the mapping $T : S \times S \rightarrow S \times S$ or written explicitly, $(V_c, U_c)(k) = T(V_c, U_c)(k)$. Denote by
\[ T_V(V_c, U_c)(k) = V_c(k) \] is the first dimension of this mapping and similarly, \( T_U(V_c, U_c)(k) = U_c(k) \) the second dimension. It is readily seen that \( T \) maps \( S \times S \) into itself since the right hand sides of (8) and (9) are maximizations of bounded, continuous functions over compact sets. Thus, the solution to these maximization problems must exist and be continuous and bounded. The mapping \( T \) is furthermore easily verified as being a contraction mapping (see the Appendix). This immediately implies existence of a unique fix point \( (V_c^*, U_c^*) \). Also, the contraction mapping property of \( T \) implies that for some closed set \( S_1 \subseteq S \), if \( T(S_1) \subseteq S_2 \subseteq S_1 \), then \( (V_c^*, U_c^*) \in S_2 \).

In the following, it will be shown that \( T \) maps the closed set of functions \( S_1 \) defined by:

\[ S_1 = \{(V_c, U_c) \in S \times S \mid V_c'(k) - U_c'(k) \leq 0 \forall k\}, \]

into the set \( S_2 \) defined by:

\[ S_2 = \{(V_c, U_c) \in S \times S \mid V_c'(k) - U_c'(k) < 0 \forall k\}. \]

Thus, by the argument above it must be that the fix point of the mapping is characterized by \( V_c' - U_c' < 0 \).

The derivatives of the mapping are:

\[ T_V'(V_c, U_c)(k) = u'(c(k; e))(1 + r) \]
\[ = \frac{1 + r}{1 + \rho} [((1 - \eta)V_c'(k_{+1}(k; e)) + \eta U_c'(k_{+1}(k; e))] + (1 + r)\lambda_V(k) \quad (10) \]
\[ T_U'(V_c, U_c)(k) = u'(c(k; u))(1 + r) \]
\[ = \frac{1 + r}{1 + \rho} U_g'(k_{+1}(k; u)) + (1 + r)\lambda_U(k) \]
\[ = \frac{1 + r}{1 + \rho} \left[ s(k_u^l) V_c'(k_u^l) + \left(1 - s(k_u^l)\right) U_c'(k_u^l) \right] + (1 + r)\lambda_U(k), \quad (11) \]

where \( k_u^l \equiv k_{+1}(k; u) \).

Assume that \( V_c'(k) - U_c'(k) \leq 0 \) for all \( k \). It will then be shown that it must be that \( T_V'(V_c, U_c)(k) - T_U'(V_c, U_c)(k) < 0 \) for any \( k \in [\underline{k}, \bar{k}] \).

Consider any given wealth level \( k \in [\underline{k}, \bar{k}] \). Suppose that \( \lambda_V(k) > \lambda_U(k) \), which is to say that, contrary to intuition, a marginal relaxation of the lower wealth bound is more valuable in the employed than in the unemployed state. By non-negativity of the Lagrange multipliers it follows that \( \lambda_V(k) > 0 \) which implies that the lower wealth bound constraint must bind in the employed state, \( k_{+1}(k; e) = \underline{k} \). The lower wealth bound may or may not bind in the unemployed state,
Thus, the assumption of $\lambda_V (k) > \lambda_U (k)$ necessarily implies $k_{+1} (k; e) \leq k_{+1} (k; u)$ which by $w > b$ implies $c(k; e) > c(k; u)$. By (10) and (11) and by strict concavity of $u(\cdot)$ it immediately follows that $T'_V (V_c, U_c) (k) - T'_U (V_c, U_c) (k) < 0$. Generally, any case for which $k_{+1} (k; e) \leq k_{+1} (k; u)$ immediately yields $T'_V (V_c, U_c) (k) - T'_U (V_c, U_c) (k) < 0$.

