THE UZAWA LUCAS MODEL

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SOCIAL PLANNER PROBLEM

In this version of the model (Cf B&S 5.2.2) there are no externalities. Hence, we go straight at the Social planner problem

$$\max \int_0^\infty \ln c_t e^{-\rho t} dt$$

$$c_t \geq 0, u_t \geq 0$$

$$\dot{K} = Y - C - \delta K$$

$$Y = K^\alpha (uH)^{1-\alpha}$$

$$\dot{H} = B(1-u)H - \delta H$$

$$K_t \geq 0, H_t \geq 0 \text{ for all t}$$

Control v.:

State v.:

The Hamiltonian

$$H\left(c,u,H,K,\lambda_K,\lambda_H,t\right)$$
$$=\ln c_t e^{-\rho t}+\lambda_K\left[K^{\alpha}\left(uH\right)^{1-\alpha}-C-\delta K\right]+\lambda_H\left[B\left(1-u\right)H-\delta H\right]$$

FOC

$$c:\frac{1}{c}e^{-\rho t}=\lambda_k$$
$$u:\lambda_k\left(1-\alpha\right)\frac{Y}{u}=\lambda_H BH$$
$$H:\lambda_K\left(1-\alpha\right)\frac{Y}{H}+\lambda_H\left[B\left(1-u\right)-\delta\right]=-\dot{\lambda}_H$$
$$K:\lambda_K\left[\alpha\frac{Y}{K}-\delta\right]=-\dot{\lambda}_K$$

Usual manipulations lead to the K-R rule

$$\frac{\dot{C}}{C} = \gamma_C = \alpha \frac{Y}{K} - \delta - \rho$$

in addition we have

$$\gamma_H = B\left(1 - u\right) - \delta$$

and

$$\gamma_K = \frac{Y}{K} - \frac{C}{K} - \delta$$

A Balanced growth path is quantities $\{Y, C, H, K, u\}_0^\infty$ and prices $\{\lambda_K, \lambda_H\}_0^\infty$ such that $\gamma_j = \gamma_i$ for all $i \neq j$ where $i, j = Y, C, H, K$; λ_K / λ_H is constant and $u =$ constant.

Our strategy is simply to impose this requirement, and derive the growth rate of the economy as well as $u = u^*$.

Step 1: Find $\alpha\left(Y/K\right)^{*}$

Use H_{u}' :

$$\frac{\lambda_{k}}{\lambda_{H}}=\frac{BuH}{\left(1-\alpha\right)Y}$$

and H_{H}'

$$\begin{aligned}-\gamma_{\lambda_{H}} &= \frac{\lambda_{K}}{\lambda_{H}}\left(1-\alpha\right)\frac{Y}{H}+B\left(1-u\right)-\delta\\ &= \frac{BuH}{\left(1-\alpha\right)Y}\left(1-\alpha\right)\frac{Y}{H}+B\left(1-u\right)-\delta=B-\delta\end{aligned}$$

Finally, imposed balanced growth: $\gamma_{\lambda_{H}}=\gamma_{\lambda_{K}}$ and use H_{K}'

$$-\gamma_{\lambda_{K}}=\alpha\left(\frac{Y}{K}\right)^{*}-\delta=-\gamma_{\lambda_{H}}=B-\delta$$

Hence

$$\alpha\left(\frac{Y}{K}\right)^{*} = B$$

Observe that this implies $\gamma_K^* = \gamma_Y^*$, and that the growth rate of C

$$\gamma_C^* = B - \delta - \rho.$$

Step 2: Show $\gamma_K^* = \gamma_C^* = \gamma_Y^*$. This follows since:

$$\gamma_K^* = \left(\frac{Y}{K}\right)^{*} - \left(\frac{C}{K}\right)^{*} - \delta$$

so $\gamma_K^* = \gamma_C^* = \gamma_Y^*$.

Step 3: get u^*

Begin by noting that (from H_c')

$$\frac{\lambda_k}{\lambda_H} = B\frac{uH}{(1-\alpha)\,Y}$$

So $\frac{\lambda_k}{\lambda_H}$ = constant requires H/Y to be constant, for u constant. Hence $\gamma_H^* = \gamma_Y^* = ... = \gamma_c^*$.

Next, use $\dot{H}$ equation

$$\gamma_H^* = B\,(1-u^*) - \delta = \gamma_Y^* = B - \delta - \rho$$

Which leads to

$$u^* = \rho/B.$$

which completes the solution for the balanced growth path

Some remarks on transitional dynamics

Consider the imbalance effect noted in the "simple" Uzawa-Lucas model. We need to figure out the transitional dynamics of Y/K; then we will know how C behaves, and as it turns out, $Q \equiv Y + \lambda_H \dot{H}$

From the production function we find that

$$\frac{Y}{K} = \left(\frac{uH}{K}\right)^{1-\alpha}$$

in addition H_u gives us

$$\frac{\lambda_k}{\lambda_H} = B\frac{uH}{(1-\alpha)\,Y} = B\frac{uH/K}{(1-\alpha)\,Y/K} = B\frac{(Y/K)^{1/(1-\alpha)}}{(1-\alpha)\,Y/K}$$
$$= \frac{B}{1-\alpha}\,(Y/K)^{\frac{\alpha}{1-\alpha}}.$$

Hence, $\gamma_{\frac{\lambda_k}{\lambda_H}} = \frac{\alpha}{1-\alpha}\gamma_{Y/K}$.

Now $-\gamma_{\lambda_H} = B - \delta$ and $-\gamma_{\lambda_K} = \alpha\frac{Y}{K} - \delta$. Hence, using this and the fact that $\gamma_{\frac{\lambda_k}{\lambda_H}} = \frac{\alpha}{1-\alpha}\gamma_{Y/K}$:

$$\gamma_{\lambda_K} - \gamma_{\lambda_H} = \frac{\alpha}{1-\alpha}\gamma_{Y/K} = -\alpha\frac{Y}{K} + \delta + B - \delta$$

so

$$\gamma_{Y/K} = \frac{1-\alpha}{\alpha}\left[B - \alpha\frac{Y}{K}\right] = (1-\alpha)\left[\frac{B}{\alpha} - \frac{Y}{K}\right]$$