Systemic risk in financial networks: a graph-theoretic approach

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Abstract

This paper puts forward a novel approach, based on the theory of network flows, for the analysis of default contagion in financial systems. We use a graph-theoretic representation of a financial network and of the flows of losses that can propagate across such a network. Existence and conditions for uniqueness of a propagation of losses and defaults are established. In so doing, we address a known problem of indeterminacy of payment flows that arise in presence of intercyclic patterns of financial obligations. Moreover, we present an algorithm that computes the propagation of losses and defaults across a financial network. Finally, we investigate the relation between some characteristics of a financial networks—such as its degree of capitalization, connectivity, concentration and the relative size of interbank exposures—and its degree of resiliency towards default contagion. We characterize the network structure that prevents the occurrence of default contagion.

JEL classification: C63, G10, G33.

Key words: financial networks, diffusion algorithms, systemic risk.

1 Introduction and motivation

This paper puts forward a novel approach, based on the theory of network flows, for the analysis of systemic risk, i.e., the risk that—in a network of financially interconnected agents—the financial distress of one or more agents is transmitted to other agents in the network, with the possibility of generating a widespread crisis. The literature has distinguished among three different
notions of systemic risk, corresponding to different possible causes of such a phenomenon.\textsuperscript{2} 1) \textit{Informational contagion} in banking systems, where depositors’ expectations about the possibility of a crisis can lead to bank runs; 2) \textit{Direct contagion} transmitted via financial links, i.e., through debt/credit relationships generated on the interbank markets— in banking systems\textsuperscript{3}—or by other kinds of contracts, such as the inter-firm credit chains.\textsuperscript{4} 3) \textit{Common exposure} to exogenous drops in the value of assets, a risk born by agents who hold the same assets or highly correlated assets.

The theoretical contributions to the analysis of systemic risk have mostly focused on informational contagion. The authors of this stream of literature have analyzed the microeconomic behaviour of banks and depositors using models based on the celebrated paper by Diamond and Dybvig (1983), in which liquidity crises due to bank runs arise from self fulfilling depositors’ expectations. Several papers—e.g., Rochet-Tirole (1996), Freixas et al. (2000)—have followed the Diamond-Dybvig approach and analyzed the possibility of bank runs caused by ‘sunspot’ variables that affect the formation of expectations. These papers investigate the effects that such crises have on social welfare and the efficacy of possible government interventions aiming at preventing them. In general, in this class of models, if the total available liquidity is sufficient to satisfy the liquidity needs expressed by ‘impatient’ consumer (that is, depositors who withdraw their money before the long term assets held by banks yield their returns), then interbank loans are an effective way to redistribute liquidity among banks and to share the liquidity risk. Conversely, if there is an aggregate shortage of liquidity, the existence of a net of financial obligations among banks creates the grounds for the diffusion of financial distress and for the occurrence of systemic crises. Other papers based of the Diamond-Dybvig approach, such as Allen and Gale (1998, 2000), present models in which financial crises arise as a consequence of downturns in the economic cycle. Recessions can cause losses in the value of the assets held by banks, losses capable of rendering them insolvent. If depositors foresee the recession, they will protect themselves from possible bank defaults by withdrawing their deposits and, in so doing, they create the conditions for the occurrence of a widespread crisis. In this class of models the crises are determined by changes in the fundamentals of the economy, rather than sunspot variables.

In some of these theoretical papers, the authors investigate, by means of stylized examples, the link existing between the structure of the net of financial obligations among banks and the resiliency of the banking system to possible liquidity and insolvency shocks. Allen and Gale (2000) show that in their example of a ‘complete’ network—a network where all banks are equal to one another, all have mutual bilateral obligations and of the same amount—is more robust than an incomplete network. In the former case, interbank loans distribute liquidity risk across banks in an effective fashion. In an incomplete network, the undesirable effects of interbank loans prevail: financial obligations act as transmission channels that diffuse contagion. Freixas et al. (2000) achieve similar results: in their examples the ‘complete’ network structure bears the smallest

\textsuperscript{2}See the review articles by Dow (2000) and by De Bandt-Hartmann (2000).
\textsuperscript{3}In banking systems, the network of debt/credit relations among banks arises from three sources: a) loans in the interbank money market, b) ‘over-the-counter’ trading in assets and derivatives, c) payment systems.
\textsuperscript{4}See Kiyotaki-Moore (2001, 2002)
risk of contagion, while an incomplete structure increases the fragility of the banking system. In this paper the authors also consider a third case: the structure known as ‘money center’, that is a star-shaped structure with one central node and several peripheral nodes connected only with the central one. For this case, though, the authors do not obtain clear cut results.

Understanding how the structure of financial networks affects their resiliency to systemic shocks is an important issue for central banks and policy makers. In most advanced nations, monetary authorities have imposed rules, known as “large exposure rules”, to limit the credit exposures of banks towards single borrowers and increase the diversification of their portfolio. Both the Basel I and the Basel II committees have recommended this sort of controls on ‘credit risk’. In setting an upper limit to single loans – usually linking the size of a loan to some measure of the capital of the lending bank – these measures also imply a growth in the number of debt/credit relations existing in a financial system, i.e., a growth in the connectivity of the financial network. In several countries, the authorities have encouraged mergers and acquisitions in the banking sector, leading to more concentrated systems with fewer and larger operators. In most cases this policy has reinforced, if not generated, two-tiers banking systems where few large operators act as money centers, i.e. each of them is connected to many small banks which, in turn, are not connected among themselves. Whether this policies have rendered financial systems more resilient to systemic shocks or not, given the structural changes that they brought along, is a question that does not have an obvious answer.

In order to address these issues, and to assess the robustness of different network structures, several authors have focused on the mechanics of default contagion, leaving the microeconomic behaviour of banks and depositors in the background. An entire stream of literature—which includes the works by Sheldon and Maurer (1998), Furfine (2003), Wells (2002), Elsinger, Lehár and Summer (2006), Upper and Worms (2004), Degryse and Nguyen (2004), Blavarg and Nirmander (2002), Cifuentes (2003), Mistrulli (2005,2006)– has analyzed national banking systems, using numerical simulations to evaluate their exposure to systemic risk.

Shin et al. (2005) and Nier et al. (2007) have analyzed generic network structures, rather that specific national ones. Shin et al. present a model where default contagion is exacerbated by the effects of ‘fire sales’. They show that if the demand for illiquid assets is not perfectly elastic, the forced and untimely sale of such assets by financially distressed operators induces further reductions in their market value, feeding further contagion.

Nier et al. build their model on a previous and unpublished version of the present paper. They adapt to a banking sector the generic financial network described below, by introducing depositors as terminal nodes (i.e., sink nodes). Using a computing device, these authors generate random banking networks, in the fashion of the random graphs à la Erdős-Rényi, and use them to run numerical simulations aiming at evaluating the exposure to systemic risk of different network structures.

The ambition of the present work is to investigate the mechanics of default contagion with analitical methods. To this end, we root our analysis into graph

\[ \text{See, on the web page of the Bank for International Settlements, the documents “Principles for the Management of Credit Risk” and the paragraphs 729 and 736 of “The New Basel Capital Accord”.} \]
theory, representing a financial system as a flow network, and introduce a propagation function designed to model the diffusion of losses and insolvencies across such a network. This is done in section 2. In section 3 we discuss the existence of such a function and in section 4 we pin down the conditions that guarantee its uniqueness. In so doing, we address a known problem of indeterminacy of payment flows that arise in presence of intercyclic patterns of financial obligations. In section 5 we present an algorithm that computes the propagation of losses and defaults across a financial network. Finally, in section 6, we investigate the relation between some characteristics of a financial networks—such as its degree of capitalization, connectivity, concentration and the relative size of interbank exposures—and its degree of resiliency towards default contagion.

