Solow vs. Ramsey: When are the implied dynamics equivalent?

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Abstract

This note shows formally that, under certain circumstances, the dynamic system of the Ramsey-Cass-Koopmans model can be expressed as the mirror image of the laws of motion guiding consumption and capital in the Solow model.

Consider a Ramsey-Cass-Koopmans (RCK) model where the dynamical system is given by:

\[ \begin{align*}
\frac{\dot{c}}{c}^{RCK} &= \frac{1}{\theta} (f'(k) - \rho - \theta x), \\
\dot{k} &= f(k) - c - (n + \delta + x) k.
\end{align*} \tag{1} \]

Consider the Solow model, we have that the following hold:

\[ \dot{k} = f(k) - c - (n + \delta + x) k \]

and

\[ c = (1 - s) f(k). \tag{3} \]

\[ \text{Lecture notes: Economic Growth, Spring 2003.} \]

\[ ^{1}\text{Everything is expressed in efficiency units of labor. In order to ease notation the } \sim \text{ has been suppressed. Hence, for } z = c, k \text{ it holds that } z \equiv Z/(AL). \text{ Time subscripts are also ignored for brevity.} \]
Clearly the law of motion for capital is the same in the two models. However, the evolution of consumption differ. Indeed, differentiating wrt time yield

\[
\frac{\dot{c}}{c} = \frac{f'(k)}{f(k)} \cdot \frac{\dot{k}}{k} \\
= \frac{f'(k)}{f(k)} \left[ \frac{f(k) - c}{k} - (n + \delta + x) \right] \\
= \frac{f'(k)}{f(k)} \left[ \frac{f(k) - (1 - s) f(k)}{k} - (n + \delta + x) \right] \\
\left( \frac{\dot{c}}{c} \right)^{\text{*}} = \frac{f'(k)}{f(k)} \left[ s \frac{f(k)}{k} - (n + \delta + x) \right]. \tag{4}
\]

In order for the RCK model to simplify to the dynamical system describing the Solow model we need to figure out under which circumstances equation (4) and (1) are indentical, i.e when \((\frac{\dot{c}}{c})^{\text{S}} = (\frac{\dot{c}}{c})^{\text{RCK}}\). To anticipate the results, we need the following to hold²

A. Production is Cobb-Douglas (CD): \(y = k^{\alpha}\).

B. The following restriction of exogenous variables and parameters hold

\[
\rho = \alpha \theta (n + \delta + x) - (\delta + \theta x)
\]

C. \(\alpha \theta > 1\).

D. \(s \equiv 1/\theta\).

In order to see that (A)-(D) does the job, start by noting that the CD technology implies:

\[
f'(k) = \alpha k^{\alpha - 1} \\
f(k) = k^{\alpha - 1}
\]

²The result stated below was first demonstrated by Kurz (1968). A nice exposition of his result is found in Groth (2003), Appendix B (in Danish); Its downloadable from: http://www.econ.ku.dk/okocg/Makro-mat-ok/kapitler-til-Ramseynote/Ramsey-kap2–jan03.pdf
and combining these:
\[
\frac{f'(k)}{f(k)} = \alpha.
\]

Using these facts in equation (4) yields
\[
\left(\frac{\dot{c}}{c}\right)^S = \alpha \left[ sk^{\alpha - 1} - (n + \delta + x) \right],
\]
while from equation (1) we get
\[
\left(\frac{\dot{c}}{c}\right)^{RCK} = \frac{1}{\theta} \left( \alpha k^{\alpha - 1} - \delta - \rho - \theta x \right) = \frac{1}{\theta} \alpha k^{\alpha - 1} - \frac{\rho + \delta + \theta x}{\theta}.
\]

Now use that \( s \equiv \frac{1}{\theta} \) along with our parameter restriction (B)
\[
\left(\frac{\dot{c}}{c}\right)^{RCK} = \alpha sk^{\alpha - 1} - \frac{\alpha \theta (n + \delta + x) - (\delta + \theta x) + \delta + \theta x}{\theta} = \alpha sk^{\alpha - 1} - \alpha (n + \delta + x) = \left(\frac{\dot{c}}{c}\right)^S.
\]

The only restriction we still haven’t used is \( \alpha \theta > 1 \). This condition is necessary to insure that in fact the utility maximization problem of the representative household is well defined. We require
\[
\rho > \rho - \left(\frac{\alpha \theta - 1}{\alpha \theta}\right) (x + \delta).
\]
Clearly, this condition is only met if $\theta \alpha > 1$.

Hence, in sum, the Solow model can be seen as a special case of the RCK model — in the specific sense that it, in the aggregate, behaves as if the Solow model were the underlying framework. The neat part is that the expression for the rate of convergence, derived in the Solow model, holds for this variant of the RCK model. But the models are still different. Most obviously, as an illustration of the limitation of the above "equivalence", recall that while dynamic inefficiency can occur in the Solow model, this will never be the outcome of RCK model, no matter what parameter constellations (i.e. as long as an optimal solution exists) we might dream up.

**Closing remarks**

B&S state (p.60) that: “the Solow model with a constant savings rate is a special case of the Ramsey model; moreover, this case corresponds to reasonable parameter values". As we saw above, the first part of the statement is true (although the two models are not "identical" - only their dynamics turn out to be the same). What about the plausibility?

First, the restriction $\alpha \theta > 1$ is not completely outlandish. If $\alpha = 1/3$ it requires a $\theta > 3$. Empirical estimates of $\theta$ varies from study to study, but generally fall in the interval 0 to 10 (Kocherlakota, 1996). Typically (growth) people regard $\theta = 4$ as plausible.

Second, concerning the restriction $\rho = \alpha \theta (n + \delta + x) - (\delta + \theta x)$. Clearly, this would only hold under very special circumstances. One take on the restriction, however, is to think of it as a restriction on growth of the labor force

$$n = \frac{\rho - \delta (\alpha \theta - 1) + \theta x (1 - \alpha)}{\alpha \theta}.$$ 

B&S usually work with $n = 0.01$ as a "reasonable" benchmark along with $\delta = 0.05, x = 0.02$ and $\rho = 0.02$. If the Solow/Ramsey "equivalence" is to hold we need that $\theta \approx 10$ which produces $n = 0.011$. This value of $\theta$ is clearly in the very high end of what appears to be empirical plausible.
References
