Advertising and price signaling of quality
in a duopoly with endogenous locations*

Philippe Bontems† Valérie Meunier‡

May 2, 2005

Abstract
This paper considers a horizontal product differentiation duopoly model in a linear city, where firms are also differentiated with respect to the quality of their products. Firms first choose their location and then simultaneously compete in prices. At the price setting stage, they may also make expenditures for purely dissipative advertising. At the location stage, there exists an uncertainty about the quality produced by each of the duopolists, quality that will be privately revealed to them before the price competition and advertising stage. We show that signaling high quality through price only is impossible whatever the degree of vertical differentiation. We then show that separating equilibria exist where only the high-quality firm advertises. An equilibrium refinement produces two kinds of separating equilibria depending on the degree of vertical differentiation and cost difference. Horizontal differentiation is minimal only when the high-quality firm is able to get a monopoly position, otherwise differentiation is maximal with an upward distortion in both prices compared to symmetric information. We also show that pooling equilibria exist but do not survive to the imposition of intuitive beliefs. Last, we get the result that asymmetric information enlarges the set of parameters for which maximal differentiation is obtained.

JEL Codes: D43, L15.

Key-words: advertising, location choice; quality; incomplete information; multi-sender signaling.

*Address for correspondence: INRA, Université des Sciences Sociales, Manufacture des Tabacs, 21 Allée de Brienne, 31000 Toulouse, France.
†University of Toulouse (INRA, IDEI): bontems@toulouse.inra.fr
‡University of Aarhus: vmeunier@econ.au.dk
1 Introduction

Markets for experience goods are characterized by the uncertainty consumers face regarding the utility they will get from purchasing and consuming these goods. In many situations, consumers do not observe the objective quality of the commodity they intend to buy. In order to avoid the standard “lemons” problem, emphasized by Akerlof (1970), producers may find incentives to transmit some information to consumers. Since the pioneering work of Spence (1973), an extensive literature has looked at the various strategies firms can employ to reveal information about their products’ quality. In particular, in their pathbreaking paper, Milgrom and Roberts (1986) show how a monopolist can signal the high quality of a new experience good by distorting his strategy (price, advertising intensity) compared to the benchmark situation of complete information.\(^1\) Hence, signaling a higher quality is costly for the monopolist because distorting strategies implies a profit reduction.

However, the literature we refer to in analyzing a price signaling game has almost exclusively focused on the monopoly case, where a producer with private information about his high quality only “compete” with his “ghost” of low quality. To our knowledge, very few papers are studying a similar price-signaling game in a competitive environment. Among these few, we can cite Kihlstrom and Riordan (1984), Bagwell (1990), Fluet and Garella (2002), Hertzendorf and Overgaard (1998, 2001a, 2001b, 2002). As Hertzendorf and Overgaard (1998) mention, this emphasis on the monopoly case does not rely on the belief that monopolized markets are more realistic, or common, neither does it stem from the intuition that a monopoly is more likely to signal its type. The lack of work on oligopolistic markets reflects the absence of a model, or a game, as tractable and coherent as the monopoly game. Each of the papers treating the oligopoly case studies a particular setting, in order to keep a balance between introducing competition and ensure tractability.

In this paper, we contribute to this emerging literature by considering the issue of signaling quality in a duopolistic setting. More precisely, we study the impact of (horizontal) product differentiation on the existence and features of separating and pooling equilibria. We analyze a model where both firms choose their locations (variety) before competing in price and dissipative

\(^1\)See also Bagwell and Riordan (1991).
advertising. One of the duopolists will have a quality advantage, which is exogenously selected by Nature, but the identity of the better quality product will remain unknown to consumers before actual consumption. An important feature of our model is that the duopolists will have to choose their locations under uncertainty, that is before knowing which quality they will be able to produce. Hence, only the probability distribution of quality is known at the location-choice stage.

This assumption about the timing can be justified as follows: consider a market on which two firms enter. They have to make a long-term decision as to the variety of their goods (for example, one can think of a choice of ingredients, components, or marketing or distribution channels) before knowing their exact quality. Then which firm produces which good is exogenously determined (weather determines the quality of grape used to process wine, and does not affect all regions identically, or a recipe ends up yielding a better quality...). Finally, both firms simultaneously compete in price. Consumers perfectly observe locations and prices but not qualities. Note that our assumption that quality is revealed to firms only after locations are chosen allows to rule out the possibility that duopolists may use location as a signal of their types.2

We believe that this framework illustrates some stylized facts observed in several sectors. For instance, quality is often variable for many food commodities as it is linked to the quality of raw inputs which is largely influenced by exogenous factors such as climatic ones (seasonal quality of fresh inputs such as fruits). Nevertheless, food processors have to take observable long term decisions such as packaging, content per unit, the type of retailing channel (supermarkets versus specialized retailers), before processing the product. To further motivate our timing, consider also the often invoked example of restaurants with no established reputation in a tourist resort. Restaurant owners might have to choose the style of cuisine (Italian versus French cuisine) before hiring the chef whose ability will determine the quality of food. We argue that this commitment to a specific variety, i.e. a choice of location, is a chance for a firm to benefit from a specific market (a niche), on which it might benefit from enough market power to be able to signal

---

2This alternative timing where duopolists may signal their type through two instruments, location and price is studied by Vettas (1999). Vettas shows that a high-quality firm will signal its type by choosing to locate closer to its rival (relative to the complete information benchmark).
through prices the true quality of the product to consumers. Nonetheless, at the time the firm chooses a variety or location, it does not know what the actual quality of his commodity will be at the final stage.

It is worth noting that recently Hertzendorf and Overgaard (2001a) (hereafter H&O) showed that signaling quality is possible in a duopolistic setting. However, their assumptions are significantly different from ours: first, both firms may signal quality through advertising expenditures as well as prices, and second goods are only vertically differentiated so that there is no location stage before the price competition game. But more importantly, H&O introduce consumers’ heterogeneity through the marginal willingness to pay (MWTP) for quality while in our model consumers are homogenous with respect to their MWTP for quality but they differ according to their ideal product in terms of location. Finally, a common assumption of H&O’s paper and ours is that consumers a priori know that the two goods on the market are vertically differentiated, and so they only have to infer from signals which firm is producing high quality.\textsuperscript{3}

In the complementary setting where consumers do not know a priori whether the offered goods are heterogeneous or homogeneous, Hertzendorf and Overgaard ((2001b), (2002)), using price signaling only, show that fully revealing equilibria no longer survive their equilibrium refinement. In a very close framework, Fluet and Garella (2002) demonstrate that advertising is necessary to obtain separation between high and low quality when quality differentials is small, but contrary to H&O, they do not use any selection criteria.\textsuperscript{4}

Our main results are as follows. We first prove that any separating equilibria involves strictly positive advertising expenditures by the high-quality firm, whatever the quality differential. Only one separating equilibrium resists the application of equilibrium refinement. It is characterized by an upward distortion in both low- and high-quality prices compared to symmetric information. Second, all pooling equilibria are pruned when imposing an appropriate version of Cho and Kreps (1987) intuitive criterion. This equilibrium selection allows to determine the equilibrium location choices. Maximal or minimal differentiation occurs depending on parameters. When there is

\textsuperscript{3}It is worth noting that Gabszewicz and Grilo (1993) studied price competition in a duopoly when consumers are uncertain about which firm sells which quality. A significant difference with our paper and H&O’s is that Gabszewicz and Grilo assume that consumers’ beliefs are exogenous, such that no inference from prices can be considered.

\textsuperscript{4}Contrary to H&O, Fluet and Garella (2002) assume that consumers differ according to their MWTP for the high quality only.
minimal differentiation, only the high-quality firm is active on the market. We also show that under asymmetric information it is possible that the most profitable firm ex post is the low-quality provider while under symmetric information the high quality firm earns always more than its competitor.

Incidentally, we also provide some insights on the impact of asymmetric information on horizontal differentiation. On this subject, Boyer et al. (1994, 1995), Bester (1998) and Vettes (1999) all conclude that incomplete information about the type of a duopolist yields less horizontal differentiation. On the contrary, in our setting, the likelihood of observing maximum differentiation increases under incomplete information. It then appears that asymmetric information allows the low-quality firm to better resist an increasingly efficient competitor. Note however that in the papers cited above, location is distorted in order to signal the type, which is not the case in our framework.

The remainder of the paper is organized as follows. Section 2 presents the model and Section 3 is devoted to the benchmark situation of complete information. Section 4 introduces asymmetric information and examines the interactions between location choices and signaling through price and advertising. Section 5 concludes.

2 The Model

Consider a continuum of consumers whose locations are uniformly distributed over the unit interval $[0, 1]$. Two firms, labeled by $i = 1, 2$, choose a location $y_i \in [0, 1]$. Without loss of generality, we assume that $y_1 \leq y_2$.