2 The financial flow network

2.1 A network of financially interconnected agents

We are interested in modelling a system composed of a set of financial operators which are directly or indirectly connected to one another by debt/credit relations. Let \( \Omega = \{ \omega_i \}, i = 1..n \), be the set of such operators. Let \( c_{ij} \in \mathbb{R}^+ \), be the amount of debt, if any, that agent \( i \) owes agent \( j \), and let \( C = \{ c_{ij} \} \), for \( i, j = 1..n \) and \( i \neq j \). Each agent in \( \Omega \) is characterized by its own balance sheet. Let \( a_i \in \mathbb{R}^+ \) be the value of the external assets owned by \( \omega_i \), i.e., assets issued by agents that do not belong to \( \Omega \) and let \( r_i \in \mathbb{R}^+ \) be the sum of the loans granted by \( \omega_i \) to other agents in \( \Omega \), i.e., \( r_i = \sum_j c_{ij} \). On the liability side of the balance sheet, let \( d_i \in \mathbb{R}^+ \) be the sum of the loans granted to \( \omega_i \) by other agents in \( \Omega \), i.e., \( d_i = \sum_j c_{ji} \); and let \( a_i + r_i - d_i \equiv v_i \in \mathbb{R} \) be the net worth of the \( i \)-th agent. Finally, let \( A = \{ a^k \}, k = 1..m \), be a set of external assets such that each \( a^k \) in \( A \) appears in the balance sheet of at least one operator in \( \Omega \), and let \( a^k_i \in \mathbb{R}^+ \) be the amount of asset \( k \) held by agent \( i \), if any.

For our purposes it is convenient to represent this financial system as a multisource network, i.e., a directed and connected graph, with some sources and a sink, whose links are endowed with non-negative capacities. More precisely, let \( N = \{ \Omega, A, t, \Lambda, S, L, \Gamma \} \) be a multisource network where:

1. \( \Omega = \{ \omega_i \} \) is the set of \( n \) nodes that represent the above defined agents;
2. \( A = \{ a^k \} \) is the set of \( m \) source nodes, i.e., nodes with no incoming links, that represent the external assets;
3. \( t \) is the sink, i.e., a terminal node with no outgoing links;
4. \( \Lambda \subseteq \Omega^2 \) is a set of ordered pairs of nodes in \( \Omega \), i.e., a set of directed links \( \{ l_{ij} \} \) representing the liabilities in \( C \), where \( l_{ij} \) starts from node \( \omega_i \) and ends in node \( \omega_j \), and \( l_{ij} \in \Lambda \) only if \( c_{ij} > 0 \);
5. \( S = \{ s^k_{ij} \} \) is a set of directed links, with start nodes in \( A \) and end nodes in \( \Omega \), that connect the external assets to their owners, where \( s^k_{ij} \in S \) only if \( a^k_i > 0 \);
6. \( L = \{ l^k_i \} \) is a set of directed links, with start nodes in \( \Omega \) and end node \( t \), that connect each node in \( \Omega \) to the sink.
7. \( \Gamma : \Lambda \rightarrow \mathbb{R}^+, S \rightarrow \mathbb{R}^+, L \rightarrow \mathbb{R}^+ \) is a map, called capacity function, that associates i) to each \( l_{ij} \) the value of the corresponding liability \( c_{ij} \), ii) to each

\[ \text{See Ahuja et al. (1993), sections 1 and 2, or Diestel (2000), ch. 6.} \]

\[ \text{For the sake of simplicity, and without danger of confusion, we use the same notation} \]

\[ \text{to indicate the nodes in the network and the objects that they represent.} \]
\(s_i^k\) the value of the corresponding asset \(a_i^k\), and iii) to each \(l_i^k\) the net worth, \(v_i\), of its start node \(\omega_i\).

The difference between this network and a generic multisource network lies in the restrictions (i)-(iii) imposed on the capacity function \(\Gamma\) and in the constraints set on the variables \(a_i^k, c_{ij}\) and \(v_i\) by the budget identities of the members of \(\Omega\). We shall refer to \(N\) as a financial flow network or, for brevity, as a network \(N\), while we shall refer to a generic multisource network simply as a network. We use the above defined network \(N\) to model the propagation of financial losses among the agents in \(\Omega\) as a flow across \(N\).

2.2 The propagation of losses and insolvency

In a network \(N\), a propagation of losses is generated either by an exogenous common shock, defined as a drop in the value of some assets in \(A\), or by an idiosyncratic shock, i.e. the insolvency of a single node, usually due to fraud. Since idiosyncratic shocks are modelled as special cases of exogenous shocks (henceforth called simply shocks), we first define the latter.

To define a shock, let \(b^k \in [0, 1]\) be a parameter that measures the fraction of the value of the asset \(a_i^k\) which is lost, and let \([b^k], k = 1...m\), be the vector composed of such parameters. A given exogenous shock is an assignment of value to this vector where at least one of its components assumes a strictly positive value. If \(b^k > 0\) then source node \(a_i^k\) is activated and sends to its direct descendants in \(\Omega\) — i.e., to the nodes \(\omega_i \in \Omega\) such that \(s_i^k \in S\) — a financial loss equal to \(b^k a_i^k\). The flow of losses out of the source nodes is a vector of scalars \([b^k a_i^k]\) and the total value of the external shock is \(\sigma = \sum_A b^k a_i^k\).

The propagation of these losses through the network is governed by the rules of limited liability and debt priority. When a node suffers a loss, this loss is first absorbed by the net worth of the node. Only the residual loss, if any, is passed over to other nodes in \(\Omega\). The losses that are offset by the equity of the agents in \(\Omega\) exit the flow of losses that circulate across the network \(N\). The sink node is the virtual bucket where, for modelling convenience, we direct such losses. For each node \(\omega_i\) in \(\Omega\), let

\[
\beta_i(\lambda_i) = \min \left( \frac{\lambda_i}{v_i}, 1 \right)
\]

be an activation function, where \(\lambda_i\) is the total loss born by the \(i\)-th node—received from source nodes and/or from other nodes in \(\Omega\). Thus the variable \(\beta_i \in (0, 1)\) measures the share of net worth lost by a node. If a node \(\omega_i\) receives a positive flow of losses, it is activated and sends to the sink an amount of its own net worth equal to \(\beta_i v_i\).

If the losses suffered by \(\omega_i\) are larger than its net worth, then this node is insolvent and sends the residual loss, \(\lambda_i - v_i\), to its creditors, i.e., to its direct descendants in \(\Omega\): \(\omega_j \in \Omega\) such that \(l_{ij} \in L\), also said children nodes of \(\omega_i\). For each node \(\omega_i\) in \(\Omega\), let

\[
b_i(\lambda_i) = \max \left( 0, \frac{\lambda_i - v_i}{d_i} \right)
\]

Of course the labelling of the sets \(\Omega, A, \Lambda, S\) and \(L\) does not alter the properties of the network.
be an insolvency function. The variable $b_i \in [0, 1]$ assumes a value of zero if the $i$-th operator is solvent, while it assumes a strictly positive value if the operator defaults. In the latter case, the assets of the insolvent node are liquidated and its creditors get a pro rata refund. We assume that this is done without incurring bankruptcy costs.\(^9\) Thus $b_i$ measures the fraction of the $i$-th agent's debt that can not be recovered through liquidation\(^10\) and the loss born by this agent is $\lambda_i = \sum_k b_k a_i^k + \sum_j b_j c_{ji}$. In case of an idiosyncratic shock, the latter is defined simply as a loss, born by a single agent, large enough to render the agent insolvent, i.e., $\lambda_i > v_i$. The node hit by an idiosyncratic shock acts as a source node, even if it is not in $A$. When the $i$-th agent becomes insolvent, a node $\omega_j$ which is a creditor of node $\omega_i$ receives from the latter a loss equal to $b_j c_{ij}$.

