Final Exam in Economic Growth
University of Copenhagen
Institute of Economics
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Abstract

The exam consists of 3 assignments. You should attempt to answer all the questions. The weights used to determine the final grade are indicated in parenthesis. While the questions are stated in English the exam may be answered either in Danish or English.

1. Shorter questions (20 percent)

Do you agree with statements A - C? Explain why or why not.

A. "The so-called "Dual" estimates of total factor productivity growth are calculated as

$$\dot{A}/A = \dot{Y}/Y - s_K \cdot \left(\dot{K}/K\right) - s_L \cdot \left(\dot{L}/L\right),$$

where $\dot{Y}/Y$ is the growth rate of GDP, $\dot{K}/K$ represents the growth rate of the capital stock, $\dot{L}/L$ is the growth rate of labor input, and s_i, $i = K, L$, represents capital and labor's share in national income, respectively."

B. "Judged from an Uzawa-Lucas model, one should expect a difference in how the growth rate of consumption changes following a natural disaster, which only destroys physical capital, compared with an epidemic, which only destroys human capital."

C. Consider Table 1 below. When focusing on the results from the intermediate sample: "The estimated parameters are well in accord with priors as to their signs and nummerical size."

TABLE I

ESTIMATION OF THE TEXTBOOK SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n + g + \delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\overline{R}^2$	0.59	0.59	0.01
s.e.e.	0.69	0.61	0.38
Restricted regression:			
CONSTANT	6.87	7.10	8.62
	(0.12)	(0.15)	(0.53)
$\ln(I/GDP) - \ln(n + g + \delta)$	1.48	1.43	0.56
	(0.12)	(0.14)	(0.36)
$\overline{R}^2$	0.59	0.59	0.06
s.e.e.	0.69	0.61	0.37
Test of restriction:			
p-value	0.38	0.26	0.79
Implied α	0.60	0.59	0.36
	(0.02)	(0.02)	(0.15)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

Source: Mankiw, Romer and Weil, 1992. A Contribution to the Empirics of Economic Growth. *Quarterly Journal of Economics*, 107, p. 414.

2. Endogenous Growth and Endogenous Policy (40 percent).

Consider an economy inhabited by L *infinitely lived agents indexed by* i*.* L is constant. *It is assumed that agents differ with respect to their initial endowment of capital,* $k^i(0)$*. In all other respects, agents are identical. The government levies a proportional tax on income, at the constant rate* τ*, which ultimately is determined by majority voting. The proceeds from taxing income are used to finance lump sum transfers to all* L *agents at the rate* $\theta(t)$*. The government balances its budget. Hence*

$$\theta(t)L = \tau \sum_i^L y^i(t) = \tau y(t) L,$$

where $y(t) \equiv \left(\sum_i^L y^i\right)/L$ *is income per capita. The average capital stock in the economy is defined as* $k(t) \equiv K(t)/L = \left(\sum_i^L k^i\right)/L$*. The problem facing agent* i *is to*

$$\max_{(c^i(t))_{t=0}^{\infty}} \int_0^{\infty} \ln c^i(t) e^{-\rho t} dt, \ \ \rho > 0$$

$$c^i(t) \geq 0,$$
$$\dot{k}^i(t) = y^i(t)(1-\tau) + \theta(t) - c^i(t), \ \ k^i(0) \text{ given},$$
$$y^i(t) = Ak^i(t),$$
$$\lim_{t\to\infty} k^i(t) e^{-(1-\tau)At} \geq 0.$$

A. Solve the above maximization problem, and show that

$$\frac{\dot{c}^i(t)}{c^i(t)} = A(1-\tau) - \rho.$$

Comment on this expression.

