On the Noncompensation for Illiquidity in Equilibrium Asset Returns

Christoph Heumann*
University of Mannheim

First Draft: March 2005
Current Draft: July 2005

Abstract

This paper studies how asset prices and traders’ portfolios are affected by illiquidity, emphasising the microstructure view of market liquidity. We set up a static CARA-Gaussian model in which a risky asset is traded under imperfect competition in an otherwise frictionless market. We find that each trader bears liquidity costs because of his price impact, which results in Pareto inefficient risk sharing. The expected return on the risky asset, however, is unaffected by illiquidity, and thus reflects only the traditional risk premium but no liquidity premium. This result is contradictory to asset pricing models on exogenous transaction costs, which argue that expected return partly represents a liquidity premium to compensate traders for the costs that they bear in illiquid markets.

Keywords: Market liquidity, Asset pricing, Market microstructure, Imperfect competition
JEL classification: G12, D43

*Chair of Finance, University of Mannheim, 68131 Mannheim, Germany, Phone: +49-621-181-1487, Fax: +49-621-181-1519, E-mail: heumann@lsdb.bwl.uni-mannheim.de. I thank Wolfgang Bühler for helpful comments. The responsibility for errors is mine alone.


1 Introduction

In traditional asset pricing theory, equilibrium prices and portfolios are attained in a frictionless world: there is perfect competition, price setting and trading takes place in a costless and instantaneous Walrasian auction, and traders have symmetric information about assets' payoffs. The focus within this familiar neoclassical framework is on the determination of risk premia as component of expected returns, which are required to compensate risk-averse investors for holding risky assets in their portfolios. One of the several frictions whose importance for asset pricing is discussed to extend the traditional setting is illiquidity. Beginning with the work of Amihud and Mendelson (1986) and Constantinides (1986), an extensive and still growing body of research suggests that illiquidity constitutes an additional factor in determining expected returns. The common reasoning is that trading in illiquid markets involves transaction costs and, as compensation for bearing these costs, investors demand a liquidity discount on prices or, equivalently, a liquidity premium on expected returns. Consequently, in illiquid markets, expected returns do not only reflect the traditional risk premium but also a liquidity premium.\(^1\)

Market liquidity is an elusive concept, however. Generally, it refers to the ease with which assets can be traded among investors; in detail, it covers several dimensions, such as tightness, depth, and immediacy, and these dimensions interact with each other and can be themselves a source of risk. Given this variety of liquidity-related aspects, there is a wide range of theoretical and empirical approaches to study the effects of illiquidity on asset prices.

On the theoretical front, Amihud and Mendelson (1986) capture the transaction cost aspect of illiquidity and predict the existence of significant liquidity premia. Constantinides (1986) takes the same view on illiquidity, but argues that transaction costs have only a small effect on asset returns compared to the risk premium. Acharya and Pedersen (2004) examine the impact of unpredicted changes in liquidity and find that liquidity risk is priced according to an asset’s expected liquidity and to the correlation of an asset’s liquidity with return and liquidity of the overall market. Huang (2003) considers the interaction of illiquidity with other frictions and derives significant liquidity premia for the case that agents are subject to borrowing constraints. Lindenberg (1979), Rudolph (1982), and Pritsker (2003) analyse asset prices when, by assumption, some investors are

\(^{1}\)We use the terms “liquidity premium” and “liquidity costs” synonymously to the just as commonly used terms “illiquidity premium” and “illiquidity costs.”
large and thus have price impact, whereas other traders are small and face perfectly li-
quid markets. They find that expected returns are determined by additional factors, which
correspond to each large investor’s initial endowment.