**Definition 1** Let $f : S, \Lambda, L \rightarrow \mathcal{R}^+$ be a map such that: $f(s_i^k) = b^k a_i^k$, $f(l_{ij}) = b_j c_{ij}$, $f(l_i^t) = \beta_i v_i$, and call this function a propagation in a network $N$.

We now proceed to establish that a propagation function is a flow in $N$. A flow over a generic network is a vector valued function, defined over the links of the network, such that: i) for all the links in the network, the scalar associated to a link does not exceed its capacity; and ii) for all the nodes in the network which are neither a source node nor a terminal node, the divergence — i.e., the difference between the total flow arriving at a node and the total flow departing from such a node — is null.

**Definition 2** Let $X := S \cup \Lambda \cup L$ and let $X^+(\omega_i)$ ($X^-(\omega_i)$) be the set of the outgoing (incoming) links of a node $\omega_i \in \Omega$. A function $\varphi : X \rightarrow \mathcal{R}$ is a flow in a network if it satisfies the following conditions:

a. $\varphi(x) \leq \Gamma(x)$, for all $x$ in $X$;  
   (Capacity constraint)

b. $\sum_{X^+(\omega_i)} \varphi(x) = \sum_{X^-(\omega_i)} \varphi(x)$, for all $\omega_i \in \Omega$;  
   (Flow conservation)

**Theorem 3** The above defined propagation function is a flow in a network $N$.

**Proof.** 1: The capacity constraint is satisfied because i) $f(s_i^k) = b^k \Gamma(s_i^k)$ for all $s_i^k$ in $S$, $f(l_{ij}) = b_i \Gamma(l_{ij})$ for all $l_{ij}$ in $\Lambda$ and $f(l_i^t) = \beta_i \Gamma(l_i^t)$, for all $l_i^t$ in $L$; and ii) $b^k, b_i, \beta_i \in [0, 1]$, for all $i \in \Omega$ and all $k \in A$. 2: The budget identity of the balance sheets of the agents in $\Omega$, together with the rules of limited liability and debt priority — encoded in (1) and (2) — ensure that any flow of losses that arrives in a node is redirected either towards the sink or, for the residual part, towards the node’s descendants in $\Omega$. In notation: $\sum_{X^-(\omega_i)} \varphi(x) = \lambda_i = \beta_i (\lambda_i) v_i + b_i (\lambda_i) d_i = \sum_{X^+(\omega_i)} \varphi(x)$, for all $\omega_i \in \Omega$. \(\blacksquare\)

A flow that satisfies conditions (a) and (b) in the above definition is said to be feasible, i.e., it exists. A flow out of the sources of a network is feasible, also said legitimate, if it entirely reaches the sink. The role played by the budget constraint of the nodes and by the rules of limited liability and debt priority in i) ensuring the feasibility of any propagation in $N$, and ii) setting the conditions for uniqueness of a propagation, are discussed in the following two sections.

\(^9\)The implications of this assumption will be discussed below.

\(^{10}\) $b_i$ is equal to what is known as the ‘loss-given-default’ generated by a failing agent.
3 Network capacity and feasibility of a propagation

In this section we discuss the feasibility of a propagation in $N$, i.e., its existence. In so doing, we also establish that the structure of a network $N$ does not affect its carrying capacity and, as a consequence, has no effect on the value of the flow of net losses that crosses the network.

Every network has an upper bound to its overall capacity to carry a flow. The carrying capacity of a network is equal to the value of the largest flow out of the sources that can cross the network and be entirely absorbed by the sink, i.e., the largest feasible flow. In general, the carrying capacity of a network is smaller or equal to the absorbing capacity of its sink. Finding the feasible flow of maximum value, for a given network, is a fundamental problem in the study of networks – known, in fact, as the maximum flow problem. This problem has been addressed by the celebrated result of Ford and Fulkerson (1956), known as the minimum cut-maximum flow theorem. Before presenting this theorem, we need to introduce the notions of a cut and of its capacity. Let $Y = \{A \cup \Omega \cup t\}$. A cut in a network $N$ is a partition of $Y$, $\{U, \overline{U}\}$, where $U$ and $\overline{U}$ are two non-empty sets such that $S \subseteq U$ and $t \in \overline{U}$. Let $X(U)$ be the set of links that cross such a partition, i.e., the union of the set of forward links going from $U$ into $\overline{U}$, $X^+(U) := \{s^k_i \in S \mid s^k_i \in X^- (\omega_i), \forall \omega_i \in \overline{U}\} \cup \{l_{ij} \in A \mid \omega_i = U, \omega_j = \overline{U}\} \cup \{l_{it} \in L \mid \omega_i \in U, \omega_t \in \overline{U}\}$, and of the set of backward links going in the opposite direction, $X^-(U) := \{l_{ij} \in A \mid \omega_i = \overline{U}, \omega_j = U\}$. The capacity of a cut, $\Gamma \{U, \overline{U}\}$, is the sum of the capacities of its forward links less the sum of the capacities of its backward links: $\Gamma \{U, \overline{U}\} = \sum_{X^+(U)} \Gamma(x) - \sum_{X^-(U)} \Gamma(x)$.

Theorem 4 (Ford and Fulkerson, 1956) In every network, the largest value of a feasible flow equals the capacity of a cut of smallest capacity.

In a network $N$ the budget identities $a_i + r_i \equiv v_i + d_i$ of the nodes in $\Omega$ imply the following:

Lemma 5 In a financial flow network $N$, the capacity of all cuts $\{U, \overline{U}\}$ equals the capacity of the cut $\{S, Y \setminus S\}$.

Proof. Let $\{U_i, \overline{U}_i\}$ be a cut in $N$ and let $\{U_{i-1}, \overline{U}_{i-1}\}$ be another cut in $N$ such that $U_{i-1} = U \setminus \omega_i$; $\omega_i \in U_i$. The set of forward links of $U_i$ is $X^+(U_i) = X^+(U_{i-1}) + \{l_{ij} \in A \mid \omega_i = U_i, \omega_j = \overline{U}_i\} + l_{it} - \{s^k_i \in S \mid a^k \in A\} - \{l_{ij} \in A \mid \omega_j = U_{i-1}\}$, while the set of backward links of $U_i$ is $X^-(U_i) = X^-(U_{i-1}) + \{l_{ij} \in A \mid \omega_j = \overline{U}_i\} - \{l_{ij} \in A \mid \omega_i = U_{i-1}\}$. Thus we can express the capacity of $\{U_i, \overline{U}_i\}$ as

$$\Gamma \{U_i, \overline{U}_i\} = \Gamma \{U_{i-1}, \overline{U}_{i-1}\} + \Gamma \{l_{ij} \in A \mid \omega_j = \overline{U}_i\} + \Gamma(l_{it})$$

$$- \Gamma \{s^k_i \in S \mid a^k \in A\} - \Gamma \{l_{ij} \in A \mid \omega_j = U_{i-1}\}$$

$$- \Gamma \{l_{ij} \in A \mid \omega_j = \overline{U}_i\} + \Gamma \{l_{ij} \in A \mid \omega_i = U_{i-1}\}$$