A consumer located in $x \in [0, 1]$ gets a utility from purchasing a unit of good $i$ that is specified as follows:

$$U(p_i, q_i) = R + q_i - p_i - t(x - y_i)^2, \text{ for } i = 1, 2,$$

where $R > 0$ is the basic utility obtained by a consumer purchasing any of the two goods, $q_i$ and $p_i$ represent respectively the quality and the price of the good. A consumer’s transportation cost of visiting a firm located in $y_i$ is quadratic, and the parameter $t$ reflects the degree of horizontal
product differentiation.\textsuperscript{5} We assume that \(R\) is sufficiently large that consumers always choose to purchase from Firm 1 or Firm 2. Notice that we also assume perfect homogeneity of consumers with respect to their valuation of quality.

We assume that \(q_1 \neq q_2\): the goods are exogenously vertically differentiated. For the sake of simplicity, we will suppose that quality can only take one of two values, \(L\) or \(H\). Therefore, we will have \(q_i = L\) and \(q_j = H\), where \(L\) and \(H\) stand for low and high quality respectively. We will denote \(\Delta \equiv H - L\), the quality discrepancy.

Both firms own the same production technology with constant marginal cost of production \(c(q_i)\), and we assume that this cost is increasing with quality. Indeed, producing high quality requires more inputs, or inputs of better quality that are more costly than the ones necessary to produce low quality. We normalize the unit cost of low quality \(c(L)\) to 0 and we denote the unit cost of high quality by \(c(H) = c > 0\). Assume that \(\Delta > c\), which basically means that the production of high quality is socially valuable. Otherwise the high-quality producer would have no demand to satisfy even when pricing at marginal cost.

We analyze the following game:

• In a first stage, firms choose their respective locations, still being uncertain of their respective type.

• Then, in a second stage, they learn the quality of the good they will produce.

• Finally, they compete in price and dissipative advertising, and consumers decide which good to purchase.

The choice of location is made under uncertainty as to the quality of the good. In stage 2, only firms learn their respective types. Consumers will not, but they know that products are vertically differentiated. In others words, consumers perfectly observe locations but remain uninformed about which brand offers the highest quality. The question is whether firms will manage to signal their types in stage 3.

There is no reason to believe a priori that one firm is more likely to produce high quality, hence we will assume that Nature draws firms’ types from a prior, commonly known, probability

\textsuperscript{5}This specification of the utility function is widely retained in the literature. See for instance Bester (1998), Vettas (1999).
distribution \( \text{Prob} (q_i = H) = 1/2 \). Since firms are \textit{ex ante} symmetric, and consumers’ preferences follow a symmetric density, incentives to horizontally differentiate will be symmetric. This implies that locations will be symmetric.\(^6\) It also means that the identity of the firm (that is, whether we are considering firm 1 or firm 2) is irrelevant when we look for firms’ optimal strategies. What matters is the quality each firm produces. Therefore, we will write \( y_1 = 1 - y_2 \equiv y \), and \( p_L \) and \( p_H \) for locations and prices of the low- and high-quality firm, respectively.

3 Benchmark case: symmetric information

This section proposes a solution to the game under symmetric information. It has two purposes. One is to introduce some notations and to derive some results that will turn out to be useful later in the paper, when solving the game under incomplete information. The other one is to show what are the forces driving firms to choose their locations and prices.

We solve the game by backward induction, starting with the last stage of the game, where firms simultaneously compete in price and advertising, for given locations \( y \) and \( 1 - y \). Here, firms as well as consumers observe the actual quality of the two goods. (Note that since advertising is purely dissipative, it has no effect on demand, such that firms will optimally choose not to advertise.) Then in a second step we will solve the location stage, where both firms rationally anticipate equilibrium prices and choose locations, not knowing their types. We analyze the case where Firm 1 produces low quality and Firm 2 high quality. As the game is symmetric, we can denote \( p_L \) and \( p_H \) the respective prices of the goods of low and high quality.

3.1 Price equilibria

Demand functions depend on the location of the marginal consumer who is indifferent between the two products. We assume that parameter values are such that, in equilibrium, the market is entirely covered. We denote \( z^B (p_L, p_H) \in (0, 1) \) the location of the marginal consumer.\(^7\) Its

\(^6\)See for example Anderson, Goeree and Ramer (1997).

\(^7\)In what follows, the superscript \( B \) denotes benchmark values of all variables.
identity is given by

\[ L - p_L - t(z^B - y)^2 = H - p_H - t(z^B - 1 + y)^2. \]

Solving this equation in \( z^B \) gives

\[ z^B(p_L, p_H) = \begin{cases} 
0 & \text{if } p_L \geq p_H - \Delta + td \\
1 & \text{if } p_L \leq p_H - \Delta - td \\
\frac{1}{2} + \frac{p_H - p_L - \Delta}{2td} & \text{otherwise.} 
\end{cases} \tag{2} \]

where \( d \equiv 1 - 2y \) denotes the distance between the two firms.

We denote profit functions \( \pi_L^B = p_L z^B \) and \( \pi_H^B = (p_H - c)(1 - z^B) \). Looking for a Nash equilibrium in price, we first obtain each firm’s best response to its competitor’s price:

\[ p_L^B(p_H) = \frac{1}{2} [p_H + td - \Delta], \]
\[ p_H^B(p_L) = \frac{1}{2} [p_L + td + \Delta + c]. \]

For further reference, we introduce the following notation:

\[ \rho \equiv \frac{\Delta - c}{t}. \]

We find that equilibrium prices are

\[ p_L^B(d) = \begin{cases} 
0 & \text{if } d \leq \frac{\rho}{3}, \\
\frac{1}{3} [3td - \Delta + c] & \text{otherwise,} 
\end{cases} \tag{3} \]

and

\[ p_H^B(d) = \begin{cases} 
\Delta - td & \text{if } d \leq \frac{\rho}{3}, \\
\frac{1}{3} [3td + \Delta + 2c] & \text{otherwise.} 
\end{cases} \]

The condition on \( d \) means that if firms are too close to each other (the distance between them is too small) the low-quality firm will not have a positive market share. In this case, the high-quality firm charges a limit price and gets the entire demand. Note that there is a market for
the low quality good if \( d > \frac{\rho}{3} \), which can happen only if \( \rho < 3 \).

Note also that if qualities offered were identical, \( \Delta = 0 \) and \( c = 0 \), then price competition would correspond here to the one in a standard Hotelling setting with quadratic transportation costs and homogeneous products (see D’Aspremont et al. (1979)). When offered qualities differ (i.e. \( \Delta > 0 \), \( c > 0 \)), each firm obtains a strictly positive demand only when firms are located sufficiently far apart. If firms are too close to each other, such that \( d < \frac{\rho}{3} \), the high-quality firm gets the whole demand. In particular if \( d = 0 \), then \( p^H_L = 0 \) and \( p^H_H = \Delta \), and the low-quality firm is excluded. This is the standard outcome in a duopoly where differentiation is only vertical and consumers have homogenous taste for quality.

Equilibrium profits, for given locations, are:

\[
\pi_L(d) = \begin{cases} 
\frac{t}{18d} \left(3d - \rho\right)^2 & \text{if } d > \frac{\rho}{3}, \\
0 & \text{otherwise}, 
\end{cases}
\]

(4)

and

\[
\pi_H(d) = \begin{cases} 
\frac{t}{18d} \left(3d + \rho\right)^2 & \text{if } d > \frac{\rho}{3}, \\
t(\rho - d) & \text{otherwise.}
\end{cases}
\]

(5)

### 3.2 Location equilibria

We now turn to the first stage of the game, where firms choose their locations while still uncertain about the quality of the good they will produce later. A firm’s expected profit, evaluated at the beginning of the game, is

\[
E\Pi(d) = \frac{1}{2} \pi_L(d) + \frac{1}{2} \pi_H(d)
\]

An equilibrium where both firms are active is \( d > \rho/3 \) maximizing \( E\Pi(d) \). Using (4) and (5), we have

\[
E\Pi(d) = \frac{1}{2} \left[ \frac{t}{18d} (3d - \rho)^2 + \frac{t}{18d} (3d + \rho)^2 \right] = \frac{t}{18d} (9d^2 + \rho^2),
\]

(6)

Differentiating (6) with respect to \( d \) yields

\[
\frac{dE\Pi(d)}{dd} = \frac{1}{2} \left(1 - \frac{\rho^2}{9d^2}\right).
\]

9
Note that for $d > \rho/3$, $\frac{d\Pi(d)}{dd} > 0$: firms have incentives to maximize distance between them. Conversely, for $d \leq \rho/3$, $\Pi(d)$ decreases in $d$ until $d = 0$. We thus have to compare $\Pi(1)$ with $\Pi(0)$:

$$\Pi(1) = \frac{t}{18}(9 + \rho^2) \geq \Pi(0) = \frac{t}{2} \rho \iff \rho \leq \frac{9 - 3\sqrt{5}}{2} \approx 1.146$$

Therefore, we can state the following result:

**Proposition 1** The outcome of the game under symmetric information involves either maximal differentiation or minimal differentiation depending on the value of $\rho$. More precisely,

(i) for $\rho \in [0, 1.146]$, there is maximum horizontal differentiation ($d = 1$) and both firms are active and get strictly positive profits,

(ii) for $\rho > 1.146$, there is minimum horizontal differentiation ($d = 0$) and only the high-quality firm is active.

