Bills and Klenow (2000): Does Schooling Cause Growth?

The first order condition (11)

Suppose the consumer has log preferences (this is the assumption in the later section on calibration anyway):

$$\begin{aligned} & \int_0^T \ln c_t e^{-\rho t} dt + \int_0^s \xi e^{-\rho t} dt \\ = & \int_0^T \ln c_t e^{-\rho t} dt - \frac{\xi}{\rho} e^{-\rho s} + \text{ stuff that diss. in max} \end{aligned}$$

where c is consumption and ξ reflects the flow utility from attending school.

So the problem is to find a sequence of consumption levels along with an optimal number of years of schooling:

$$\left(\{c\}_{t=0}^T, s\right) = \arg\max \int_0^T \ln c_t e^{-\rho t} dt - \frac{\xi}{\rho} e^{-\rho s}$$

Subject to:

$$\int_s^T e^{-rt} w(t) h(t) dt - \int_0^T e^{-rt} c(t) dt - \mu \int_0^s w(t) h(t) e^{-rt} dt = 0$$

and

$$h(t) = E_t e^{f(s)+g(t-s)} \text{ for all } t > s.$$

where $E(t)$ is an externality (taken as given by the individual; captures the influence by teachers).

Lagrange

$$\int_0^T \ln c_t e^{-\rho t} dt - \frac{\xi}{\rho} e^{-\rho s} + \lambda \left(\int_s^T e^{-rt} w(t) h(t) dt - \int_0^T e^{-rt} c(t) dt - \int_0^s \mu w(t) h(t) e^{-rt} dt \right)$$

FOC: Wrt consumption at any given point in time

$$c_t : \frac{1}{c_t} e^{-\rho t} = \lambda e^{-rt} \text{ for all } t$$

in particular "time" s:

$$\frac{1}{c(s)} e^{-\rho s} = \lambda e^{-rs} \Leftrightarrow \lambda = c(s)^{-1} e^{-(\rho - r)s}$$

Optimal schooling (here I'm applying Leibnitz' rule for differentiation of an integral):

$$s : \xi e^{-\rho s} + \lambda \left\{ \left[-e^{-rs} w(s) h(s) \right] + \right.$$

$$\left.\int_s^T [f'(s) - g'(t-s)]\, e^{-rt} w(t)\, h(t)\, dt - \mu w(s)\, h(s)\, e^{-rs} \right\} = 0$$

(note that $\int_0^s \mu w(t) \frac{\partial h(t)}{\partial s} e^{-rt} dt = 0$ since $h(t)$ only is defined for $t > s$ ... more schooling does not change the alternative costs associated with schooling). Moving on:

$$\xi e^{-(\rho - r)s} + \lambda [-(1+\mu)\, w(s)\, h(s)] + \int_s^T [f'(s) - g'(t-s)]\, e^{-r(t-s)} w(t)\, h(t)\, dt = 0$$

$$\xi e^{-(\rho - r)s} = \lambda \left\{ [(1+\mu)\, w(s)\, h(s)] - \int_s^T [f'(s) - g'(t-s)]\, e^{-r(t-s)} w(t)\, h(t)\, dt \right\}$$

substituting for λ

$$\xi e^{-(\rho - r)s} = c(s)^{-1}\, e^{-(\rho - r)s} \left\{ [(1+\mu)\, w(s)\, h(s)] - \int_s^T [f'(s) - g'(t-s)]\, e^{-r(t-s)} w(t)\, h(t)\, dt \right\}$$

So

$$\xi c(s) = [(1+\mu)\, w(s)\, h(s)] - \int_s^T [f'(s) - g'(t-s)]\, e^{-r(t-s)} w(t)\, h(t)\, dt$$

which is the expression shown in the text of the paper.