Given the AK-structure of the model there are no transitional dynamics. Moreover it holds that

$$\gamma \equiv \frac{\dot{c}^i(t)}{c^i(t)} = A(1-\tau) - \rho = \frac{\dot{k}^i(t)}{k^i(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{y}(t)}{y(t)} = \frac{\dot{\theta}(t)}{\theta(t)} \ \forall i, t.$$

B. Show that the level of consumption for individual i is given by

$$c^i(t) = \left[\tau A \sigma^i + \rho\right] k^i(0)\, e^{\gamma t},$$

where $\sigma^i \equiv k(t)/k^i(t)$, which is constant through time.

The problem of choosing the preferred tax rate for individual i is therefore to

$$\max_{\tau} \int_0^{\infty} \ln c^i(t)\, e^{-\rho t} dt, \;\; \rho > 0$$

s.t.

$$c^i(t) = c^i(0)\, e^{\gamma t} = \left[\tau A \sigma^i + \rho\right] k^i(0)\, e^{\gamma t}$$
$$\gamma = A(1-\tau) - \rho.$$

C. (i) Solve the problem for individual i of choosing the preferred tax rate τ^i. (*Hint:* use the fact that $\int_0^{\infty} \ln\left(c^i(0)\, e^{\gamma t}\right) e^{-\rho t} dt = \frac{1}{\rho}\left(\ln c^i(0) + \frac{\gamma}{\rho}\right)$ and proceed to solve the static maximization problem). **(ii)** Explain why and how τ^i depends on σ^i, A and ρ in the manner indicated by the formula you find.

D. Assuming majority voting over taxes, and full participation at elections, what will be the implemented tax rate? Proceed to derive the long-run growth rate of income per capita. Comment on the results.

3. R&D-Based Growth (40 percent)

Consider an economy where final output of firm i, Y_i, is produced using the following technology

$$Y_i = A L_i^{1-\alpha} \sum_{j=1}^{N} x_{ij}^{\alpha}, \;\; A > 0.$$

A is a parameter, L_i is labor input and x_{ij} is input of intermediate good j. It is assumed that the labor force is constant through time, $L(t) = L = \sum_i L_i$. The factor prices for labor and intermediate good j are w and P_j, respectively. In the final goods sector perfect competition prevails. The price of final output is normalized to 1. Profit maximization imply that demand for intermediate good j for firm i satisfies

$$P_j = \alpha A L_i^{1-\alpha} x_{ij}^{\alpha-1} \text{ for all } j.$$

Moreover, aggregate demand for intermediate good j, $X_j = \sum_i x_{ij}$, is

$$P_j = \alpha A L^{1-\alpha} X_j^{\alpha-1}.$$

The intermediate goods sector consists of $j = 1, ..., N$ firms. Each firm operates as monopolist in their market for intermediate good j, since they all hold a patent of infinite duration. Each firm uses a technology which involves spending one unit of output so as to produce one unit of intermediate good. Accordingly, profits for intermediate goods firm j are given by

$$\Pi_j = P_j X_j - X_j.$$

A. Write down the relevant profit maximization problem for firm j in the intermediate good sector, and proceed to show that the optimal monopoly price and quantity of X_j are

$$P_j = 1/\alpha,$$

and

$$X_j = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L,$$

respectively, and that the value of holding a patent on production of intermediate good j is

$$V^j(t) = \int_{v=t}^{\infty} \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L e^{-\int_{\omega=t}^{v} r(\omega) d\omega} dv.$$

Comment on these results.

Research is conducted by spending units of output. Specifically, spending η units of output produces, deterministically, 1 new idea. It is assumed that free entry prevails in the R&D sector. Consequently $V(t) = \eta$ must hold in equilibrium.

The households maximize discounted utility, and are infinitely lived. The per-period utility function is $u(c(t)) = \left(c(t)^{1-\theta} - 1\right)/(1-\theta)$. *The optimal evolution of consumption is*

$$\dot{c}(t)/c(t) = \frac{1}{\theta}(r - \rho).$$

Since the model, structurally, is an "AK-model" there are no transitional dynamics. Hence $\dot{c}(t)/c(t) = \dot{Y}(t)/Y(t) = \dot{N}(t)/N(t) \equiv \gamma$ *at all points in time.*

B. Show that the growth rate of income per capita is given by

$$\gamma = \frac{1}{\theta}\left[\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{2}{1-\alpha}}A^{\frac{1}{1-\alpha}}\frac{L}{\eta} - \rho\right],$$

and explain why the various variables affects the growth rate in the manner indicated.