Since: i) $\Gamma \{l_{ij} \in A \mid \omega_j = \overline{U}_i\} + \Gamma \{l_{ij} \in A \mid \omega_j = U_{i-1}\} = d_i$$

ii) $\Gamma \{l_{ij} \in A \mid \omega_j = \overline{U}_i\} + \Gamma \{l_{ij} \in A \mid \omega_j = U_{i-1}\} = r_i$$

iii) $\Gamma \{s^k_i \in S \mid a^k \in A\} = a_i$, and iv) $\Gamma(l_{it}) = v_i$, by the budget identity $a_i + r_i \equiv v_i + d_i$ we obtain that $\Gamma \{U_i, \overline{U}_i\} = \Gamma \{U_{i-1}, \overline{U}_{i-1}\}$. This procedure can be iterated for all pairs of cuts $\{U_i, \overline{U}_i\} \{U_{i-1}, \overline{U}_{i-1}\}$ in $N$, starting from $\{U_{i-1}, \overline{U}_{i-1}\} = \{S, Y \setminus S\}$.
This lemma, coupled with the maximum flow-minimum cut theorem, delivers the following proposition:

**Theorem 6** The largest value of a feasible propagation defined in a network $N$ equals the largest possible flow out of the source nodes.

This result, in turn, implies that, in a financial flow network $N$, any flow of losses out of the source nodes ends up entirely into the sink. Looking at the propagation as a sequence in time of defaults and passing of credit losses, this result tells us that the propagation stops when and only when the total loss of net worth suffered by the agents in $\Omega$ equals the initial shock.

**Corollary 7** Any propagation in $N$ stops when and only when the total amount of net worth lost by the nodes in $\Omega$ equals the value of the external shock:

$$\sum_{i} \beta_i v_i = \sum_{k} b^k a^k.$$ 

This is an irrelevance result: If the defaults do not involve bankruptcy costs as we have so far assumed — the total loss of net worth suffered by the agents in a financial network does not depend on the structure of the network. This also implies that a flow of losses entering the network can be neither restricted nor magnified. There can be no systemic bottlenecks capable of restraining the propagation of a shock — as shown above by lemma 5.

### 4 Cycles and nominal indeterminacy of a propagation

The interdependence of obligations that constitutes the fabric of a financial network can create problems of indeterminacy to the propagation function defined above, i.e., under some circumstances the propagation induced by a given shock is not unique. In what follows, we pin down the conditions that create such indeterminacy and discuss its implications.

This problem was first pointed out by Eisenberg and Noe (2001). These authors investigate the existence and uniqueness of a vector of payments that clears a network of interdependent financial claims, where the capability of an agent to repay in full his debts depends on the solvency of his own debtors which, in turn, depend on the solvency of their debtors, and so forth. They express such a vector as a function defined on a lattice, a function that complies with the requirements of limited liability and debt priority. Using a fixed point argument, Eisenberg and Noe establish the uniqueness of the clearing vector under a restrictive assumption, namely the absence of connected subgraphs of defaulting agents with null cash flow to honor their obligations. They explain this problem with the following example. Suppose the system contains two defaulting nodes, 1 and 2, both without any operating cash flow. Suppose also that each node has nominal liabilities of 1.00 to the other node. In this case, the flow of payments that goes from node 1 towards node 2 depends on the payments that node 1 receives from node 2, and viceversa. Thus, any vector $x(1,1), x \in [0,1]$, is a clearing payment vector between the two nodes at hand.

To discuss this issue, we need to introduce a few more notions. The structure of linkages that connect the nodes in a network is composed by directed paths and cycles. A directed path in a network is a sequence of distinct and pairwise
adjacent nodes, \((\omega_1 \omega_2 \cdots \omega_1 \cdots \omega_T)\), with start node \(\omega_1\) and end node \(\omega_T\), where each pair \((\omega_i, \omega_{i+1})\) in the sequence is connected by a link going from \(\omega_i\) to \(\omega_{i+1}\).

A directed path for which the start node and the end node coincide is called a cycle. Correspondingly, a path flow and a cycle flow are flows defined over a path and over a cycle, respectively. The value of a flow along a path (cycle) is equal to the smallest link flow in the path (cycle). The upper bound of a path (cycle) flow is set by the capacity of a path (cycle), i.e., by the capacity of the link of smallest capacity along the path (cycle).

It is the presence of cycle flows in a network \(N\) that can create problems of indeterminacy of a propagation. The example by Eisenberg and Noe applies also to a cycle \((\omega_1 \omega_2 \omega_3 \omega_1)\) of defaulting agents in a network \(N\). Suppose that node \(\omega_1\) defaults and sends a loss to node \(\omega_2\) which, in turn, defaults and sends a loss to \(\omega_3\) which, in turn, defaults and sends a loss back to \(\omega_1\) which now passes a further loss to \(\omega_2\), and so forth. In this case we have a cycle flow. We show below that, under some conditions, this recursive process does not converge to a unique value of the corresponding cycle flow and the latter can take on any value between zero and the capacity of the cycle. Conversely, a propagation that does not entail cycle flows – as it is the case for any propagation that takes place in an acyclic network \(N\) – does not pose problems of non-uniqueness:

**Theorem 8** If the network \(N\) is acyclic, a propagation in \(N\) is uniquely defined.

**Proof.** i) Let \(P^1(\omega_i) = \{\omega_j | j \in \Lambda\}\) be the set of parent nodes of \(\omega_i\), let \(P^2(\omega_i)\) be the set of the parent nodes of the parent nodes of \(\omega_i\) and so forth for \(P^3(\omega_i), P^4(\omega_i), \ldots, P^n(\omega_i)\). The union of such sets, \(P(\omega_i) = \bigcup_{j=1}^{n} P^j(\omega_i)\), forms the set of the predecessors of \(\omega_i\), i.e. the set of nodes \(\omega_j\) in \(\Omega\) s.t. there exists a directed path from \(\omega_j\) to \(\omega_i\). For each \(i \in \Omega\), \(P^i(\omega_i)\) and \(b_i\) are both uniquely defined functions of \(\lambda_i\) which, in turn, is a uniquely defined function of the insolvency functions \(b_j\) of the nodes in \(P^1(\omega_i)\) which, in turn, are both uniquely defined functions of the losses suffered by the nodes in \(P^1(\omega_i)\) which, in turn, are uniquely defined functions of the values taken on by the insolvency functions of the nodes in \(P^2(\omega_i)\) and so forth up to the source nodes of \(N\). In sum, \(\lambda_i, \beta_i, b_i\) and \(b_i\) are functions of the values taken on by the insolvency functions of the nodes in \(P(\omega_i)\). Since, in absence of cycles, we have that \(\omega_i \notin P(\omega_i)\) for all \(i \in \Omega\), \(\lambda_i, \beta_i, b_i\) and \(b_i\) are obtained through the non-recursive iteration of uniquely defined functions, thus they are uniquely defined as well.

Thus, to establish the conditions for uniqueness of a propagation, we can focus our attention on cyclic networks and on the cycle flows that they can support. We now show that if a propagation in \(N\) contains cycles of defaulting agents, i.e. if it embeds cycle flows, the propagation is not uniquely defined if one or more of such cycles are composed by agents who have no obligations towards agents that do not belong to their cycle.

In order to restrict the investigation to cycle flows, we resort to a property of network flows, known as conformal decomposition,\(^{11}\) that enables us to decompose a flow in a network into the sum of simpler components:

**Theorem 9** (Conformal Decomposition Theorem) Any positive flow, in a network with \(n\) nodes and \(l\) links, is composed by the sum of at the most \(n + l\) cycle flows and simple path flows, where the latter are flows defined on directed paths starting with a source node and ending with the sink.