On the empirical side, Amihud and Mendelson (1986) and Eleswarapu (1997) find a
significant positive effect of bid-ask spreads in explaining cross-sectional stock returns,
and Brennan and Subrahmanyam (1996) show similar results using price impact to proxy
for illiquidity. Chen and Kan (1996), Barclay, Kandel and Marx (1998), and Easley,
Hvidkjaer and O’Hara (2002), in contrast, do not find a reliable relation between bid-ask
spreads and returns. Pástor and Stambaugh (2003) consider the impact of liquidity risk on
stock returns and show a significant positive effect of systematic liquidity risk (liquidity
betas) on returns. Empirical studies on liquidity premia, however, are subject to the
problem to adjust asset returns for the traditional risk premium. As Berk (1995) argues
in the context of size-related asset pricing anomalies, variables that are related to risk
 premia may possess explanatory power on expected returns, not because these variables
are priced, per se, but rather because they proxy for mis-specifications of risk. If this
argument also applies to market liquidity, then findings of liquidity premia are ambiguous
and may, at worst, basically represent significant but unexplained residual terms in stock
returns, disguised under the elusive concept of illiquidity.

As discussed by O’Hara (2003) and Easley and O’Hara (2003), the major problem for
understanding the different facets of market liquidity and their role in asset pricing is that
the existing literature on liquidity can be divided into two research areas that hitherto
coexist largely unrelated to one another. On the one hand, the literature on liquidity
premia cited above analyses the impact of liquidity costs on asset returns and traders’
portfolios. In these asset pricing models, however, illiquidity is incorporated as exoge-
nous transaction cost in the broadest sense, often more akin to a tax on assets’ payoffs
than to a lack of assets’ tradeability. On the other hand, market microstructure theory is
concerned, among other things, with explaining market liquidity in the context of trader
interaction under explicit trading rules. Illiquidity thereby arises from a couple of un-
derlying trading frictions, e.g. asynchrony in order arrival or asymmetric information,
which result in problems of inventory control (Garman (1976), Stoll (1978), and Grossman
and Miller (1988)) or adverse selection (Glosten and Milgrom (1985) and Kyle (1985))
to market intermediaries. In microstructure models, in turn, there is always some exoge-
nous trading from non-optimising agents, obstructing the analysis of risk-return relations
and traders’ portfolios. This dichotomy of liquidity-related models into asset pricing and
microstructure makes it difficult to identify what aspect of illiquidity is reflected in asset prices and to understand, beyond the common “investors-demand-compensation” intuition, how trading frictions carry over to expected returns and portfolios in the trading process at all.

The contribution of this paper is to examine the role of illiquidity in asset pricing within a theoretical framework that integrates the asset pricing and the microstructure view, thereby determining asset prices and traders’ portfolios as well as market liquidity endogenously. Liquidity aspects are not incorporated, however, by the presence of an optimising and dealing market intermediary, but rather by dropping the traditional assumption of perfect competition (price-taking behavior) among traders. Without this assumption, agents do not a priori have the possibility to buy and sell arbitrary quantities of assets at given prices; instead, each trader has price impact, which arises from the interaction with other market participants and the trading rules.

We set up a static CARA-Gaussian model with a single risky asset in which each trader submits a demand function for execution against the demand functions submitted by other traders. Price impact results from the condition of market-clearing and each investor’s awareness that other agents submit downward-sloping demand functions because of their risk aversion. If, for example, a trader considers to sell some units of the asset, he has to move the price downward for each (additional) unit to sell, simply to make other market participants willing to buy (additional) units of the asset. Since the demand function of each trader both provides liquidity for other traders and is affected by the liquidity provided to him, trading decisions are interdependent. The resulting equilibrium is a Nash equilibrium in trading strategies as in the model on information aggregation under imperfect competition by Kyle (1989).\(^2\)

We derive equilibrium for the case of identical risk aversion for all traders and obtain two main results. First, price impact is a cost because agents are forced to trade off an improvement in portfolio structure against an improvement in price. As a consequence, investors trade lower quantities than under perfect competition, thereby impeding Pareto efficient risk sharing. Second, the expected return on the risky asset is not affected by illiquidity and, in particular, does not reflect a liquidity premium to compensate traders for bearing liquidity costs. This result differs from asset pricing models with exogenous

\(^2\)Other applications of strategic trading with price-quantity functions are given, e.g., by Wilson (1979) to divisible good auctions, by Jackson (1991) to information aggregation with costly information acquisition, by Pagano (1989) to market fragmentation, and by Klemperer and Meyer (1989) to oligopoly theory.
transaction costs, which discuss whether illiquidity is priced to a significant extent relative
to risk, but do not question the existence of liquidity premia, per se. The intuition for
our result is as follows. If illiquidity is modelled from the microstructure view, such that
buyers and sellers bear liquidity costs on trading, then buyers demand a price discount
and sellers demand a price premium, and these effects cancel each other out. That is,
both sides of the market demand compensation for illiquidity, but nobody is willing to
pay that compensation.