As it turns out, a social planner would choose a faster pace of consumption growth. Specifically, maximizing the discounted utility of the representative agent, subject to the resource constraints of the economy, yields

$$\gamma^{sp} = \frac{1}{\theta}\left[\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{1}{1-\alpha}}A^{\frac{1}{1-\alpha}}\frac{L}{\eta} - \rho\right]$$

C. (i) Comment on the difference between the two growth rates. **(ii)** Could an appropriate subsidy to R&D, financed by lump-sum taxes, make the market outcome socially optimal?

Solutions to
Final exam in Economic Growth
Spring 2003

1. Shorter questions.

1A. No (Dual TFP estimates are discussed in Hseih, 1999). The stated formula relates to "primal" TFP estimates. The dual counter part uses factor prices to calculate TFP growth rates. Specifically

$$\dot{A}/A = s_L \cdot \frac{\dot{w}}{w} + s_R \frac{\dot{r}}{r},$$

where w and r represents the real wage, and the real rate of interest. Theoretically primal and dual estimates are equivalent of course. However, due to problems associated with the measurement of capital (and rental rates) the two measures may diverge, as evident from the debate between Young and Hsieh on the "miraculous" East-asian growth performance

1B. Yes (Barro and Sala-i-Martin Ch. 5). Assume that the economy initially is on a balanced growth path. Consumption per capita grows at a constant rate, given by the Keynes-Ramsey rule

$$\gamma_C^* = \frac{1}{\theta}\left(r^* - \rho\right),$$

where r^* signifies the constant real rate of return on capital investments. A reduction in the capital stock will increase the real rate of return on capital investments. This follows from diminishing returns to physical capital in the production function. Consequently, in transition, the growth of consumption will be higher than γ_C^*. A once-and-for-all reduction in the stock of human capital will however reduce the real rate of return on capital investments, since physical and human capital are complements in the production of output. In transition, therefore, the growth rate of consumption will be lower that its long-run level. In this sense, recovery will be faster following a natural disaster, compared with an epidemic.

1C. On balance, No (Mankiw, Romer and Weil, 1992; from now on MRW). (The argument put forward by the student is more important than whether the student ultimately chooses to say "No" or "Yes").

The following observations can be made. The structural equation being estimated is

$$\ln\left(\frac{Y(t)}{L(t)}\right) = \beta_0 + \beta_1 \ln s + \beta_2 \ln(n+\delta+g) + \varepsilon,$$

where $\beta_0 = \ln A(0) + gt$, $\beta_1 = \alpha/(1-\alpha)$, $\beta_2 = -\beta_1$ and ε is assumed to be noise.

The Solow model fits the data well in that β_1 and β_2 are of the right sign predicted by the basic solow model; a higher investment rate, and a lower rate of population growth implies a higher growth rate, ceteris paribus. Moreover, the restriction $\beta_1 = -\beta_2$ cannot be rejected, which is also consistent with the predictions of the basic Solow model.

