Learning and Asset Prices under Ambiguous Information*

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Abstract

We propose a new continuous-time framework to study asset prices under learning and ambiguity aversion. In a partial information Lucas economy with time-additive power utility, a discount for ambiguity arises if and only if the relative risk aversion is below one. Then, ambiguity increases equity premia and volatilities, and lowers interest rates. In our setting, ambiguity does not resolve asymptotically and, for low risk aversion, it is consistent with the qualitative predictions of the equity premium, the low interest rate, and the excess volatility puzzles.

Keywords: Financial Equilibria, Learning, Knightian Uncertainty, Ambiguity Aversion, Model Misspecification, Robust Decision Making.

JEL Classification: C60, C61, G11.
This paper studies the equilibrium asset pricing implications of learning when the distinction between risk and ambiguity (Knightian uncertainty) aversion matters. Ambiguity refers to situations in which investors do not rely on a single probability law to describe the relevant random variables. Ambiguity aversion means that investors dislike ambiguity about the probability law of asset returns. In a continuous-time economy, we study the joint impact of learning and ambiguity aversion on asset prices and learning dynamics. More specifically, we tackle the problem of asset pricing under learning and ambiguity aversion in a continuous-time Lucas (1978) exchange economy, where economic agents have partial information about the ambiguous dynamics of some aggregate endowment process. We develop a new continuous-time setting of learning under ambiguity aversion that allows us to study analytically the conditional and unconditional implications for equilibrium asset prices.

It is an open issue, whether ambiguity aversion gives a plausible explanation for salient features of asset prices when learning is accounted for. For instance, can the equity premium puzzle be still addressed in a model of ambiguity aversion as new data are observed and more data-driven knowledge about some unobservable variable becomes available? The answer to this question depends on the ability of investors to learn completely the underlying probability laws under a misspecified belief. Rational models of Bayesian learning\(^1\) cannot address such issues, because they are based on a single-prior/single-likelihood correct specification assumption about the beliefs that define the learning dynamics. Therefore, to study asset prices under learning and ambiguity aversion we have to consider settings where a possible misspecification of beliefs and the corresponding learning dynamics is explicitly addressed. In our model, agents learn only some global ambiguous characteristics of the underlying endowment process, parameterized by a finite set of relevant ambiguous states of the economy. Moreover, extending Epstein and Schneider (2004), we account for a set of multiple likelihoods in the description of the local ambiguous properties of the underlying endowment process, conditional on any relevant state of the economy. Since we allow for multiple likelihoods, ambiguity is not resolved in the long run in our model, even when the underlying endowment process is not subject to changes in regime.

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Using the exchange economy framework, we are able to compute analytically equilibrium equity premia, equity expected returns and volatilities, interest rates and price dividend ratios. Since we allow also for external exogenous signals about the unobservable expected growth rate of the aggregate endowment, we are also able to study the relation between asset prices, information noisiness and ambiguity. However, our main focus will be on studying how learning under ambiguity aversion affects the functional form of the equilibrium variables and, more specifically, if it worsens existing asset pricing puzzles. For instance, while there is plenty of evidence that settings of ambiguity aversion do help in explaining the equity premium and the low interest rate puzzles, we also know that in a pure setting of learning the equity premium can be even more than a puzzle (see, e.g., Veronesi (2000)). Does the combination of learning and ambiguity aversion help in giving a reasonable explanations for the equity premium puzzle? Similarly, we know that pure settings of learning can explain excess volatility and volatility clustering of asset returns. At the same time, simple constant opportunity set models of ambiguity aversion do not affect substantially expected equity returns and equity volatility; see, e.g., Maenhout (2004) and Sbuelz and Trojani (2002). Does the combination of learning and ambiguity aversion still generate excess volatility and volatility clustering?

All the above questions can be addressed directly in our model. First, we find that in a Lucas economy learning under ambiguity aversion implies an equilibrium discount for ambiguity, if and only if relative risk aversion is smaller than one or, equivalently, if the elasticity of intertemporal substitution (EIS) is above one. Under low risk aversion, learning and ambiguity aversion increase conditional equity premia and volatilities. Second, learning and ambiguity aversion imply lower equilibrium interest rates, irrespective of risk aversion. Thus, with low risk aversion, we get both a higher equity premium and a lower interest rate. This is a promising feature of our setting, from the perspective of explaining simultaneously the equity premium and the risk free rate puzzles without an ad hoc use of preference parameters. Third, under learning and ambiguity aversion the theoretical equilibrium relation between excess returns and conditional variances is highly time-varying. This feature can generate estimated relations between excess returns and equity conditional variances having undetermined sign over time. Moreover, it can imply huge time varying biases in the naively estimated risk-return trade-off using, e.g., regression methods. Fourth, we show that estimates of the EIS based on standard
Euler equations for equity returns are strongly downward biased in a setting of learning and ambiguity aversion. Therefore, under learning and ambiguity aversion, an EIS above one can be consistent with an observed estimated EIS clearly below one. Finally, since in our setting ambiguity does not resolve asymptotically, we can show explicitly that asset pricing relations under ambiguity aversion but no learning can be interpreted as the limit of an equilibrium learning process under ambiguity aversion.

The paper is organized as follows. The next section reviews the relevant literature on learning and ambiguity. Section 2 introduces our setting of learning under ambiguity aversion. The properties of the optimal learning dynamics are studied in Section 3. Section 4 characterizes and discusses conditional asset pricing relations under learning and ambiguity aversion. Section 5 concludes.

1. Background

Distinguishing between ambiguity aversion and risk aversion is economically and behaviorally important. As the Ellsberg (1961) paradox illustrates, investors behave inherently different under ambiguity and risk aversion. Moreover, ambiguity itself is pervasive in financial markets. Gilboa and Schmeidler (1989) suggest an atemporal axiomatic framework of ambiguity aversion where preferences are represented by Max-Min expected utility over a set of multiple prior distributions. More recently, authors have attempted to incorporate ambiguity aversion also in an intertemporal context. These approaches have been largely inspired by the Gilboa and Schmeidler (1989) Max-Min expected utility setting. Epstein and Wang (1994) study some asset pricing implications of Max-Min expected utility in a discrete-time infinite horizon economy. A discrete-time axiomatic foundation for that model has been provided later in Epstein and Schneider (2003), showing that a dynamically consistent conditional version of Gilboa and Schmeidler (1989) preferences is represented by means of a recursive Max-Min expected utility criterion over a set of multiple distributions. Chen and Epstein (2002) extend that setting to continuous time. A second setting of intertemporal ambiguity aversion based on an alternative form of Max-Min expected utility preferences is proposed in Hansen, Sargent and Tallarini
(1999, in discrete time) and Anderson, Hansen and Sargent (2003, in continuous time). Their setting applies robust control theory to economic problems.

Continuous-time models of full information economies with ambiguity aversion have been recently proposed to give plausible explanations for several important characteristics of asset prices. Examples of such models include, among others, Gagliardini et al. (2004; term structure of interest rates), Epstein and Miao (2003; home bias), Liu et al. (2004; option pricing with rare events), Maenhout (2004; equity premium puzzle), Routledge and Zin (2001; liquidity), Sbuelz and Trojani (2002; equity premium puzzle), Trojani and Vanini (2002, 2004; equity premium puzzle and stock market participation) and Uppal and Wang (2003; home bias). By construction, the above models exclude any form of learning. Investors observe perfectly the state variables determining their opportunity set, but they are not fully aware of the probability distribution of the state variables. Consequently, some form of conservative worst case optimization determines their optimal decision rules.

Only more recently the issue of learning under ambiguity aversion has been addressed by a few authors. In a production economy subject to exogenous regime shifts driven by a two-state Markov chain, Cagetti et. al. (2002) apply robust filtering theory to provide numerical evidence for the impact of learning and ambiguity aversion on the aggregate capital stock, equity premia and price dividend ratios. Using numerical methods, they provide evidence that ambiguity aversion increases precautionary saving in a way that is similar to the effect of an increased subjective time preference rate, leading to an increase in the capital stock. Moreover, the equity premium increases substantially due to ambiguity aversion and price dividend ratios turn out to be lower. Our model differs from their setting in several aspects. First, we work with an exchange Lucas economy that allows us to solve the model in closed form under an arbitrary number of possible states for the underlying dividend drift. Therefore, we are able to study theoretically and in great detail all relevant asset pricing relations and their dependence on model parameters. For instance, we show that in the partial information exchange Lucas economy ambiguity aversion can fail to increase equity premia, if standard risk aversion is too high. Moreover, in contrast to Cagetti et al. (2002), we allow for external public signals, in excess of dividends, and for heterogeneous ambiguity sizes across the relevant states of the
economy. These extensions have nontrivial implications for the resulting asset pricing relations. For instance, the effect of ambiguity aversion on price-dividend ratios cannot be easily mapped, as in Cagetti et al. (2002), into an adjustment of the subjective time preference rate. Moreover, ambiguity premia caused by extraneous signals have an important role in the determination of equity premia.

Epstein and Schneider (2004) highlight in a simple discrete-time setting that learning about an unknown parameter under multiple likelihoods can fail to resolve ambiguity asymptotically, even when the underlying state process is not subject to regime shifts. Epstein and Schneider (2004) use a similar learning model under ambiguity to highlight the impact of an ambiguous signal precision on asset prices. Using numerical methods, they provide evidence in some very stylized discrete-time settings with risk-neutral investors that an ambiguous quality of information, defined in terms of a set of possible values of the signal precision parameter, can generate skewed asset returns and returns excess volatility. In our more general continuous-time setting, we compute equilibria analytically for (i) general power utility investors and (ii) when both fundamental and extraneous ambiguous signals on the underlying state of the economy are present. The first point is key to understand the impact of ambiguity aversion and risk aversion on equity premia because, as we show, positive ambiguity premia can arise only for moderate risk aversions. The second aspect is important, to emphasize the distinction between ambiguous fundamental signals, which affect both the underlying opportunity set and expectations of the economy’s growth rate, and ambiguous extraneous signals which, instead, only affect the expected growth rate of the economy. Finally, Knox (2004) proposes an axiomatic setting of learning about a model parameter under ambiguity aversion, however without studying the general equilibrium asset pricing implications.

2. The Model

We start with a continuous-time Lucas economy. The drift rate in the diffusion process for the dividend dynamics is unobservable. Investors learn about the “true” drift through the observation of dividends and a second distinct signal. In contrast to most other models of
rational learning, we explicitly allow for a distinction between noisy and ambiguous signals. For a purely noisy signal, the distribution conditional on a given parameter value is known. For ambiguous signals, the distribution conditional on a given parameter value is unknown or at least not uniquely identified. This distinction broadens the notion of information quality. In many situations, it is plausible that agents are aware of a host of poorly understood or unknown factors that obscure the interpretation of a given signal. Such obscuring factors can depend on economic conditions or on some specific aspects of a given state of the economy.

In our model, signals on the state of the economy are ambiguous and can be interpreted differently, depending on whether agents condition on good or bad economic information. This feature is modeled by a set of multiple likelihoods on the underlying dividend dynamics. The size of such sets of multiple likelihoods can depend on the state of the economy. Disentangling the properties of noisy and ambiguous signals across the possible relevant states of the economy gives the model builder a more realistic way to specify a learning behavior with multiple beliefs.

Our objective is to characterize equilibrium asset returns in the presence of noisy signals on ambiguous states of the economy. To this end, we develop an equilibrium model of learning under ambiguity aversion consisting of the following key ingredients:

1. A parametric reference model dynamics for the underlying dividend process and the unobservable dividend drift. The reference model is explicitly treated as an approximation to the reality, rather than an exact description of it. Therefore, economic agents possess some motivated specification doubts. Specification doubts arise, e.g., when agents are aware that, based on an empirical specification analysis, they choose the reference model from a set of statistically close models. When agents have to learn the unobservable drift of the dividend dynamics, taking into account such specification doubts is an important modeling device. We introduce the reference model in Section 2.1.

2. A set of multiple likelihoods on the dynamics of the unobservable dividend drift. We use these multiple likelihoods to compute a set of multiple ahead beliefs about the unknown dividend drift dynamics. This set of beliefs represents the investor’s ambiguity on the dynamic structure of the unobservable expected dividend growth rate. The set of multiple likelihoods can also be interpreted as a description of a class of alternative specifications
to the reference model, which are statistically close and therefore difficult to distinguish from it. We introduce the set of multiple likelihoods in Sections 2.2 and 2.3.

3. An intertemporal Max-Min expected utility optimization problem.\(^2\) The Max-Min problem models the agents’ optimal behavior given their attitudes to risk and ambiguity and under the relevant set of multiple ahead beliefs. We formulate the optimization problem in Section 2.4.

Given the three key ingredients above, a set of standard market clearing conditions on good and financial markets closes the model. After solving the model, equilibrium asset prices under learning and ambiguity aversion follow.

### 2.1. The Reference Model Dynamics

We consider a Lucas (1978) economy populated by CRRA investors with utility function

\[
    u(C_t, t) = e^{-\delta t \frac{C_t^{1-\gamma}}{1-\gamma}},
\]

where \(\gamma > 0\). The representative investor has a parametric reference model that describes in an approximate way the dynamics of dividends \(D\)

\[
    \frac{dD}{D} = E_t \left( \frac{dD}{D} \right) + \sigma_D dB_D,
\]

(1)

where \(\sigma_D > 0\) and \(E_t \left( \frac{dD}{D} \right)\) is the unobservable drift of dividends at time \(t\). Investors further observe a noisy unbiased signal \(e\) on \(E_t \left( \frac{dD}{D} \right)\) with dynamics

\[
    de = E_t \left( \frac{dD}{D} \right) + \sigma_e dB_e,
\]

(2)

where \(\sigma_e > 0\). The standard Brownian motions \(B_D\) and \(B_e\) are independent.

The parametric reference model to describe the dividend drift dynamics is a rough approximation of the reality. It implies a simple geometric Brownian motion dynamics for dividends with a constant drift that can take one of a finite number of candidate values.

**Definition 1** The reference model dividend drift specification is given by

\[ \frac{1}{dt} E_t (dD/D) = \theta, \]  

for all \( t \geq 0 \), where \( \theta \in \Theta := \{\theta_1, \theta_2, ..., \theta_n\} \) and \( \theta_1 < \theta_2 < ... < \theta_n \). The representative investor has some prior beliefs \((\hat{\pi}_1, .., \hat{\pi}_n)\) at time \( t = 0 \) on the validity of the candidate drift values \( \theta_1, ..., \theta_n \).

In a single-likelihood Bayesian framework, Definition 1 implies a parametric single-likelihood model for the dividend dynamics, where the specific value of the parameter \( \theta \) is unknown. The only relevant statistical uncertainty about the dynamics in equation (1) is parametric. Therefore, in a single-likelihood Bayesian setting, a standard filtering process leads to asymptotic learning of the unknown constant dividend drift \( \theta \) in the class \( \Theta \) of candidate drift values. Moreover, the equilibrium asset returns dynamics can be determined and the pricing impact of learning can be studied analytically, as for instance in Veronesi (2000).