Now, consider the alternative case of $k_{+1} (k; e) > k_{+1} (k; u)$ which by the above argument implies that $\lambda_V (k) \leq \lambda_U (k)$. Thus, it follows that $k^l_u \equiv k^l (k_{+1} (k; u)) \leq k_{+1} (k; u) < k_{+1} (k; e)$. Subtract (11) from (10):

$$T'_V (V_c, U_c) (k) - T'_U (V_c, U_c) (k) = T'_V (V_c, U_c) (k) - \frac{1 + r}{1 + \rho} \left[ s(k^l_u) V'_c (k^l_u) + (1 - s(k^l_u)) U'_c (k^l_u) \right] - (1 + r) \lambda_U (k)$$

$$< T'_V (V_c, U_c) (k) - \frac{1 + r}{1 + \rho} \left[ s(k^l_u) V'_c (k_{+1} (k; e)) + (1 - s(k^l_u)) U'_c (k_{+1} (k; e)) \right] - (1 + r) \lambda_U (k)$$

$$= \frac{1 + r}{1 + \rho} \left[ 1 - s(k^l_u) - \eta \right] \left[ V'_c (k_{+1} (k; e)) - U'_c (k_{+1} (k; e)) \right] + (1 + r) (\lambda_V (k) - \lambda_U (k))$$

$$\leq 0,$$

where the strict inequality follows from strict concavity of $V_c (\cdot)$ and $U_c (\cdot)$ and that $k^l_u < k_{+1} (k; e)$. The weak inequality follows from the assumption that the period length is sufficiently small so that $1 - s(k^l_u) - \eta \geq 0$, the assumption that $V'_c (k) - U'_c (k) \leq 0$ for all $k$ and finally that $\lambda_V (k) \leq \lambda_U (k)$.

Thus, it has been shown that for a sufficiently small period length, $T(S_1) \subseteq S_2$ and therefore the fix point of $T$ must be characterized by $V'_c (k) - U'_c (k) < 0$. ■

### 4.2 Wealthier Workers Search Less

With the results of lemmas 1-3 in hand, we can now characterize how search effort is affected by wealth by applying (7) and remembering that the effort function $e(s)$ is assumed to be strictly convex:

**Proposition 1** Search effort increases as wealth decreases; $s'(k) < 0$ for all $k$. 

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Thus, under assumption 1 and 2, the wealthier an unemployed worker is the less incentive she has to get back into employment. Since by assumption search costs are not affected by the level of wealth holdings, we obtain the result that wealthier workers will search less.

It is worthwhile noting that the special case of assumption 2 for which \( u(c) = \log(c) \) is consistent with a balanced growth steady state in the model if modified to allow for a positive growth trend in the income process and wealth bounds, see King, Plosser, and Rebelo (1988) for details.\(^{10}\) They also put forth a non-separable utility function class that is consistent with a balanced growth steady state in our search and savings model, \( v(c,s) = \frac{1}{1-\sigma} c^{1-\sigma} e(1-s) \), where \( e(\cdot) \) is increasing and concave for \( \sigma < 1 \) and decreasing and convex for \( \sigma > 1 \). We conjecture that this class of utility function will yield a monotonically decreasing relationship between wealth and search intensity in the case of \( \sigma < 1 \).\(^{11}\)

5 Consumption Smoothing, Self Insurance, and Duration Dependence

This section shifts the focus to the savings choice and the implied relationship between unemployment spell duration and the unemployment hazard rate. We also consider the role of savings as self-insurance against income fluctuations.

If wealth is steadily reduced as an unemployment spell progresses and there is a negative relationship between wealth and search, then search effort will be increasing in the duration of the spell. This is referred to as positive duration dependence. While it is a very intuitive result, it should be noted that it is a direct extension of the negative relationship between search intensity and wealth that was established in Proposition 1. In fact, the duration dependence results can be reversed by having the utility and search cost structure chosen accordingly. This is discussed in more detail below.

Again we continue the analysis under assumption 1 and 2. Beginning with the consumption decisions, it is seen by the following proposition that savings indeed provide insurance against income fluctuations but that the insurance is imperfect:

\(^{10}\)While their model describes a consumption-labor supply choice, the basic methodology translates in a straightforward manner.