But the estimate value for α is too high. Assuming perfect competition, the parameter α should reflect capitals' share in total income. Hence, a resonable prior, based on national accounts data, is therefore roughly 1/3. As can be seen from the table, the implied α is around 0.6. Over-all therefore, all is not well. The student may wish to point out that a likely reason for capital's share to be overestimated is omitted variable bias. MRW argue that "human capital" is precisely the variable being omitted, and goes on to show that once proxies for human capital are entered into the regression equation, the implied value for capital's share declines to rougly 1/3. Doubts can however be raised as for the assumption of common levels of technology (which Islam 1995 shows is doubtful) and more broadly whether the same model fits all countries in the world (Durlauf and Johnson, 1995)

2. Endogenous Growth and Endogenous Policy

Essentially the model is a basic AK model (Barro and Sala-i-Martin Ch. 4). Introducing heterogenous agents into this enviornment, and determining policy through voting, is developed in Alesina and Rodrik (1994). However, whereas the Alesina and Rodrik model uses productive government investments, the model below introduces pure redistribution which simplifies the algebra. Accordingly the exact model is new to the students which compensates for the fact that the formal structure is much easier to work with.

2A. The Hamiltonian is

$$H\left(c^i, k^i, \lambda, t\right) = \ln c^i(t)\, e^{-\rho t} + \lambda\left[Ak^i(t)(1-\tau) + \theta(t) - c^i(t)\right]$$

The first order conditions are

$$H_c = 0 : \frac{1}{c^i} e^{-\rho t} = \lambda$$

$$H_k = -\dot{\lambda} : \lambda A(1-\tau) = -\dot{\lambda},$$

and the transversaility condition. Standard manipulations of H_c and H_k leads to

$$\frac{\dot{c}^i}{c^i} = A(1-\tau) - \rho,$$

which represents the Keynes-Ramsey rule, which states that at all points in time the marginal rate of substitution must equal the marginal rate of transformation. The growth rate of consumption is increasing in the after-tax return on capital investments. Hence, if τ is increased, consumers will tend to cut savings implying slower growth in consumption.

2B. Using the balanced growth property along with the Ramsey rule and the law of motion for the capital stock for individual i implies immediately:

$$\frac{\dot{c}^i}{c^i} = A(1-\tau) - \rho = \frac{\dot{k}^i}{k^i} = A(1-\tau) + \frac{\theta(t) - c^i(t)}{k^i(t)}.$$

Using that budget balance implies

$$\theta(t) = \tau A k(t)$$

we have

$$\rho = \tau A \sigma^i - \frac{c^i}{k^i}.$$

Rearrangeing terms leads to the stated result. Note that σ^i reflect the inverse capital share of individual i. Hence, a "large" σ^i indicate that the agent is relatively poor.

2C. Using the hint it follows that the problem of individual i is to choose

$$\tau^i = \arg\max \frac{1}{\rho}\left(\ln\left[\tau^i A\sigma^i + \rho\right] + \ln k^i(0) + \frac{\gamma}{\rho}\right),$$

st

$$\gamma = A\left(1-\tau^i\right) - \rho.$$

The first order condition is therefore

$$\begin{aligned}
\frac{\partial c^i(0)}{\partial \tau}\frac{1}{c^i(0)} &= \frac{\partial\gamma/\partial\tau}{\rho} \\
&\Updownarrow \\
\frac{A\sigma^i}{\tau^i A\sigma^i + \rho} &= \frac{A}{\rho} \\
&\Updownarrow \\
\rho\sigma^i &= \tau^i A\sigma^i + \rho \\
&\Updownarrow \\
\tau^i &= \frac{\rho}{A}\left(\frac{\sigma^i - 1}{\sigma^i}\right),
\end{aligned}$$

The tax rate preferred by individual i depends in the individuals share of capital, σ^i. As is clear, the poorer the agent (a larger σ^i) implies that the individual prefers a higher tax, and thus more redistribution. The preferred tax rate may be negative, corresponding to a subsidy (if the individual is endowed with above average wealth - $\sigma^i < 1$). A larger level of productivity, A, or a lower rate of time preference, will imply a lower preferred tax rate.