Our results are related to the model of Gărleanu and Pedersen (2004). In a dynamic
model with risk-neutral agents, they derive bid-ask spreads from future asymmetric in-
formation about an asset’s payoff. They find that illiquidity due to adverse selection
generates allocation inefficiencies, but is not, per se, reflected in the expected return. The
intuition for their result is the same as in our model: if both buyers and sellers are subject
to illiquidity, then the effects on the expected return offset each other.

The remainder of this paper is organised as follows. Section 2 outlines the assumptions
of the model. Section 3 derives the equilibrium and section 4 discusses market liquidity,
asset prices, and traders’ portfolios in equilibrium. Section 5 concludes.

2 A Description of the Economy

We describe trading in a static framework. At the beginning of the period, a riskless asset
and a risky asset are traded among traders. Taking the riskless asset as numeraire with
perfectly elastic supply, let \( p \) be the price at which the risky asset is traded. At the end
of the period, assets pay off an exogenous liquidation value, and each trader consumes his
terminal wealth. Each unit of the riskless asset pays \( 1 + r \), where \( r \) is the riskfree rate of
interest, and each unit of the risky asset yields a normally distributed liquidation value
\( \tilde{v} \). The total number of outstanding shares of the risky asset is denoted by \( \bar{X} \).

In the economy, there are \( I \) rational traders (indexed by \( i = 1, \ldots, I \)), who have
identical information about the probability distribution of the risky payoff \( \tilde{v} \). At the
beginning of the period, each trader is endowed with \( \bar{y}^i \) units of the riskless asset and \( \bar{x}^i \)
units of the risky asset. If \( y^i \) and \( x^i \) are the corresponding portfolio holdings at the end
of the period, trader \( i \)’s terminal wealth is given by

\[
\tilde{w}^i = (1 + r)y^i + \tilde{v}x^i.
\]

Traders’ preferences are assumed to satisfy the von Neumann-Morgenstern axioms. Each
trader has a utility function

\[ u^i(w^i) = -\exp(-\rho^i w^i), \quad \rho^i > 0, \]

where \( \rho^i \) is the coefficient of absolute risk aversion (and its inverse, \( 1/\rho^i \), is referred to as the risk tolerance). With normally distributed terminal wealth, maximizing expected utility is equivalent to maximizing

\[ E[\tilde{w}^i] - \frac{\rho^i}{2} \text{Var}[\tilde{w}^i] = (1 + r) y^i + E[\tilde{v}] x^i - \frac{\rho^i}{2} (x^i)^2 \text{Var}[\tilde{v}]. \]

The market for the risky asset is organised as call auction. Traders simultaneously submit orders to an auctioneer, who sets a single price to clear the market and settles all orders at this price. The auctioneer himself does not not trade, and his price-setting activity is not based on optimising behavior. For each trader \( i \), an order is allowed to be any non-increasing, continuously differentiable function mapping the price of the risky asset into units of the risky asset to buy: \( \Theta^i : \mathbb{R} \to \mathbb{R} \), such that \( \theta^i = \Theta^i(p) \). (Negative values indicate units of the risky asset to sell.) The auctioneer sets the price and allocates shares of the risky asset according to the following rules. If there exists a unique price \( p' \) that satisfies the market-clearing condition

\[ \sum_{i=1}^{I} \Theta^i(p) = 0, \quad (1) \]

then all orders are executed at this price, and the risky asset is allocated such that \( x^i = \bar{x}^i + \Theta^i(p') \) for all \( i \). If there does not exist a unique market-clearing price, then no price is announced, no order is executed, and \( x^i = \bar{x}^i \) holds for all \( i \). This last rule is required to fully describe our trading game for all feasible order choices of traders; it does not drive any of the results in the equilibrium discussed below.