\(^{11}\) See, inter alia, Bertsekas (1991), sect. 1.1, or Ahuja et al. (1993), sect. 3.5.
This property allows us to analyze cycle flows in isolation from the propagation in which they take place. Let \( f(x) \) be a given propagation in \( N \) and suppose that it entails one or more cycles of defaulting agents. Let \( \phi(l_{ij}) \) be the function that associates to each link in such cycles the value of the corresponding cycle flow and zero to the other links in \( N \). By the conformal decomposition property, we can subtract the cycle flows from the propagation to which they belong and obtain an acyclic propagation \( \tilde{f}(x) = f(x) - \phi(l_{ij}) \). From theorem 8 we know that \( \tilde{f}(x) \) is uniquely defined, thus we can state the following:

**Lemma 10** A propagation in \( N \) containing cycle flows is uniquely defined iff such cycle flows are uniquely defined.

**Theorem 11** A cycle flow in a propagation in \( N \) is uniquely defined iff at least one node in the cycle has liabilities towards nodes that do not belong to the cycle.

**Proof.** Let \( \gamma = (\omega_1\omega_2\cdots\omega_i\cdots\omega_1) \) be a cycle of defaulting agents in a given propagation in \( N \). For each \( \omega_i \) in \( \gamma \) we have \( \lambda_i = b_k a^k_i + \sum_{j \not\in \gamma} b_j c_{ji} + b_{i-1} c_{i-1,i} \), where - of course - \( \omega_{i-1} \in \gamma \). Invoking the conformal decomposition property, we take \( \sum_{j \not\in \gamma} b_j c_{ji} \) as a datum and express \( b_i(\lambda_i) \) as a linear function of \( b_{i-1} \):

\[
b_i(b_{i-1}) = \frac{b_k a^k_i + \sum_{j \not\in \gamma} b_j c_{ji} - v_i}{d_i} + b_{i-1} \frac{c_{i-1,i}}{d_i}
\]

rewritten as

\[
b_i d_i - b_{i-1} c_{i-1,i} = b_k a^k_i + \sum_{j \not\in \gamma} b_j c_{ji} - v_i
\]

For a cycle composed of \( h \) nodes, we have a system composed of \( h \) of such linear equations. The matrix of the coefficients of this system of equations is

\[
\begin{bmatrix}
d_1 & -c_{h1} & 0 & \cdots & 0 \\
0 & d_2 & -c_{12} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \vdots \\
-c_{h-1,h} & 0 & 0 & 0 & d_h
\end{bmatrix}
\]

It can be check by inspection that the determinant of this \( h \times h \) matrix is equal to \((d_1 \times d_2 \times \cdots \times d_h) + (-1)^{h+1}(c_{h1} \times c_{12} \times \cdots \times c_{h-1,h})\). If no node in \( \gamma \) is indebted towards one or more nodes that do not belong to \( \gamma \), then \( d_i = c_{i,i+1} \) for all \( i = 1, \ldots, h \). In this case this determinant is equal to zero and the solution of the above system of equations is indeterminate: the cycle flow at hand can take on any value in the interval between zero and the capacity of the cycle \( \gamma \). Conversely, if at least one node in \( \gamma \) has liabilities towards nodes which are not in \( \gamma \), then \( d_i > c_{i,i+1} \) for at least one node in \( \gamma \) and the above determinant is strictly positive. This is sufficient to ensure existence and uniqueness of the \( h \)-dimensional vector \([b_i]\) that provides the solution of the above system of equations and that characterizes the cycle flow in \( \gamma \).

The indeterminacy arising when the condition stated in the above theorem is not met is a sheerly nominal indeterminacy: any positive reimbursement along a cycle of obligations, under the above conditions, would simply be a clearing transaction with no consequences on the rest of the propagation. In absence of
external creditors, the values taken on by the propagation on the other links in \( N \) are not affected by the value taken on by the cycle flow at hand. Moreover note that, for a cycle with no external creditors, the loss arriving into the cycle must be maximal:

\[
\sum_i b_k a_i^k + \sum_{j \notin \gamma} b_{c_{ji}} = \sum_i v_i
\]

otherwise at least one of the nodes in the cycle has a strictly positive net worth, i.e., it does not default. Note also that if a cycle with external creditors receives a maximal loss, the \( b_i \)'s of the nodes involved in such a cycle flow are all equal to unity, i.e., also the cycle flow is maximal. Thus, in conformity with the latter case and in order to get rid of the above described indeterminacy, in point 3 of the algorithm presented in the next section we set the value of a cycle flow that satisfies the condition in Theorem 11 as equal to the maximal cycle flow.

5 Computing the propagation

Here we put forward an algorithm that computes the value of a propagation \( f(x) \) in \( N \) through the iterated application – node by node, along the directed paths and cycles of \( N \) – of the activation and insolvency functions defined above. This procedure yields a sequence of passing of financial losses starting from the source nodes and ending in the sink. When the flow out of the sources has been completely absorbed by the sink, the flow conservation property is satisfied, the algorithm stops and the algorithm delivers the pair of \( n \) dimensional vectors \( \{[\beta_i], [b_i]\} \) which identify the propagation at hand for any given value of the shock vector \( [b_k] \).

We add a superscript \( t = 1, 2, 3, \ldots \) to the variables involved in the computation – namely \( \lambda_i^t, b_i^t, \beta_i^t \) – to indicate the value taken on by these variables at each iteration of the algorithm. Recall that \( \lambda_i = \sum_k b_k a_i^k + \sum_j b_{c_{ji}} \) and let

\[
[\lambda_i]_{1 \times n} = [b_k]_{1 \times m} [a_i^k]_{m \times n} + [b_{c_{ji}}]_{1 \times n} [c_{ji}]_{n \times n}
\]

(3)

be the vector of the losses born by the agents in \( \Omega \). Further, let \( \Omega = \{\omega_i \in \Omega | b_i > 0\} \) be the set of insolvent agents. The algorithm is the following:

1. For the given value assignment of the vector \([b_k]\), compute \([\lambda_i^1] = [b_k] [a_i^k] + [b_{c_{ji}}] [c_{ji}]\), starting with \( t = 1 \) and setting \( b_i^0 = 0 \);
2. Compute \([\beta_i^t] = [\beta_i(\lambda_i^1)]\) and \([b_i^1] = [b_i(\lambda_i^1)]\) according to (1) and (2);
3. If \( |\Omega^t| \geq 2 \), then (i) search for cycles of nodes in \( \Omega^t \), and (ii) set \( b_i = 1 \) for all nodes in each of such cycles if a) the cycle flow is maximal, and b) the nodes in the cycle have no debts with nodes outside the cycle;
4. If \( \sum_{i \in \Omega} \beta_i^t v_i = \sum_A b_k a_k \), then stop. If \( \sum_{i \in \Omega} \beta_i^t v_i < \sum_A b_k a_k \), then start again from point 1.

The values of the vectors \([\lambda_i^t], [\beta_i^t], [b_i^t]\) are strictly increasing in \( t \) as long as there are nodes in \( \Omega \) with strictly positive divergence, i.e., as long as there exists at least one \( i \in \Omega \) such that \( \lambda_i^t > \beta_i^{t-1} v_i + b_{c_{ji}}^{t-1} d_{ji} \), which, in turn, implies \( \sum_{i \in \Omega} \beta_i^t v_i < \sum_A b_k a_k \). Conversely, the repeated iteration of the algorithm yields
stationary values of the vectors at hand once the flow out of the sources has been entirely absorbed by the sink, i.e., when $\sum_{i} \beta_i v_i = \sum_{h} b_h a_h$. By Corollary 7 this condition is eventually achieved, then the divergence of all nodes in $\Omega$ is null and neither the losses arriving at a node nor the losses departing from a node can grow anymore. In numerical simulations where only the number of defaults is counted, while the total debt deflation is not considered—such as the simulations performed by Nier et al.—point 3 can be skipped without altering the results.