In the sequel, we strongly depart from such a Bayesian asset pricing settings, by allowing for the possibility of a misspecification in the reference model of Definition 1. Relevant misspecifications will be of a general nonparametric form, so that they cannot be consistently detected even by means of parametric Bayesian model selection approaches. More precisely, we generalize Definition 1 in order to account for specification doubts about the unobservable dividend drift dynamics (1).

### 2.2. Multiple Likelihoods

In reality, a Bayesian (single likelihood) specification hypothesis of the type given in Definition 1 is very restrictive. It assumes that even when dividend drifts are unobservable the investor can identify a parametric model that is able to describe exactly, in a probabilistic sense, the
relevant dividend drift dynamics. More realistically, we propose a model of learning where economic agents have some specification doubts about the given parametric reference model. Such a viewpoint is motivated by considering that any empirical specification analysis provides a statistically preferred model only after having implicitly rejected several alternative specifications that are statistically close to it. Even if such alternative specifications to the reference model are statistically close, it is well possible that they can quantitatively and qualitatively affect the optimal portfolio policies derived under the reference model’s assumptions.\textsuperscript{3} To avoid the negative effects of a misspecification on the optimal policies derived from the reference model, it is desirable to work with consumption/investment optimal policies that account explicitly for the possibility of model misspecifications. This approach should ensure some degree of robustness of the optimal policies against misspecifications of the reference model dynamics.

We address explicitly specification doubts by modelling agents’ beliefs, conditional on any possible reference model drift $\theta$, by means of a set of multiple likelihoods. To define these sets, we restrict ourselves to absolutely continuous misspecifications of the geometric Brownian motion processes in equations (1) and (3). By Girsanov’s theorem, the likelihoods implied by absolutely continuous probability measures can be equivalently described by a corresponding set of drift changes in the model dynamics in equations (1) and (3).

Let $h(\theta)\sigma_D$ be a process describing the dividend drift change implied by such a likelihood function. We assume that $h(\theta) \in \Xi(\theta)$, where $\Xi(\theta)$ is a suitable set of standardized change of drift processes that will be defined more precisely later on (see Assumption 2 below). Under such a likelihood, the prevailing dividend dynamics are

$$\frac{dD}{D} = E^{h(\theta)}_t \left( \frac{dD}{D} \right) + \sigma_D dB_D, \tag{4}$$

with signal dynamics

$$de = E^{h(\theta)}_t \left( \frac{dD}{D} \right) + \sigma_e dB_e. \tag{5}$$

\textsuperscript{3}The importance of this issue has been early recognized, e.g., by Huber (1981) in his influential introduction to the theory of robust statistics and has been further developed, e.g., in econometrics to motivate several robust procedures for time series models. See Krishnakumar and Ronchetti (1997), Sakata and White (1999), Ronchetti and Trojani (2001), Mancini et al. (2003), Gagliardini et al. (2004) and Ortelli and Trojani (2004) for some recent work in the field.
Ambiguity on $D$’s dynamic arises as soon as for some $\theta \in \Theta$ the set $\Xi(\theta)$ contains a drift distortion process $h(\theta)$ different from the zero process. In this case, several possible functional forms of the drift in equation (4) are considered, together with the reference model dynamics in equations (1) and (3). The set of possible drifts implied by the multiple likelihoods in $\Xi(\theta)$ represents the relevant beliefs of an agent who does not trust completely the reference model dynamics.

2.3. A Specific Set of Multiple Likelihoods

Compared with the Bayesian single likelihood specification hypothesis in Definition 1, an agent with multiple likelihood beliefs is less ambitious. More precisely, we have the following assumption.

**Assumption 1** The "true" dividend drift specification is given by

$$
\frac{1}{dt} E_t^{h(\theta)} \left( \frac{dD}{D} \right) = \theta + h(\theta, t) \sigma_D ,
$$

for all $t \geq 0$, some $\theta \in \Theta$ and some $h(\theta) \in \Xi(\theta)$. The representative investor has some beliefs $(\hat{\pi}_1, \ldots, \hat{\pi}_n)$ at time $t = 0$ on the a priori plausibility of the different sets $\Xi(\theta_1), \ldots, \Xi(\theta_n)$ of candidate drift processes.

Under Assumption 1, the representative agent recognizes that a whole class $\Xi(\theta)$ of standardized drift changes is statistically hardly distinguishable from a zero drift change, i.e., from the reference model dynamics with drift $\theta$ given in Definition 1. If $\Xi(\theta) = \{0\}$ for all $\theta \in \Theta$, the Bayesian setup follows. Then, agents would be concerned only with the pure noisiness of a signal about the parameter value $\theta$. Therefore, the distinction between ambiguity and noisiness is absent in a pure Bayesian setting.

The size of the set $\Xi(\theta)$ describes the degree of ambiguity associated with any possible reference model dividend drift $\theta$. The broader the set $\Xi(\theta)$, the more ambiguous are the signals about a specific dividend drift $\theta + h(\theta) \sigma_D \in \Xi(\theta)$. Such ambiguity reflects the fact that there are aspects of the unobservable dividend drift dynamics which agents think are
hardly possible, or even impossible, to ever know. For example, the representative agent is aware of the problem that identifying the exact functional form for a possible mean reversion in the dividend drift dynamics is empirically a virtually infeasible task. Accordingly, the agent tries to understand only a limited number of features on the dividend dynamics.

In our setting, we represent this limitation by a learning model about the relevant neighborhood $\Xi(\theta)$, rather than by a learning process on the specific form of $h(\theta)$. Therefore, the learning problem under multiple beliefs becomes one of learning the approximate features of the underlying dividend dynamics across a class of model neighborhoods $\Xi(\theta)$, $\theta \in \Theta$. Hence, the representative agent has ambiguity about some local dynamic properties of equity returns, conditional on some ambiguous local macroeconomic conditions, and tries to infer some more global characteristics of asset returns in dependence of such ambiguous macroeconomic states. We could also model ambiguity about the set of initial priors $(\hat{\pi}_1, ..., \hat{\pi}_n)$ by introducing a corresponding set of multiple initial priors. However, the main implications from our analysis would not change, because the choice of the initial prior only affects the initial condition in the relevant dynamics of $\Pi$. Finally, since the size of the set $\Xi(\theta)$ can depend on the specific value of $\theta$, our setting allows also for degrees of ambiguity that depend on economic conditions.

We next specify the set $\Xi(\theta)$ of multiple likelihoods relevant for our setting. The set $\Xi(\theta)$ contains all likelihood specifications that are statistically close (in some appropriate statistical measure of model discrepancy) to the one implied by the reference model dynamics. This feature makes more precise the general principle that $\Xi(\theta)$ should contain only models for which agents have some well motivated specification doubt, relative to the given reference model dynamics. The relevant reference model misspecifications are constrained to be small and are thus hardly statistically detectable. Moreover, the set $\Xi(\theta)$ contains any misspecification which is statistically close to the reference model. This property defines a whole neighborhood of slight but otherwise arbitrary misspecifications of the reference model distributions. This is the starting point to develop optimal consumption/investment policies that are robust to general small misspecifications of the given state dynamics. A set of multiple likelihoods satisfying the above requirement is given below.

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Footnote: Shepard and Harvey (1990) show that in finite samples, it is very difficult to distinguish between a purely iid process and one which incorporates a small persistent component.
Assumption 2 For any \( \theta \in \Theta \) we define \( \Xi(\theta) \) by:

\[
\Xi(\theta) := \left\{ h(\theta) : \frac{1}{2}h^2(\theta,t) \leq \eta(\theta) \text{ for all } t \geq 0 \right\},
\]

(7)

where \( \eta(\theta_1), \ldots, \eta(\theta_n) \geq 0 \). Moreover, for any \( i \neq j \) if follows:

\[
\Xi(\theta_i) \cap \Xi(\theta_j) = \emptyset.
\]

(8)

Under Assumption 2, the discrepancy between the reference model distributions under a drift \( \theta \) and those under any model implied by a drift distortion process \( h(\theta) \in \Xi(\theta) \) can be constrained to be statistically small. Anderson, Hansen and Sargent (2003) show that the relative entropy between the reference model probability law and the one under any alternative candidate specifications can be constrained to be small. Relative entropy is a statistical measure of model discrepancy that can be used to bound model detection error probabilities to imply a relatively high probability of an error in model choice. In this sense, a moderate bound \( \eta(\theta) \) implies for any likelihood in the set \( \Xi(\theta) \) a small statistical discrepancy relative to a reference model dynamics with drift \( \theta \). Moreover, since (7) does not make any specific assumption on a parametric structure for \( h(\theta) \), the neighborhood \( \Xi(\theta) \) is nonparametric and contains all likelihood models that are compatible with the bound defined by (7).

Condition (8) means that economic agents have ambiguity only about candidate drifts within neighborhoods, but not between neighborhoods. In other words, different macroeconomic conditions can be mapped into disjoint sets of likely drift dynamics. Such a situation arises when the degree of ambiguity \( \eta(\theta) \) in the economy is not too high, relative to the distance between reference model drifts \( \theta \). Therefore, we focus on situations in which ambiguity in the economy is moderate.

2.4. Ambiguity Aversion and Intertemporal Max-Min Expected Utility

Denote by \( \mathcal{F}(t) \) the information available at time \( t \), containing all possible realizations of dividends and signals. \( P \) is the price of the risky asset in the economy, \( r \) the instantaneous
interest rate and \( \eta(\theta) \) the function that describes the amount of ambiguity relevant to investors, in dependence of a reference model drift \( \theta \). The representative investor determines consumption and investment plans \( C(t) \) and \( w(t) \) by solving the intertemporal Max-Min expected utility optimization problem

\[
(P) : \max_{C,w} \inf_{h(\theta)} \mathbb{E} \left[ \int_0^\infty u(C,s) \, ds \bigg| \mathcal{F}(0) \right],
\]

subject to the dividend and wealth dynamics

\[
\begin{align*}
    dD &= (\theta + h(\theta) \sigma_D) \, D \, dt + \sigma_D \, D \, dB \\
    dW &= W \left[ w \left( \frac{dP + D \, dt}{P} \right) + (1-w) \, r \, dt \right] - C \, dt,
\end{align*}
\]

where for any \( \theta \in \Theta \) the standardized drift distortion is such that \( h(\theta) \in \Xi(\theta) \) and Assumption 2 holds. In (9), the representative agent has to select, in excess of an optimal consumption/investment policy, an optimal worst case belief \( h \) out of the admissible class \( \Xi(\theta) \). The fact that such an optimal belief is determined endogenously as a function of investors’ preferences, is a sharp difference with the standard Bayesian setting, where beliefs are fixed a priori by a parametric assumption on the unobservable dynamics of the underlying dividend drift process.

An equilibrium in our economy is a vector of processes \( (C(t), w(t), P(t), r(t), h(\theta, t)) \) such that the optimization problem \( (P) \) is solved and markets clear, i.e., \( w(t) = 1 \) and \( C(t) = D(t) \).

3. Multiple Filtering Dynamics under Ambiguity

Learning under ambiguity requires constructing a set of standard Bayesian ahead beliefs for \( E^h_t(\frac{dD}{D}) \), in dependence of any likelihood \( h(\theta) \in \Xi(\theta) \). In this section, we first study the dynamic properties of such Bayesian ahead beliefs under different hypotheses on the relation between the underlying true dividend drift dynamics and any corresponding Bayesian prediction for \( E^h_t(\frac{dD}{D}) \), where \( h(\theta) \in \Xi(\theta) \). In a second step, we show how the equilibrium optimal worst case belief of Problem (9) is obtained.
3.1. Bayesian Learning and Likelihood Misspecification

For a given likelihood model $h(\theta) \in \Xi(\theta)$, let $\pi_i(t)$ be the investor’s belief that the drift rate is $\theta_i + h(\theta_i)\sigma_D$, conditionally on past dividend and signal realizations, i.e.,

$$
\pi_i(t) = \Pr\left( \frac{1}{dt} E_t^{h(\theta)} \left( \frac{dD}{D} \right) = \theta_i + h(\theta_i)\sigma_D \left| \mathcal{F}(t) \right. \right).
$$

The distribution $\Pi(t) := (\pi_1(t), \ldots, \pi_n(t))$ summarizes investors beliefs at time $t$, under a given likelihood $h(\theta) \in \Xi(\theta)$. Given such beliefs, investors can compute the expected dividend drift at time $t$,

$$
\frac{1}{dt} E_t^{h(\theta)} \left( \frac{dD}{D} \right) \left| \mathcal{F}(t) \right. = \sum_{i=1}^n \left( \theta_i + h(\theta_i)\sigma_D \right) \pi_i(t) = m_{\theta,h},
$$

where

$$
m_{\theta,h} = m_{\theta} + m_{h(\theta)} , \quad m_{\theta} = \sum_{i=1}^n \theta_i \pi_i(t) , \quad m_{h(\theta)} = \sum_{i=1}^n h(\theta_i) \pi_i(t) \sigma_D .
$$

The filtering equations implied by any given likelihood $h(\theta) \in \Xi(\theta)$ are standard and are given in the next lemma; see, e.g., Lipster and Shiryayev (1997).

**Lemma 1** Suppose that at time zero investors’ beliefs are represented by the prior probabilities $\tilde{\pi}_1, \ldots, \tilde{\pi}_n$. Under a likelihood $h(\theta) \in \Xi(\theta)$, the dynamics of the optimal filtering probabilities vector $\pi_1, \ldots, \pi_n$ is given by

$$
d\pi_i = \pi_i(\theta_i + h(\theta_i)\sigma_D - m_{\theta,h}) \left( k_D d\tilde{B}_D^h + k_e d\tilde{B}_e^h \right) ; \quad i = 1, \ldots, n ,
$$

where

$$
d\tilde{B}_D^h = k_D (dD/D - m_{\theta,h}dt) , \quad d\tilde{B}_e^h = k_e (de - m_{\theta,h}dt) ,
$$

$k_D = 1/\sigma_D$, $k_e = 1/\sigma_e$. In this equation, $\left( \tilde{B}_D^h, \tilde{B}_e^h \right)$ is a standard Brownian motion in $\mathbb{R}^2$, under the likelihood $h(\theta) \in \Xi(\theta)$ and with respect to the filtration $\{\mathcal{F}(t)\}$. To study how a likelihood misspecification affects the dynamic properties of the perceived beliefs, it is useful to express (12) in terms of the original Brownian motions $B_D$ and $B_e$. This
description helps to highlight how a likelihood misspecification can fail to imply consistency of a Bayesian learning process.

**Corollary 1** Let \( h(\theta) \in \Xi(\theta) \) be an admissible likelihood and \( \theta_l + h_D\sigma_D, \ l \in \{1, .., n\}, \) be the true dividend drift process. It then follows:

\[
d\pi_i = \pi_i (\theta_i + h(\theta_i)\sigma_D - m_{\theta,h}) [k(\theta_i + h_D\sigma_D - m_{\theta,h}) dt + k_DdB_D + k_e dB_e] \quad ; \quad i = 1, .., n .
\]

where \( k = k_D^2 + k_e^2 .\)

Expression (13) gives the dynamics of the posterior probability \( \pi_i \) for the general case where the likelihood \( h(\theta) \) might be different from the true underlying drift distortion \( h_D, \) i.e., the case where the likelihood \( h(\theta) \) might be misspecified. The case of a correctly specified likelihood arises when \( h(\theta_l) = h_D .\) In this case, simple inspection of the dynamics (13) shows that the underlying dividend drift \( \theta_l + h_D\sigma_D \) will be eventually learned.