3 Equilibrium with Imperfect Competition

In the following analysis, traders do not act as price-takers, but are aware of the fact that their individual trading affects the price in consequence of the market-clearing condition; it is this awareness that makes competition imperfect and raises the question for market liquidity in our model. More precisely, each trader chooses a trading strategy to maximise his expected utility of terminal wealth subject to his budget constraint and the trading rules, taking as given the strategies of all other traders (rather than the price). The resulting equilibrium is a Nash equilibrium in trading strategies.
We assume that all aspects of the model are common knowledge and focus on Nash equilibria in pure strategies. In our trading game, such an equilibrium is a profile of orders \(\Theta^1, \ldots, \Theta^I\) such that for each trader \(i\), \(\Theta^i\) is \(i\)'s optimal order given that each other trader \(h \neq i\) chooses the order \(\Theta^h\). Furthermore, we confine attention to equilibria in linear orders. Let the conjectured order for each trader \(i\) be denoted \(\Theta^i(p) = \alpha^i - \beta^i p\). (2)

According to the trading rules, the coefficient \(\beta^i\) is restricted to be non-negative for each trader \(i\). An order \(\alpha^i > 0 (\leq 0)\), \(\beta^i = 0\) is a market order, which specifies to buy (sell) a fixed number of shares no matter what the price. A price-contingent order \(\beta^i > 0\) assigns to each quantity a price at which trader \(i\) is willing to buy that quantity; since these orders can be thought of as schedules of many limit orders, we refer to them below, somewhat loosely, as limit orders. If trader \(i\) employs the strategy \(\alpha^i = \beta^i = 0\), then \(i\) does not submit any order at all. From the market-clearing condition (1) and the restrictions on each trader’s order, a unique market-clearing price

\[
p = \frac{\sum_{i=1}^I \alpha^i}{\sum_{i=1}^I \beta^i}
\]

exists if and only if at least one trader \(i\) submits a limit order.

First, we derive each trader’s individually optimal order given the conjectured orders of all other traders. For this purpose, we define for each trader \(i\) a supply curve as the negative sum of the orders of all other traders \(h \neq i\),

\[
-\sum_{h \neq i} \Theta^h(p) = -\sum_{h \neq i} \alpha^h + p \sum_{h \neq i} \beta^h;
\]

which gives the aggregate number of shares that all traders \(h \neq i\) offer to sell at a given price. The supply curve can be thought of as order book against which trader \(i\) submits his own order to buy shares. Since each trader’s order both depends on his supply curve and affects the supply curve of other traders, all orders will be determined interdependently in equilibrium.

Consider the case that at least one trader \(h \neq i\) submits a limit order \(\beta^h > 0\), so that the supply curve for trader \(i\) is upward-sloping in \((p, -\sum_{h \neq i} \theta^h)\) space; this case will provide the equilibrium discussed in the next section. If \(i\) trades against an upward-sloping supply curve, he can implement his optimal order by choosing quantities \(\theta^i\), since from the market-clearing condition each quantity is related to a unique price according to

\[
\theta^i = -\sum_{h \neq i} \Theta^h(p).
\]

7
Consequently, from (3) and (4), it is possible to write the market-clearing price as function of the number of shares trader $i$ demands to buy,

$$p = \hat{p}^i + \lambda^i \theta^i,$$

where $\lambda^i = \left(\sum_{h \neq i} \beta^h\right)^{-1}$ is the slope and $\hat{p}^i = \lambda^i \sum_{h \neq i} \alpha^h$ is the intercept of the supply curve of trader $i$. Equation (5) illustrates the role of market liquidity in the trading decision of trader $i$. The intercept gives the market-clearing price if trader $i$ does not submit an order but holds passively his initial endowment. If $\lambda^i > 0$, the market is not perfectly liquid, and $i$ cannot buy and sell all he wants at a given price. Instead, there is an implicit bid-ask spread insofar as the price to buy a certain number of shares exceeds the price to sell the same number of shares. Furthermore, trader $i$ has price impact in that the price for a buy or sell order depends on the size of the order. The reciprocal of $\lambda^i$ is the Kyle-measure of market depth for trader $i$; we apply this term to analyse market liquidity.\footnote{The Kyle (1985) lambda follows from learning private information from order flow by risk-neutral market makers, and the Kyle (1989) lambda results from inferring private information from prices by risk-averse investors in a Walrasian auction trading mechanism. In our model, in contrast, illiquidity is not driven by asymmetric information.}