The values of equilibrium prices and profits are contained in Table 1. Ex post, firms are asymmetric: one of them will have the advantage of producing higher quality. Nevertheless, the uncertainty concerning the quality of the good they will produce implies that firms are *ex ante* symmetric, such that they have identical incentives to differentiate. The parameter $\rho \equiv \frac{\Delta - c}{t}$ indicates a relative quality advantage: the higher it is, the more likely firms will want to take the chance of locating in the middle of the market, with the hope of eliminating competition at the final stage.
4 Signaling quality

Back to the incomplete information framework we are interested in, recall that consumers do not ascertain quality before purchase. At the last stage of the game, consumers have to choose between commodities offered by firm 1 and firm 2. They observe locations and are aware these were chosen before firms learned their types, and they know that goods are of different qualities. But consumers do not know which firm produces which quality. Their prior belief about firm 1 producing high quality is \( \mu_0 = 1/2 \). The question is whether firms can signal their types, and if so, how the possibility of signaling at the last stage of the game influences their previous choice of locations.

4.1 Strategies, beliefs, and equilibrium definition

Firms first choose locations, before they learn their types. Each firm knows it will produce high quality with probability 1/2. Then firms privately learn their types. They each offer a price for their good and simultaneously determine their expenditures in dissipative advertising. Finally, consumers, observing these variables, try to infer some information about which firm produces which quality and make their choice. Strategies for firms are:

**Location** Firm 1 chooses \( y_1 = y \) and Firm 2 \( y_2 = 1 - y \).

**Price and advertising expenditures** Each firm \( i, i = 1, 2 \), can have one of two types. Firm \( i \)'s strategy is a pair \((p_{iL}, p_{iH}, a_{iL}, a_{iH})\). Since the game is symmetric, \((p_{iL}, p_{iH}, a_{iL}, a_{iH}) = (p_{jL}, p_{jH}, a_{jL}, a_{jH})\). We will hence be interested in the strategy \((p_L, p_H, a_L, a_H)\), identical for both firms.

We assume that advertising expenditures \( a_i \) enter firm \( i \)'s profit function as a fixed cost. For given locations, having observed prices and advertising strategies \(((p_1, a_1), (p_2, a_2))\), consumers update their belief about which firm produces which quality. Denote \( \mu((p_1, a_1), (p_2, a_2)) \in [0, 1] \) the probability that firm 1 produces high quality conditional on \(((p_1, a_1), (p_2, a_2))\) being observed.

The game under study is a game with imperfect information and observed actions, for which we are looking for perfect Bayesian equilibria (PBE). We limit our analysis to pure strategy
equilibria. These can be of two types: separating or pooling. In a separating equilibrium, firms choose different pairs of price and advertising expenditure that truly reveal their type. Strategies are such that consumers correctly infer the true quality of each good. Conversely, in a pooling equilibrium, both firms charge the same price and choose the same level of advertising, which does not allow consumers to infer any more information than the one they have \textit{a priori}.

\textbf{Definition 1} A vector \( \{y, p_L, a_L, p_H, a_H, \mu(,.)\} \) characterizes a pure strategy perfect Bayesian equilibrium if:

(a) For any given \( y \), \((p_L, a_L) = \arg \max_{p,a} \pi_L (p, a, p_H, a_H, \mu(,.)). \)

(b) For any given \( y \), \((p_H, a_H) = \arg \max_{p} \pi_H (p, a, p_L, a_L, \mu(,.)). \)

(c) If \((p_H, a_H) \neq (p_L, a_L)\) then \( \mu((p_H, a_H), (p_L, a_L)) = 1 = \mu((p_L, a_L), (p_H, a_H)) \).

(d) If \((p_H, a_H) = (p_L, a_L)\) then \( \mu((p_H, a_H), (p_L, a_L)) = \mu((p_L, a_L), (p_H, a_H)) = \frac{1}{2}. \)

(e) \( y \in \arg \max E\Pi(y). \)

Conditions (a) and (b) require that each firm maximizes its profit, given the its rival’s strategy and consumers posterior beliefs. Requirements (c) and (d) state that beliefs must be consistent with the structure of the game and firms’ strategies. Namely, when firms choose different strategies, consumers will correctly infer which firm produces which quality. Conversely, if strategies are identical, consumers cannot update their beliefs and must revert to their prior ones. Note that those requirements on beliefs only refer to observations of equilibrium strategies. There remains to define beliefs off the equilibrium path, for any deviation by any of the two firms. We will specify those beliefs later. Finally, Condition (e) requires each firm to choose its location optimally, anticipating equilibrium prices and advertising strategies.

4.2 Separating equilibria

When consumers observe a pair of strategies \((p_Q, a_Q), (p_K, a_K)\), they update their beliefs such that \( \mu \equiv \mu((p_Q, a_Q), (p_K, a_K)) = \text{Prob} (Q = H|p_Q, a_Q, p_K, a_K). \) The indifferent consumer,
located in $z^\ast$ is such that

$$R + \mu H + (1 - \mu)L - pQ - t(z^\ast - y)^2 = R + (1 - \mu)H + \mu L - p_K - t(z^\ast - 1 + y)^2$$

which implies that

$$z^\ast(p_Q, p_K, \mu) = \begin{cases} 0 & \text{if } p_Q \geq p_K + td - (1 - 2\mu)\Delta \\ 1 & \text{if } p_Q \leq p_K - td - (1 - 2\mu)\Delta \\ \frac{1}{2} - \frac{p_Q - p_K + (1 - 2\mu)\Delta}{2d} & \text{otherwise.} \end{cases} \quad (7)$$

Profit functions are denoted\(^8\)

$$\pi_Q(p_Q, a_Q, p_K, \mu) = (p_Q - c(Q))z^\ast(p_Q, p_K, \mu) - a_Q$$

$$\pi_K(p_K, a_K, p_Q, \mu) = (p_K - c(K))(1 - z^\ast(p_Q, p_K, \mu)) - a_K$$

where $\mu = \mu((p_Q, a_Q), (p_K, a_K))$.

The necessary conditions for the strategy profile $(p_L, a_L, p_H, a_H)$ to be a separating profile are:

$$\pi_L(p_L, a_L, p_H, 0) \geq \pi_L\left(p_H, a_H, p_H, \frac{1}{2}\right) \quad (IC_L)$$

$$\pi_H(p_H, a_H, p_L, 0) \geq \pi_H\left(p_L, a_L, p_L, \frac{1}{2}\right) \quad (IC_H)$$

$$(p_L, a_L) \in \arg \max_{p, a} \pi_L(p, a, p_H, 0). \quad (8)$$

Conditions (IC\(_L\)) and (IC\(_H\)) are implied by (a), (b) and (d) of Definition 1.\(^9\) Namely, by construction of the belief system, each firm has the possibility of mimicking the strategy of its rival, and then confuse consumers. Condition 8 stipulates that in a separating equilibrium the low-quality firm, facing the worst possible beliefs and being perfectly identified by consumers, should optimize accordingly.

Replacing profit functions by their values, we easily obtain after straightforward computa-

\(^8\)Note that we have suppressed the variable $a_i$ in $\pi_j$ since $\pi_j$ only depends on $a_i$ through beliefs $\mu$.

\(^9\)Obviously, the conditions that would be obtained when instead firm 1 produces high-quality and firm 2 low-quality are similar to these ones, given the symmetry of the game.
tions a more simple characterization of these no-mimicking conditions:

\[
\begin{align*}
a_H & \geq \max(a_H(p_H, p_L, a_L), 0) \quad \text{(IC}_L) \\
0 & \leq a_H \leq \pi_H(p_H, p_L, a_L) \quad \text{(IC}_H)
\end{align*}
\]

where

\[
a_H(p_H, p_L, a_L) = \frac{1}{2}p_H + a_L - p_L z^*(p_L, p_H, 0)
\]

and

\[
\pi_H(p_H, p_L, a_L) = (p_H - c)(1 - z^*(p_L, p_H, 0)) - \frac{1}{2}(p_L - c) + a_L.
\]

The expenditure \( a_H \) represents the minimum level of dissipative advertising for the high-quality firm that deters a low-quality firm from mimicking its strategy, while \( \pi_H \) is the highest level of advertising sustainable for a high-quality firm in a separating equilibrium, above which the high-quality firm would rather mimic the low-quality firm’s strategy. Hence, on the one hand, advertising expenditures must be sufficiently high to prevent the low-quality firm from mimicking, and on the other hand, they must not be so high that separation is no longer profitable for the high-quality firm.