Each iteration of this algorithm computes the passing of losses from a set of nodes in $N$ to their children nodes, i.e., to their direct descendants. In absence of cycles, the length of the longest possible path in $N$ is equal to $n$ and so is the largest possible number of iterations in the algorithm. Conversely, in presence of cycle flows, the algorithm converges asymptotically to the solution values by computing progressively smaller augmentations of the cycle flows. Since the values at hand are sums of money, this problem can be easily overcome by setting an approximation of, say, one cent of a euro. In this case the algorithm stops in a finite number of iterations.

6 Network structures and resiliency to contagion

The recent numerous studies on systemic risk reflect the growing interest of central banks and policy makers in the evaluation of the risk of default contagion in financial systems and in policies capable of reducing it. Central banks and policy makers are increasingly concerned with the stability of national and international financial systems and see the possibility of contagion of financial distress within and across such systems as a major threat. Moreover, central banks seem very interested in preventing the failure of banks—regardless whether theoretical economists may judge such failures as an efficient way to achieve a better allocation of resources—even if they do not state such objective in clear terms for the obvious incentive problems that this would create. As mentioned above, some regulations and policies adopted by the authorities have an impact on the pattern of financial obligations that arise in financial systems. For these reasons, many of the above mentioned scholars have studied the relative resiliency to default contagion of different national banking systems. In so doing, all of them have resorted to numerical simulations. In this section we investigate this issue applying the above described flow network representation of a financial system. This approach enables us to support our claims on the basis of analytical results, rather than numerical simulations. Following Nier et al., we discuss the implications of variations in four parameters that characterize a network structure, namely: 1) the capitalization of its members; 2) its degree of connectivity; 3) its degree of concentration, and 4) the size of the interbank exposures. As a measure of default contagion we will use the total number of

12 As Upper (2007, page 2 and 3) puts it “Unfortunately, analytical results on the relationship between market structure and contagion have been obtained only for a limited number of highly stylised structures of interbank markets, which are of limited use when it comes to assessing the scope for contagion in realworld banking systems[...]. Given the scarcity of theoretical results, researchers have increasingly turned to computer simulations to study contagion.”
defaults, when possible, and the contagion threshold, which we define as the largest flow of losses that can cross a given network without causing secondary defaults.

6.1 Capitalization

The rule of limited liability implies that the capacity of each node in $N$ to absorb losses without defaulting is equal to its capital endowment, i.e., its net worth. Thus, any financial network becomes more resilient to default contagion as the capitalization of one or more of its members increases. A formal statement of this conclusion would be superfluous, this point is evident and does not require to be supported by any analytical or numerical result.

6.2 Connectivity

Graph theory presents several measures of graph connectivity. It is convenient, given our purposes, to define the degree of connectivity of a network as its total number of links, i.e., the cardinality of the set $\Lambda$, relative to the numerosity of its members, i.e., the cardinality of the set $\Omega$: i.e., the ratio $\varphi = |\Lambda|/|\Omega|$. The question we want to address is the following: given a network $N$, the addition of links to the network, ceteris paribus, makes it more or less resilient to default contagion? There is no general answer to this question, it depends on how the precise pattern of obligations in $N$ is modified by the addition of the extra links and on the magnitude of the shock. The network becomes more robust, in terms of contagion threshold, if the extra links help to distribute the flows of losses among the nodes in $N$ in proportion to their absorbing capacity.

Consider the case of a node $A$ with outdegree $m$, i.e., with $m$ direct descendants (creditors), and suppose that the net worth of $A$ and of all its creditors is equal to $v$. An increase in $m$ has a twofold effect. On one hand, the contagion threshold of the loss born by node $A$, $\lambda_A$, equal to $v(m + 1)$, grows linearly in $m$ (there is no default contagion for $\lambda_A \leq v(m + 1)$). On the other hand, as $m$ grows, so does the number of defaults for large enough shocks, i.e., for $\lambda_A > v(m + 1)$.

The beneficial effect of connectivity is guaranteed only for ‘relatively’ small idiosyncratic shocks and only under fairly restrictive conditions, namely that a) the flows of losses unfold, across $N$, along directed trees;\footnote{A tree is an acyclic graph.} and b) all nodes have the same absorbing capacity and the same outdegree.

Definition 12 Let a tree network be a network $N$ such that every node in $N$ is the vertex of a directed spanning tree.

Conjecture 13 Let $N_A$ and $N_B$ be two tree networks equal in everything but the set of links, where $\Lambda_A \supset \Lambda_B$, and both composed by agents endowed with the same amount of net worth. Let $m_A$ be the outdegree common to all nodes in $A$ and $m_B$ be the common outdegree of all nodes in $B$ where $m_A > m_B$. Then, for an idiosyncratic shock $\lambda \leq v(m_A + 1)$, the number of defaults occurring in network $N_A$ are smaller or equal to the ones occurring in network $N_B$, while the converse is true for idiosyncratic shocks larger than $v(m_A + 1)$.
To visualize the pros and cons of diversification, consider the stylized network depicted in figure 1 below, which is a tree network both with and without the dotted link. For any idiosyncratic shock that hits the starting node of the dotted link, the addition of such a link directs part of the losses on a portion of the network previously not involved in the propagation. In this case the burden of losses is born by a larger number of nodes, reducing the impact of the flow on each single node. However, if the impact on each node is large enough to cause its insolvency, then the increase in connectivity increases the total number of defaults.

Things get more complicated if we introduce cycles and closed paths in the picture, whose presence in the structure of networks generates an ‘uneven’ allocation of losses among the nodes. Consider the example of a closed path depicted in figure 2. The addition of the dotted link going from node B to node E creates a closed path and, as a consequence, it reinforces the flow of losses that can reach node E while diminishes the flows that can reach nodes C and D. If nodes C, D and E have the same absorbing capacity, the addition of the dotted link renders this simple network more prone to default contagion. The same reasoning applies to cycles, as shown the example in figure 3. In this case the addition of the dotted link creates the (A, B, D) cycle that, in case of default of nodes A, B and D, redirects losses towards the nodes A, B, C, and D, while it reduces the losses born by node E.
Networks with a low degree of connectivity are more likely to be tree networks or, at least, to embed subnetworks which comply with the definition of a tree network. Conversely, networks with a high degree of connectivity are more likely to embed cycles and closed paths. Thus, it is plausible to think that an increase of connectivity that takes place in networks with a low $|A|/|\Omega|$ ratio is more likely to bring about the benefits of a higher diversification of lending and borrowing behaviours, reducing the risk of default contagion, for relatively small shocks. Adding links to networks with a high degree of connectivity, instead, certainly increases the number of cycles and closed paths and, in so doing, reduces the scope of the benefits of diversification.

Interestingly, the benefits of diversification prevail again when the connectivity of a network is maximal. We show now that, under some conditions, the network structure which is most resilient to default contagion is the one with the highest possible connectivity. In a financial system $N$ where i) each agent lends to every other agent in the system a sum proportional to its own total intra-network exposure, i.e., the values of the liabilities in $C$ are $c_{ij} = 1/(n - 1)r_j$, for all $i \neq j$, and ii) all agents in the system have the same credit/equity ratio, there is no default contagion for any exogenous shock vector $[b^k]$ defined below.