**Lemma 1** Case 1: If $\beta^h > 0$ for some trader $h \neq i$, then the optimal order for trader $i$ is given by

$$\Theta^{i,*}(p) = \frac{E[\bar{v}] - \rho_i \text{Var}[\bar{v}] \bar{x}^i - (1 + r) p}{(1 + r) \lambda^i + \rho_i \text{Var}[\bar{v}]}.$$  

Case 2: If $\beta^h = 0$ for all $h \neq i$ and $\sum_{h \neq i} \alpha^h = 0$, then trader $i$ is indifferent between all feasible trading strategies. Case 3: If $\beta^h = 0$ for all $h \neq i$ and $\sum_{h \neq i} \alpha^h > 0 (<0)$, then the optimal order for trader $i$ is characterised by $\alpha^i/\beta^i = +\infty (-\infty)$. \n
**Proof:** The problem for trader $i$ is to determine $y^i$ and $x^i$ to maximise

$$(1 + r)y^i + E[\bar{v}] x^i - \frac{\rho_i}{2} (x^i)^2 \text{Var}[\bar{v}]$$  

subject to his budget condition $\bar{y}^i + px^i = y^i + px^i$ and the trading rules. First consider Case 1, where $\beta^h > 0$ for at least one trader $h \neq i$. Then, according to (5), there exists a unique market-clearing price $p = \hat{p}^i + \lambda^i (x^i - \bar{x}^i)$. Plugging the price into the budget condition yields the trading constraint of trader $i$ as

$$y^i = \bar{y}^i - \hat{p}^i (x^i - \bar{x}^i) - \lambda^i (x^i - \bar{x}^i)^2.$$
Plugging (8) into (7) for $y_i$ and differentiating for $x_i$ gives the first-order condition

$$E[\tilde{v}] - \rho^i x^{i,*} \Var[\tilde{v}] = (1 + r)\tilde{p}^i + 2(1 + r)\lambda^i (x^{i,*} - \bar{x}^i).$$  \hfill (9)

Using the market-clearing price to substitute for $\tilde{p}^i$ on the right-hand side of (9) yields the optimal demand to hold shares of the risky asset

$$x^{i,*} = \frac{E[\tilde{v}] + (1 + r)\lambda^i \bar{x}^i - (1 + r)p}{(1 + r)\lambda^i + \rho^i \Var[\tilde{v}]}. \hfill (10)$$

The optimal order to buy shares is then given by $\Theta^{i,*} = x^{i,*} - \bar{x}^i$. For the second-order condition

$$-2(1 + r)\lambda^i - \rho^i \Var[\tilde{v}] < 0$$
to hold, it is sufficient that $\beta^h > 0$ for some trader $h \neq i$, given the assumptions on trader $i$’s preferences and on traded assets. Now consider the cases where $\beta^h = 0$ for all $h \neq i$. Then, trader $i$ does not trade against an upward-sloping supply curve and thus cannot choose different quantities of the risky asset at different prices. In Case 2, where $\sum_{h \neq i} \alpha^h = 0$, the orders of all other traders $h \neq i$ aggregate to zero at each price. Consequently, trader $i$ cannot trade anyway and hence is indifferent between all trading strategies. In Case 3, where $\sum_{h \neq i} \alpha^h > 0 (< 0)$, the orders of all other traders $h \neq i$ constitute an aggregate (non-zero) market order. Then, trader $i$ obtains maximum expected utility by submitting a limit order which sells (buys) against this market order and pushes the price to $+\infty$ ($-\infty$). The appendix spells out the details of the last two cases, which involve the trading rules for situations in which a unique market-clearing price does not exist.