**Corollary 2** If the likelihood \( h(\theta) \) is correctly specified, i.e., if \( h_D = h(\theta_l) \) for some \( \theta_l \in \Theta, \) then \( \pi_l \underset{t \to \infty}{\to} 1, \) almost surely.

Corollary 2 shows that consistency of a Bayesian learning process is inherently linked to the correct specification of the given likelihood. Intuitively, consistency cannot be generally expected under a misspecified likelihood \( h(\theta) .\) To illustrate the basic point, we can study the resulting learning dynamics in a simplified setting with only two possible dividend drift states.

**Example 1** Consider the simplified model structure:

\[
\Theta = \{\theta_1, \theta_2\} \quad , \quad h(\theta_1) = h(\theta_2) = 0 .
\]

Let \( \theta_1 + h_D\sigma_D \) be the true underlying dividend drift process. Then, equation (13) implies the learning dynamics:

\[
d\pi_1 = \pi_1 (1 - \pi_1) (\theta_1 - \theta_2) [k(\theta_1 + h_D\sigma_D - m_{h,\theta}) dt + k_DdB_D + k_e dB_e] . \quad (14)
\]
From Example 1, we see immediately that if

\[ \theta_1 + h_D \sigma_D < m_{\theta,h} \quad (\theta_1 + h_D \sigma_D > m_{\theta,h}) \]

then \( \pi_1 \to 1 \) \( (\pi_1 \to 0) \) almost surely as \( T \to \infty \). Under these conditions investors will therefore "learn" asymptotically a constant dividend drift process \( \theta_1 \) \( (\theta_2) \) even if the true one \( \theta_1 + h_D \sigma_D \) is possibly time varying in a nontrivial and unpredictable way. This remark implies that we will always have \( \pi_1 \to 1 \) \( (\pi_1 \to 0) \) as \( T \to \infty \) for all settings where the true drift \( \theta_1 + h_D \sigma_D \) is uniformly lower than \( \theta_1 \) \( (\text{higher than } \theta_2) \). In the more general case with \( \theta_1 + h_D \sigma_D \) between \( \theta_1 \) and \( \theta_2 \), both outcomes are possible \( (\text{i.e., either } \pi_1 \to 1 \text{ or } \pi_1 \to 0) \). Figure 1 illustrates this point.

**Insert Figure 1 about here**

In Figure 1, we plot two different trajectories of \( \pi_1 \) under a dividend drift process such that

\[
\theta_1 + h_D(t) \sigma_D = \begin{cases} 
(\theta_1 + \theta_2)/2 + a & t \in (k, k+1], \\
(\theta_1 + \theta_2)/2 - a & t \in (k+1, k+2] 
\end{cases},
\]  

where \( k \in \mathbb{N} \) is even and \(|a| \leq (\theta_1 + \theta_2)/2\). Process (15) describes a simple deterministic and piecewise constant dividend drift misspecification. More complex (possibly nonparametric) misspecifications can be considered, but the main message of Figure 1 would not change.

Figure 1 shows that under a dividend drift process (15) a Bayesian investor could converge to infer asymptotically both \( \theta_1 \) and \( \theta_2 \) as the dividend drift process that generated asset prices, even if the true drift process is always strictly between \( \theta_1 \) and \( \theta_2 \). In Panel (A), we plot two possible posterior probabilities trajectories when no shift arises \( (a = 0) \). In Panel (B), we add two alternative trajectories implied by \( a = 0.015 \) when a yearly deterministic shift in the underlying parameters is present. The only attainable stationary points in the dynamics (14) are the points \( \pi_1 = 1 \) and \( \pi_1 = 0 \). Any value \( \pi_1 \in (0,1) \) such that

\[ \theta_1 + h_D \sigma_D = m_{h,\theta} \]
makes the drift, but not the diffusion, equal to zero in the dynamics (14). Consequently, $\pi_1$ will never stabilize asymptotically in regions such that $m_{h,\theta} \approx \theta_1 + h_D\sigma_D$. An asymptotic behavior such that $m_{h,\theta} \approx \theta_1 + h_D\sigma_D$ would be ideally more natural, if the goal is to approximate adequately $\theta_1 + h_D\sigma_D$ by means of $m_{h,\theta}$, even under a misspecified likelihood. However, this behavior will never arise under the given misspecified likelihood. Richer, but qualitatively similar, patterns emerge when the set of possible states of the economy is enlarged or when the form of introduced misspecification in the likelihood is more complex.

The above discussion highlights that a Bayesian investor will not be able to evaluate exactly the utility of a consumption/investment strategy, because she will never identify exactly the underlying dividend drift process, even asymptotically. We therefore work with a setting of learning where investors explicitly exhibit some well founded specification doubts about the given reference model.

3.2. Learning under Ambiguity Aversion

Which learning behavior should agents adopt in an ambiguous environment? Since agents are not particularly comfortable with a specific element of $\Xi(\theta)$, they base their beliefs on the whole set of likelihoods $\Xi(\theta)$. By Corollary 1, this approach generates a whole class $\mathcal{P}$ of indistinguishable dynamic dividend drift prediction processes given by

$$\mathcal{P} = \{m_{\theta,h} : h(\theta) \in \Xi(\theta)\} ,$$

where the dynamics of any of the corresponding posterior probabilities $\pi_1, .., \pi_n$ under the likelihood $h(\theta)$ is given by

$$d\pi_i = \pi_i(\theta_i + h(\theta_i)\sigma_D - m_{\theta,h})\left(k_Dd\tilde{B}_D^h + k_ed\tilde{B}_e^h\right) , \quad i = 1, .., n .$$

The set $\mathcal{P}$ of dynamic dividend drift predictions represents investor’s ambiguity on the true dividend drift process, conditional on the available information generated by dividends and signals. As expected, the larger the size of the set of likelihoods $\Xi(\theta)$ (i.e., the ambiguity about
the dividend dynamics) the larger the size of the set $\mathcal{P}$ of dynamic dividend drift prediction processes.

Using the set $\mathcal{P}$ of dynamic dividend drift predictions, the continuous-time optimization problem (9) can be written as a full information problem where the relevant dynamics are defined in terms of the filtration $\{\mathcal{F}(t)\}$. Indeed, since all beliefs implied by likelihoods $h(\theta) \in \Xi(\theta)$ are absolutely continuous, all relevant processes $(\tilde{B}_D^h, \tilde{B}_e^h)'$ generate the same filtration $\{\mathcal{F}(t)\}$ and the dynamic budget constraint associated with Problem (9) can be equivalently formulated in terms of $m_{\theta,h}$ and $(\tilde{B}_D^h, \tilde{B}_e^h)'$. See also Miao (2001) for a related discussion.

In equilibrium, the relevant problem then reads

\[
(P) : \quad J(\Pi, D) = \inf_{h(\theta)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \frac{D(t)^{1-\gamma}}{1-\gamma} \left| \mathcal{F}(0) \right| \right], \tag{16}
\]

subject to the dynamics

\[
dD = m_{\theta,h} D dt + \sigma_D D d\tilde{B}_D^h, \tag{17}
\]
\[
d\pi_i = \pi_i (\theta_i + h(\theta_i) \sigma_D - m_{\theta,h}) \left( k_D d\tilde{B}_D^h + k_e d\tilde{B}_e^h \right), \tag{18}
\]

where for any $\theta \in \Theta$ we have $h(\theta) \in \Xi(\theta)$ and Assumption 2 holds. The key difference with a standard (single-likelihood) equilibrium Bayesian setting of learning is that in (16) investors are requested to select optimally a worst case forecast procedure for the unknown dividend drift. Such a worst case belief selection generates an endogenous systematic discrepancy between the reference model belief and the one applied by investors to evaluate risky assets under ambiguity. The worst case belief selection has a direct impact on the equilibrium level of asset prices, because it affects investors’ relevant belief for pricing future asset pay-offs, by enforcing a conservative max-min utility behavior. A more indirect effect arises, however, for the equilibrium equity return dynamics under the reference model belief. In particular, the systematic bias between reference model belief and investors’ worst case belief affects reference model equilibrium equity premia in a nontrivial way.
We first study in Proposition 1 the direct impact of ambiguity aversion on the price of equity and equilibrium interest rates, by presenting the solution to Problem (16). The implications of ambiguity aversion for the dynamics of asset returns are analyzed in Section 4.

Proposition 1 Let \( \hat{\theta}_i := \delta + (\gamma - 1) \theta_i + \gamma (1 - \gamma) \frac{\sigma^2}{\hat{D}_i^2} \) and assume that
\[
\hat{\theta}_i + (1 - \gamma) \sqrt{2 \eta(\theta_i) \sigma_D} > 0 \quad , \quad i = 1, \ldots, n \quad .
\] (19)

Then, we have:

1. The normalized misspecification \( h^*(\theta) \) solving Problem (16) is given by
\[
h^*(\theta_i) = -\sqrt{2 \eta(\theta_i)} \quad , \quad i = 1, \ldots, n \quad .
\] (20)

2. The equilibrium price function \( P(\Pi, D) \) for the risky asset is given by:
\[
P(\Pi, D) = D \sum_{i=1}^{n} \pi_i C_i
\] (21)

where
\[
C_i = 1/(\hat{\theta}_i + (1 - \gamma) \sqrt{2 \eta(\theta_i) \sigma_D}) \quad , \quad i = 1, \ldots, n \quad .
\] (22)

3. The equilibrium interest rate \( r \) is:
\[
r = \delta + \gamma m_{\theta,h^*} - \frac{1}{2}\gamma (\gamma + 1) \sigma_D^2 \quad ,
\] (23)

where \( m_{\theta,h^*} = m_{\theta} + m_{h^*(\theta)} \) is such that
\[
m_{h^*(\theta)} = \sum_{i=1}^{n} h^*(\theta_i) \pi_i \sigma_D = -\sum_{i=1}^{n} \sqrt{2 \eta(\theta_i)} \pi_i \sigma_D \quad .
\] (24)

Each constant of the form (22) is proportional to investors’ expectation of discounted lifetime dividends, conditional on a constant dividend drift process \( \theta_i - \sqrt{2 \eta(\theta_i) \sigma_D} \). The drift pro-
cess $\theta_i - \sqrt{2\eta(\theta_i)}\sigma_D$, is the worst case drift misspecification $\theta_i + h^*(\theta_i)\sigma_D$ selected from the neighborhood $\Xi(\theta_i)$. More specifically, we have:

$$C_i = E^{h^*(\theta_i)}\left[\int_s^\infty e^{-\delta(t-s)} \left(\frac{D(t)}{D(s)}\right)^{1-\gamma} dt\right] = \frac{1}{D(s)} E^{h^*(\theta_i)}\left[\int_s^\infty \frac{u_c(D(t), t)}{u_c(D(s), s)} D(t) dt\right],$$

(25)

where $E^{h^*(\theta_i)}[\cdot]$ denotes expectations under a geometric Brownian motion process for $D$ having drift $\theta_i - \sqrt{2\eta(\theta_i)}\sigma_D$.

A high $C_i$ implies that investors are willing to pay a high price for the ambiguous state $\Xi(\theta_i)$. Since the state is not observable, they weigh each $C_i$ by the posterior probability $\pi_i$ to get the price (21) of the risky asset under learning and ambiguity aversion. We remark that $C_i$ is a function of both investors’ ambiguity aversion, via the parameter $\eta(\theta_i)$, and investor’s relative risk aversion $\gamma$.

It is easy to see that (25) can be equivalently written as:

$$C_i = E\left[\int_s^\infty e^{-(\delta+(1-\gamma)\sqrt{2\eta(\theta_i)}\sigma_D)(t-s)} \left(\frac{D(t)}{D(s)}\right)^{1-\gamma} dt|\theta = \theta_i\right],$$

(26)

where $E[\cdot|\theta = \theta_i]$ denotes reference model expectations conditional on a constant drift $\theta = \theta_i$. Therefore, the impact of ambiguity aversion on the price of the ambiguous state $\Xi(\theta_i)$ is equivalent to the one implied by a corrected time preference rate

$$\delta \rightarrow \delta + (1-\gamma)\sqrt{2\eta(\theta_i)}\sigma_D$$

(27)

under the reference model dynamics. Adjustment (27) depends on the amount of ambiguity of the ambiguous state $\Xi(\theta_i)$, relative risk aversion $\gamma$ and dividend growth volatility $\sigma_D$. Cagetti et al. (2002) produce numerical evidence that ambiguity aversion decreases the aggregate capital stock in way that is similar to the effect of an increased subjective discount rate. (27) suggests that their finding can be rationalized theoretically in our setting. However, in our general case of an heterogeneous degree of ambiguity $\eta(\theta)$, the final effect of ambiguity on equity prices cannot be mapped into an adjustment of one single time preference rate. Moreover, as shown
in Section 4, heterogeneous ambiguity structures generate model predictions that are consistent with the well-known puzzles even more easily than homogenous ambiguity structures.

The dependence of the price $C_i$ on an arbitrary ambiguity parameter $\eta(\theta_i)$ is summarized next.

**Corollary 3** The price of any ambiguous state $\Xi(\theta)$ is a decreasing function in the degree of ambiguity $\eta(\theta)$ if and only if $\gamma < 1$. In such a case, $C_i$ is a convex function of $\eta(\theta_i)$, which is uniformly more convex for smaller risk aversion $\gamma$.

From Corollary 3, the marginal relative price of ambiguity is negative if and only if relative risk aversion $\gamma$ is less than 1.\(^5\) In the opposite case, if $\gamma > 1$, one obtains the somewhat counterintuitive implication\(^6\) that the price of an ambiguous state is higher than the one of an unambiguous one.

To understand this apparently paradoxical finding, recall that in the determination of $C_i$ the representative investor discounts worst-case future dividends through their marginal utility. In equilibrium, a lower dividend growth rate implies a lower expected future consumption growth and a lower discount rate. Since for high risk aversions the last effect dominates, a lower expected dividend growth deriving from a conservative belief under ambiguity implies a lower discount rate and a higher price for ambiguous states.

For $\gamma > 1$, settings of learning and ambiguity aversion with high risk aversions deliver low (negative) equity premia and low volatilities, together with high and highly variable interest rates. That is, imposing high risk aversions worsens the well-known asset pricing puzzles when

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\(^5\)There is some recent work in the literature on the question of which are typical risk aversion parameters of agents that are confronted with an ambiguous environment. Suggestive in this context is the evidence collected by the experimental work of Wakker and Dennehe (1996; e.g., the graphs on p. 1143), who estimate a virtually linear utility function with a utilities elicitation procedure that is robust to the presence of ambiguity. In the same experiments, the utility functions estimated by procedures that are not robust to the presence of ambiguity were clearly concave. These findings suggest that the high risk aversions estimated in some experimental research can be also due to some pronounced deviations from expected utility. In our setting, such deviation from expected utility arises from Max-Min expected utility preferences reflecting ambiguity aversion.