To define default contagion, let $[b^k]$ be a given shock vector and let $D$ be the set of agents in $\Omega$ who default as a consequence of this shock, $D := \{\omega_i \in \Omega \mid \sum_k b^k a^i_k + \sum_j b_{ji} c_{ji} \geq v_i\}$. Further, let $D'$ be the set of agents that suffer a loss of value of their external assets large enough to cause their default, i.e., $D' := \{\omega_i \in \Omega \mid \sum_k b^k a^i_k \geq v_i\}$ and let $D'' = D \setminus D'$ be the set of defaulting agents who would be solvent if they had not received losses from their debtors in $\Omega$. There is no default caused by contagion if the set $D''$ is empty. We label as fundamental the defaults in $D''$ and as secondary the ones in $D''$.

In order to isolate the contagion caused by a propagation of losses among the agents in $\Omega$ from the contagion caused by common exposures to exogenous shocks (i.e., the third type of systemic risk listed in the introduction), we restrict our attention to external shocks $[\tilde{b}^k]$ that partition the set $\Omega$ into a pair $\{\hat{D}', \Omega \setminus \hat{D}'\}$, where $\hat{D}' := \{\omega_i \in \Omega \mid \sum_k \tilde{b}^k a^i_k \geq v_i\}$ as above, and such that $\sum_k \tilde{b}^k a^i_k = 0$ for all nodes $\omega_i \in \Omega \setminus \hat{D}'$.

**Theorem 14** In a network $N$ where i) $\Gamma(l_{ij}) = 1/(n - 1)r_j$ for all $l_{ij} \in A$, and ii) $\rho = r_j/v_i$ for all $\omega_j \in \Omega$, the set of secondary defaults is:
and such that deposits of equal amount in each bank and banks do not lend money to all proportional to the absorbing capacities of the lenders. Note that the conditions also the intra-network borrowing of each agent is as diversified as possible and possible diversification of its own intra-network exposure. As a consequence, descendants in nodes default, send to the sink their entire net worth and send to their direct descendants in $\Omega \setminus \tilde{D}'$ the residual loss. In notation: $\sum_k \bar{b}^k a^k = \sum_i \beta_i v_i + \sum_i \sum_j b^*_i c_{ij}$, for $i \in \tilde{D}'$ and $j \in \Omega \setminus \tilde{D}'$, where the values of the variables $b^*_i$ are the ones computed, for the shock vector at hand, by the first round of the algorithm presented above. Aggregating the balance sheets of the nodes in $\Omega$, we have that $\sum_k a^k = \sum_i v_i$, thus $\sum_k \bar{b}^k a^k \leq \sum_i v_i$. Since $\sum_i \beta_i v_i = \sum_i v_i$, for $i \in \tilde{D}'$, we have
\[
\sum_i \sum_j b^*_i c_{ij} \leq \sum_j v_j, \quad \text{for } i \in D' \text{ and } j \in \Omega \setminus D'.
\] (4)
where the inequality holds strictly if $\sum_k \bar{b}^k a^k < \sum_i v_i$. In other words, the loss overflowing from the primary defaults towards the agents in $\Omega$ who have not yet defaulted, if ever, is smaller or equal to the sum of the net worth of such agents. We now seek to establish that, if $\sum_k \bar{b}^k a^k < \sum_i v_i$, then $\sum_i b^*_i c_{ij} < v_j$, for all $i \in D'$ and $j \in \Omega \setminus D'$, i.e., no node in $\Omega \setminus D'$ suffers a loss large enough to cause its default. Conditions (i) and (ii) imply that $c_{ij} = \frac{1}{n-1} v_j \rho$, for all $i \neq j$. Substituting $c_{ij}$ in (3), and assuming $\sum_k \bar{b}^k a^k < \sum_i v_i$, we have
\[
\sum_j \sum_i b^*_i \frac{1}{n-1} \rho v_j < \sum_j v_j,
\]
thus
\[
\rho < \frac{n-1}{\sum_i b^*_i}
\]
Substituting $c_{ij}$ and $\rho$ in $\sum_i b^*_i c_{ij}$ we have
\[
\sum_i b^*_i c_{ij} = \sum_i b^*_i \frac{1}{n-1} v_j \rho < \sum_i b^*_i \frac{1}{n-1} v_j \sum_i \sum \frac{1}{n-1} v_j = v_j.
\]
thus no node in $\Omega \setminus D'$ defaults and the computation of the flow induced by $[\bar{b}^k]$ stops at the first round, assigning value $b^*_i$ to the insolvency functions of the nodes in $\tilde{D}'$.

(b) By Corollary 7, if $\sum_k \bar{b}^k a^k = \sum_i v_i$ the net worth of all nodes in the network is lost as a consequence of the exogenous shock, thus all nodes in $\Omega$ default. If $a_i > 0$ for all $\omega_i \in \Omega$, then, by the definition of $[\bar{b}^k]$, $D' = \Omega$ and $\tilde{D''}$ is empty. If $a_i = 0$ for some $\omega_i \in \Omega$, then such nodes are in $\tilde{D''}$ since they are not in $\tilde{D}'$ by its definition.

Conditions (i) and (ii) imply that each agent in $N$ applies the maximum possible diversification of its own intra-network exposure. As a consequence, also the intra-network borrowing of each agent is as diversified as possible and proportional to the absorbing capacities of the lenders. Note that the conditions (i) and (ii) do not apply to a banking system, where depositors do not have deposits of equal amount in each bank and banks do not lend money to all.

(a) empty for any propagation induced by a shock vector $[\bar{b}^k]$ defined above and such that $\sum_k \bar{b}^k a^k < \sum_i v_i$;
(b) non empty if and only if $\sum_k \bar{b}^k a^k = \sum_i v_i$ and $a_i = 0$ for some $\omega_i \in \Omega$.

Proof. (a) As the flow of external losses reaches the nodes in $\tilde{D}'$, these nodes default, send to the sink their entire net worth and send to their direct descendants in $\Omega \setminus \tilde{D}'$ the residual loss. In notation: $\sum_k \bar{b}^k a^k = \sum_i \beta_i v_i + \sum_i \sum_j b^*_i c_{ij}$, for $i \in \tilde{D}'$ and $j \in \Omega \setminus \tilde{D}'$, where the values of the variables $b^*_i$ are the ones computed, for the shock vector at hand, by the first round of the algorithm presented above. Aggregating the balance sheets of the nodes in $\Omega$, we have that $\sum_k a^k = \sum_i v_i$, thus $\sum_k \bar{b}^k a^k \leq \sum_i v_i$. Since $\sum_i \beta_i v_i = \sum_i v_i$, for $i \in \tilde{D}'$, we have
\[
\sum_i \sum_j b^*_i c_{ij} \leq \sum_j v_j, \quad \text{for } i \in \tilde{D}' \text{ and } j \in \Omega \setminus \tilde{D}'.
\] (4)
where the inequality holds strictly if $\sum_k \bar{b}^k a^k < \sum_i v_i$. In other words, the loss overflowing from the primary defaults towards the agents in $\Omega$ who have not yet defaulted, if ever, is smaller or equal to the sum of the net worth of such agents. We now seek to establish that, if $\sum_k \bar{b}^k a^k < \sum_i v_i$, then $\sum_i b^*_i c_{ij} < v_j$, for all $i \in \tilde{D}'$ and $j \in \Omega \setminus \tilde{D}'$, i.e., no node in $\Omega \setminus \tilde{D}'$ suffers a loss large enough to cause its default. Conditions (i) and (ii) imply that $c_{ij} = \frac{1}{n-1} v_j \rho$, for all $i \neq j$. Substituting $c_{ij}$ in (3), and assuming $\sum_k \bar{b}^k a^k < \sum_i v_i$, we have
\[
\sum_j \sum_i b^*_i \frac{1}{n-1} \rho v_j < \sum_j v_j,
\]
thus
\[
\rho < \frac{n-1}{\sum_i b^*_i}
\]
Substituting $c_{ij}$ and $\rho$ in $\sum_i b^*_i c_{ij}$ we have
\[
\sum_i b^*_i c_{ij} = \sum_i b^*_i \frac{1}{n-1} v_j \rho < \sum_i b^*_i \frac{1}{n-1} v_j \sum_i \frac{1}{n-1} v_j = v_j.
\]
thus no node in $\Omega \setminus \tilde{D}'$ defaults and the computation of the flow induced by $[\bar{b}^k]$ stops at the first round, assigning value $b^*_i$ to the insolvency functions of the nodes in $\tilde{D}'$.