The intuition is as follows. The marginal benefits (MB) of an increase in the tax rate is $\frac{\partial c^i(0)}{\partial\tau}\frac{1}{c^i(0)}$, while marginal costs (MC) are given by $\frac{\partial\gamma/\partial\tau}{\rho}$. Fundamentally, a higher tax rate implies a higher level of consumption as of time zero, but slower growth of the same. The Figure illustrates two consumption paths for two different levels of taxation, all else equal. As can be seen, the two paths intersect. Hence, redistribution implies an intertemporal reallocation of consumption, from tomorrow to today. Hence, only if the consumer is impatient, $\rho > 0$, should this be attractive. This is why only $\rho > 0$ can be consistent with $\tau^i > 0$. Second, MB is increasing in σ^i. The poorer the agent, the larger the relative gain in consumption "today". This is why poorer agents prefer more redistribution. "A" affects both costs and benefits. MC are rising in A, since the reduction in the growth rate is increasing in the size of A. MB are on net increasing in A, but there are two countervailing forces. On the one hand, a higher level of A implies a higher level of consumption, which means smaller MB from futher increases

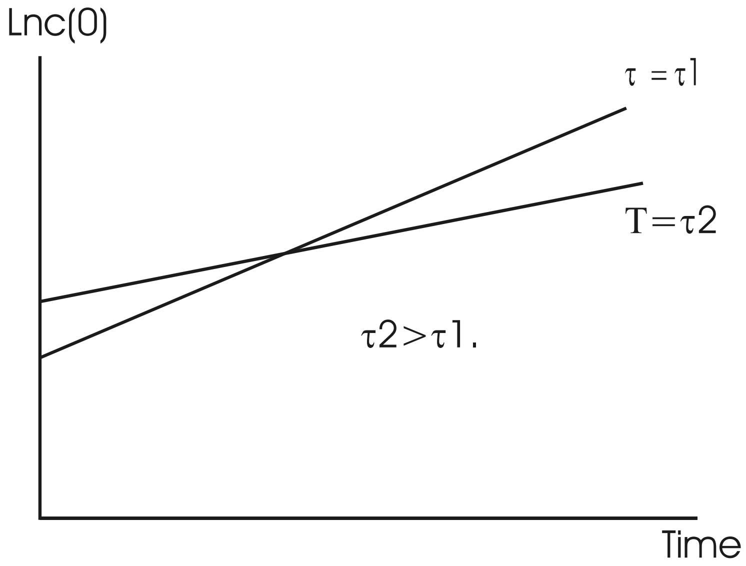

Figure 1: The time path for log consumption for varying levels of τ.

due to diminishing marginal utility. On the other hand, a larger A implies more redistribution of any given level of taxation, which makes redistribution somewhat more attractive. It is worth noting that a negative tax may be preferred by the agent (i.e. a capital subsidy) if $\sigma^i < 1$.

2D. Assuming majority voting, the outcome of elections should be the tax rate preferred by the median voter. Given full participation the median voter will be equal to the person with median wealth. Accordingly, the implemented tax rate is

$$\tau = \frac{\rho}{A}\left(1 - \frac{1}{\sigma^m}\right),$$

where $\sigma^m = k/k^m$. One may observe that $\sigma^m > 1$ is an empirically reasonable assumption (i.e. empirically the distribution of wealth is skewed to the right). Substituting for τ in the Keynes-Ramsey rule yields

$$\begin{aligned}\gamma &= (1-\tau)\,A - \rho \\ &= A\left(1 - \frac{\rho}{A}\left(1 - \frac{1}{\sigma^m}\right)\right) - \rho \\ \gamma &= A - \rho - \rho\left(1 - \frac{1}{\sigma^m}\right).\end{aligned}$$

Hence, in reduced form a more unequal distribution of wealth (strickly speaking measured by *skewness* of the distribution if wealth) should be associated with a lower growth rate. Thus "inequality is bad for growth". Evidence presented in Alesina and Rodrik indicate that this prediction is bourne out in the data. However, the authors' do not test for the avocated mechanisms of the model (i.e. the tax/growth relationsship and the link between inequality and taxes). A final observation is that if $\sigma^m = 1$ (equivalent to a symmetrical distribution for wealth) the "optimal" tax rate is zero (and redistribution is zero), and the associated growth rate $\gamma = A - \rho$ corresponds to the choice of an altruistic social planner who maximizes the utility of a representative agent. The intuition should be clear: in the present model there are no externalities, hence the market solution – under laissez-faire – is pareto optimal.