\(^6\)Relatively, e.g., to the basic intuition provided by the standard (static) Ellsberg (1961) paradox.
learning under ambiguity aversion is considered. Therefore, we focus in the sequel on settings with moderate risk aversions.\footnote{Under power utility, Assumption 3 is equivalent to an elasticity of intertemporal substitution (EIS) $1/\gamma > 1$. Hansen and Singleton (1982) and Attanasio and Weber (1989) estimated an EIS well above one. Hall (1988) considered aggregation effects and estimated an EIS well below one. Similar low estimates are obtained in Campbell (1999). Recent work focusing on the consumption of households participating in the stock or the bond market has suggested that such investors have much larger EIS than individuals which do not hold stocks or bonds. For instance, Vissing-Jorgensen (2002) estimate an EIS well above one for individuals holding portfolios of stocks and bonds in Euler equations for treasury bills. Attanasio and Vissing-Jorgensen (2003) also estimate large EIS for stockholders, when using Euler equations for treasury bills and after-tax returns. Attanasio, Banks, and Tanner (2002) find with UK data EIS larger than one for Euler equations including treasury bills and equity returns in an econometric model where ownership probabilities are jointly estimated.}

**Assumption 3** The representative agent in the model has a relative risk aversion parameter $\gamma < 1$.

### 3.2.1. Price Dividend Ratios and Interest Rates

Under Assumption 3, we obtain from Proposition 1 a few nice and simple implications for the behavior of the price dividend ratio $P/D$ in the model. They are summarized by the next result.\footnote{Finding a) in Corollary 4 is a direct implication of (21) and (26). Finding b) follows from the convexity of $C_i$ in (21) as a function of $\theta_i - \sqrt{2\eta(\theta_i)\sigma_D}$.}

**Corollary 4** Under Assumption 3 we have the following:

a) The price dividend ratio $P/D$ is a decreasing convex function of the amount of ambiguity $(\eta(\theta_1), \ldots, \eta(\theta_n))$ in the economy. Moreover, $P/D$ is a uniformly more convex function for lower risk aversion $\gamma$.

b) A mean preserving spread $\hat{\Pi}$ of $\Pi$ implies

$$\hat{P}/D > P/D$$

that is, the price dividend ratio $P/D$ is increasing in the amount of uncertainty of the economy.
Under Assumption 3, the impact of a higher ambiguity on price dividend ratios (Finding a) has a different sign than the one of a higher uncertainty in the economy (Finding b). This is a distinct prediction of ambiguity aversion for the behavior of $P/D$.

In Proposition 1, the equilibrium interest rate is given by equation (23). The effect of learning and ambiguity aversion on equilibrium interest rates is always negative, since $r$ is a decreasing convex function of $(\eta(\theta_1), \ldots, \eta(\theta_n))$. The special case of an equilibrium interest rate $r_{NA}$ under no ambiguity, as in Veronesi (2000), is obtained by setting $\eta(\theta) = 0$ for all $\theta \in \Theta$ in (23),

$$r_{NA} = \delta + \gamma m_\theta - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 .$$

Hence,

$$r - r_{NA} = \gamma m_{h^*(\theta)} < 0 \iff \text{there exists } \theta \in \Theta \text{ such that } \eta(\theta) > 0 .$$

The case with no uncertainty about the true model neighborhood arises under a degenerate distribution $\Pi$, implying $m_\theta + m_{h^*(\theta)} = \theta_l - \sqrt{2\eta(\theta_l)}\sigma_D$ for some $\theta_l \in \Theta$ and

$$r = \delta + \gamma \left( \theta_l - \sqrt{2\eta(\theta_l)}\sigma_D \right) - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 . \quad (28)$$

The interest rate (28) is the equilibrium interest rate of an economy with ambiguity but no learning. Hence, even in the case of an asymptotic learning about $\Xi(\theta_l)$, the asset pricing impact of ambiguity on interest rates does not disappear. Asymptotically, the representative agent still has ambiguity about which is the precise drift $\theta_l + h(\theta_l)\sigma_D \in \Xi(\theta_l)$ that generated the dividend dynamics, even if she learned that the relevant model neighborhood is $\Xi(\theta_l)$. Therefore, a premium for this residual ambiguity persists asymptotically.

For the case where the asymptotic distribution of $\Pi$ is nondegenerate, the contribution $m_{h^*(\theta)}$ of ambiguity aversion to the level of interest rates is a weighted sum of the contributions of ambiguity aversion under the single model neighborhoods $\Xi(\theta_1), \ldots, \Xi(\theta_n)$, and it is time varying. The weights in $m_{h^*(\theta)}$ are given by the posterior probabilities $\Pi$. Moreover, the dynamics of $\Pi$ depend on \(\eta(\theta)\), via the worst case likelihood $h^*(\theta)$ that has been optimally selected by our ambiguity averse investor; see again equation (18). Settings of ambiguity
aversion but no learning have been studied in Sbuelz and Trojani (2002), under an *exogenous* dynamics for the state variables driving the time varying degree of aggregate ambiguity. In our model, instead, time varying aggregate ambiguity arises fully *endogenously*, because its \( \Pi \)-dependent dynamics cannot be determined irrespective of the ambiguity parameter \( \eta \) in the economy.

### 3.2.2. Endogenous Learning Dynamics

The normalized worst case drift distortion (20) in Proposition 1 determines the description of the endogenous relevant \( \Pi \)-dynamics under ambiguity aversion. We focus on a description under the reference model dynamics from the perspective of an outside observer knowing:

a) that the dividend dynamics indeed satisfies the reference model in (1) and (2).

b) the specific value of the parameter \( \theta \).

Despite the fact that the true dynamics are those under the reference model, misspecification doubts coupled with ambiguity aversion force investors to follow a different learning dynamics than the optimal Bayesian one under the reference model’s likelihood. This is highlighted by the next Corollary, which follows directly from Corollary 1 and Proposition 1.

**Corollary 5** Under the reference model in (1) and (2), the filtered probabilities dynamics of a representative agent solving the equilibrium optimization problem (16) are:

\[
d\pi_i = \pi_i \left( \theta_i - \sqrt{2\eta(\theta_i)}\sigma_D - m_{\theta,h^*} \right) \left[ k(\theta - m_{\theta,h^*}) dt + k_D dB_D + k_e dB_e \right] \tag{29}
\]

Equation (29) gives us a way to study the learning dynamics realized under ambiguity aversion. We observe that ambiguity aversion can imply a tendency to overstate the probability of good states, relatively to the probabilities implied by a learning dynamics of a Bayesian
investor. To emphasize this point, we consider for simplicity the case of a constant ambiguity aversion \( \eta(\theta_1) = \ldots = \eta(\theta_n) = \eta \), implying filtered probability dynamics given by

\[
d\pi_i = \pi_i(\theta_i - m_\theta) \left[k \left( \theta - m_\theta + \sqrt{2\eta} \sigma_D \right) dt + k_D dB_D + k_e dB_e \right].
\]

For \( \eta = 0 \), the dynamics (30) are those of a standard (single likelihood) Bayesian learner. Under ambiguity aversion, the drift

\[
k\pi_i (\theta_i - m_\theta) \left( \theta - m_\theta + \sqrt{2\eta} \sigma_D \right) dt
\]

of probability \( \pi_i \) is larger for above average reference model drifts \( \theta \) \( (\theta - m_\theta > 0) \) and lower for below average reference model drifts \( \theta \) \( (\theta - m_\theta < 0) \). Therefore, investors subject to ambiguity aversion will tend to “learn” more rapidly a large reference model drift than a low reference model drift. Unconditionally, this will imply learning dynamics where the a posteriori expected reference model drift \( m_\theta \) under ambiguity aversion will be higher than the one of a Bayesian investor, i.e., the learning dynamics under ambiguity aversion will imply an optimistic tendency to overstate the a posteriori reference model drifts relatively to a standard Bayesian prediction. Such a tendency is more apparent for large precision parameters \( k \).

Figure 2 illustrates the above features for a setting with three possible neighborhoods \( \Xi(\theta_1), \Xi(\theta_2), \Xi(\theta_3) \). We plot the posterior probabilities \( \pi_1 \) implied by Corollary 5 for the "bad" state \( \Xi(\theta_1) \) in Panel (A) and those for the good state \( \Xi(\theta_3) \) (the probabilities \( \pi_3 \)) in Panel (B). The "true" underlying state is \( \Xi(\theta_2) \).

**Insert Figure 2 about here**

In Panel (A), the uniformly higher probabilities \( \pi_1 \) arise in the absence of ambiguity (the straight line corresponding to \( \eta = 0 \)), while for the largest ambiguity aversion parameter \( \eta = 0.05 \) the uniformly lowest posterior probabilities arise. Hence, the ambiguity averse investor systematically understates the probability of the "bad" state \( \theta_1 \). Similar features, but with opposite direction, arise for the probabilities \( \pi_3 \) of the "good" state \( \theta_3 \) in Panel (B).
4. Conditional Asset Returns

Given the worst case dividend drift $\theta_i - \sqrt{2\eta(\theta_i)}\sigma_D$ conditional on the ambiguous state $\Xi(\theta_i)$, we obtain the equilibrium equity excess return $R$ dynamics under learning and ambiguity aversion, defined by

$$dR = \frac{dP + Ddt}{P} - rdt$$

We first study the direct effect of learning and ambiguity aversion on $R$—dynamics, by describing it with respect to the filtered Brownian motions $\tilde{B}_D^{h^*}, \tilde{B}_e^{h^*}$, implied by the selected optimal worst case likelihood belief $h^*(\theta)$ of Proposition 1. This description provides the dynamics of $R$ under the worst case scenario $h^*(\theta) \in \Xi(\theta)$ in our economy. In this sense, the resulting expected excess return on equity can be interpreted as the worst case equity premium in the economy. The indirect impact of learning and ambiguity aversion on $R$—dynamics arises because of the differences between the likelihood belief under the reference model dynamics and the optimal worst case belief adopted by ambiguity averse investors to compute asset prices. Under the reference model likelihood belief, such a discrepancy determines an additional ambiguity premium component for misspecification in the $R$—dynamics. We can analyze this further important effect of learning and ambiguity aversion, by describing $R$—dynamics with respect to the filtered Brownian motions $\tilde{B}_D, \tilde{B}_e$ implied by the reference model belief for dividends in Definition 1. This description provides the correct $R$—dynamics from the perspective of an outside observer (for instance an econometrician), which (i) believes in the reference model of Definition 1 as an approximate description of the dividend dynamics and (ii) knows that investors in the economy are ambiguity averse. The resulting expected excess return on equity identifies the structure of equity premia under learning and ambiguity aversion. We summarize our findings in the next proposition.

Proposition 2 (i) If the dividend dynamics are those implied by the optimal worst case dynamics of the likelihood $h^*(\theta)$ in Proposition 1, the equilibrium return process $R$ under ambiguity aversion has dynamics

$$dR = \mu_R^{wc}dt + \sigma_Dd\tilde{B}_D^{h^*} + \nu_{\theta,h^*} \left( k_Dd\tilde{B}_D^{h^*} + k_ed\tilde{B}_e^{h^*} \right)$$

(32)
where
\[
\mu_{\text{wc}}^R = \gamma \left( \sigma_D^2 + V_{\theta,h^*} \right), \quad V_{\theta,h^*} = \sum_{i=1}^{n} \frac{\pi_i C_i \left( \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \right)}{\sum_{i=1}^{n} \pi_i C_i} - m_{\theta,h^*}, \quad (33)
\]
and with Brownian motion increments with respect to the filtration \{F(t)\} given by:
\[
d\tilde{B}_D^h = k_D \left( \frac{dD}{D} - m_{\theta,h^*} dt \right), \quad d\tilde{B}_e^h = k_e (d e - m_{\theta,h^*} dt).
\]

(ii) If the dividend dynamics are those under the reference model, the equilibrium excess return process \(R\) under ambiguity aversion has dynamics
\[
dR = \mu_R dt + \sigma_D d\tilde{B}_D + V_{\theta,h^*} \left( k_D d\tilde{B}_D + k_e d\tilde{B}_e \right), \quad (34)
\]
where
\[
\mu_R = \mu_{\text{wc}}^R - m_{h^*(\theta)} (1 + kV_{\theta,h^*})
\]
and with Brownian motion increments with respect to the filtration \{F(t)\} given by:
\[
d\tilde{B}_D = k_D \left( \frac{dD}{D} - m_{\theta} dt \right), \quad d\tilde{B}_e = k_e (d e - m_{\theta} dt).
\]

We can analyze in more detail how learning under ambiguity aversion affects the conditional structure of asset returns. We first study the sign and comparative statics for quantities \(m_{h^*(\theta)}\) and \(V_{\theta,h^*}\) arising in the dynamics (32) and (34). In a second step, we discuss the impact of learning under ambiguity aversion on equity premia and volatilities. Quantity
\[
m_{h^*(\theta)} = m_{\theta,h^*} - m_{\theta} = - \sum_{i=1}^{n} \sqrt{2\eta(\theta_i)} \pi_i \sigma_D
\]
is a conservative correction to the reference model’s a posteriori expectations \(m_{\theta}\). It accounts for misspecification doubts in the a posteriori expectations for the growth rate of the economy and is always negative. Quantity \(V_{\theta,h^*}\) reflects the difference between the worst case expected growth rate of the economy, \(m_{\theta,h^*}\), and its value adjusted counterpart. \(V_{\theta,h^*}\) is larger either when agents have more diffuse beliefs about \(\Xi(\theta_1), \ldots, \Xi(\theta_n)\) or when they value the asset very
differently across the different states. These differences in valuation depend on the heterogeneity of the worst case growth rate \( \theta - \sqrt{2\eta(\theta)\sigma_D} \) across such states. Under Assumption 2, it follows

\[
\theta_1 - \sqrt{2\eta(\theta_1)\sigma_D} < \theta_2 - \sqrt{2\eta(\theta_2)\sigma_D} < \cdots < \theta_n - \sqrt{2\eta(\theta_n)\sigma_D}.
\]

Therefore, we can use similar arguments as in the proof of Lemma 3 in Veronesi (2000) to obtain the following characterization of \( V_{\theta,h^*} \) in our setting of learning under ambiguity aversion.

**Lemma 2** Let Assumption 2 be satisfied. It then follows:

1. \( V_{\theta,h^*} \) is a decreasing function of \( \gamma \).
2. The following statements are equivalent:
   
   (a) Assumption 3 holds.
   
   (b) \( V_{\theta,h^*} > 0 \).
   
   (c) For any mean preserving spread \( \tilde{\Pi} \) of \( \Pi \) it follows

\[
\tilde{V}_{\theta,h^*} > V_{\theta,h^*},
\]

where "∽" denotes quantities under \( \tilde{\Pi} \).