Recall from Condition (8) that in a separating equilibrium, the low-quality firm faces the worst possible beliefs and is perfectly identified by consumers. Therefore, this firm will maximize its profit accordingly and consequently, its best response to its competitor is identical to the one under symmetric information. This easily yields to the following result.

**Lemma 1** In any separating equilibrium profile, the low-quality firm’s strategy is \( p_L = p^B_L(p_H) = \frac{1}{2}[p_H + td - \Delta] \) and \( a_L = 0 \).

Intuitively, the low-quality firm has no better choice than playing its symmetric-information best price response and hence is not willing to spend money in wasteful advertising. Using Lemma 1, we are now able to characterize the set of potential separating equilibria \((p_H, a_H, p_L, a_L) = (p_H, a_H, p^B_L(p_H), 0)\), using the following necessary condition on \( p_H \) and \( a_H \),

\[
\max(a_H(p_H, p^B_L(p_H), 0), 0) \leq a_H \leq \pi_H(p_H, p^B_L(p_H), 0) \quad \text{(9)}
\]
together with the non-negativity condition for firms’ demands,

\[ 0 \leq z^*(p^B_L(p_H), p_H, 0) \leq 1, \]

(10)

and the condition \( p_H \geq c \) for positive margin.

**Lemma 2** The set \( \Omega \) of price \( p_H \) and advertising expenditure \( a_H \) such that the necessary conditions (9) and (10) for separating equilibria are satisfied is non empty and defined as follows:

\[ a_H(p_H, p^B_L(p_H), 0) \leq a_H \leq \pi_H(p_H, p^B_L(p_H), 0) \]

and

\[ d \leq \rho \iff p_H \in [\Delta - td, \Delta + td] \]

\[ d > \rho \iff p_H \in [2c + td - \Delta, \Delta + td] \]

**Proof:** See Appendix A. \( \blacksquare \)

Given this characterization, we are now able to state the main result on the existence of separating equilibria.

**Proposition 2** Any pair of strategies such that \((p_H, a_H) \in \Omega \) and \((p_L, a_L) = (\frac{1}{2}[p_H + td - \Delta], 0) \)

can be paired with a system of beliefs to form a separating equilibrium.

**Proof:** See Appendix B. \( \blacksquare \)

The proof of Proposition 2 shows that considering beliefs that satisfy Definition 1 and the following requirements:

\[ \mu(p, a, p_H, a_H) = 0 \text{ for any } (p, a) \notin \{(p_H, a_H), (p_L, a_L)\} \]

\[ \mu(p_L, a_L, p, a) = 1 \text{ for any } (p, a) \notin \{(p_H, a_H), (p_L, a_L)\} \]

is sufficient to support any strategy \((p_H, a_H), (p_L, a_L)\) such that \((p_H, a_H) \in \Omega \) and \((p_L, a_L) = (\frac{1}{2}[p_H + td - \Delta], 0)\) as part of a separating equilibrium. With such beliefs, consumers infer from
any deviation that the deviating firm offers low quality and that the non-deviating firm offers high quality.

Figures 1, 2 and 3 present the region of separating equilibria in the space \((p_H, a_H)\) according to the different values of parameter \(\rho\). An important consequence of our result is that purely dissipative advertising is absolutely necessary for the high-quality firm in a separating equilibrium. This holds whatever the gap between qualities or the distance between firms. This contrasts with the results obtained by H&O and Fluet and Garella (2002), where it is suggested that when the qualities differential is sufficiently high, price signaling alone is sufficient to get separation.

An intuition for this result unfolds as follows. If advertising is not possible, one can show that, to prevent imitation by the low-quality firm, the high-quality firm would have to distort its price downward (one can observe this by extrapolating the curves \(a_H\) and \(\pi_H\) in Figures 1, 2 and 3 until they cross the horizontal axis). But such a low price would then allow the high-quality firm to appropriate the entire demand, leaving its competitor with zero profit. As the low-quality firm gets nothing, it has nothing to loose from choosing any other price: Its opportunity cost of deviating is null. Therefore, its incentive compatible constraint not to deviate cannot be satisfied, since it is always worthwhile imitating any price the high-quality firm could choose and have a chance to get a positive share of the demand.

4.3 Pooling equilibria

We now turn to the study of pooling equilibria. In a pooling equilibrium, both firms set the same price \(p\) and the same advertising expenditure \(a\). Consumers are therefore unable to infer any information about quality, so that their posterior beliefs are identical to their prior ones: \(\mu(p, a, p, a) = \frac{1}{2}\). Firms will therefore split the market equally. Let us denote \(\Gamma\) the set of pooling equilibrium strategies \((p, a)\). We define out-of-equilibrium beliefs as being pessimistic, such that given a putative pooling equilibrium strategy profile \(((p, a), (p, a))\), consumers infer from any unilateral deviation \((p', a') \neq (p, a)\) that the deviating firm produces low quality: \(\mu(p', a', p, a) = 1 - \mu(p, a, p', a') = 0\). Hence, the set \(\Gamma\) is characterized by the following necessary
and sufficient conditions:

\[
\pi_L \left( p, a, p, \frac{1}{2} \right) \geq \max_{p', a'} \pi_L \left( p', a', p, 0 \right) \tag{11}
\]

\[
\pi_H \left( p, a, p, \frac{1}{2} \right) \geq \max_{p', a'} \pi_H \left( p', a', p, 1 \right) \tag{12}
\]

Studying conditions (11) and (12) gives rise to the following Proposition.

**Proposition 3** For \( d \geq c^2/16\Delta t \), the set \( \Gamma \) is non empty and defined by

\[
0 \leq a \leq \min \left( a_H(p, p, 0), a_H(p - c, p - c, 0) \right)
\]

\[
td + \Delta + c - 2\sqrt{\Delta td} \leq p \leq td + \Delta + 2\sqrt{\Delta td}.
\]

Any \((p, a)\) \(\in\) \(\Gamma\) can be supported as a pooling equilibrium with the following system of beliefs:

\[
\mu(p, a, p, a) = \frac{1}{2} \text{ and } \mu(p', a', p, a) = 1 - \mu(p, a, p', a') = 0 \text{ for any } (p', a') \neq (p, a).
\]

**Proof:** See Appendix C. \(\Box\)

When there is a sufficiently high distance between firms, there is an infinity of pooling equilibria, some of which are characterized by positive advertising expenditures. On the other hand, when \( c^2/16\Delta t > 1 \), there does not exist any pooling equilibrium. However, note that in any pooling equilibrium, firms share equally the (constant) market demand. Hence, all pooling equilibria can be ranked according to a Pareto-dominance criterion, so that any pooling equilibrium \((p, a)\) is Pareto dominated for both firms if there exists another pooling equilibrium involving lower advertising expenditures and a higher price. This remark suggests that a good candidate for the most plausible pooling equilibrium is the one with the highest price and zero advertising expenditures, that is \( \left( td + \Delta + 2\sqrt{\Delta td}, 0 \right) \).

4.4 Selecting an equilibrium

As we ultimately want to solve the first stage of the game, i.e. the location stage, we need to select a plausible equilibrium for the prices and advertising competition subgame. Given the results contained in the previous two subsections, we are left with two different regimes. First, when the gap between qualities is sufficiently large \((\Delta \geq c^2/16t)\) only separating equilibria exist.
and second, when there is sufficient distance between firms \(d \geq \frac{c^2}{16\Delta t}\), pooling and separating equilibria co-exist.

### 4.4.1 Refining pooling equilibria

We show in this subsection that pooling equilibria do not survive provided one imposes intuitive beliefs off the equilibrium path (Schultz 1999, Bagwell and Ramey 1991). The idea behind this refinement is as follows: If \((p', a') \neq (p, a)\) is a deviation which is equilibrium dominated for the low-quality firm but not for the high-quality firm, then consumers should not put a positive weight on the possibility that such a deviation might come from a low-quality firm. Formally, if \((p', a') \neq (p, a)\) is such that

\[
\pi_L(p', a', p, 1) \leq \pi_L(p, a, p, \frac{1}{2}) \quad (13)
\]

\[
\pi_H(p', a', p, 0) \geq \pi_H(p, a, p, \frac{1}{2}) \quad (14)
\]

then we must have \(\mu(p, a, p', a') = 1 - \mu(p', a', p, a) = 0\).

