The World Distibution of Wealth and the Solow Model*

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Abstract

This note adapts the Stiglitz (1969) analysis to a fully integrated world economy. We derive a law of motion for the variance of wealth per capita and income per capita across countries, and characterizes the evolution of world wealth and income inequality.

1 Introduction

To study the predictions of the neoclassical growth model for global convergence (or divergence) we will develop a model of the world economy, comprising a large number of individual countries. The fundamental question is whether the model may allow us to generate "divergence" of income per capita, as we can observe in the data (Pritchett, 1997). The model is essentially a reinterpretation of Stiglitz (1969).

Conceptually we are thinking about a world where all people are participating in the same labor market. Also, capital is invested in a common market. In this sense the world is "fully integrated". By allowing for perfect integration we are effectively "stacking the deck" against divergence, since we thereby neglegt the possibility of differences in factor prices. As a result, the sole difference between countries lies in their different initial endowments of wealth. As in the standard Solow model, all markets are competitive.

2 Analysis

Suppose the world comprises $c = 1, ...C$ countries. Each country comprises L_{ct} individuals. We assume the global distribution of population is stable, so $L_{ct+1}/L_{ct} = 1 + n$ for all c. Time is discrete, $t = 1, 2, ..$ and extends into the infinite future. Technological

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change is ignored, and capital does not depreciate. Both assumptions are made soley for brievty.[1]

Total output is given by $Y_t = F(K_t, P_t)$, where K_t is the total (world) capital stock and P_t is the total (world) labor force (/population) – $P_t = \sum_c L_{ct}$. F features the usual properties including constant returns to scale.

The wealth of country c at time $t+1$ is given by

$$A_{c,t+1} = A_{c,t} + s_{ct} L_{c,t},$$

where s_t^i is per capita savings of country c. We ignore inequality within each country. The *per capita* wealth endowment as of time $t+1$ given by

$$a_{c,t+1} = \frac{a_{ct} + s_{ct}}{1+n},$$

where we have used the assumption that the population is growing at the same rate in all countries. Per capita savings are in turn given by:

$$s_{ct} = sr_t a_{ct} + sw_t + \bar{s}, \bar{s} \gtreqless 0,$$

where s is the *marginal* savings rate, r_t is the real rate of return and w is the real wage, which we assume, for now, is identical across individuals in the world. We deviate from a standard Solow model in allowing for a non-constant *average* savings rate. Notice in particular, if $\bar{s} < 0$ the average savings rate is increasing in income. This is a realistic feature, bearing the empirically observed positive correlation between income per capita and the average savings rate (/investment rate) in mind.

Putting the two equations together we have the following path for per capita wealth over time in country c:

$$a_{ct+1} = \frac{1 + sr_t}{1+n} a_{ct} + \frac{sw_t + \bar{s}}{1+n}. \quad (1)$$

Using a standard result for variances we get an equation governing the evolution of the variance of wealth per person:

$$\sigma_{at+1} = \left(\frac{1 + sr_t}{1+n}\right)^2 \sigma_{at} \Leftrightarrow \frac{\sigma_{at+1}}{\sigma_{at}} = \left(\frac{1 + sr_t}{1+n}\right)^2.$$

Obviously, the variance of wealth (and in this sense inequality) will be growing over time, $\frac{\sigma_{at+1}}{\sigma_{at}} > 1$, iff $sr_t > n$. And vice versa.

[1] Still, under the neoclassical view, technology better not be important in accounting for divergence since the model offers no explanation for the evolution of technology.

Now, due to the fact that savings are a linear function of income aggregation is a fairly straight forward exercise. Define the world stock of capital as $K_t = \sum_c A_{ct}$, which implies that the per capita stock of capital ($K/P = k$) is

$$k_t = \frac{\sum_c A_{ct}}{P_t}$$

Aggregating across countries

$$K_{t+1} = \sum_c A_{ct+1} = (1 + sr_t) \sum_c A_{ct} + sw_t P_t + \bar{s} P_t$$

or in per capita terms

$$k_{t+1} = \frac{[1 + sr_t] k_t + sw_t + \bar{s}}{1+n}.$$

Since

$$r_t k + w_t = f'(k_t) k_t + (f(k_t) - f'(k_t) k_t) = f(k_t) = y_t,$$

by constant returns to scale in the production function, it follows that the evolution of the world per capita capital stock is given by