(b) By Corollary 7, if $\sum_k \bar{b}^k a^k = \sum_i v_i$ the net worth of all nodes in the network is lost as a consequence of the exogenous shock, thus all nodes in $\Omega$ default. If $a_i > 0$ for all $\omega_i \in \Omega$, then, by the definition of $[\bar{b}^k]$, $D' = \Omega$ and $\tilde{D''}$ is empty. If $a_i = 0$ for some $\omega_i \in \Omega$, then such nodes are in $\tilde{D''}$ since they are not in $\tilde{D}'$ by its definition.

Conditions (i) and (ii) imply that each agent in $N$ applies the maximum possible diversification of its own intra-network exposure. As a consequence, also the intra-network borrowing of each agent is as diversified as possible and proportional to the absorbing capacities of the lenders. Note that the conditions (i) and (ii) do not apply to a banking system, where depositors do not have deposits of equal amount in each bank and banks do not lend money to all.
depositors. We need to adapt the above theorem to extend its insight to banking systems. Following Nier et al.,\textsuperscript{14} we introduce depositors in the picture by distinguishing them from the other nodes in the system. In so doing we resort to the simplifying assumption that depositors do not hold any external asset and do not have financial obligations (such as mortgages): a depositor balance sheet is of the type \( r_i = v_i \). Thus depositors’ nodes have only one outgoing link, directed towards the sink.

**Definition 15** A banking network \( N_b \) is a financial network \( N \) where a subset of nodes, \( \Omega_d \subset \Omega \), is such that \( r_i = v_i \), for all \( \in \Omega_d \).

We show now that in a banking network where interbank loans comply with conditions (i) and (ii), the contagion threshold is the highest possible one, that is, it is equal to the total absorbing capacity of the banks which are not in \( D \).

Let \( \Lambda_b \) be the set of links that connect banks to other banks in \( N \), i.e., \( \Lambda_b \subset \Lambda \) s.t. \( l_{ij} \in \Lambda_b \) for all \( i, j \in \Omega_b \).

**Theorem 16** In a banking network \( N_b \) where \( \Gamma(l_{ij}) = 1/(n-1)r_j \) for all \( l_{ij} \in \Lambda_b \), and \( \rho = r_j/v_i \) for all \( \omega_j \in \Omega_b \) there are no secondary defaults for any shock \([\mathcal{B}]\) whose magnitude is smaller or equal to the sum of the total net worth of the banks in \( N_b \) and of the losses born by the depositors of the primarily defaulting agents.

**Proof.** To be added. It goes along the same lines as the proof of the above theorem, exploiting the fact that the overflow of losses that goes from the primary defaults towards the rest of the banking sector (i.e., the initial shock minus the losses absorbed by the shareholders and depositors of the primarily defaulting banks) is distributed in proportion to the absorbing capacities of the lenders.

A complete structure is auspicious, for a banking system, only for shocks that generate an intra-banks overflow of losses which is smaller than the absorbing capacity of the banks which are not in the set of primary defaults, i.e., \( \sum_{\Omega_b \setminus \mathcal{B}} v_i \).

Shocks larger than this would cause the collapse of the entire banking system, if the size of interbank exposures allows such a shock to be transmitted, while a complete melt-down would be avoided in a network with a lower degree of connectivity.

**Corollary 17** In a banking network \( N_b \) where \( \Gamma(l_{ij}) = 1/(n-1)r_j \) for all \( l_{ij} \in \Lambda_b \), and \( \rho = r_j/v_i \) for all \( \omega_j \in \Omega_b \) there are no secondary defaults for any shock \([\mathcal{B}]\) if the overall sum of interbank liabilities is smaller or equal to the total net worth of the banks in \( N_b \).

Note the these results hold for any complete network structure, as defined above. This means that the other characteristics of a network—such as the degree of concentration of the system, the presence of money centers, etc.—do not alter these results.

\textsuperscript{14}We do not assume, as Nier et al. do, that customer deposits enjoy a priority, with respect to interbank deposits, in the reimbursements that follow the liquidation of a bank. Such an assumption is neither supported by any national legislation about bankruptcy procedures nor by the fact that, in most countries, losses born by customer deposits are partially or totally refund by third parties. Moreover, this assumption is not innocuous: it determines a bias towards contagious defaults in the numerical simulations that Nier et al. present in their paper.
6.3 Concentration
Since the beginning of the 90’s many national banking systems have undergone a process of progressive concentration, often with the active support of the monetary authorities. The emergence of systems characterized by fewer and larger operators—that usually act as money centers—has occurred, and is still occurring, through mergers and acquisitions. The impact of these operations on the resiliency of a financial network is ambiguous. In as much as the capital endowment of an operator grows with its size, so does its capacity to absorb losses and to avoid default: the larger the operator, the larger the minimum flow of losses that causes its default. Moreover, it is plausible that also the size of the total intra-network borrowing of an operator grows with its size. In turn, in systems where lenders diversify their intra-network exposure, this implies that the number of direct descendants (i.e., lenders) of a node in a network grows with the size of the balance sheet of the node. Thus, in case of default of a large operator, the overflow of losses is born by a large number of creditors. Whether this effect is auspicious or not depends in the size of the flow of losses that comes from or through the node. On one hand, the contagion threshold associated with an idiosyncratic shock of a node increases in the number of its creditors. On the other hand, the number of direct secondary defaults that the failure of a large operator can cause also grows in the number of its creditors—an effect that can render a large bank “too large to fail”, in the eyes of the monetary authorities.

6.4 Size of interbank loans
In a banking network, the size of the interbank market, i.e., the total interbanks exposures, can vary with respect to the size of the economy, hence with respect to the total collection of deposits. As the overall ratio of interbank deposits to customer deposits grows, so does the portion of losses that, in a propagation, is directed towards other banks in the network. For this reason a banking network becomes more exposed towards default contagion as, ceteris paribus, the size of its overall interbank exposures grows.

**Theorem 18** Let \( N^A \) and \( N^B \) be two banking networks equal in everything but the size of the interbank loans, and let the weights of their interbank links be such that \( c_{ij}^A = \delta c_{ij}^B \) for all \( i,j \) in \( \Omega \), where \( \delta \) is a real number larger than unity and where \( \{ c_{ij}^A \} \) and \( \{ c_{ij}^B \} \) are the sets of the weights of the interbank links in \( N^A \) and \( N^B \), respectively. Then, for any given shock, the number of defaults occurring in network \( N^A \) are larger or equal to the ones occurring in network \( N^B \), and the contagion threshold of network \( N^A \) is smaller or equal to the one of network \( N^B \).

**Proof.** The proof is trivial and is omitted. ■

7 Conclusions
To be added.
References