It means that if there exists a deviation to \((p', a')\) such that, with the best possible beliefs (or the most optimistic beliefs), the low-quality firm could not benefit while the high-quality firm could, then consumers should understand that this deviation can only come from the high-quality firm and should thus infer \(\mu(p, a, p', a') = 1 - \mu(p', a', p, a) = 0\). This contradicts the system of beliefs that supports pooling equilibria in Proposition 3, where we assumed that consumers infer that any deviating firm is of low-quality.

Let us show that such a deviation exist. First, recall that for any \((p, a) \in \Gamma\), there exists another pooling equilibria \((p, 0) \in \Gamma\) that yields the same variable profit for both firms but without the drawback of spending money on advertising. Hence, it is preferred by both firms and we can restrict our search to the deviation \((p', 0)\), which is profitable for the high-quality firm but not for the low-quality one, with respect to any \((p, 0) \in \Gamma\). Given this remark, conditions
(13) and (14) require that:

\[ p' \left( \frac{1}{2} - \frac{p' - p - \Delta}{2td} \right) \leq \frac{1}{2}p \]  \hspace{1cm} (15)

\[ (p' - c) \left( \frac{1}{2} + \frac{p' - p + \Delta}{2td} \right) \geq \frac{1}{2}(p - c). \]  \hspace{1cm} (16)

Let us determine the best deviation for the high-quality firm. It is determined as

\[ p' = \arg \max_{\tilde{p}} \pi_H (\tilde{p}, 0, p, 0) = (\tilde{p} - c) \left( \frac{1}{2} + \frac{p - \tilde{p} + \Delta}{2td} \right) \]

with \( p' = \frac{1}{2} (p + td + \Delta + c) \). The corresponding deviating profit is \( \pi_H (p', 0, p, 0) = (p + td + \Delta - c)^2 / 8td \).

It is then easy to check that this deviation is more profitable than sticking to the pooling equilibria:

\[ \pi_H (p', 0, p, 0) - \frac{1}{2}(p - c) = \frac{(p + td + \Delta - c)^2}{8td} - \frac{1}{2}(p - c) \]

\[ = \frac{1}{8td} \left[ (p - td + \Delta - c)^2 + 4td\Delta \right] > 0 \]

so that (16) is verified with \( p' = \frac{1}{2} (p + td + \Delta + c) \). It remains to check whether (15) is also verified for \( p' = \frac{1}{2} (p + td + \Delta + c) \). Replacing we get

\[ \pi_L (p', 0, p, 1) - \frac{1}{2}p = \frac{(p + td + \Delta + c)(p + td + \Delta - c)}{8td} - \frac{1}{2}p \]

Note that \( \pi_L (p', 0, p, 1) - \frac{1}{2}p \) is a quadratic convex function of \( p \), strictly positive at \( p = 0 \) and increasing with \( p \geq 0 \). It then follows that deviating to \( p' \) is not profitable to the low-quality firm. Consequently, all pooling equilibria are non intuitive in the sense that they are supported by non intuitive beliefs.

4.4.2 Refining separating equilibria

Given the preceding result, we are left with only separating equilibria. Following H&O, we now restrict out-of-equilibrium beliefs using the refinement REDE (Resistance to Equilibrium Defections) for separating equilibria.
Definition 2 Consider a profile \((p_L, a_L, p_H, a_H)\) such that \((p_H, a_H) \in \Omega\) and \((p_L, a_L) = (p_H^B(p_H), 0)\) as defined in Proposition 2. Consider also an alternative profile \((p'_L, a'_L, p'_H, a'_H)\) ≠ \((p_L, a_L, p_H, a_H)\) where \((p'_L, a'_L) ≠ (p'_H, a'_H)\). An equilibrium profile \((p_L, a_L, p_H, a_H)\) is resistant to equilibrium defections (REDE) if beliefs satisfy \(\mu((p'_L, a'_L), (p'_H, a'_H)) = 0\) whenever

(1) \((p'_L, a'_L) = (p_H^B(\tilde{p}_H), 0)\) for some \((\tilde{p}_H, \tilde{a}_H) \in \Omega\) and,

(2) \((p'_H, a'_H) \in \Omega\).

The idea behind this refinement is as follows. Suppose that consumers expect to see \((p_H, a_H)\) from the high-quality producer and \((p_H^B(p_H), 0)\) from the low-quality producer. Suppose instead that they observe \((p'_L, a'_L, p'_H, a'_H)\). If \((p'_L, a'_L)\) is consistent with some alternative separating equilibrium play of a low-quality firm, and if \((p'_H, a'_H)\) is consistent with some (possibly different) separating equilibrium play of a high-quality firm, then consumers have enough information to infer that \((p'_H, a'_H)\) is played by the high-quality firm with probability one. When the refinement REDE is silent, the beliefs are still specified as being pessimistic as in Proposition 2.

Consider the low-quality firm first. The imposition of the REDE criterion on out-of-equilibrium beliefs has no consequences for that producer. Indeed, any unilateral (non mimicking) deviation from a putative equilibrium profile \((p_L, a_L, p_H, a_H)\) leads consumers to infer that the deviating firm produces low quality. But the equilibrium has been constructed such that those deviations are suboptimal. From the low-quality firm perspective, this inference remains unchanged under REDE. The same is true for any mimicking deviation.

On the contrary, for the high-quality firm, the REDE criterion has a large impact. Indeed, any deviation from \((p_H, a_H) \in \Omega\) to another \((p'_H, a'_H) \in \Omega\) should leave beliefs unchanged according to REDE. Moreover, any deviation to \((p'_H, a'_H) \notin \Omega\) would lead consumers to infer that the deviating firm offers low quality and hence will not be considered by the high-quality firm. It results that for \((p_L, a_L, p_H, a_H)\) to survive the REDE criterion, \((p_H, a_H)\) must be a best response to \((p_L, a_L) = (p_H^B(p_H), 0)\) in the set \(\Omega\). This allows to establish the next Proposition.

For this, let us define \(D_1 = \{d \mid 1 \geq d > \rho\}\) and note that if \(\rho > 1\), then \(D_1 = \emptyset\). Similarly, let us denote \(D_2 = \{d \mid \rho \geq d > \rho/3\}\) and \(D_3 = \{d \mid \rho/3 \geq d \geq 0\}\). Also define \(D = D_1 \cup D_2 \cup D_3\).
Proposition 4 There is a unique separating equilibrium profile \((p^*_L, a^*_L, p^*_H, a^*_H)\) that satisfies the REDE refinement criterion where 
\((p^*_L, a^*_L) = (p^*_L(p^*_H), 0), a^*_H = a^*_H(p^*_H, p^*_L, 0)\) and

(i) when \(d \in D_1 \neq \emptyset\), then \(p^*_H = \Delta + td\),
(ii) when \(d \in D_2 \neq \emptyset\), then \(p^*_H = c + 2td\),
(iii) when \(d \in D_3 \neq \emptyset\), then \(p^*_H = \Delta - td\).

Proof: See Appendix D.

4.5 Location equilibria

Making use of the results from the preceding subsection, we first note that the type of separating equilibrium depends in particular on which set \(D_i\) \((i = 1, 2\) or 3\) the parameter \(\rho\) belongs to. Computing the expected profit \(\Pi = \frac{1}{2} \pi^*_H + \frac{1}{2} \pi^*_L\) and looking for its maximum with respect to \(d\), we get the following Proposition.

Proposition 5 Whenever \(\Delta \geq c\), the equilibrium is separating and involves either maximal or minimal horizontal differentiation. More precisely,

(i) when \(0 \leq \rho \leq 1\), horizontal differentiation is maximal, \(d^* = 1\), with \(p^*_H = \Delta + t\),
(ii) when \(1 < \rho \leq 3 - \sqrt{2}\), horizontal differentiation is also maximal, \(d^* = 1\), but with \(p^*_H = c + 2t\),
(iii) and when \(3 - \sqrt{2} < \rho\), horizontal differentiation is minimal, \(d^* = 0\), with \(p^*_H = \Delta\) and only the high-quality firm is active.

Proof: See Appendix E.

Tables 2 and 3 summarize the equilibrium values for prices, advertising, locations and ex-post profits. From an ex-post perspective, a high-quality firm benefits from its type only in regions \(\Sigma_2\) and \(\Sigma_3\). Indeed, only in region \(\Sigma_1\) (i.e. when \(\rho\) is sufficiently low), the low-quality firm earns more than its competitor who faces positive advertising expenditures. This feature of the separating equilibrium contrasts with the situation of symmetric information where the high-quality firm always has a higher (ex-post) payoff than its competitor. Hence, when \(\rho\) is low
enough, e.g. when the quality differential is weak, the most profitable firm is the low-quality one. Moreover, the comparison between ex-post profits of the low-quality firm under symmetric and asymmetric information reveals that this firm benefits strictly from asymmetric information whenever there is maximal differentiation and obtains the same (zero) payoff when there is minimal differentiation (region $\Sigma_3$).