$$k_{t+1} = \frac{k_t + sf(k_t) + \bar{s}}{1+n},$$

or equivalently:

$$k_{t+1} - k_t = \frac{1}{1+n}(sf(k) + \bar{s} - nk_t).$$

The steady state (which is not nessesarily unique if $\bar{s} < 0$) is characterized by

$$sf(k^*) = nk^* - \bar{s}.$$

Figure 1 illustrates the Phase diagram, assuming $\bar{s} < 0$ and chosen such that there exists two steady states. Geometrically it should be clear that there exist a $k = \tilde{k}$ such that $sf'\left(\tilde{k}\right) = n$. In other words, at $\tilde{k}$, it holds that $sr = n$. To the left of $\tilde{k}$, $sr > n$ (due to diminishing returns), whereas the opposite is the case for $k > \tilde{k}$. Moreover, by concavity of f it follows that $\tilde{k} < k_h^*$; where k_h^* is the "high" steady state, which is stable. Observing that, over time, per capita income of the world economy has been increasing in practise, we confine attention to paths of k_t consistent with the intitial condition $k_l^* < k_0 < k_h^*$; where k_l^* is the lower (unstable) steady state.

The first major result is this. When the world economy is at the stable (non-trivial) steady state, $k = k_h^*$ it must be the case that $\frac{\sigma_{t+1}^a}{\sigma_t^a} < 1$. This means that *in the limit all*

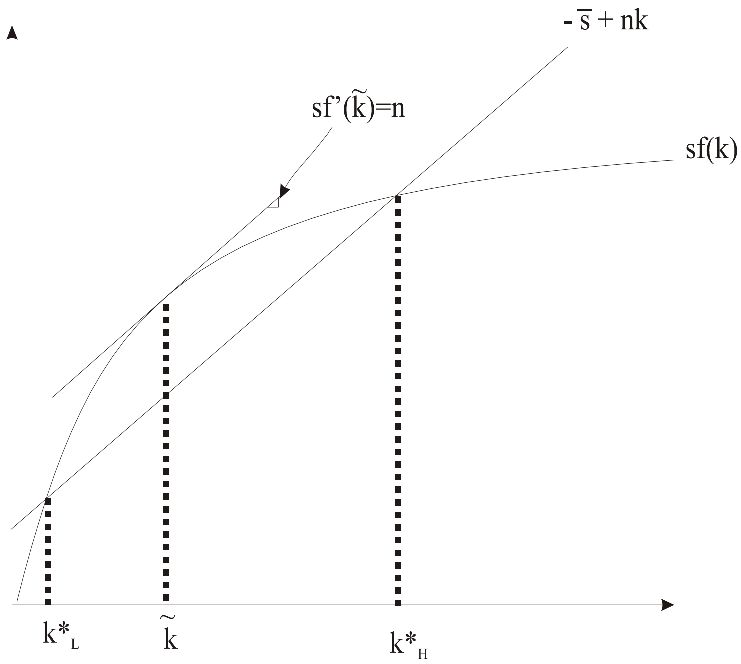

Figure 1: Phase diagram: The world economy

inequality disappears regardless of the initial distribution of wealth, and even though the average savings rate of the rich is higher! More generally, we have that according to this (slightly modified) Solow model, the distribution of wealth should converge, in the long run, *if all countries have similar structural characteristics.* Its precisely the "conditional convergence" result, now applied to the country specific distribution of wealth. Of course, s and n are not the same in practise. But sweeping this under the carpet for a moment allows us to see another, and perhaps slightly more striking result.

Suppose $k_l^* < k_0 < \tilde{k} < k_h^*$. If initially $k_0 < \tilde{k}$ it will be the case that $\frac{\sigma_{t+1}^a}{\sigma_t^a} > 1$. However, eventually k will grow beyond the level $\tilde{k}$, as the world economy approaches its steady state k_h^*. Hence, eventually $\frac{\sigma_{at+1}}{\sigma_{at}} < 1$.

