Exercises in Growth Theory and Empirics*

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Exercise 6: Productive government investments and exogenous growth

Consider the following growth model for a closed economy with a government sector. Output is produced using the following production function:

$$Y = K^{\alpha} G^{\gamma} (AL)^{1-\gamma-\alpha},$$

where K is private capital, G is public capital (infrastructure, say), A is an index for the level of technology which grows at the exogenous rate x while L is labor input. Private agents save a fraction s of their disposable income, $Y - T$, where T are taxes payed to the government:

$$S^p = s(Y - T),$$

the remaining part is consumed. The government, in turn, invests a fraction σ of total tax revenue, the rest is consumed. So

$$I^g = \sigma T$$

Private capital expands over time according to

$$\dot{K} = S^p - \delta K,$$

whereas public capital increases over time in accordance with

$$\dot{G} = I^g - \delta G.$$

From now on we assume that are proportional to private income:

$$T = \tau Y,$$

where $\tau \in (0, 1)$ is the tax rate.

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Question 1

Show that the the law of motion for government capital in efficiency units of labor ($g = G/AL$) is given by

$$\dot{g} = \sigma\tau k^{\alpha} g^{\gamma} - (n + x + \delta)\, g$$

while private capital per efficiency units of labor ($k \equiv K/AL$) evolves according to

$$\dot{k} = s\,(1-\tau)\, k^{\alpha} g^{\gamma} - (n + x + \delta)\, k.$$

Question 2.

Show that income in efficiency units of labor, in the steady state, is given by

$$y^{*} = \left((1-\tau)^{\frac{\alpha}{1-\alpha-\gamma}}\, \tau^{\frac{\gamma}{1-\alpha-\gamma}}\right) \left(s^{\frac{\alpha}{1-\alpha-\gamma}} \sigma^{\frac{\gamma}{1-\alpha-\gamma}}\right) (n + x + \delta)^{-\frac{\alpha+\gamma}{1-\alpha-\gamma}}.$$

(*Hint*: you might find it useful to begin by rewriting the production function so as to yield: $y = \left(\frac{k}{y}\right)^{\frac{\alpha}{1-\alpha-\gamma}} \left(\frac{g}{y}\right)^{\frac{\gamma}{1-\alpha-\gamma}}$, and then proceed to figure out what $\frac{k}{y}$ and $\frac{g}{y}$ are in steady state.)

Question 3.

What's the relationsship between y^{*} and τ? Explain.

Question 5.

(a) Suppose both the private and government investment rate rises by one percentage point. How many percent will y^{*} increase ? (b) Using plausible parameter values for α and γ, what's the numerical value of this elasticity? (Hint: Consult Aschauer,1989 for "γ") Comment on whether its larger or smaller than what one would expect based on a standard Solow model, and explain why this is so.

Question 6.

Show that the rate of convergence to steady state is given by

$$\beta = -\,(1 - \alpha - \gamma)\,(n + \delta + x)\,.$$

How does this result depart from the Solow model? How does it compare with empirical estimates?

Exercise 7: The dynamics of the savings rate in the Ramsey-Cass-Koopmans model

Solve exercise 2.3 in Barro and Sala-i-Martin.

Exercise 8: Various questions for review (mainly on endogenous growth)

True or false? Explain.

1. In one sector growth models in general: "Unless the aggregate production function is lineare in reproducible factor(s) of production, endogenous growth cannot be obtained theoretically"

2. In one sector growth models with optimizing infinitely lived households: "If we assume that the aggregate production function exhibits constant returns to the reproducible factor(s) of production then that is sufficient for endogenous growth".

3. The implied rate of convergence is faster in the Ramsey-Cass-Koopmans model, compared with the Solow model.

4. The "learning rate" says how much cost decline when output increases by 1 percent.

5. If endogenous growth is driven by learning-by-doing, the market outcome will be socially inoptimal.

Suppose a researcher uses the following structural production function $Y = K^{\alpha} (AL)^{1-\alpha}$ and performs growth accounting on a large cross-section of countries. Assume each economy in the sample exhibits endogenous growth due productive government investments, financed by production taxes. Then:

6. One would expect that TFP growth rates should be negatively correlated with the tax rate in the individual countries.
7. One would expect that the correlation between TFP growth and rate of growth of the capital stock should be virtually zero, which is inconsistent with available evidence.

Exercise 9: Productive government investments, endogenous growth, Congestion and Scale effects[1]

Consider the following growth model for a closed economy with a government sector. Firm i uses the following technology to produce output:

$$Y_i = K_i^{\alpha} L_i^{1-\alpha} \hat{G}^{\pi}, \tag{1}$$

where K_i and L_i represents the capital stock and labor input of firm i respectively, while

$$\hat{G} = \frac{G}{K^{\lambda} L^{\phi}}. \tag{2}$$

K represents the aggregate capital stock, and L the total labor force whereas G is public capital (infrastructure, say). Let r denote the return on capital investments, and w the real wage. For simplicity it is assumed that capital does not depreciate. G is financed by taxing *household* income.

Question 1

What would be a reasonable interpretation of equation (2)?

Question 2

Solve the profitmaximization problem of firm i, and proceed to show that it implies that aggregate output per worker can be written

$$y = k^{\alpha} \hat{G}^{\pi},$$

where $k = K/L$ and $y = Y/L$.

The representative household maximizes

$$U_0 = \int_0^{\infty} \frac{c^{1-\theta}}{1-\theta} L e^{-\rho t} dt$$

s.t.

$$c_t \geq 0$$
$$\dot{K} = (1-\tau) rK + (1-\tau) wL - cL,$$
$$\lim_{t \to \infty} K_t e^{-\int_{s=0}^{t} r_s ds} \geq 0.$$

$\tau \in (0, 1)$ is the income tax rate. The labor force, L, is assume to be constant over time. Note that, to simplify notation, we have already imposed the requirement that (in a closed economy) wealth per person in the representative household equals the capital stock per capita.

[1] Based on Glomm and Ravikumar (1997) "Productive government expenditure and Long-run growth". Journal of Economic Dynamics and Control, 21, p. 183-204

Question 3

Derive the Keynes-Ramsey rule.

Question 4.

Show that the above tax scheme, along with constant returns to capital and labour implies

$$G = \tau Y.$$

Question 5.

Use the above result to derive the aggregate production function.

Question 6

Under what parameterrestriction will r be constant at all points in time?

Assuming this condition holds, the model reduces to an "AK model". Hence, there are no transitional dynamics, and the model exhibits balanced growth.

Question 7

Show that the growth rate of income per capita, γ, is given by

$$\gamma = \frac{1}{\theta}\left[(1-\tau)\left(1+(\lambda-1)\pi\right)\tau^{\frac{\pi}{1-\pi}}L^{\frac{\pi(1-(\lambda+\phi))}{1-\pi}} - \rho\right],$$

and explain why the relationsship between γ and L is ambigious. Is it possible to have both endogenous growth, and no scale effects?