In regions $\Sigma_1$ and $\Sigma_2$, it is easy to check that prices of both goods are higher than their counterparts under symmetric information. Hence, not only does signaling high-quality involve advertising, but it also involves an upward distortion in the price ($p^*_H > p^*_B$). The upward distortion on the low-quality price then follows from the fact that prices are strategic complements. This has consequences on the way the market is shared between products: under symmetric information, the low-quality firm has a greater market share than the competitor ($z^B > 1/2$), while under asymmetric information, we get $z^* = 1/2$ for region $\Sigma_1$ and $z^* < 1/2$ for region $\Sigma_2$. The market share of the low-quality firm is thus decreasing when the high-quality product becomes more and more interesting (i.e. increasing $\rho$).

The pattern of prices as $\Delta$ evolves differs from the one under symmetric information. When
the quality differential increases, the price of the low-quality product is rapidly decreasing before falling down to zero under symmetric information. Under asymmetric information, while starting from the same value (when $\rho = 0$, $p^*_L = p^*_B = t$), $p^*_L$ first remains constant before slowly decreasing towards zero. The evolution of advertising $a_H$ depends on the region we consider. While it is increasing with the quality differential $\Delta$ in region $\Sigma_1$, it is clearly decreasing in $\Delta$ in region $\Sigma_2$. Hence the cost of advertising is non monotonic in the quality gap $\Delta$. This is reminiscent of results obtained by H&O in a different setting.

In region $\Sigma_3$, $\Delta$ is so large that the low-quality firm is not active at the last stage of the game, and ex ante, both firms locate at the center of the market. Recall that under symmetric information, minimal differentiation was obtained for $\rho \geq 1.146$. As $3 - \sqrt{2} > 1.146$, we can conclude that asymmetric information reduces the set of parameters for which there is minimal differentiation and only the high-quality firm is active. Intuitively, asymmetric information allows the low-quality firm to resist an increasingly efficient high-quality firm for a larger set of parameters values. As prices are the same under symmetric and asymmetric information (in
both cases, \( p_H = \Delta \), consumers do not suffer from uncertainty over quality. However, the high-quality firm earns less because of the necessity to signal its type through costly advertising, which could reduce the incentives to enter on the market. Last, the cost of advertising is increasing in the quality differential in region \( \Sigma_3 \).

### 5 Conclusion

In this paper we have analyzed a horizontal product differentiation duopoly model in a linear city, where firms are also differentiated with respect to the quality of their products. Firms first choose their location and then simultaneously compete in prices and advertising expenditures. At the location stage, there exists an uncertainty about the quality produced by each of the duopolists, quality that will be privately revealed to them before the price and advertising competition stage.

We have shown that signaling high quality through price only is impossible, whatever the degree of vertical differentiation. Indeed, separating equilibria only exist when the high-quality firm advertises. An equilibrium refinement produces two kinds of separating equilibria depending on the degree of vertical differentiation and cost difference. Horizontal differentiation is minimal only when the high-quality firm is able to get a monopoly position, otherwise differentiation is maximal with an upward distortion in both prices compared to symmetric information. We have
also shown that pooling equilibria exist but do not survive to the imposition of intuitive beliefs.

An obvious extension of the present paper would be to study an alternative timing where location choice can act as a signal of quality as in Vettas (1999). Moreover, we have assumed that qualities are perfectly correlated. Another extension would be to relax this assumption and assume uncorrelated qualities, such that consumers do not a priori know if products are vertically differentiated. Finally, it would be interesting to generalize the present setting to include the possibility that firms can make (potentially observable) decisions like investment early in the game, that could affect the realization of quality which would remain private information to the firms.
References

[1]


Appendix

A Proof of Lemma 2

Let us start with condition (10). From (7) and Lemma 1, we have

\[ z^*(p_B^L(p_H), p_H, 0) = \frac{1}{2} - \frac{1}{2} \left( \frac{(p_H + td - \Delta) - p_H + \Delta}{2td} \right) = \frac{1}{4td} (p_H + td - \Delta) \]

and \( 0 \leq z^*(p_B^L(p_H), p_H, 0) \leq 1 \) is then equivalent to the following condition

\[ \max(c, \Delta - td) \leq p_H \leq \Delta + 3td. \]

This gives us a lower and upper bound for \( p_H \) such that demands are non negative for both firms. In addition, given the definition of \( a_H \) and Lemma 1, we get

\[ a_H (p_H, p_B^L(p_H), 0) = \frac{1}{2} p_H - \frac{1}{8td} (p_H + td - \Delta)^2. \]

Hence, \( a_H \) is an inverted parabola function of \( p_H \), which appears to be symmetric on the interval \( I = [\Delta - td, \Delta + 3td] \) and that has its maximum in the middle of \( I, \Delta + td \).

Last, given the definition of \( \pi_H \) and Lemma 1, we obtain

\[ \pi_H (p_H, p_B^L(p_H), 0) = \frac{1}{4td} (p_H - c)(3td - p_H + \Delta) - \frac{1}{4}(p_H + td - \Delta - 2c) \]

which is also an inverted parabola. Its maximum is obtained at \( p_H = \frac{\Delta + c}{2} + td \), which is lower than \( \Delta + td \), since by assumption \( c < \Delta \).\(^{10}\)

Let us compute the difference between \( \pi_H \) and \( a_H \):

\[ \pi_H - a_H = \frac{1}{4td} (p_H - c)(3td - p_H + \Delta) - \frac{1}{4}(p_H + td - \Delta - 2c) - \frac{1}{2} p_H + \frac{1}{8td} (p_H + td - \Delta)^2 \]

\[ = \frac{1}{8td} (p_H - td - \Delta) (2c - p_H + td - \Delta). \]

This is again an inverted parabola which is positive only between the roots \( \Delta + td \) and \( 2c + td - \Delta \).

\(^{10}\) Recall that \( \Delta > c \) ensures that the production of the high-quality product is socially valuable.
Given that \( c < \Delta \), the upper root is \( \Delta + td > 0 \).

Hence the set of admissible prices \( p_H \), that is high-quality prices satisfying the necessary conditions for a separating equilibrium, is non empty and such that

\[
\max(c, \Delta - td, 2c + td - \Delta) \leq p_H \leq \Delta + td.
\]

More precisely, as we have the following inequalities:

\[
2c + td - \Delta > c \iff d > \rho = \frac{\Delta - c}{t} \\
\Delta - td > c \iff d < \rho \\
2c + td - \Delta > \Delta - td \iff d > \rho.
\]

It follows that when \( \rho \geq d \), the set of admissible \( p_H \) is the interval \([\Delta - td, \Delta + td]\) while when \( \rho < d \), the set of admissible \( p_H \) is \([2c + td - \Delta, \Delta + td]\). This completes the proof.

\section*{B Proof of Proposition 2}

Given a definition of out-of-equilibrium beliefs, we check that neither firm has incentives to deviate from a putative equilibrium \((\tilde{p}_H, \tilde{a}_H, \tilde{p}_L, \tilde{a}_L)\). Let us assume that beliefs are such that

\[
\mu(p, a, \tilde{p}_H, \tilde{a}_H) = 0 \text{ for any } (p, a) \notin \{(\tilde{p}_H, \tilde{a}_H), (\tilde{p}_L, \tilde{a}_L)\} \\
\mu(\tilde{p}_L, \tilde{a}_L, p, a) = 1 \text{ for any } (p, a) \notin \{(\tilde{p}_H, \tilde{a}_H), (\tilde{p}_L, \tilde{a}_L)\}.
\]

Hence, if one firm is playing according to a separating equilibrium strategy (whether the high- or low-quality firm) and the other one is deviating by not mimicking, then the latter is immediately perceived as a low-quality producer by consumers. These beliefs support the largest possible set of equilibria.

We first check that the low-quality firm has no incentives to deviate. Indeed, it is clear from the above definition of beliefs that any deviation to \((p, a) \neq (\tilde{p}_H, \tilde{a}_H)\) by this firm does not affect consumers perception of its type. From the necessary conditions for the existence
of separating equilibria, the best response to these beliefs and the rival’s action is \((\tilde{p}_L, \tilde{a}_L) \in \arg\max_{p, a} \pi_L(p, a, \tilde{p}_H, 0)\), implying that \((p, a)\) is suboptimal. In addition, deviating by mimicking the high-quality behavior \((\tilde{p}_H, \tilde{a}_H)\) leads to \(\mu(\tilde{p}_H, \tilde{a}_H, \tilde{p}_H, \tilde{a}_H) = \frac{1}{2}\). However, by virtue of necessary condition (IC\(_L\)), this deviation is suboptimal.

