Planner Problem in the Model of Public Goods featuring Congestion: Math.

Carl-Johan Dalgaard
Department of Economics
Economic Growth, Spring 2007.

The problem at hand (time indices suppresed):

$$\max_{\{c,G\}_0^\infty} \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

$$c_t \geq 0,\ G \geq 0,$$

$$\dot{K} = Y - C - G - \delta K,\ K_0 \text{ given}$$

$$Y = Kf\left(\frac{G}{Y}\right),$$

$$K_t \geq 0 \text{ for all } t.$$

Hamiltonian

$$H(c, G, \lambda, t) = \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} + \lambda\left(Kf\left(\frac{G}{Y}\right) - C - G - \delta K\right)$$

First order conditions

$$c : \theta c^{-\theta} e^{-\rho t} = \lambda$$

$$G : \lambda \frac{\partial Y}{\partial G} = \lambda \Leftrightarrow \frac{\partial Y}{\partial G} = 1$$

$$K : \lambda\left(\frac{\partial Y}{\partial K} - \delta\right) = -\dot{\lambda}.$$

Standard calculations leads to

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\frac{\partial Y}{\partial K} - \delta - \rho\right],$$

i.e. the Keynes-Ramsey rule. But what about $\partial Y / \partial G$ and, equally important, $\partial Y / \partial K$?

Total differentiation of the production function yields:

$$dY = dKf\left(\frac{G}{Y}\right) + Kf'\left(\frac{G}{Y}\right)\left(\frac{dGY - GdY}{Y^2}\right)$$

Isolate dY:

$$\begin{aligned} dY &= dKf\left(\frac{G}{Y}\right) + \frac{Kf'\left(\frac{G}{Y}\right)}{Y} dG - \frac{Kf'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}{Y} dY \\ dY\left(1 + \frac{Kf'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}{Y}\right) &= dKf\left(\frac{G}{Y}\right) + \frac{Kf'\left(\frac{G}{Y}\right)}{Y} dG \end{aligned}$$

noting that

$$Y = Kf\left(\frac{G}{Y}\right)$$

implies that:

$$dY\left(\frac{f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}\right) = f\left(\frac{G}{Y}\right)dK + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}dG$$

$$dY = \frac{f\left(\frac{G}{Y}\right)^2}{f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}dK + \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}dG.$$

First, we thereby have

$$\frac{\partial Y}{\partial G} = 1 \Leftrightarrow \frac{f'\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)} = 1$$

or

$$\begin{aligned} f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right) &= f'\left(\frac{G}{Y}\right) \\ f\left(\frac{G}{Y}\right) &= f'\left(\frac{G}{Y}\right)\left(1-\left(\frac{G}{Y}\right)\right). \end{aligned}$$

Denote the G/Y ratio which fulfills this equation by τ^{sp}

$$f'\left(\tau^{sp}\right)\left(1-\tau^{sp}\right) = f\left(\tau^{sp}\right).$$

Next, regarding dY/dK :

$$\begin{aligned} \frac{\partial Y}{\partial K} &= \frac{f\left(\frac{G}{Y}\right)f\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)+f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)} \\ &\Updownarrow \\ \frac{\partial Y}{\partial K} &= \frac{f\left(\frac{G}{Y}\right)}{1+\frac{f'\left(\frac{G}{Y}\right)\left(\frac{G}{Y}\right)}{f\left(\frac{G}{Y}\right)}}. \end{aligned}$$

Using that $\tau = \tau^{sp}$

$$\frac{\partial Y}{\partial K} = \frac{f\left(\tau^{sp}\right)}{1+\frac{f'\left(\tau^{sp}\right)\tau^{sp}}{f\left(\tau^{sp}\right)}}$$

and that from $dY/dG = 1$

$$f'\left(\tau^{sp}\right)\left(1-\tau^{sp}\right) = f\left(\tau^{sp}\right) \tag{1}$$

it follows that

$$\begin{aligned} \frac{\partial Y}{\partial K} &= \frac{f\left(\tau^{sp}\right)}{1+\frac{f'\left(\tau^{sp}\right)\tau^{sp}}{f'\left(\tau^{sp}\right)\left(1-\tau^{sp}\right)}} \\ \frac{\partial Y}{\partial K} &= \frac{\left(1-\tau^{sp}\right)f\left(\tau^{sp}\right)}{1-\tau^{sp}+\tau^{sp}} = \left(1-\tau^{sp}\right)f\left(\tau^{sp}\right). \end{aligned}$$

If we insert this into the Keynes-Ramsey rule

$$\gamma = \frac{1}{\theta}\left(\left(1-\tau^{sp}\right) f\left(\tau^{sp}\right) - \delta - \rho\right).$$

where τ^{sp} fulfills, to repeat

$$f'\left(\tau^{sp}\right)\left(1-\tau^{sp}\right) = f\left(\tau^{sp}\right).$$

The latter condition is exatly equal to the condition ensuring "maximum" growth in the market economy, since the market solution is

$$\gamma = \frac{1}{\theta}\left(f\left(\tau\right)\left(1-\tau\right) - \delta - \rho\right)$$

and

$$\frac{\partial \gamma}{\partial \tau} = -f\left(\tau\right) + \left(1-\tau\right) f'\left(\tau\right) = 0$$

thus

$$\left(1-\tau^{*}\right) f'\left(\tau^{*}\right) = f\left(\tau^{*}\right). \tag{2}$$

Comparing equation (1) with equation(2) immediately yield the insight that $\tau^{sp} = \tau^{*}$.

Botton line: The market solution (where τ is selected such that it maximizes growth, $\tau = \tau^{*}$) equals the planners solution, where $\tau = \tau^{sp}$.