\(^{11}\)The cross-derivative \( v'_{cs}(c,s) \) is such that marginal search costs are increasing in wealth for \( \sigma < 1 \). Combined with the result that gains to search are decreasing in wealth, one would expect an overall negative effect on search from an increase in wealth.
Proposition 2  For all $k$, consumption increases as wealth increases, $c' (k; e) > 0$ and $c' (k; u) > 0$, and consumption when employed is strictly greater than consumption when unemployed, $c(k; e) > c(k; u)$. Furthermore, for all $k > k$ it must be that $b + rk < c(k; u) < c(k; e)$ and $k+1(k; u) < k$.

Proof. Lemma 1 and the envelope conditions (6) immediately yield that consumption must be increasing in wealth both when employed and unemployed. The conclusion that $c(k; u) < c(k; e)$ follows directly from lemma 3 and the envelope conditions (6).

The result that $b+rk < c(k; u)$ follows from the fact that for $r < \rho$ it must be that $k+1(k; u) < k$ for all $k > k$. This is seen from the following argument: In Lemma 2 it was established that for an interior solution to the lottery, $U_g(k+1(k; u)) = U'(k^h(k+1(k; u)))$. Now let $k^h_u \equiv k^h(k+1(k; u))$.

By the first order and envelope conditions, it then follows that:

$$U'_c(k) = \frac{1 + r}{1 + \rho} \left[ s \left( k^h_u \right) V'_c \left( k^h_u \right) + \left( 1 - s \left( k^h_u \right) \right) U'_c \left( k^h_u \right) \right] + (1 + r) \lambda_U(k). \quad (12)$$

By the period length being sufficiently small, it is given in Lemma 3 that $V'_c(k) - U'_c(k) < 0$ for all $k$. Re-writing (12), it follows that

$$U'_c(k) - \frac{1 + r}{1 + \rho} U'_c \left( k^h_u \right) - (1 + r) \lambda_U(k) = \frac{1 + r}{1 + \rho} s \left( k^h_u \right) \left[ V'_c \left( k^h_u \right) - U'_c \left( k^h_u \right) \right]. \quad (13)$$

Thus, it must be that the right hand side of (13) is strictly negative. Now suppose, contrary to the claim, that $k+1(k; u) \geq k$. This first of all implies that $\lambda_U(k) = 0$ since the lower bound must not have been binding. Furthermore, it implies that $k^h_u \geq k$. But since $U'_c(\cdot)$ is strictly concave and $r < \rho$ this must mean that the left hand side of (13) is positive, yielding a contradiction of the inequality. This establishes that $b + rk < c(k; u) < c(k; e)$ when $k > k$. If $k = k$, an unemployed worker is no longer able to insure against the low income state and will simply consume the income $b + rk$.

Hence, for $r < \rho$ unemployed workers always dis-save as long as their wealth is above the minimum level, $k$. The employed worker faces two opposing savings motives: Since $r < \rho$ there is a negative real return on savings and thus the speculative savings motive dictates that savings be reduced. However, the precautionary savings motive dictates an increase in savings during employment to insure against low income shocks. For low wealth levels, the precautionary motive dominates and for sufficiently high wealth levels, the speculative savings motive eventually begins to dominate. In the special case where $r = \rho$, only the precautionary motive manifests itself and by an argument analogous to the proof of Proposition 2, one can show that $k+1(k; e) \geq k$ for all $k$. 


Proposition 2 shows that savings do allow for some insurance against income fluctuations but that the insurance is imperfect since the consumption path is not perfectly smooth over states. This is because transitions between income states are never fully anticipated and the income process is a jump process.

As a direct extension of the arguments made in the proof of proposition 2, it can be shown that the search intensity of an unemployed worker will exhibit positive duration dependence throughout the unemployment spell. This result follows from the basic relationship between the choice of search intensity and the worker’s wealth as established in proposition 1.

**Proposition 3** The worker’s search intensity exhibits positive duration dependence.

**Proof.** The result follows directly from two observations. 1) $k_{t+1}(k;u) < k$, that is, an unemployed worker will monotonically decrease wealth. This was shown in the proof of Proposition 2. 2) In Proposition 1, it was established that the search intensity is decreasing in the worker’s wealth. Thus, it must be that the search intensity increases as the unemployment spell progresses to the point where the worker’s wealth reaches the lower bound. After this point, the search intensity remains constant.