Figure 1 illustrates the effect of market liquidity on the individually optimal order of trader $i$ for the case that $\beta^h > 0$ for some trader $h \neq i$. Given the initial endowment at point $A$, the curve $TC$ represents the trading constraint (8) for $\lambda^i > 0$, and the dashed straight line is the constraint for the limiting case of a perfectly liquid market as $\lambda^i \rightarrow 0$ (perfect competition). If $\lambda^i > 0$, trader $i$ acts as a monopsonist over his supply curve and recognises that the price moves against him when he trades an additional unit of the risky asset. Hence, his trading aggressiveness is restricted not only by his risk aversion but also by his price impact. The optimal order equates marginal expected utility of the risky asset with marginal cost, including liquidity costs (point B). The figure shows that liquidity costs from imperfect competition decrease trading volume and expected utility for trader $i$ compared with the individual optimum in the perfectly competitive case (point $C$).
Now we turn to the derivation of Nash equilibrium. For analytical tractability, we assume that all traders have identical risk aversion and submit orders with identical slope, i.e., we assume $\rho^i = \rho$ for all $i$ and $\beta^i = \beta$ for all $i$. Traders are allowed to have different initial endowments of the risky asset, however.

**Proposition 1** Consider the case of linear orders, $\rho^i = \rho$ for all $i$, and $\beta^i = \beta$ for all $i$. If $I \geq 3$, then there exist two equilibria. The first equilibrium is given by the coefficients

$$
\alpha^i = \frac{(I - 2)}{(I - 1)} \left( \frac{E[\tilde{v}]}{\rho \text{Var}[\tilde{v}]} - \tilde{x}^i \right) \quad \text{and} \quad \beta^i = \frac{(I - 2)}{(I - 1)} \frac{(1 + r)}{\rho \text{Var}[\tilde{v}]} > 0
$$

(11)

for all $i$. The second equilibrium is given by the coefficients

$$
\alpha^i = 0 \quad \text{and} \quad \beta^i = 0
$$

(12)

for all $i$. If $I = 2$, then only the second of those two equilibria exists.

**Proof:** We derive Nash equilibria from the individually optimal orders given in Lemma 1. The first equilibrium follows from Case 1, where, for each trader $i$, $\beta^h > 0$ for at least one trader $h \neq i$. By equating coefficients from the optimal order (6) with those from the proposed order (2) for each trader $i$, equilibrium is determined as the solution to the
equation system

\[
\alpha^i = \frac{E[\tilde{v}^i] - \rho^i \text{Var}[\tilde{v}] \bar{x}^i}{(1 + r) \left( \sum_{h \neq i} \beta^h \right)^{-1} + \rho^i \text{Var}[\tilde{v}]} \quad i = 1, \ldots, I \tag{13}
\]

\[
\beta^i = \frac{(1 + r)}{(1 + r) \left( \sum_{h \neq i} \beta^h \right)^{-1} + \rho^i \text{Var}[\tilde{v}]} \quad i = 1, \ldots, I. \tag{14}
\]

Given the assumptions on \(\rho^i\) and \(\beta^i\), equation (14) can be solved immediately. Plugging the solution for \(\beta\) into (13) yields the solution for \(\alpha^i\). The second solution to (14), \(\beta^i = 0\), does not yield an equilibrium from Case 1, since it violates the condition that for each trader \(i\), \(\beta^h > 0\) for at least one trader \(h \neq i\). The second equilibrium follows from Case 2, where, for each trader \(i\), \(\beta^h = 0\) for all \(h \neq i\) and \(\sum_{h \neq i} \alpha^h = 0\). Then, each trader \(i\) is indifferent between all feasible trading strategies, including the possibility of not submitting an order at all by choosing \(\alpha^i = \beta^i = 0\). This gives the second equilibrium.