In particular, quantity \( V_{\theta,h^*} \) is positive and increasing with respect to mean preserving spreads of \( \Pi \) if and only if \( \gamma < 1 \). As we discuss below, the positivity of \( V_{\theta,h^*} \) is crucial in order to avoid theoretical asset pricing relations that are clearly inconsistent with the equity premium puzzle predictions.

To study the impact of ambiguity aversion on \( V_{\theta,h^*} \), we compute in the next proposition comparative statics with respect to the standardized worst case drift ambiguity sizes

\( \sqrt{2\eta(\theta_1)}, \ldots, \sqrt{2\eta(\theta_n)} \) in a neighborhood of \( \eta(\theta) = 0 \), i.e., the pure Bayesian learning setting.
Proposition 3  
(i) The comparative statics of $V_{\theta,h^*}$ with respect to the ambiguity parameter $\sqrt{\eta(\theta_i)}$ is:

$$
\frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta(\theta_i)}} \bigg|_{\eta(\theta) = 0} = - \left[ \frac{\pi_i C(\theta_i)}{\sum_{j=1}^{n} \pi_j C(\theta_j)} - \pi_i + (1 - \gamma) \frac{\pi_i C(\theta_i)^2}{\sum_{j=1}^{n} \pi_j C(\theta_j)} (\theta_i - m_\theta - V_\theta) \right] \sigma_D ,
$$

where for any $i = 1,\ldots,n$ coefficient $C(\theta_i)$ is the value of $C_i$ in Proposition 1 for $\eta(\theta_i) = 0$ and $V_\theta$ is the value of $V_{\theta,h^*}$ for $\eta(\theta_1) = \ldots = \eta(\theta_n) = 0$.  

(ii) Let Assumption 3 be satisfied. If both conditions

$$
\frac{\pi_i C(\theta_i)}{\sum_{j=1}^{n} \pi_j C(\theta_j)} \geq \pi_i \quad ; \quad \theta_i - m_\theta \geq V_\theta
$$

are satisfied, then:

$$
\frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta(\theta_i)}} \bigg|_{\eta(\theta) = 0} \leq 0 .
$$

(iii) Let Assumption 3 be satisfied. If ambiguity is homogenous ($\eta(\theta_i) = \eta$ for $i = 1,\ldots,n$), it follows

$$
\frac{\partial V_{\theta,h^*}}{\partial 2\eta} \bigg|_{\eta = 0} < 0 .
$$

To understand the meaning of Proposition 3 and condition (35), define for any $i = 1,\ldots,n$ the value adjusted probability of the reference model drift $\theta_i$ in the absence of ambiguity,

$$
\tilde{\pi}_i = \frac{\pi_i C(\theta_i)}{\sum_{j=1}^{n} \pi_j C(\theta_j)} .
$$

Then, (35) can be rewritten as

$$
\tilde{\pi}_i \geq \pi \quad ; \quad \theta_i - m_\theta \geq \tilde{m}_\theta - m_\theta ,
$$

where "∼" denotes quantities under $\tilde{\Pi}$. Condition (39) is intuitive. It requires that the value weighted probability $\tilde{\pi}_i$ of $\theta_i$ and the reference model drift $\theta_i$ itself are larger (or smaller) than,
respectively, the posterior probability \( \pi_i \) and the \( \tilde{\Pi} \)-value weighted mean of \( \theta \). If (39) is valid with ”>”, then adding ambiguity to state \( \theta_i \) implies, under Assumption 3, \( V_{\theta,h^*} < V_{\theta} \). The opposite holds if (39) is valid with ”<”. Under Assumption 3, \( C(\theta_i) \) in an increasing convex function of \( \theta_i \). Therefore, (39) will tend to hold with ”>” for large values of \( \theta_i \) and with ”<” for low values of \( \theta_i \).\(^9\) Inequality (36) of Proposition 3 then implies that asymmetric ambiguity structures \( \eta(\theta) \) tend to decrease (increase) \( V_{\theta,h^*} \), when ambiguity is sufficiently large for high (low) reference model drift states. When ambiguity is homogenous, however, no increase in \( V_{\theta,h^*} \) arises when extending the Bayesian learning setting to incorporate ambiguity aversion; this is the content of finding (iii) in Proposition 3.

4.1. Equity Premia

From equation (33), the equilibrium equity premium \( \mu_R \) is given by

\[
\mu_R = \frac{\gamma (\sigma_D^2 + V_{\theta,h^*})}{A} - \frac{m_{h^*}}{B} - \frac{m_{h^*} k V_{\theta,h^*}}{C}.
\] (40)

\( \mu_R \) is the sum of three conceptually different equity premium contributions (A), (B) and (C). (A) is the equity premium part deriving from standard risk exposure, i.e., the standard risk premium. It can also be interpreted as the worst case equity/risk premium in our economy. The sum (B)+(C) is the equity premium part caused by exposure to ambiguity, i.e., the ambiguity premium. (B) is the part of the ambiguity premium caused by misspecifications in the dynamics for \( D \). (C), instead, is the part of the ambiguity premium caused by misspecifications in the dynamics for the posterior probabilities \( \Pi \).

A typical pattern for the equity premium \( \mu_R \) is presented in Figure 3, Panel (D), for different levels of the risk aversion parameter and under an homogeneous degree of ambiguity \( \eta(\theta) = \eta = 0.01 \).

\[\text{Insert Figure 3 about here}\]

\(^9\) For instance, under Assumption 3 condition (39) is always satisfied with ”<” by \( \theta_1 \) and with ”>” by \( \theta_n \).
In Figure 3, the equity premium $\mu_R$ is a monotonically decreasing function of risk aversion. However, for low risk aversion such a feature seems to be well compatible with the predictions of the equity premium puzzle. E.g., for moderate risk aversions $\gamma$ between 0.2 and 0.4, the equity premium ranges from about 8% to about 5%. This effect arises despite the small size of the ambiguity parameters used; see Panel (B) of Figure 3.

In the next sections, we study the interpretation and the contribution of the risk and ambiguity premium components (A), (B), (C) to the total premium $\mu_R$.

4.1.1. Premia for Risk

The term (A) in (40) is the equity premium perceived by an investor under the optimal worst case likelihood $h^*(\theta)$ selected in Proposition 1. More precisely, from Proposition 2 we have:

$$\mu_{R}^{wc} = \gamma \left( \sigma_D^2 + V_{\theta,h^*} \right) = \gamma Cov_{t}^{h^*}(dR, dD/D) = \gamma Cov_{t}(dR, dD/D)$$

where $Cov_{t}^{h^*}$ ($Cov_{t}$) denotes conditional covariances under the worst case likelihood $h^*$ (under the reference model likelihood) and the last equality arises because worst case and reference model likelihoods are absolutely continuous. Therefore, (A) has the dual interpretation of being:

(a) The total equity/risk premium arising under the worst case likelihood belief,

(b) The part of equity premium deriving from pure risk exposure under the reference model likelihood belief.

In particular, (41) emphasizes the fact that under learning and ambiguity aversion the covariance term $Cov_{t}(dR, dD/D)$ captures only the fraction (A) of the whole equity premium $\mu_R$ under the reference model dynamics.

Is it natural to expect that the risk premium (A) will be actually quite small in our economies. Indeed, $\gamma \sigma_D^2$ is the risk premium under ambiguity aversion but no learning (see, e.g., Maenhout (2004) and Trojani and Vanini (2002)). It is increasing in $\gamma$, but for realistic risk aversion parameters it is typically a very small number. Moreover, Lemma 2 implies that
quantity $V_{\theta,h}^*$ is decreasing in risk aversion. Therefore, (A) as a function of risk aversion is bounded and is negligible for practical purposes.

A typical profile of the risk premium (A) as a function of $\gamma$ is plotted in Panel (C) of Figure 3 (circled curve), together with the equity/risk premium function implied by a setting of pure Bayesian learning (crossed curve). The equity/risk premium function prevailing in Panel (C) under a pure Bayesian learning setting almost coincides with the risk premium function under ambiguity aversion. The risk premium (A) under learning and ambiguity aversion is in general different from the risk premium arising under pure learning, because in general the term $V_{\theta,h}^*$ in (41) is different from the corresponding one prevailing in a setting of pure learning when $\eta(\theta) = 0$. However, for realistic structures of the ambiguity function $\eta(\theta)$ we always found the two risk premia to be numerically very similar. Therefore, as in the pure Bayesian setup, the risk premium is bounded from above as a function of risk aversion and Mehra and Prescott (1985) equity premium puzzle is even more puzzling in such a setting, because equity premia cannot be matched by risk premia, even for very high risk aversions. However, the equity premium (40) under learning and ambiguity aversion also consists of the ambiguity premium (B)+(C). As we show below, this component is crucial, in order to obtain model predictions that can be consistent with the equity premium puzzle.

4.1.2. Premia for Ambiguity

Under ambiguity aversion, the equity premium (40) depends on the ambiguity premia (B) and (C), which are both positive under Assumption 3. The sum of (B)+(C) represents a premium for ambiguity in the reference model dynamics, deriving from the discrepancy between the reference model likelihood belief and the worst case likelihood belief optimally selected by the ambiguity averse representative investor. More specifically, recall that the worst case return dynamics (32) depends on two filtered random shocks

$$d\tilde{B}_D^{h^*} = d\tilde{B}_D - k_D m_{h^*}(\theta) dt$$
$$d\tilde{B}_e^{h^*} = d\tilde{B}_e - k_e m_{h^*}(\theta) dt$$

(42)
By construction, \((\tilde{B}^h_D, \tilde{B}^h_e)\) is a \(\{\mathcal{F}(t)\}\)–Brownian motion under the worst case likelihood belief \(h^*(\theta)\), while it is a \(\{\mathcal{F}(t)\}\)–Brownian motion with drift under the reference model likelihood belief. The differences

\[
\lambda_D^A := \frac{(d\tilde{B}^h_D - d\tilde{B}_D)}{dt} = -k_D m_{h^*(\theta)} ,
\]

\[
\lambda_e^A := \frac{(d\tilde{B}^h_e - d\tilde{B}_e)}{dt} = -k_e m_{h^*(\theta)} ,
\]

are the market prices of ambiguity for \(d\tilde{B}^h_D\) and \(d\tilde{B}^h_e\) shocks, respectively, prevailing under the reference model belief. Such market prices of ambiguity arise because ambiguity averse investors apply a worst case learning approach to price assets, which understates systematically the expected dividend drift prevailing under the reference model. The filtered shocks \(d\tilde{B}^h_D\) and \(d\tilde{B}^h_e\) influence the worst case dynamics (32) in two distinct ways: via the isolated impact of \(d\tilde{B}^h_D\) on the worst case filtered dynamics for dividends and via the joint impact of \(d\tilde{B}^h_D\) and \(d\tilde{B}^h_e\) on the worst case filtered dynamics for the posterior probabilities \(\Pi\). More precisely, by setting \(h = h^*\) in (17), (18) and using (42) we obtain, for the optimal joint \((D, \Pi)\)–filtered dynamics:

\[
\frac{dD}{D} = m_{\theta,h^*} dt + \sigma_D d\tilde{B}^h_D ,
\]

\[
\frac{d\pi_i}{\pi_i} = (\theta_i + h(\theta_i) \sigma_D - m_{\theta,h^*}) \left( k_D d\tilde{B}^h_D + k_e d\tilde{B}^h_e \right) ; \quad i = 1, \ldots, n .
\]

The ambiguity premium for exposure to shocks in any of the \(\pi_i\) dynamics is

\[
k_D(d\tilde{B}^h_D - d\tilde{B}_D)/dt + k_e(d\tilde{B}^h_e - d\tilde{B}_e)/dt = k_D \lambda_D^A + k_e \lambda_e^A = -k \mu_{\theta,h^*} ,
\]

where \(k = k_D^2 + k_e^2\). From the worst case \(R\)–dynamics,

\[
dR = \mu_R^{wc} dt + \sigma_D d\tilde{B}^h_D + V_{\theta,h^*} \left( k_D d\tilde{B}^h_D + k_e d\tilde{B}^h_e \right) ,
\]

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we see that ambiguity premia for shocks $d\tilde{h}^\ast_D$ in the $D$–dynamics get multiplied by dividend volatility $\sigma_D$. Similarly, ambiguity premia for shocks $k_D d\tilde{h}^\ast_D + k_e d\tilde{h}^\ast_e$ in the $\Pi$–dynamics get multiplied by $V_{\theta,h^\ast}$. It then follows:

$$
-\mu_{\theta,h^\ast} = \sigma_D \lambda_D^A ; \quad -\mu_{\theta,h^\ast}kV_{\theta,h^\ast} = V_{\theta,h^\ast}(k_D \lambda_D^A + k_E \lambda_e^A)
$$

that is:

(a) The equity premium component (B) in (40) is the equilibrium ambiguity premium for misspecification in the $D$–dynamics,

(b) the equity premium component (C) in (40) is the equilibrium ambiguity premium for misspecification in the $\Pi$–dynamics.

Component (B) is not zero also under a degenerate $\Pi$–distribution, i.e., in the absence of learning. Under a non degenerate $\Pi$–distribution, it is affected by $\Pi$ only when ambiguity $\eta(\theta)$ is not homogenous and to the extent that $\Pi$ affects the posterior mean of $\sqrt{\eta(\theta)}\sigma_D$. Indeed, under an homogenous ambiguity $\eta(\theta) = \eta > 0$ it follows

$$
-\mu_{\theta,h^\ast} = \sqrt{2\eta}\sigma_D.
$$

i.e., the ambiguity premium in a full information economy with ambiguity averse agents, as obtained, e.g., in Trojani and Vanini (2002). Therefore, component (B) is naturally interpreted as a pure premium for ambiguity. In contrast to the findings in the above risk premia analysis, the contribution of the ambiguity premium component (B) to the equity premium is given by a first order effect of ambiguity that is proportional to dividend volatilities $\sigma_D$. Moreover, since (B) is not zero even for degenerate $\Pi$–distributions, it will not disappear asymptotically, even in the case of an asymptotic learning about the underlying true model neighborhood $\Xi(\theta)$. In contrast to (B), the ambiguity premium part (C) is zero under a degenerate $\Pi$–distribution, implying that the asymptotic level of the ambiguity premium in the case of asymptotic learning is fully determined by component (B). Such an asymptotic ambiguity premium rewards the representative agent for the residual ambiguity about the precise drift that generated the dividend dynamics, out of a relevant neighborhood $\Xi (\theta_l)$. In such a case, the ambiguity
premium (48) in a full information economy with ambiguity aversion is obtained in our setting as the limit of a sequence of ambiguity premia in partial information economies with ambiguity averse agents.

Under Assumption 3, component (C) in equation (40) is not zero if and only if \( \Pi \) is not degenerate, i.e., if ambiguity averse investors did not yet fully learn the underlying model neighborhood \( \Xi(\theta) \). Therefore, (C) can be interpreted as an ambiguity premium component due to the joint presence of ambiguity aversion and learning in our economy. From Lemma 2, the ambiguity premium (C) is positive and decreasing in \( \gamma \). Moreover, in contrast to the ambiguity premium (B), it depends on the signal precision parameter \( k_e \). To highlight (C)'s dependence on the ambiguity parameter \( \eta(\theta) \), we can make use of the second order asymptotics provided by the next Lemma.