For an interest rate less than the worker’s rate of time preferences, we have shown that an unemployed worker will monotonically decrease wealth and as such, the search intensity will strictly increase during spells of unemployment, up to the point where the lower bound on wealth has been reached.

6 Conclusion

In this paper we study a risk averse worker who moves back and forth between employment and unemployment and thus faces a joint consumption smoothing and job search (leisure smoothing) problem. A main insight of the paper is that both of the choices are affected in fundamental ways by allowing the analysis to treat them as interrelated problems. Under the assumption that utility is additively separable and that wealth is bounded we then derive a number of results regarding the worker’s search and consumption choices. Some of these are quite intuitive and reassuring as to the usefulness of our modelling approach. For instance, consumption increases with wealth both when the worker is employed and unemployed. Also, it is found that precautionary savings
are built up during employment spells and run down during unemployment spells. Furthermore, our result that savings will not smooth consumption perfectly over states and time extend similar results in Deaton (1991) to also hold in the case of endogenous income processes, here coming from endogenous search. The fact that insurance is less than perfect suggests that there may be room for welfare improving labor market policies such as unemployment benefit programs and the like.

Our main result is to identify conditions under which the worker’s search effort choice and thus her re-employment prospects are inversely related to wealth, so that, an unemployed worker is expected to search less the higher is her wealth. This implies for instance, that as an unemployment spell stretches out and savings are reduced, the probability that the individual will find a job increases. We conjecture that the results of our analysis can be carried over directly to conclusions about reservation wages instead of search intensities. The analysis in Gomes, Greenwood, and Rebelo (2001) suggests that this might very well be true.

If one is interested in the savings decision from an insurance point of view, the case where the worker need not worry about insurance once employment is found (i.e. $\eta = 0$) is of course of limited interest. So even though many of the analytical complications in the analysis above follow from the assumption that $\eta > 0$, it is crucial in this respect. Several recent papers deal with the insurance aspect, for instance, Wang and Williamson (1999), Acemoglu and Shimer (1999), and Lentz (2002) where optimal unemployment insurance schemes are derived in models where the workers are insured against unemployment via both their own savings as well as unemployment benefits. A similar problem is analyzed in Pissarides (2000) where instead of unemployment insurance, forms of employment protection such as severance pay and advance notice of job termination are the insurance mechanisms to supplement workers’ own savings. However, due to the analytical difficulties with the setup, these papers obtain analytical results only by assuming away the wealth effects on the search decisions or simply obtain results numerically instead. The present paper should facilitate further analytical work on search models with job separation and where wealth is allowed to affect search behavior.
7 Appendix

Proof of Lemma 1. The value function $V_g (k)$ is the concavification of $V (k)$. Formally, let $\text{epi}(V)$ be the epigraph of $V (k)$, i.e., $\text{epi}(V) = \{(k, V) \in [k, \bar{k}] \times \mathbb{R} | V \leq V (k)\}$, and let $\text{conv}(\text{epi}(V))$ be its convex hull. The concavification of $V (k)$ is then simply the upper bound of the convex hull of $V$’s epigraph, i.e., the sup $\{V \in \mathbb{R} | (k, V) \in \text{conv}(\text{epi}(V))\}$ which is exactly $V_g (k)$. Similarly, $U_g (k)$ is the concavification of $U(k)$. By concavity of $V_g$ and $U_g$ and by strict concavity of $u (c)$ it follows that $V_c (k)$ and $U_c (k)$ are strictly concave. To see this for $V_c (k)$, choose some $k_0, k_1 \in [k, \bar{k}]$. Denote $\hat{k}_0 \equiv k_{i+1} (k_0; c)$ and $\hat{k}_1 \equiv k_{i+1} (k_1; c)$. Furthermore, define $k_\lambda \equiv \lambda k_0 + (1 - \lambda) k_1$ and $\hat{k}_\lambda \equiv \lambda \hat{k}_0 + (1 - \lambda) \hat{k}_1$ for some $\lambda \in [0, 1]$. Note that $\hat{k}_\lambda$ is not necessarily the optimal choice of next period’s wealth given $k_\lambda$. Furthermore, it must be that $\hat{k}_\lambda$ is in the feasible set of choices of next period’s wealth levels given $k_\lambda$. To see this, note that $\hat{k}_0 \leq \min \left[(1 + r) k_0 + w, \bar{k}\right]$ and $\hat{k}_1 \leq \min \left[(1 + r) k_1 + w, \bar{k}\right]$. Hence, it must be that $\hat{k}_\lambda \leq \min \left[(1 + r) k_\lambda + w, \bar{k}\right]$. To show strict concavity, one must show that for all $k_0, k_1 \in [k, \bar{k}]$, it must be that $V_c (k_\lambda) > \lambda V_c (k_0) + (1 - \lambda) V_c (k_1)$. Thus, we get the following,