In words: If the world is converging to the stable steady state from below, we may have a period of *rising* inequality (i.e. "divergence"), ultimately followed by declining inequality.[2] The important implication is this: *Even if* the distribution of wealth per

[2] This finding mirrors the famous Kuznets (1955) hypotheis, which suggest a similar pattern for the *personal* distribution of income should arise within economies.

capita becomes perfectly egalitarian in the long run, we may have an extensive period of divergence occurring!

But what about *income* inequality? As it turns out all we said about wealth inequality carries over to income per capita inequality in a qualitative sense. To see this clearly, observe that income per capita in country c is given by

$$y_{ct} = w + r_t a_{c,t}$$

so the variance of y_c is

$$\sigma_{yt} = r_t^2 \sigma_{at}.$$

Consequently, the law of motion for income inequality is (substituting for σ_{at+1} and σ_{at})

$$\sigma_{yt+1} = \left(\left[\frac{r_{t+1}}{r_t} \right] \frac{1 + sr_t}{1 + n} \right)^2 \sigma_{yt}.$$

The issue is whether we can be sure that $\frac{\sigma_{yt+1}}{\sigma_{yt}} > 1$ can arise for some levels of k. The following argument tells us that it can.

Suppose that the economy is at $k_l^* -$ the low (unstable) steady state. By virtue of a steady state, the real rate of return is constant, since the capital stock per capita is constant. So $\frac{r_{t+1}}{r_t} = 1$, but clearly $\frac{1+sr}{1+n} > 1$ (look at Figure 1: $sf'(k_l^*) > n$). Suppose next, that we are at the stable steady state k_h^*. Here $\frac{r_{t+1}}{r_t} = 1$ also, but $\frac{1+sr_t}{1+n} < 1$. Consequently, somewhere between k_l^* and k_h^* there exist a factor intensitiy $\tilde{k}_{new}$ where $\left[\frac{r_{t+1}}{r_t}\right] \frac{1+sr_t}{1+n} = 1$. Below $\tilde{k}_{new}$ income inequality is rising, to the right of $\tilde{k}_{new}$ is declining. Hence, the only difference between the evolution of income and wealth inequality is that the critical level of k, where divergence comes to an halt, differs. But qualitatively, the path is the same, contingent on chosing the initial capital stock appropriately. That is, we may see divergence for a while, but ultimately this tend is replaced by one of convergence.

Still, as pointed out perfect convergence does not occur if the structural characteristics of the household, or the national state under the broader interpretation, differ. The next subsection provides an example.

2.1 An Extension: Differences in Productivity

Imagine now that income of a citizen of country c is given by

$$y_c = ra_c + w\varepsilon_c,$$

where ε_c could reflect human capital or perhaps differences in labor supply (participation). Either way we would now have the following equation governing the evolution of wealth per person in country c

$$a_{ct+1} = \frac{a_{ct} + sra_{ct} + sw\varepsilon_c + \bar{s}}{1+n}.$$

Assume, for simplicity, that the expected value for $\varepsilon : E(\varepsilon_c) = 1$, and that $VAR(\varepsilon_c) = \sigma_\varepsilon$,and constant. In that case we would have the following equation governing the variance for wealth:

$$\sigma_{at+1} = \left(\frac{1+sr}{1+n}\right)^2 \sigma_{at} + \frac{s}{1+n}\sigma_\varepsilon.$$

On average, however, everything works as above since $E(\varepsilon_c) = 1$.

$$k_{t+1} - k_t = \frac{1}{1+n}\left(sf(k) + \bar{s} - nk_t\right)$$

Assuming the economy is approaching the stable steady state from below, we will ultimately end up with a stable distribution of wealth as well, $\sigma_{at+1} = \sigma_{at} = \sigma_a^*$:

$$\sigma_a^* = \frac{(1+n)s}{(n - sf'(k^*))(2 + n + sf'(k^*))}\sigma_\varepsilon.$$

Recall, that $sf'(k^*) < n$. Hence, in this case there will not be complete equalization of wealth in the long run.

References

[1]Kuznets, S, 1955. Economic Growth and Income Inequality American Economic Review, Vol 45, 1-28.

[2]Pritchett, L, 1997. Divergence, Big Time. Journal of Eonomic Perspectives.

[3]Stiglitz, J., 1969. Distribution of Income and wealth across individuals. Econometrica, 37, p. 382-97