**Lemma 3** The following second order asymptotics for \(-m_h^* V_{\theta,h^*}\) around \( \eta(\theta) = 0 \) holds:

\[
-m_h^* V_{\theta,h^*} = -m_h^* \left[ V_\theta + \sum_{i=1}^n \frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta(\theta_i)}} \bigg|_{\eta(\theta)=0} \sqrt{2\eta(\theta_i)} \sigma_D \right] + o(\|\eta(\theta)\|)
\]

\[
= \sum_{k=1}^n \pi_k \sqrt{2\eta(\theta_k)} \sigma_D \left[ V_\theta + \sum_{i=1}^n \frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta(\theta_i)}} \bigg|_{\eta(\theta)=0} \sqrt{2\eta(\theta_i)} \sigma_D \right] + o(\|\eta(\theta)\|)
\]

(49)

From Lemma 3, the first order impact of ambiguity aversion on the ambiguity premium (C) is given by the term

\[-km_h^* V_\theta = kV_\theta \sum_{k=1}^n \pi_k \sqrt{2\eta(\theta_k)} \sigma_D .\]

Therefore, for moderate ambiguity sizes, the term \( m_h^* \) determines the size of (C) as a function of \( \eta(\theta) \). Given a posterior distribution \( \Pi \), we can therefore expect ambiguity structures implying largest ambiguity premia (B) to imply also the largest premium component (C). Given \( m_h^* \), the first order quantitative effect of ambiguity on the ambiguity premium (C) is larger when either (i) the risk premium in a comparable purely Bayesian economy is large (when \( V_\theta \)

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is large) or (ii) the signal precision parameter $k$ is larger. From Lemma 3, the second order effect of ambiguity on the premium $(C)$ is given by the term

$$-km_h^* \sum_{i=1}^{n} \frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta(\theta_i)}} \bigg|_{\eta(\theta)=0} \sqrt{2\eta(\theta_i)} \sigma^D.$$  

(50)

Hence, given a first order impact of ambiguity on $(C)$ (i.e., given $m_h^*$), the second order effect of ambiguity is determined by its first order effect on $V_{\theta,h^*}$, which has been characterized by the comparative statics in Proposition 3. From Proposition 3 and the following discussion, we expect the sign of (50) to be negative for homogenous ambiguity structures or for asymmetric ambiguity structures that associate a larger ambiguity with favorable economic states. Asymmetric ambiguity structures that associate a larger ambiguity with unfavorable economic states, instead, tend to imply an opposite sign in (50). Figure 4 illustrates these features under a parameter choice and a prior structure $\Pi$ identical to the one in Figure 3.

**Insert Figure 4 about here**

Panels (A1)-(A4) present different forms for the ambiguity function $\eta(\theta)$. For comparability, we choose these functions in a way that preserves the same weighted entropy measure $\frac{1}{2} \sum \pi_i h^{*2}(\theta_i)$ as in Figure 3. Panels (B)-(D) plot the corresponding equity premia $\mu_R$ and risk premia $\mu_{\mu R}^{nc}$ as a function of risk aversion $\gamma \in [0,1]$. In Figure 4, the size of $-m_h^*$ is 0.27%, 0.19%, 0.23% and 0.23% in Panels (B)-(D), respectively, implying very small pure ambiguity premium components (B) in all panels. The risk premium component (A) is also very small: in all plots it is always below 0.25%.

In Figure 4, it is shown that the ambiguity premium $(C)$ under learning and ambiguity aversion is quantitatively quite substantial, for moderate risk aversions $\gamma < 1$. For instance, in Panel (D), it is above 8% for risk aversions of about 0.2 and it is above 5% for risk aversions of about 0.4. For all practical purposes, the key equity premium component is therefore the ambiguity premium $(C)$, i.e., the ambiguity premium part due to the joint presence of ambiguity aversion and learning.
The prior distribution $\Pi$ underlying Figure 4 is the symmetric one plotted in Panel (A) of Figure 3. Such a prior distribution puts low probabilities on reference model states above/below average and higher probabilities on more central states. Therefore, in Figure 4 the pure ambiguity premium $-m_{h^*}$ is lower for ambiguity structures putting large ambiguity sizes on external reference model states (as for instance, the ambiguity structure in Panel (A2)). According to Lemma 3 and the following discussion, we then expect higher equity premia for situations in which the first order impact $(-m_{h^*}V_{\theta})$ of ambiguity aversion on $-m^*V_{\theta,h^*}$ is larger. This intuition is confirmed by Panels (B), (C), (D) of Figure 3, where larger value of $-m_{h^*}$ are associated with higher equity premium functions. In the comparison of Panels (D) and (E), instead, the first order impact on the premium is identical, since $-m_{h^*}$ is the same in both settings. However, from Lemma 3 and Proposition 3, we expect the second order impact of ambiguity on the premium to be larger for the ambiguity structure in Panel (A-4), a conjecture that is confirmed by a comparison of the equity premia plotted in Panels (D) and (E).

### 4.1.3. Is there an Ambiguity Premium for Unprecise Signals?

In our Lucas economy, the risk premium (A) is quantitatively negligible, even for large risk aversions. For such large risk aversions, more precise signals tend to increase, rather than decrease, the equity risk premium. Moreover, when signals are unprecise, the risk premium is bounded from above (Veronesi (2000), Proposition 3). In other words: there is no quantitatively relevant risk premium for unprecise signals.

Under ambiguity aversion, however, we have shown that the key equity premium component is the ambiguity premium (C). For low risk aversions and when signals are unprecise, the resulting equity premium is quantitatively very significant. Therefore, a key question arises: Is there an ambiguity premium for unprecise signals? Sufficient conditions for a positive answer to this question are provided by the following simple corollary.
Corollary 6  Let Assumption 3 be satisfied and suppose that function $\sqrt{\eta(\theta)}$ is a convex function of $\theta$. Then, for any mean preserving spread $\tilde{\Pi}$ of $\Pi$ it follows:

$$-\tilde{m}_h^\star(1 + k\tilde{V}_{\theta,h}^\star) > -m_h^\star(1 + kV_{\theta,h}^\star),$$  \hspace{1cm} (51)

where $\sim$ denotes quantities under $\tilde{\Pi}$.

Corollary 6 states that under a convex ambiguity function $\sqrt{\eta(\theta)}$ there is always an ambiguity premium (B)+(C) for information noisiness.\(^\text{10}\) This finding implies that when public signal realizations are less precise the expected excess return is higher, because of an ambiguity premium for misspecification in the dividend and posterior probabilities $\Pi$ dynamics under the reference model. When realized signal precision is low, the posterior probabilities $\Pi$ are more diffuse, implying larger market prices of ambiguity $\lambda_D^A$ and $\lambda_e^A$ in (43) and (44). Quantity $V_{\theta,h}^\star$ in (51) also increases under a less imprecise signal (see again Lemma 2). $V_{\theta,h}^\star$ is the covariance between equity returns $R$ and signals $e$. Indeed, the relevant signal dynamics are

$$de = m_{\theta,h}^\star dt + \sigma_e d\tilde{B}_h^\star = m_{\theta,h}^\star dt + \sigma_e (d\tilde{B}_e - m_{\theta,h}^\star dt),$$  \hspace{1cm} (52)

implying, from (34):

$$V_{\theta,h}^\star = Cov_t(dR, de).$$  \hspace{1cm} (53)

Therefore, the increased ambiguity premium under less precise signals follows from (i) higher ambiguity market prices of risk and (ii) higher equilibrium covariances between equity returns and public signals. The higher covariance (53) under unprecise signals arises because less precise a posteriori dividend drift predictions $m_{\theta,h}^\star$ imply a lower sensitivity of investors’ hedging demand to signals. Therefore, positive (negative) signals tend to generate a positive (negative) excess demand for equity and a positive covariance between equilibrium equity

\(^{10}\)More generally, in the case where, e.g., $\sqrt{\eta(\theta)}$ is a concave function, the final result depends on the strength of the effects implied by the ambiguity premium (C) components $-m_h^\star$ and $V_{\theta,h}^\star$. In all our numerical examples, we found the effect caused by changes in $V_{\theta,h}^\star$ to dominate. This evidence supports the hypothesis that, for practical purposes, ambiguity premia for information noisiness arise also more generally than under the conditions of Corollary 6.
returns and signals. This discussion emphasize the important role of imprecise signals in determining the level of the ambiguity premium.

4.2. Equity Volatility

From Proposition 2, the volatility of stock returns is given by

\[ \sigma^2_R = \sigma^2_D + V_{\theta,h^*} (2 + kV_{\theta,h^*}) \]  \hspace{1cm} (54)

Lemma 2 implies that \( \sigma^2_R \) is a U-shaped function of risk aversion \( \gamma \), having a minimum \( \sigma^2_R = \sigma_D \) at \( \gamma = 1 \). Moreover, under a setting of pure ambiguity aversion (i.e., for a degenerate \( \Pi \)) we also have \( V_{\theta,h^*} = 0 \) and, hence, \( \sigma_R = \sigma_D \). \( \sigma_D \) is the return volatility in a setting of ambiguity aversion without learning, as obtained for instance in Maenhout (2004) and Trojani and Vanini (2002) under a constant opportunity set dynamics.

To obtain non trivial sizes of equity returns volatility, it is therefore important to introduce learning in the model, in excess of ambiguity, and to ensure \( \gamma \neq 1 \). From Section 4.1, however, the only such parameter choice than can be consistent with sizable equity premia is \( \gamma < 1 \), that is Assumption 3. Therefore, adding learning to a setting of ambiguity aversion is crucial in order to obtain qualitative model predictions that can be consistent with the excess volatility puzzle. Only under Assumption 3, however, such predictions can be consistent also with the equity premium puzzle.

Given a non degenerate learning setting with posterior probabilities \( \Pi \), it is interesting to analyze the additional contribution of ambiguity aversion to equity returns volatilities, when Assumption 3 is satisfied. In order to study the behavior of \( \sigma_R \) as a function of \( \eta(\theta) \) we can again make use of Proposition 3, where the dependence of \( V_{\theta,h^*} \) on \( \eta(\theta) \), in a neighborhood of the purely Bayesian Lucas economy, has been characterized. From that proposition, we expect higher equilibrium volatilities for asymmetric ambiguity structures \( \eta(\theta) \) that associate a higher concern for ambiguity with less favorable states of the economy.
Figure 5 highlights the effect of homogenous and heterogenous ambiguity structures on equilibrium equity return volatility.

Insert Figure 5 about here

In Panels (A-1)–(A-3) of Figure 5, we plot three different heterogenous ambiguity structures in a setting with five possible reference model drift states. In Panels (B)–(C), we present the equity return volatility $\sigma_R$ as a function of risk aversion $\gamma$. For each such panel, we plot volatility in a setting of pure learning (dotted lines), in a setting of pure ambiguity (dash dotted lines), in a setting of heterogenous ambiguity given, respectively, by Panels (A1)–(A3) on the left for Panels (B)–(D) on the right (straight lines) and finally in a setting of homogenous ambiguity (dashed lines). As expected, in the presence of learning all volatility functions are U–shaped and attain a minimum at $\gamma = 1$, where $\sigma_R = \sigma_D$. The volatility for the pure ambiguity setting is constant at $\sigma_D$ and is very small. Large differences compared to all other volatility curves arise, already outside small neighborhoods of the point $\gamma = 1$. For instance, in Panel (D) the value of $\sigma_D$ is about 3%, while the volatility for $\gamma = 0.6$ in the setting with learning and homogenous ambiguity is about 18%. This evidence emphasizes the dominant role of learning, as opposed to ambiguity aversion, in generating interesting model volatility predictions for the excess volatility puzzle.

Different ambiguity structures $\eta(\theta)$ can imply higher or lower volatilities than in the pure Bayesian setting. Consistently with Proposition 3, the ambiguity structure in Panel (A-3) delivers the highest volatility curve, while the one in Panel (A-2) implies the lowest volatilities. Homogenous ambiguity structures always deliver lower volatilities than those of a pure Bayesian learning setting. Asymmetric ambiguity structures of the type in Panel (A-3) tend to link less favorable economic states with a higher ambiguity.
4.3. Time Varying Theoretical Risk/Return Relations

From Proposition 2 and equation (54), the relations between risk or equity premia and the conditional variance of returns are given by

\[ \mu_{wc}^R = \gamma \sigma^2_R - \gamma V_{\theta,h^*} (1 + kV_{\theta,h^*}) \] (55)

and

\[ \mu_R = \left( \gamma - \frac{m_{h^*(\theta)}}{V_{\theta,h^*}} \right) \sigma^2_R - \gamma V_{\theta,h^*} (1 + kV_{\theta,h^*}) + m_{h^*(\theta)} \left( 1 + \frac{\sigma_D}{V_{\theta,h^*}} \right), \] (56)

respectively. In particular, when \( \Pi \) is non-degenerate equation (56) implies a truly positive but time varying theoretical relation between the equity premium \( \mu_R \) and the conditional variance \( \sigma^2_R \). Such a time varying relation is due to the ambiguity premium component (C) and derives from the interaction of learning and ambiguity aversion. The true relation between the risk premium \( \mu_{wc}^R \) and the conditional variance \( \sigma^2_R \) is, instead, linear and constant. Both relations (55) and (56) are biased by an heteroscedastic error term having non zero conditional mean. More precisely, since the dominating term in such errors is

\[ -\gamma V_{\theta,h^*} (1 + kV_{\theta,h^*}) < 0 \]

both relations are biased downwards. Figure 6 illustrates the theoretical relation between risk or equity premia and equity return conditional variances.

The theoretical (time varying) equity premium "sensitivity" to changes in \( \sigma^2_R \) is huge, compared to the one of the risk premium, which in turn is given by the risk aversion coefficient \( \gamma \). Moreover, ambiguity premia derive by definition from model misspecification, rather than from covariances between asset returns and economic state variables. Therefore, we can expect them to be very difficult to identify by, for instance, regression methods.
Figure 7 highlights this point by plotting the time series of estimated parameters in a sequence of rolling regressions of $R$ on $\sigma_R^2$.

**Insert Figure 7 about here**

As expected, highly time varying regression estimates arise. Such estimates may even indicate a switching sign in the estimated relation between $\mu_R$ and $\sigma_R^2$ over different time periods. More importantly, the estimated (time varying) coefficients don’t even approximately identify correctly the equity premium ”sensitivity” to changing variances under learning and ambiguity aversion. For instance, estimated parameters for $\gamma = 0.9$ in Figure 7 are never above 0.3, while the theoretical ”sensitivities” of equity premia to $\sigma_R^2$ in Figure 6 are above 10 for all ambiguity aversion parameters.