We now show that the high-quality firm has no incentives to deviate. First, deviating by mimicking the low-quality firm’s equilibrium strategy leads to \(\mu(\tilde{p}_L, \tilde{a}_L, \tilde{p}_L, \tilde{a}_L) = \frac{1}{2}\) and from necessary condition (IC\(_H\)), this deviation is suboptimal because this strategy is dominated by playing \((\tilde{p}_H, \tilde{a}_H)\).

Second, if the high-quality producer deviates to \(p \neq \tilde{p}_L\), then by virtue of the definition of out-of-equilibrium beliefs, consumers believe that the rival is offering the high-quality product as \(\mu(\tilde{p}_L, \tilde{a}_L, p, a) = 1\). It then follows obviously that the high-quality firm has no incentives to spend money in useless advertising, so that \(a_H = 0\). In such a deviation, the demand \(D_{L/H}\) addressed to the high-quality firm when perceived by consumers as offering a low-quality product is given by:

\[
D_{L/H} = 1 - z^*(\tilde{p}_L, p, 1) = 1 - \left(\frac{1}{2} - \frac{\tilde{p}_L - p - \Delta}{2td}\right)
\]

and the corresponding (concave) payoff is:

\[
\pi_{L/H}^d(p) = (p - c)D_{L/H} = (p - c) \left(\frac{1}{2} + \frac{\tilde{p}_L - p - \Delta}{2td}\right).
\] (17)

The best deviation \(p^d(\neq \tilde{p}_L)\) from \(\tilde{p}_H\) is such that \(p^d \in \arg\max_p \pi_{L/H}^d(p)\) which leads to:

\[
p^d = \frac{td}{2} + \frac{\tilde{p}_L + c - \Delta}{2}.
\] (18)

Recall from Lemma 1 that \(\tilde{p}_L = \frac{1}{2}[\tilde{p}_H + td - \Delta]\). Substituting this expression into (18) and the derived \(p^d\) in (17), we can finally write the payoff from the best (non-mimicking) deviation:

\[
\pi_{L/H}^d(p^d) = \frac{1}{32td}(\tilde{p}_H + 3(td - \Delta) - 2c)^2
\]
which is defined only for $\bar{p}_H \geq p^1_H \equiv 3(\Delta - td) + 2c$ to ensure positive demand and margin.

We now prove that playing the best deviation strategy $(p^d, 0)$ is dominated by playing the putative equilibrium strategy $(\bar{p}_H, \bar{a}_H)$. For this, it suffices to prove that $(p^d, 0)$ is dominated by the mimicking strategy $(\bar{p}_L, \bar{a}_L) = \left( \frac{1}{2}(\bar{p}_H + td - \Delta), 0 \right)$. Indeed, given that mimicking a low quality producer is already suboptimal from the necessary conditions for separation, it is then true that the best non-mimicking strategy is also dominated.

The profit obtained when a high-quality producer mimicks the low-quality producer is

$$\pi_H \left( \bar{p}_L, \bar{a}_L, \bar{p}_L, \frac{1}{2} \right) = \left( \frac{1}{2}\bar{p}_H + td - \Delta \right) \cdot \frac{1}{2}$$

which is only defined for $\bar{p}_H \geq p^2_H \equiv 2c + \Delta - td$. Now, computing the difference between $\pi^d_{L/H}(p^d)$ and $\pi_H \left( \bar{p}_L, \bar{a}_L, \bar{p}_L, \frac{1}{2} \right)$, we obtain a convex quadratic form in $\bar{p}_H$ which roots are given by $2c + td + 3\Delta \pm 4\sqrt{td\Delta}$. This difference is thus non positive between the roots.

There are two cases depending on whether $d$ is lower or greater than $\Delta/t$. First, when $d \leq \Delta/t$, then $\Delta - td \geq 0$. Hence, $p^2_H \leq p^1_H$, which means that the profit from mimicking is higher than the profit from deviating, $\pi^d_{L/H}$, whenever $\bar{p}_H$ is lower than the upper root $(2c + td + 3\Delta + 4\sqrt{td\Delta})$. But recall from Lemma 2 that the maximal admissible $\bar{p}_H$ for a separating equilibrium to exist is $\Delta + td$, which is clearly lower than this upper root. It follows that for the admissible values of $\bar{p}_H$, the mimicking strategy is always preferred to the best (non mimicking) deviation strategy.

Second, when $d \geq \Delta/t$, then $\Delta - td \leq 0$. In that case, recall that from Lemma 2 the set of admissible $\bar{p}_H$ for separation is $[2c + td - \Delta, \Delta + td]$. We have just shown that the maximal value $\Delta + td$ is lower than the upper root. It remains to show that the lowest admissible value $2c + td - \Delta$ is higher than the lower root $2c + td + 3\Delta - \sqrt{td\Delta}$:

$$2c + td - \Delta - \left(2c + td + 3\Delta - 4\sqrt{td\Delta}\right) = 4\sqrt{\Delta}(\sqrt{td} - \sqrt{\Delta}) \geq 0$$

as $d \geq \Delta/t$. Hence, once again, for the admissible values of $\bar{p}_H$, the mimicking deviation is always preferred to the best (non mimicking) deviation. This concludes the proof.
C Proof of Proposition 3

Let us start with condition (11). The deviating profit is

$$\max_{p', a'} \pi_L (p', a', p, 0) = \max_{p', a'} (p' z^*(p', p, 0) - a')$$

with $$z^*(p', p, 0) = \left( \frac{1}{2} - \frac{p' - p + \Delta}{2td} \right)$$ from (7). Solving this maximization problem, we find that the optimal deviation is $$(p', a') = \left( \frac{1}{2} (p + td - \Delta), 0 \right)$$. Consequently, after substituting, the optimal profit from deviating for a low-quality firm is

$$\max_{p', a'} \pi_L (p', a', p, 0) = \frac{1}{8td} (p + td - \Delta)^2$$

which is only defined for $$p \in [\max(c, \Delta - td), \Delta + 3td]$$.

The pooling equilibrium profit for a low-quality firm is

$$\pi_L (p, a, p, 1) = \frac{1}{2} p - a.$$  

Hence, condition (11) reduces to:

$$0 \leq a \leq a_H (p, p, 0) = \frac{1}{2} p - \frac{1}{8td} (p + td - \Delta)^2.$$

(19)

The function $$a_H (p, p, 0)$$ is an inverted parabola function of $$p$$ which is positive on the interval between the roots, $$[\alpha, \beta] \equiv [td + \Delta - 2\sqrt{\Delta td}, td + \Delta + 2\sqrt{\Delta td}]$$.

Looking at condition (12), it is easy to see that it yields a similar region in space $$(p, a)$$ but horizontally translated to the right by a quantity $$c$$. Indeed, we have:

$$\pi_H \left( p, a, p, \frac{1}{2} \right) = \frac{1}{2} (p - c) - a$$

$$\max_{p', a'} \pi_H (p', a', p, 1) = (p' - c) \left( 1 - z^*(p', p', 1) \right) - a'$$

11 This definition set ensures that $$p' \geq 0$$ and that $$z^*(p', p, 0) \in [0, 1]$$. 

31
with \( z^*(p, p', 1) = \left( \frac{1}{2} - \frac{p - p' - \Delta}{2td} \right) \) from (7). Solving, we obtain that,

\[
\left( \frac{1}{2} (p + td + c - \Delta), 0 \right) \in \arg \max_{p', a'} \pi_H (p', a', p, 1)
\]

and consequently, \( \max_{p', a'} \pi_H (p', a', p, 1) = \frac{1}{8td} (p + td - c - \Delta)^2 \) which is only defined for \( p \in [\max(c, \Delta - td + c), \Delta + 3td + c] \). Finally, condition (12) reduces to:

\[
0 \leq a \leq a_H (p - c, p - c, 0) = \frac{1}{2} (p - c) - \frac{1}{8td} (p - c + td - \Delta)^2.
\] (20)

Similarly, the function \( a_H (p - c, p - c, 0) \) defines an inverted parabola function of \( p \), positive on the interval between the roots \( [\gamma, \delta] = [td + \Delta + c - 2\sqrt{\Delta td}, td + \Delta + c + 2\sqrt{\Delta td}] \). We are done if we can prove that the two regions defined by (19) and (20) in the space \((p, a)\) intersect. Because \( \alpha < \gamma \) and \( \beta > \delta \), this is true if and only if \( \beta \geq \gamma \):

\[
\begin{align*}
\alpha &< \gamma \quad \text{and} \quad \beta > \delta, \\
td + \Delta + c - 2\sqrt{\Delta td} &\leq td + \Delta + 2\sqrt{\Delta td}
\end{align*}
\]

which reduces to \( d \geq c^2/16\Delta t \). Finally, note that \( \gamma = td + \Delta + c - 2\sqrt{\Delta td} \) is greater than \( c \), since

\[
\begin{align*}
\gamma &> c, \\
td + \Delta + c - 2\sqrt{\Delta td} - c &= (\sqrt{\Delta} - \sqrt{td})^2 > 0
\end{align*}
\]

This ensures that the high-quality firm gets a positive margin at any pooling equilibrium. This concludes the proof.