$$
\lambda V_c (k_0) + (1 - \lambda) V_c (k_1) = \lambda \left[u \left( (1 + r) k_0 + w - \hat{k}_0 \right) + \frac{V_g (\hat{k}_0)}{1 + \rho} \right] + (1 - \lambda) \left[u \left( (1 + r) k_1 + w - \hat{k}_1 \right) + \frac{V_g (\hat{k}_1)}{1 + \rho} \right]
$$

$$
< u \left( (1 + r) k_\lambda + w - \hat{k}_\lambda \right) + \frac{V_g (\hat{k}_\lambda)}{1 + \rho} \leq V_c (k_\lambda),
$$

where the strict inequality follows from strict concavity of $u (\cdot)$ and concavity of $V_g (\cdot)$. The weak inequality follows from optimality. Thus, it must be that $V_c (\cdot)$ is strictly concave. A similar argument applies to $U_c (\cdot)$. $V (\cdot)$ is a convex combination of two strictly concave functions. Consequently, $V (k)$ must be strictly concave.

Note that $U (k)$ need not be concave. By the envelope theorem,

$$
U' (k) = s (k) V_c' (k) + (1 - s (k)) U_c' (k).
$$

This implies that:

$$
U'' (k) = s (k) V_c'' (k) + (1 - s (k)) U_c'' (k) + s' (k) \left[V_c' (k) - U_c' (k)\right].
$$
By the first order condition of the optimal search choice, $e'(s(k)) = V(k) - U(k)$, it follows that:

$$U''(k) = s(k) V''(k) + (1 - s(k)) U''(k) + \frac{[V'_c(k) - U'_c(k)]^2}{e''(s(k))}. \quad (14)$$

The first two terms on the right hand side of (14) are negative but the last term is positive. Thus, concavity of $U(\cdot)$ does not follow directly. It should be noted that simulations of this model suggest that the last term rarely dominates the two negative terms and as such, $U(\cdot)$ is concave for these simulations. In this case, lotteries are never used. However, as an analytical result one may have to allow for lotteries in the case where $U(\cdot)$ is not concave.

**Proof of Lemma 2.** First we have that $V'(k) = V(k)$ for all $k$, since $V(k)$ is concave by Lemma 1, and thus $V'_g(k) = V'(k)$ for all $k$. Turning to $U'_g(k)$ note that for interior choices of $k^l$ and $k^h$ it must be that the optimal solution is characterized by $U'(k^l) = U''(k^h)$. To see this suppose that $U'(k^l) < U'(k^h)$. The original problem is slightly changed to make $k^h$ and $\alpha$ control variables. This is equivalent to the original problem for $k^l = (k - (1 - \alpha) k^h) / \alpha$. Then $U_g(k)$ can be written as

$$U_g(k) = \alpha U \left( \frac{k - (1 - \alpha) k^h}{\alpha} \right) + (1 - \alpha) U \left( k^h \right). \quad (15)$$

Now consider a slight increase in $k^h$ holding $\alpha$ fixed:

$$\frac{\partial U_g(k)}{\partial k^h} = -\alpha U' \left( k^l \right) \frac{1 - \alpha}{\alpha} + (1 - \alpha) U' \left( k^h \right) > 0$$