### 4.4. Biases in EIS estimates

Our model uses a time additive power utility function to obtain simple closed form solutions for the desired equilibria. Such a choice imposes a specific relation between standard risk aversion and EIS. Risk aversions less than one in our setting have to be associated with EIS above one. However, for an empirical study of the effect of learning under ambiguity aversion in our model it is important to realize that the assumption $\gamma < 1$ does not imply necessarily large estimated EIS using standard methods. Recently, Bansal and Yaron (2004) argued in a setting with Epstein and Schneider (1989) preferences that neglecting fluctuating economic uncertainty leads to a severe downward bias in estimates of the EIS using standard Euler equations. In our setting, fluctuating economic uncertainty arises endogenously, via the learning process of our representative agent. Indeed, one by-product of learning in our context is to induce a stochastic volatility in the a posteriori expected dividend growth in the model. Similarly to the effects noted by Bansal and Yaron (2004) in their full-information model, stochastic volatility of expected dividend growth can induce a large downward bias in a least-squares regression of consumption growth on asset returns, when using Euler equations including equity returns. Such regressions are typically used to estimate the EIS in applied empirical work.\textsuperscript{11} In our

\textsuperscript{11}See, e.g., Hall (1988), Vissing-Jorgensen (2002), and Attanasio and Vissing-Jorgensen (2003), among others.

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setting of learning under ambiguity aversion, it will be therefore important to account explicitly for such biases, in order to estimate accurately the EIS.

To understand the main reason for a negative bias in the estimation of the EIS, we consider for simplicity a pure setting of learning with no ambiguity aversion, that is $\eta(\theta) = 0$. From Proposition 1 and 2 we have:

$$ r = \delta + \gamma m_\theta - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2, \quad \mu_R = \gamma \left( \sigma^2_D + V_\theta \right), $$

where $m_\theta = E(dD/D|\mathcal{F}_t)$. Hence:

$$ E(dP/P + D/P|\mathcal{F}_t) = r + \mu_R = \delta + \gamma E(dD/D|\mathcal{F}_t) - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 + \gamma \left( \sigma^2_D + V_\theta \right), $$

and, solving for $E(dD/D|\mathcal{F}_t)$:

$$ E(dD/D|\mathcal{F}_t) = a + b \cdot E(dP/P + D/P|\mathcal{F}_t) - V_\theta, \quad (57) $$

where $a = -\delta/\gamma + \frac{1}{2} (\gamma - 1) \sigma_D^2$ and $b = 1/\gamma$. Equation (57) defines a correctly specified theoretical linear regression equation if and only if the random term $V_\theta$ is 0. This in turn can happen only if no learning is present ($\Pi$ is degenerate) or $\gamma = 1$ (log utility). In all other cases, the error term

$$ d\varepsilon_t := dD/D - a - b \cdot E(dP/P + D/P|\mathcal{F}_t) $$

will be correlated with the regressor $dP/P + D/P$ in a least-squares regression of $dD/D$ on $dP/P + D/P$. Under Assumption 3, such correlation induces a downward bias in the estimation of the EIS $1/\gamma$ in a least-squares regression of aggregate consumption growth $dD/D$ on total equity returns $dP/P + D/P$. Since $V_\theta$ is decreasing in relative risk aversion, we can expect the bias to be larger for lower $\gamma$ values. Figure 8 illustrates these features.

Insert Figures 8 about here

In Figures 8, we observe a very large bias in the least-squares estimates of the EIS $1/\gamma$ in a regression of $dD/D$ on $dP/P + D/P$. As expected, the bias is larger for lower values of
\(\gamma\). For instance, for \(\gamma = 0.5\) the mean estimate of \(1/\gamma\) is between 0.2 and 0.4, depending on the amount of ambiguity in the economy. The resulting downward bias in the estimation of the EIS is about 80%. For \(\gamma = 0.7\) mean EIS estimates range between about 0.35 and 0.6. Interestingly, such estimated values of the EIS are compatible with those obtained, e.g., in Vissing-Jorgensen (2002, Table 2A) and Attanasio and Vissing-Jorgensen (2003, Table 1A) for Euler equations including stock returns.

5. Conclusions and Outlook

We derived asset prices in a simple continuous-time partial information Lucas economy with ambiguity aversion and time additive power utility. Learning and ambiguity aversion imply an equilibrium discount for ambiguity only for relative risk aversions below one or, equivalently, elasticities of intertemporal substitution (EIS) above one. Equilibrium interest rates are lower irrespective of risk aversion. For low risk aversions, the qualitative model predictions show that our setting can potentially also account for the equity premium and volatility puzzles. **Introducing both ambiguity aversion and learning is crucial: model settings where either learning or ambiguity aversion are absent imply quantitatively negligible equity premia or volatilities, respectively.** In our model, ambiguity aversion implies only a partial asymptotic learning about a neighborhood of a priori statistically indistinguishable beliefs. This result motivates explicitly settings of ambiguity aversion but no learning as the limit of equilibria under learning and ambiguity aversion. We also find a highly time varying theoretical equilibrium relation between excess returns and conditional variances under learning and ambiguity aversion. This feature can generate estimated relations between excess returns and equity conditional variances having undetermined sign over time. Moreover, it can imply huge time varying biases in the naively estimated risk-return trade-off using, e.g., regression methods. Finally, standard EIS estimates based on Euler equations for equity returns are strongly downward biased in a setting of learning and ambiguity aversion. Therefore, EIS well above one in the model can be consistent with observed (biased) estimated EIS well below one, when ambiguity aversion is not explicitly incorporated in the estimated model.
We end with two final comments and a few suggestions for future research. First, the time additive power utility function in our model allows us to obtain simple closed form solutions for the desired equilibria at the cost of constraining the relation between risk aversion and EIS. The specific relation between standard risk aversion and EIS in our framework could be weakened by using a setting of learning under ambiguity aversion with Epstein and Schneider (1989)-type preferences. Disentangling risk aversion and EIS would allow for an additional degree of freedom which could be used, e.g., to generate higher worst case equity premia in our model. However, it is intriguing to note that under ambiguity aversion relative risk aversion parameters too far away from risk neutrality may even be behaviorally inappropriate. Provocative in this context is the experimental evidence collected by Wakker and Deneffe (1996) who estimated a virtually linear utility function when using a utilities elicitation procedure robust to the presence of ambiguity. In such experiments, utility functions estimated by procedures that are not robust to the presence of ambiguity were clearly concave. Moreover, the basic intuition derived from our model is likely to hold also under more general preferences that disentangle risk aversion and EIS. Investors with high relative risk aversions increase their hedging demand when they expect low consumption growth. This demand counterbalances the negative price pressure deriving from negative dividend news. Under ambiguity aversion, investors tend to understate actual consumption growth. Highly risk adverse investors therefore increase further their hedging demand for equity. Since the supply of the risky asset is fixed and the riskless bond is in zero net supply, the higher demand increases the price of the risky asset relative to dividends. At the same time, for low EIS lower expected consumption growth because of ambiguity aversion induces a large substitution from today’s to tomorrow’s consumption which smooths out consumption. Such an excess saving demand increases further the price of equity relative to dividends and lowers the equilibrium interest rate. From a more general perspective, we can thus interpret the assumption of a high EIS and a low risk aversion in our model just as a condition ensuring that the elasticity of the total demand for risky assets with respect to changes in expected consumption growth is positive.

A second simplifying building block of our model is a geometric Brownian motion reference model dynamics for the underlying dividend process. This choice allowed us to highlight in a simple framework important issues related to (i) the failing consistency of a Bayesian learning
process under a likelihood misspecification and (ii) the asymptotic persistence of ambiguity under learning and ambiguity aversion. The bias of standard Bayesian learning procedures under a misspecification of the underlying dividend dynamics gave us a natural motivation for the conservative Max-Min expected utility learning approach followed in the paper. Richer learning dynamics could be studied, including for instance reference models with business cycles and regime changes in the underlying dividend process. Such extensions are interesting venues for future work on learning under ambiguity aversion.

Finally, a key issue in future research is the real data estimation of models accounting explicitly for learning under ambiguity aversion. In contrast to other settings, as for instance Cagetti et al (2002) and Chen and Epstein (2004), our model offers fully analytical asset pricing relations. Therefore, its empirical implementation promises to be a (more) feasible task. The qualitative predictions studied in the paper for the well known asset pricing puzzles seem to point toward economies with low risk aversions and some degree of ambiguity aversion. The empirical study of the model will have to (i) verify its quantitative predictions for the well known asset pricing puzzles and (ii) estimate the structure of the ambiguity function $\eta(\theta)$. Given estimates of $\eta(\theta)$, it is then possible to consistently verify if the amount of ambiguity needed to fit the data is too large. In the latter case, a more general specification of the assumed reference model will be necessary.

6. Appendix

In this Appendix we provide the proofs to the propositions and corollaries in the paper.

**Proof of Corollary 1.** The statement of the corollary follows easily by noting that under a $h_D$—distorted dynamics we have:

$$d\tilde{B}_D^h = dB_D + k_D (\theta + h_D \sigma_D - m_{\theta,h}) dt \quad , \quad d\tilde{B}_e^h = dB_e + k_e (\theta + h_D \sigma_D - m_{\theta,h}) dt .$$
Proof of Proposition 1. We have for any likelihood $h(\theta) \in \Xi(\theta),$

\[
V^{h(\theta)}(\Pi, D) = E^{h(\theta)} \left[ \int_s^\infty e^{-\delta(t-s)} \frac{D_t^{1-\gamma}}{1-\gamma} dt \left| \mathcal{F}(s) \right. \right]
\]

\[
= E^{h(\theta)} \left[ \int_s^\infty e^{-\delta(t-s)} \frac{D(t)^{1-\gamma}}{1-\gamma} dt \left| \pi_1(s) = \pi_1, \ldots, \pi_n(s) = \pi_n, D(s) = D \right. \right]
\]

\[
= \frac{D^{1-\gamma}}{1-\gamma} \sum_{i=1}^n \pi_i E^{h(\theta)} \left[ \int_s^\infty e^{-\delta(t-s)} \left( \frac{D(t)}{D(s)} \right)^{1-\gamma} dt \left| \tilde{\theta} = \tilde{\theta}_i \right. \right],
\]

where $\tilde{\theta} = \theta + h(\theta) \sigma_D,$ $\tilde{\theta}_i = \theta_i + h(\theta_i) \sigma_D.$ Therefore, for any vector $\Pi$:

\[
J(\Pi, D) = \inf_{h(\theta)} V^{h(\theta)}(\Pi, D)
\]

\[
\geq \frac{D^{1-\gamma}}{1-\gamma} \sum_{i=1}^n \pi_i \inf_{h(\theta_i)} E^{h(\theta_i)} \left[ \int_s^\infty e^{-\delta(t-s)} \left( \frac{D(t)}{D(s)} \right)^{1-\gamma} dt \left| \tilde{\theta} = \tilde{\theta}_i \right. \right].
\]

Conditional on $\tilde{\theta}_i,$ the $h(\theta) -$drift misspecified dynamics are

\[
dD = (\theta_i + h(\theta_i) \sigma_D) Ddt + \sigma_D dB_D.
\]

Therefore, Assumption 2 implies that we can focus on solving the problem

\[
(P_i) : \begin{cases} V^i(D) = \inf_{h(\theta_i)} E^{h(\theta_i)} \left( \int_s^\infty e^{-\delta(t-s)} \frac{D(t)^{1-\gamma}}{1-\gamma} dt \left| D(s) = D \right. \right) \\
\frac{1}{2} h(\theta_i)^2 \leq \eta(\theta_i)
\end{cases}
\]

subject to the dividend dynamics

\[
dD = (\theta_i + h(\theta_i) \sigma_D) Ddt + \sigma_D dB_D.
\]

The Hamilton Jacobi Bellman (HJB) equation for this problem reads

\[
0 = \inf_{h(\theta_i)} \left\{ -\delta V^i + \frac{D^{1-\gamma}}{1-\gamma} + (\theta_i + h(\theta_i) \sigma_D) D \cdot V^i_D + \frac{1}{2} \sigma_D^2 D^2 V^i_{DD} + \lambda \left( \frac{1}{2} h(\theta_i)^2 - \eta(\theta_i) \right) \right\},
\]

(58)
where \( \lambda \geq 0 \) is a Lagrange multiplier for the constraint \( \frac{1}{2} h(\theta_i)^2 \leq \eta(\theta_i) \). This implies the optimality condition

\[
h(\theta_i) = -\frac{\sigma_D^2}{\lambda} V_D^i.
\]

Slackness then gives

\[
\frac{\sigma_D^2}{\lambda^2} (V_D^i)^2 = 2\eta(\theta_i),
\]

implying

\[
h(\theta_i) = -\sqrt{2\eta(\theta_i)} \text{sgn}(\sigma_D D V_D) = -\sqrt{2\eta(\theta_i)}.
\]

This proves the first statement. To prove the second statement, notice first that

\[
V^i (D) = \frac{D^{1-\gamma}}{1-\gamma} E \left[ \int_s^\infty e^{-\delta(t-s)} \left( \frac{D(t)}{D(s)} \right)^{1-\gamma} dt \bigg| \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \right].
\]

Conditionally on \( \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \), the solution of the dividend dynamics gives

\[
\left( \frac{D(t)}{D(s)} \right)^{1-\gamma} = \exp \left\{ (1-\gamma) \left( \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D - \frac{\sigma_D^2}{2} \right) (t-s) + (1-\gamma) \sigma_D (B_D(t) - B_D(s)) \right\},
\]

implying, under the given assumptions,

\[
E \left[ \int_s^\infty e^{-\delta(t-s)} \left( \frac{D(t)}{D(s)} \right)^{1-\gamma} dt \bigg| \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \right] = \frac{1}{\theta_i + (1-\gamma) \sqrt{2\eta(\theta_i)} \sigma_D}.
\]

where

\[
\tilde{\theta}_i = \delta - (1-\gamma) \theta_i + \gamma (1-\gamma) \frac{\sigma_D^2}{2} > 0.
\]

We thus obtain for the price of any risky asset with dividend process \( (D(t))_{t \geq 0} \):

\[
\frac{P(t)}{D(t)} = \sum_{i=1}^n \pi_i E \left[ \int_t^\infty e^{-\delta(s-t)} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} ds \bigg| \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \right],
\]

or equivalently

\[
P(t) \rho(t) = \sum_{i=1}^n \pi_i E \left[ \int_t^\infty \rho(s) D(s) ds \bigg| \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)} \sigma_D \right].
\]

50
where \( \rho(t) = u_c(D(t), t) = e^{-\delta t}D(t)^{-\gamma} \). This proves the second statement of the proposition. Writing this equation in differential form and applying it to the risky asset paying a "dividend" \( D = r \) we obtain:
\[
rdt = -\sum_{i=1}^{n} \pi_i E_t \left[ \frac{d\rho}{\rho} \right] \bar{\theta} = \theta_i - \sqrt{2\eta(\theta_i)\sigma_D} \left( \delta + \gamma \left( m_\theta + m_{h(\theta)}\sigma_D \right) - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 \right) dt ,
\]
i.e., the third statement of the proposition, concluding the proof.