In the first equilibrium, each trader submits a limit order, a unique market-clearing price exists, and there is (almost always) trade; we refer to this equilibrium as trading equilibrium. The assumption of identical \(\beta^i\) for all \(i\) is used to prove that there does not exist another trading equilibrium. We conjecture that uniqueness of the trading equilibrium holds also without this assumption, but do not attempt to prove this here. The trading equilibrium does not exist if there are only two traders. In this case, no trader is willing to submit any order, because it would be exposed to the total market power of the remaining trader. This result holds also if the two traders are not restricted to identical risk aversion and to submit orders with identical slope. The presence of a third trader, however, generates sufficient competition to make each trader willing to become active in the market and for the trading equilibrium to exist. The properties of the trading equilibrium are discussed in the next section.

In the second equilibrium, traders do not submit any orders, but hold passively their initial endowments; we call this equilibrium the passive equilibrium. It can be shown that, for each trader \(i\), the strategy played in the passive equilibrium is weakly dominated by the strategy played in the trading equilibrium. A formal way to rule out the passive equilibrium is to introduce noise traders who exogenously submit a random order \(\tilde{\mu} \sim N(0, \sigma^2_\mu)\) and thus provide a (non-zero) market order with probability one. Then, if \(I \geq 3\), the presence of noise traders attracts limit orders from all optimising traders and thereby
enforces the trading equilibrium.\footnote{The introduction of noise traders may thus be seen as a rough version of (normal form) trembling-hand refinement.} In our symmetric information model, however, there is no further role for noise traders, and therefore we do not carry out this refinement formally.

4 Characteristics of the Trading Equilibrium

In this section, we examine the characteristics of the trading equilibrium. First, we relate the price impact of traders from imperfect competition to the concept of market liquidity known from market microstructure theory. Then, we consider how illiquidity carries over to the market-clearing price and the allocation of the risky asset. In particular, we are interested in whether the price of the risky asset reflects—beyond the traditional risk premium—a liquidity premium that confirms the notion of compensation for liquidity costs that are borne by traders.

Proposition 2 For each trader $i$, market liquidity is given by

$$\lambda^{-1} = \frac{(1 + r)(I - 2)\rho^{-1}}{\text{Var} \left[ \tilde{v} \right]}.$$ 

Proof: The result follows immediately from the definition of $\lambda^i$ and the solution to $\beta$. Because of the assumptions of identical $\rho^i$ and $\beta^i$ for all $i$, market liquidity is identical for all traders, too.

The rationale of this result is straightforward. If, for instance, trader $i$ wishes to sell shares of the risky asset, he has to make other traders willing to buy these shares for the market to clear. Since orders of other agents are decreasing in price, $i$ moves the price downward with his proposal to sell. The price impact (per unit of the risky asset) is the smaller, and accordingly the degree of market liquidity is the higher, the larger the number of other traders and their risk tolerance and the smaller the risk of the asset. These results on market liquidity correspond to the inventory-based microstructure approaches of Stoll (1978), Ho and Stoll (1981) and Grossman and Miller (1988). In their models, risk-averse market makers provide trading immediacy by accommodating stochastic orders from other market participants, and illiquidity (in terms of bid-ask spreads or price concessions) arises to compensate market makers for their willingness to change the risk-return profile of their inventories. In our model, in contrast, market liquidity is not determined by the activity
of the market intermediary (the auctioneer), but stems from the mutual submission of limit orders from traders themselves. Since these orders, however, reflect traders’ portfolio decisions, our setup captures the risk-aversion based aspect of market liquidity from the microstructure literature nevertheless.

Now, we turn to the market-clearing price of the risky asset and to traders’ portfolios, and study how these variables are affected by illiquidity.

**Proposition 3** The market-clearing price and each trader’s portfolio holdings of the risky asset are given by
\[
p = \frac{1}{(1+r)} \left( E[\bar{v}] - \frac{\bar{X}Var[\bar{v}]}{I\rho^{-1}} \right)
\]
and
\[
x_i = \frac{(I-2)\bar{X}}{(I-1)} + \frac{\bar{x}_i}{(I-1)} \quad \text{for all } i.
\]

**Proof:** The market-clearing price follows from the market-clearing condition (1), Proposition 1, and the fact that \(\sum_{i=1}^{I} \bar{x}_i \equiv \bar{X}\). The number of shares bought or sold by trader \(i\) are determined by plugging the market-clearing price into the equilibrium order of \(i\). Portfolio holdings are then given by \(x_i = \bar{x}_i + \theta_i\) for all \(i\). ||

The market-clearing price in the imperfectly competitive trading equilibrium is identical to the equilibrium price in neoclassical asset pricing under perfect competition for the CARA-Gaussian setting; it is given by the expected payoff per share minus the standard risk premium, discounted at the riskfree rate of interest. In particular, the market-clearing price does not reflect an additional liquidity discount, or equivalently, the expected return does not yield an additional liquidity premium.