Thus, it cannot be that $U'(k^l) < U'(k^h)$ in the optimal solution. A similar argument applies to $U'(k^l) > U'(k^h)$. Thus, for interior choices of $k^l$ and $k^h$, the optimal choice of lottery must imply that $U'(k^l) = U'(k^h)$. Now, turn to the formulation of the problem where $k^l$ and $k^h$ are the control variables. The first order condition associated with the choice of $k^l$ having been unemployed in the previous period is given by:

$$U'(k^l) k^h - k - \frac{1}{k^h - k^l} U \left( k^h \right) - \frac{k^h - k}{(k^h - k^l)^2} U \left( k^l \right) - \frac{k - k^l}{(k^h - k^l)^2} U \left( k^h \right) = 0 \quad \hat{=} \quad \uparrow$$

$$U'(k^l) \left( k^h - k \right) - U \left( k^h \right) - \frac{k^h - k}{k^h - k^l} U \left( k^l \right) - \frac{k - k^l}{k^h - k^l} U \left( k^h \right) = 0.$$
Via the envelope condition (4) this can be re-written as,

\[ U'(k^l) (k^h - k) - U(k^h) = \frac{k^h}{k^h - k^l} U(k^l) - \frac{k^l}{k^h - k^l} U(k^h) + k \frac{U(k^h) - U(k^l)}{k^h - k^l} \]

\[ = \frac{k^h}{k^h - k^l} U(k^l) - \frac{k^l}{k^h - k^l} U(k^h) + k U'_g(k). \]  

(16)

The first order condition with respect to the choice of \( k^h \) is given by:

\[ \frac{k^h}{k^h - k^l} U'(k^h) + \frac{1}{k^h - k^l} U(k^l) + \frac{k^h - k}{(k^h - k^l)^2} U(k^l) + \frac{k^l - k}{(k^h - k^l)^2} U(k^h) = 0 \]

\[ \uparrow \]

\[ U'(k^h) (k - k^l) + U(k^l) + \frac{k^h - k}{k^h - k^l} U(k^l) + \frac{k^l - k}{k^h - k^l} U(k^h) = 0. \]

Again using the envelope condition (4) this can be expressed as:

\[ U'(k^h) (k - k^l) + U(k^l) + \frac{k^h - k}{k^h - k^l} U(k^l) - \frac{k^l}{k^h - k^l} U(k^h) + k U'_g(k) = 0. \]

(17)

Inserting this into (16), one gets:

\[ U'(k^l) (k^h - k) - U(k^h) - k U'_g(k) = -U'(k^h) (k - k^l) - U(k^l) - k U'_g(k) \]

\[ \uparrow \]

\[ U'(k^l) (k^h - k) + U'(k^h) (k - k^l) = U(k^h) - U(k^l). \]

Using the result from above that \( U'(k^l) = U'(k^h) \) it follows that

\[ U'(k^l) = \frac{U(k^h) - U(k^l)}{k^h - k^l} = U'_g(k), \]

which then establishes the second part of the lemma. 

**Proof that \( T \) is a contraction (Lemma 3).** One way of establishing this is by appealing to Blackwell’s two sufficient conditions for a contraction mapping; monotonicity and discounting.\(^{12}\)

First, monotonicity is established: Choose some \( V_1^c(k) \geq V_2^c(k) \) and \( U_1^c(k) \geq U_2^c(k) \) for all \( k \). Then Blackwell’s sufficient conditions state that it must be that \( T(V_1^c, U_1^c)(k) \geq T(V_2^c, U_2^c)(k) \) for all \( k \). By examination of (8) and (9) this is seen to be trivially satisfied. The discounting condition states that it must be that \( T(V_c + \lambda, U_c + \lambda)(k) \leq T(V_c, U_c)(k) + \lambda \beta \), for some \( 0 < \beta < 1 \) and some \( \lambda \geq 0 \). It is seen from (8) that \( T_{V}(V_c + \lambda, U_c + \lambda)(k) = T_{V}(V_c, U_c)(k) + \lambda / (1 + \rho) \). Also, it follows from (9) that \( T_{U}(V_c + \lambda, U_c + \lambda)(k) = T_{U}(V_c, U_c)(k) + \lambda / (1 + \rho) \). Thus, discounting is satisfied for \( \rho > 0 \), which, by definition, is given. Therefore, it has been established by Blackwell’s sufficient conditions that \( T(V_c, U_c) \) is a contraction mapping. 

\(^{12}\)For a proof of Blackwell’s sufficient conditions see for example Stokey and Lucas (1989).
References