**Proof of Proposition 2.** The statement (i) follows by applying the proof of Proposition 2 in Veronesi (2000) to the \( D \)-dynamics (17) under the worst case likelihood \( h^*(\theta) = -\sqrt{2\eta(\theta)} \). The statement (ii) follows by expressing the \( R \)-dynamics obtained in (i) with respect to the filtered Brownian motions \( \bar{B}_D, \bar{B}_e \) under the reference model.

**Proof of Proposition 3.** Let \( \theta := (\theta_1, .., \theta_n)' \) and
\[
C(x) = \frac{1}{\delta + (\gamma - 1)x - \gamma(1 - \gamma)\sigma_D^2} .
\]
To simplify notations, we define:
\[
V(\theta) = \sum_i \pi_i \theta_i \left[ \frac{C(\theta_i)}{\sum_j \pi_j C(\theta_j)} - 1 \right] .
\]
(60)

To prove the first and the second statement of the proposition, we compute the gradient \( \nabla V := (V_{\theta_1}, .., V_{\theta_n})' \), where for brevity \( V_{\theta_k} := \partial V/\partial \theta_k, k = 1, .., n \). It then follows:
\[
V_{\theta_k}(\theta) = \pi_k \left[ C(\theta_k) \sum_j \pi_j C(\theta_j) - 1 \right] + \sum_i \pi_i \theta_i \left[ \frac{C'(\theta_i)\delta_{ki}}{\sum_j \pi_j C(\theta_j)} - C(\theta_i) \frac{\pi_k C'(\theta_k)}{(\sum_j \pi_j C(\theta_j))^2} \right]
= \frac{\pi_k C(\theta_k)}{\sum_j \pi_j C(\theta_j)} - \pi_k + \theta_k \frac{\pi_k C'(\theta_k)}{\sum_j \pi_j C(\theta_j)} - \sum_i \pi_i \theta_i C(\theta_i) \frac{\pi_k C'(\theta_k)}{\sum_j \pi_j C(\theta_j)} - \frac{\pi_k C'(\theta_k)}{\sum_j \pi_j C(\theta_j)} \left[ \theta_k - m_\theta - V(\theta) \right] ,
\]
where $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ else. From the explicit expression for $C(x)$,

$$C'(\theta_k) = (1 - \gamma)C(\theta_k)^2,$$

implying

$$V_{\theta_k} = \frac{\pi_k C(\theta_k)}{\sum_j \pi_j C(\theta_j)} - \pi_k + (1 - \gamma) \frac{\pi_k C(\theta_k)^2}{\sum_j \pi_j C(\theta_j)} [\theta_k - m_\theta - V(\theta)].$$

Under Assumption 3, the conditions

$$\frac{\pi_k C(\theta_k)}{\sum_j \pi_j C(\theta_j)} \geq \pi_k; \quad \theta_k - m_\theta \geq V(\theta) \quad (61)$$

imply $V_{\theta_k} \geq 0$. Since

$$\left. \frac{\partial V_{\theta,k}}{\partial \sqrt{2\eta(\theta)}} \right|_{\eta(\theta)=0} = \left. \frac{\partial V(\theta - \sqrt{2\eta(\theta)\sigma_D})}{\partial \sqrt{2\eta(\theta)}} \right|_{\eta(\theta)=0} = -V_{\theta_k}(\theta)\sigma_D, \quad (62)$$

where $\sqrt{2\eta(\theta)} := (\sqrt{2\eta(\theta_1)}, .., \sqrt{2\eta(\theta_n)})'$, condition (61) also implies the sign of (62). To prove the third statement, we calculate

$$\left. \frac{\partial V_{\theta,k}}{\partial \sqrt{2\eta}} \right|_{\eta(\theta)=0} = \sum_{k=1}^{n} \left. \frac{\partial V(\theta - \sqrt{2\eta(\theta)\sigma_D})}{\partial \sqrt{2\eta(\theta)}} \right|_{\eta(\theta)=0} = -\sum_{k=1}^{n} \left\{ \frac{\pi_k C(\theta_k)}{\sum_j \pi_j C(\theta_j)} - \pi_k + (1 - \gamma) \frac{\pi_k C(\theta_k)^2}{\sum_j \pi_j C(\theta_j)} [\theta_k - m_\theta - V(\theta)] \right\} \sigma_D$$

$$= -(1 - \gamma) \sum_{k=1}^{n} \frac{\pi_k C(\theta_k)^2}{\sum_j \pi_j C(\theta_j)} [\theta_k - m_\theta - V(\theta)] \quad (63)$$

It is therefore sufficient to study the sign of

$$\sum_{k=1}^{n} \frac{\pi_k C(\theta_k)^2}{\sum_j \pi_j C(\theta_j)^2} [\theta_k - m_\theta - V(\theta)] = \sum_{k=1}^{n} \frac{\pi_k C(\theta_k)^2}{\sum_j \pi_j C(\theta_j)^2} [\theta_k - m_\theta] - V(\theta).$$
Since $C(x)$ is increasing and convex,

$$
\sum_{k=1}^{n} \frac{\pi_k C(\theta_k)^2}{\sum_{j=1}^{n} \pi_j C(\theta_j)^2} [\theta_k - m_\theta] - V(\theta) > 0
$$

and

$$
\left. \frac{\partial V_{\theta,h^*}}{\partial \sqrt{2\eta}} \right|_{\eta=0} < 0
$$

under the given assumptions, concluding the proof. ■

**Proof of Lemma 3.** We compute second order asymptotics for the function

$$
H(\eta(\theta_1), \ldots, \eta(\theta_n)) := H(\eta) := -m_{h^*} V_{\theta,h^*}
$$

We first have:

$$
\partial_k H = -V_{\theta,h^*} \partial_k m_{h^*} - m_{h^*} \partial_k V_{\theta,h^*}
$$

and

$$
\partial_i \partial_k H = -V_{\theta,h^*} \partial_i \partial_k m_{h^*} - \partial_k m_{h^*} \partial_i V_{\theta,h^*} - \partial_i m_{h^*} \partial_k V_{\theta,h^*} - m_{h^*} \partial_i \partial_k V_{\theta,h^*}
$$

where subscripts $k, i$ denote partial derivatives with respect to the arguments $\sqrt{2\eta(\theta_k)}, \sqrt{2\eta(\theta_i)}$.

Using the explicit expression of $m_{h^*}$ and since $m_{h^*} = 0$ for $\eta(\theta) = 0$, it follows:

$$
\left. \partial_k H \right|_{\eta(\theta)=0} = -V_{\theta} \partial_k m_{h^*} \big|_{\eta(\theta)=0} = V_{\theta} \pi_k \sigma_D
$$

$$
\left. \partial_i \partial_k H \right|_{\eta(\theta)=0} = -\partial_k m_{h^*} \partial_i V_{\theta,h^*} \big|_{\eta(\theta)=0} - \partial_i m_{h^*} \partial_k V_{\theta,h^*} \big|_{\eta(\theta)=0} + \partial_i \partial_k V_{\theta,h^*} \big|_{\eta(\theta)=0} \sigma_D
$$
A second order Taylor expansion of $H$ at $\eta(\theta) = 0$ then gives, up to term or order $o(\|\eta(\theta)\|)$:

$$H(\eta(\theta)) = \sum_{k=1}^{n} V_{\theta} \pi_k \sqrt{2\eta(\theta_k)} \sigma_D$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left\{ \pi_k \partial_i V_{\theta,h^*} |_{\eta(\theta)=0} + \pi_i \partial_k V_{\theta,h^*} |_{\eta(\theta)=0} \right\} \sqrt{2\eta(\theta_k)} \sqrt{2\eta(\theta_i)} \sigma_D^2$$

$$= -m_h^* V_\theta + \sum_{i=1}^{n} \sum_{k=1}^{n} \pi_k \partial_i V_{\theta,h^*} |_{\eta(\theta)=0} \sqrt{2\eta(\theta_k)} \sqrt{2\eta(\theta_i)} \sigma_D^2$$

$$= -m_h^* V_\theta + \sum_{i=1}^{n} \partial_i V_{\theta,h^*} |_{\eta(\theta)=0} \sqrt{2\eta(\theta_i)} \sigma_D \sum_{k=1}^{n} \pi_k \sqrt{\eta(\theta_k)} \sigma_D$$

$$= -m_h^* [V_\theta + \sum_{i=1}^{n} \partial_i V_{\theta,h^*} |_{\eta(\theta)=0} \sqrt{2\eta(\theta_i)} \sigma_D] \quad (70)$$

\[ \blacksquare \]

**Proof of Corollary 6.** From Lemma 2, $V_{\theta,h^*}$ is increasing in mean preserving spreads $\bar{\Pi}$, under the given assumptions. Moreover,

$$-m_h^* = \sum_{i=1}^{n} \pi_i \sqrt{\eta(\theta_i)} \sigma_D \quad (71)$$

is also increasing in mean preserving spreads, because of the assumed convexity of $\sqrt{2\eta(\theta)}$ as a function of $\theta$. This concludes the proof. \[ \blacksquare \]
References


Figures

Fig. 1. Posterior probabilities dynamics. The panels display trajectories for the probability $\pi_1$ given in equation (14) of Example 1. Panel (A) shows two trajectories for $\pi_1$ with $a = 0$ in equation (15). We plot the same trajectories with the same random seed in Panel (B) (dashed lines) and add the trajectories (solid lines) where we assume $a = 0.015$. The switching in equation (15) is deterministic and occurs every year (see the dotted vertical lines in Panel (B)). Further parameters are $\Theta = \{0.0075, 0.0275\}, \sigma_D = 0.0375, \sigma_e = 0.015$. 
Fig. 2. The effect of ambiguity aversion on the prevailing posterior probabilities dynamics. We assume three possible states and the filtered probabilities dynamics in equation (30) with parameters set equal to $\sigma_D = 0.0375$, $\sigma_e = 0.015$, $\Theta = \{0.0023, 0.0173, 0.0323\}$, $\theta = 0.0173$, and a set of discretized normal priors $\Pi(0) = \{0.3085, 0.3829, 0.3085\}$. Panel (A) plots the probability dynamics of the "bad" state $\theta_1$ for three different levels of a homogenous ambiguity parameter $\eta = \{0, 0.025, 0.05\}$. The dynamics under the intermediate level of ambiguity $\eta = 0.025$ are represented by the dotted line. In Panel (B), we plot the dynamics of the posterior probabilities for the "good" state $\theta_3$ for the same levels of ambiguity (these graphs are based on the same random seed as the one used in Panel (A)).
Fig. 3. Risk premium and ambiguity premium under homogenous ambiguity. Panel (A) plots the set of probabilities $\Pi$ relevant for the figure, while Panel (B) plots the different relevant reference model states $\theta_1, \ldots, \theta_n$. The true reference model dividend drift state is marked with a square and has been set equal to the posterior expected value $\sum \pi_i \theta_i$. We use a small amount of homogenous ambiguity $\eta = 0.01$. The size of the ambiguous neighborhoods $\Xi(\theta_1), \ldots, \Xi(\theta_n)$ is highlighted by the ellipses centered at $\theta_1, \ldots, \theta_n$ in Panel (B). Further, we set $\delta = 0.05$, $\sigma_D = 0.0375$ and $\sigma_e = 0.015$. With these parameters the resulting risk premium $\mu_{wc}^R$ and the equity premium $\mu_R$ are plotted in Panel (C) and (D) as functions of $\gamma$. 
Fig. 4. Risk premium and ambiguity premium under heterogenous ambiguity. Panels (A-1)–(A-4) plot different entropy preserving distributions of ambiguity around the reference model dividend drift states $\theta_1, \ldots, \theta_5$, i.e., Panels (A-1)–(A-3) are such that the weighted entropy measure $\frac{1}{2} \sum \pi_i h^2(\theta_i)$ is equal to 0.01, as in Figure 3. Panels (B)–(E) plot the equity premium $\mu_R$ and the risk premium $\mu_{R}^{\text{wc}}$ implied by the different distributions of ambiguity in Panels (A-1)–(A-4) as a function of risk aversion $\gamma \in [0, 1]$. For comparability, we also give the size of $-m^*$ implied by each plot. All other parameters have been fixed as in Figure 3.
Fig. 5. Equity volatility. Panels (A-1)–(A-3) plot different entropy preserving distributions of ambiguity around the reference model dividend drift states $\theta_1, \ldots, \theta_5$, i.e., Panels (A-1)–(A-3) are such that the weighted entropy measure $\frac{1}{2} \sum \pi_i h^{*2}(\theta_i)$ is equal to 0.01, as in Figure 3. Panels (B)–(D) plot the resulting volatility $\sigma_R$ (straight curves) implied by the different distributions of ambiguity in Panels (A-1)–(A-3). For comparability, we plot in each graph the quantities prevailing under an homogenous ambiguity parameter $\eta = 0.01$ (dashed curves), the pure learning setting, i.e, $\eta(\theta_i) = 0$, $i = 1, \ldots, n$ (dotted curve), and the pure ambiguity case arising under a degenerate $\Pi$ (dash dotted line). All other parameters have been fixed as in Figure 3.
Fig. 6. Theoretical time varying return/volatility trade-off. For different parameters $\gamma = 0.3, 0.5, 0.7, 0.9$, we plot the theoretical (time varying) coefficient $b_t = \gamma - m_{h^*}(\theta)/V_{h^*,\theta}$ in the theoretical expected excess return and variance relation

$$\mu_R = b_t \sigma_R^2 + c_t,$$

where

$$c_t = -\gamma V_{\theta,h^*}(1 + kV_{\theta,h^*}) + m_{h^*}(1 + \sigma_{D}^2/V_{h^*,\theta}).$$

In all panels, we plot $b_t$ as a function of time for homogenous ambiguity structures $\eta(\theta) = \eta$, where $\eta = 0.0017, 0.0033$. The horizontal flat straight lines correspond to the (constant) risk premium coefficient $\gamma$. 
Fig. 7. Rolling regression analysis. For different parameters $\gamma = 0.3, 0.5, 0.7, 0.9$ we plot the time variation of the estimated parameter $b$ in a rolling regression of $R$ on $\sigma^2_R$. The rolling regressions are based on sample sizes of 50 observations simulated from a model with three reference model drift states $\Theta = \{0.0025, 0.0175, 0.0325\}$ and under an homogenous degree of ambiguity $\eta = 0.01$. The true dividend is $\theta = 0.0175$. Further parameters are $\delta = 0.05$, $\sigma_D = 0.0375$, $\sigma_e = 0.015$. 
Fig. 8. EIS Regression analysis. For different parameters $\gamma = 0.3, 0.5, 0.7, 0.9$, we present the box plots of the estimated parameters in 1000 regressions of $dD/D$ on $dP/P + P/D$. The regressions are based on sample sizes of 365 observations simulated from a model with three reference model drift states $\Theta = \{0.0025, 0.0175, 0.0325\}$ and under three homogenous degrees of ambiguity $\eta = 0, 0.005, 0.01, 0.015$. The true dividend is $\theta = 0.0175$. Further parameters are $\delta = 0.05$, $\sigma_D = 0.0375$, $\sigma_e = 0.015$. 