**D Proof of Proposition 4**

From Lemma 1, in any separating equilibrium where the high-quality firm sets a price \( p_H \), the low-quality firm plays \((p, a) = (p_L^B(p_H), 0)\) and earns

\[
\pi_L^*(p_H) = \frac{1}{2} (p_H + td - \Delta) z^*(p_L^B(p_H), p_H, 0) = \frac{1}{8td} (p_H + td - \Delta)^2
\]
which is increasing in \( p_H \). Consequently, the low-quality firm gets its highest payoff when the high-quality firm sets the highest price compatible with separation, that is \( p_H = \Delta + td \). Hence, we obtain that

\[
\max_{p_H} \pi^s_L(p_H) = \frac{td}{2} < \pi^v_L.
\]

An obvious corollary of Proposition 2 is that, from the high-quality producer’s point of view, any separating equilibrium profile \((p_H, a_H, p^B_L(p_H), 0)\) with \( a_H > a_H(p_H, p^B_L(p_H), 0) \) is dominated by the separating equilibrium profile \((p_H, a_H(p_H, p^B_L(p_H), 0), p^B_L(p_H), 0)\), while the low-quality firm is indifferent as the price \( p_H \) is unchanged. Clearly, for a given price \( p_H \), there is no incentives for the high-quality producer to spend more than the minimum level \( a_H(p_H, p^B_L(p_H), 0) \) of dissipative advertising required for separation. Given this remark, we study the payoff of the high-quality firm, given by

\[
\pi^s_H(p_H) = (p_H - c) \left( 1 - z^\ast(p^B_L(p_H), p_H, 0) \right) - a_H(p_H, p^B_L(p_H), 0)
\]

and replacing, we get

\[
\pi^s_H(p_H) = (p_H - c) \left( 1 - z^\ast(p^B_L(p_H), p_H, 0) \right) - \frac{1}{2}p_H + p^B_L(p_H)z^\ast(p^B_L(p_H), p_H, 0)
\]

\[
= \frac{1}{4td} (p_H - c) (3td - p_H + \Delta) - \frac{1}{2}p_H + \frac{1}{8td} (p_H + td - \Delta)^2.
\]

It can easily be checked that \( \pi^s_H(p_H) \) is an inverted parabola with its maximum at \( p_H = c + 2td \). We then have to rank \( c + 2td \) with the lower and upper bounds of admissible prices \( p_H \). Given that we have the following inequalities:

\[
\begin{align*}
c + 2td &> \Delta - td \iff d > \rho / 3 \\
c + 2td &> 2c + td - \Delta \iff d + \rho > 0 \text{ which is always true} \\
c + 2td &> \Delta + td \iff d > \rho,
\end{align*}
\]

we are left with three possible regimes.

- for \( d > \rho \), the set of admissible prices \( p_H \) is \([2c + td - \Delta, \Delta + td]\) and as \( c + 2td > \Delta + td \),
clearly $\pi_H(p_H)$ is increasing on this interval. The highest payoff for the high-quality firm is then obtained when $p_H = \Delta + td$, i.e.

$$\pi_H(\Delta + td) = \frac{td - c}{2}.$$ 

- for $\rho \geq d > \rho/3$, the set of admissible prices $p_H$ is $[\Delta - td, \Delta + td]$ and $c + 2td$ belongs to this interval. Consequently, the highest payoff for the high-quality firm is obtained for $p_H = c + 2td$, i.e.

$$\pi_H(c + 2td) = \frac{1}{2}(\Delta - 2c - td) + \frac{1}{8td}(c + 3td - \Delta)^2.$$ 

- for $\rho/3 \geq d \geq 0$, the set of admissible prices $p_H$ is still $[\Delta - td, \Delta + td]$, but $\pi_H(p_H)$ is now decreasing on this interval because $c + 2td < \Delta - td$. Consequently, the highest payoff for the high-quality firm is obtained when $p_H = \Delta - td$, i.e.

$$\pi_H(\Delta - td) = \frac{1}{2}(\Delta - td) - c.$$ 

This concludes the proof.

### E Proof of Proposition 5

From Proposition 4, we have for:

- $d \in D_3$, $p_H^* = \Delta - td$ so that $\pi_H^* = \frac{1}{2}(\Delta - td) - c$. In addition, $p_L^* = p_L^B(\Delta - td) = \frac{1}{2}[\Delta - td + td - \Delta] = 0$ and hence $\pi_L^* = 0$. We thus have

$$E\Pi(d) = \frac{1}{4}(\Delta - td) - \frac{c}{2}$$

which is decreasing in $d$ over the set $D_3$.

- $d \in D_2$, $p_H^* = c + 2td$ and $\pi_H^* = \frac{1}{2}(\Delta - 2c - td) + \frac{1}{8td}(c + 3td - \Delta)^2$. Moreover, $p_L^* =$
\[ p^B_L(c + 2td) = \frac{1}{2} |c + 3td - \Delta| \] and we have \[ \pi^*_L = \frac{1}{8td} (c + 3td - \Delta)^2 \]. Hence,

\[
\begin{align*}
EII(d) &= \frac{1}{4} (\Delta - 2c - td) + \frac{1}{8td} (c + 3td - \Delta)^2 \\
&= \frac{1}{8td} \left( 7t^2 d^2 + 2t(c - 2\Delta)d + (\Delta - c)^2 \right).
\end{align*}
\]

Studying this function for positive values of \( d \) reveals that it is convex, first decreasing, reaching a unique minimum at \( d = \rho/\sqrt{7} \), and then increasing.

- \( d \in D_1, \ p^*_H = \Delta + td \) and \( \pi^*_H = \frac{td - c}{2} \). Moreover, \( p^*_L = p^B_L(\Delta + td) = td \) and \( \pi^*_L = \frac{td}{2} \).

Hence, we have for \( d \in D_1 \),

\[
EII(d) = \frac{td - c}{4} + \frac{td}{2} = \frac{td}{2} - \frac{c}{4}
\]

which is clearly increasing in \( d \).

Given that the function \( EII(d) \) is continuous over \([0, 1] \cap D\), it is clear from those results that \( EII(d) \) is first decreasing then increasing. Hence, the maximum of \( EII(d) \) is obtained either for \( d = 0 \) or for \( d = 1 \). Computing these values, we have

\[
\begin{align*}
EII(0) &= \frac{\Delta}{4} - \frac{c}{2} \\
EII(1) &= \begin{cases} \\
\frac{t}{2} - \frac{c}{2} & \text{for } \rho < 1 \\
\frac{1}{4t} (7t^2 + 2t(c - 2\Delta) + (\Delta - c)^2) & \text{for } 1 \leq \rho < 3 \\
\frac{1}{4}(\Delta - t) - \frac{c}{2} & \text{for } \rho \geq 3
\end{cases}
\end{align*}
\]

We easily obtain that \( EII(0) > EII(1) \) whenever \( \rho \geq 3 \). Hence, differentiation is minimal in equilibrium, with only the high-quality firm being active. For \( \rho < 1 \), we have

\[ EII(0) - EII(1) = \frac{\Delta}{4} - \frac{c}{2} - \left( \frac{t}{2} - \frac{c}{4} \right) = \frac{t}{4}(\rho - 2) < 0 \]

so that maximal differentiation prevails. For \( 1 \leq \rho < 3 \), we have

\[ EII(0) - EII(1) = \frac{\Delta}{4} - \frac{c}{2} - \frac{1}{8t} (7t^2 + 2t(c - 2\Delta) + (\Delta - c)^2) = \frac{t}{4}(3\rho - \frac{1}{2}\rho^2 - \frac{7}{2}) \].
This quadratic form in $\rho$ is non positive for $1 \leq \rho < 3 - \sqrt{2}$ and positive for $3 - \sqrt{2} \leq \rho < 3$.

Consequently, we have maximal differentiation ($d = 1$) for $0 \leq \rho < 3 - \sqrt{2}$ and minimal differentiation for $\rho \geq 3 - \sqrt{2}$. This completes the proof.
Figure 1: Set $\Omega$ when $\frac{\rho}{4} \leq d \leq \rho$. 
Figure 2: Set $\Omega$ when $0 \leq d \leq \frac{\Delta}{4}$
Figure 3: Set $\Omega$ when $d > \rho$. 