3. R&D-based growth

This assigment covers the bulk of Chapter 6 in B&S.

3 A

The problem is to

$$\max_{P_j, X_j} \Pi_j = P_j X_j - X_j.$$

st

$$P_j = \alpha A L^{1-\alpha} X_j^{\alpha-1}.$$

One may proceed to solve this problem directly, or by recalling that the optimal quantity to found where MR=MC.

Total revenue:

$$P_j X_j = \alpha A L^{1-\alpha} X_j^{\alpha}$$

Marginal revenue:

$$MR_j = \alpha^2 A L^{1-\alpha} X_j^{\alpha-1}$$

Marginal costs:

$$MC_j = 1$$

Hence the optimal quantity of good j is

$$MR_j = MC_j \Rightarrow \alpha^2 A L_i^{1-\alpha} X_j^{\alpha-1} = 1$$

$$\begin{aligned} X_j^{\alpha-1} &= \frac{1}{\alpha^2 A L^{1-\alpha}} \\ &\Updownarrow \\ X_j &= \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L. \end{aligned}$$

The monopoly price is therefore (using the demand curve):

$$\begin{aligned} P_j &= \alpha A L^{1-\alpha} \left(\alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L\right)^{\alpha-1} \\ &= \frac{1}{\alpha}. \end{aligned}$$

Since $\alpha < 1$ it follows that $MC < P$, where α parameterizes the mark-up.

The value of patent is the discounted value of profits:

$$V_j(t) = \int_{v=t}^{\infty} \Pi_j(v) e^{-\int_{\omega=t}^{v} r(\omega) d\omega} dv$$

Now since its clearly the case that the price and quantity of intermediate good j are constant across firms, so are profits:

$$\begin{aligned} \Pi_j &= (P_j - 1) X_j \\ &= \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L. \end{aligned}$$

Hence

$$V_j(t) = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L \int_{v=t}^{\infty} e^{-\int_{\omega=t}^{v} r(\omega) d\omega} dv = V(t).$$

3 B.

Using the fact that

$$V(t) = \left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L \int_{v=t}^{\infty} e^{-\int_{\omega=t}^{v} r(\omega) d\omega} dv = \eta$$

it follows that r is constant. Solving the integral yields

$$\eta = \frac{\left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L}{r}.$$

Since the model features balanced growth, it follows that

$$\begin{aligned} \gamma &= \frac{1}{\theta}(r - \rho) \\ \gamma &= \frac{1}{\theta}\left(\frac{\left(\frac{1-\alpha}{\alpha}\right) \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} L}{\eta} - \rho\right). \end{aligned}$$

The interpretation required are given in B&S Ch. 6.1.4.

3 C.

It may be instructive to write the growth rate in the following way

$$\gamma = \frac{1}{\theta}\left(\left(\frac{1-\alpha}{\alpha}\right)\frac{\bar{x}}{\eta} - \rho\right)$$

where $\bar{x}$ is the quantity of (each) intermediate produced in equilibrium. The key thing to know is that the solution for x in the market economy is lower than the socially optimal level. This is due to the monopoly distortion. As a result, the market real rate of interest is lower than its socially optimal level.

An R&D subsidy will effectively reduce the costs of doing research (η). By choosing the subsidy appropriately, one may in fact produce the socially optimal real rate of return. But the market economy will still feature an inefficiently low *level* of GDP, since

$$Y = AL^{1-\alpha}\bar{x}^{\alpha}N.$$

The subsidy does not remedy the monopoly distortion. As a result, a subsidy to R&D is **not** an appropriate policy instrument in the current case (B&S p. 222 ff).