The allocation of the risky asset, in contrast, is different from the (Pareto optimal) allocation under perfect competition, where in equilibrium each trader holds a risk tolerance weighted proportion of the total number of outstanding shares, which for identical risk aversion for all traders simplifies to
\[
x^c = \frac{\bar{X}}{I}
\]
for all \(i\). With imperfect competition, traders trade towards the Pareto optimal allocation, but do not achieve this allocation completely. If, for instance, trader \(i\) is endowed with few units of the risky asset, \(\bar{x}_i < x^c\), he buys additional units at the market-clearing price and obtains the portfolio holdings \(x_i\), with \(\bar{x}_i < x_i < x^c\). Consequently, liquidity costs from imperfect competition reduce trading volume and impede efficient risk-sharing.
among traders. Only for the case that $\bar{x}^i = x^c$ for all $i$, the Pareto optimal allocation is achieved. In this special case, however, there is no trade in the trading equilibrium (although traders submit their limit orders nonetheless).

As an example, Figure 2 illustrates the results on the market-clearing price and on trading volume for the case that $I = 3$, $\bar{x}^1 < x^c < \bar{x}^2$, and $\bar{x}^3 = x^c$. In equilibrium under imperfect competition, each trader’s order is characterised—from (11)—by

$$p = \frac{E[\hat{v}] - \rho \text{Var}[\hat{v}] \bar{x}^i}{(1 + r)} - \frac{(I - 1) \rho \text{Var}[\hat{v}]}{(I - 2)} \theta^i.$$  \hspace{1cm} (15)

In this example, however, trade takes place only between $i = 1$ and $i = 2$, i.e., at the market-clearing price $p^*$, trades are given by $\theta^1 = -\theta^2$ and $\theta^3 = 0$. In the figure, the orders under imperfect competition for traders $i = 1, 2$ are represented by the solid lines. \(^5\)

In comparison, in equilibrium under perfect competition, each trader’s order is similar to (15) except that the assumption of price-taking behavior implies that $\frac{(I-1)}{(I-2)} \to 1$. \(^6\)

Thus, each trader’s order intersects the price axis at the same point as his order under imperfect competition, but is less steeply sloped, as shown by the dashed lines in the figure. Consequently, if traders act as if there were no liquidity costs from imperfect

\(^5\)The order for trader $i = 3$, which is parallel to $\Theta^{1,*}$ and $\Theta^{2,*}$ and intersects the price axis at the market-clearing price, is omitted.

\(^6\)This can be seen from equation (6) for $\lambda^i \to 0$. 

Figure 2: Effect of price impact on the market-clearing price and on trading volume.
competition, the market clears at the same price $p^*$, such that $\theta^{1,c} = -\theta^{2,c}$ and $\theta^{3,c} = 0$, but traders $i = 1, 2$ trade a higher volume.

The result on inefficient risk-sharing delivers additional insights into the nature of market liquidity. In our model, the submission of a limit order by trader $i$ produces a positive externality by increasing market liquidity for all other traders $h \neq i$. In return, other traders trade more aggressively and submit less steeply sloped orders (in $(p, \theta^h_1)$ space), which offers a more liquid market to trader $i$. Thus, for each individual trader, providing liquidity for other traders generates a positive feedback effect of increased liquidity for that individual trader himself. Despite this feedback effect, however, in equilibrium the market is not sufficiently liquid to achieve a Pareto optimal allocation, which reflects the positive-externality property of market liquidity.\(^7\)

The result on the market-clearing price is the main contribution of this paper; it contradicts the common intuition that, in illiquid markets, expected returns reflect liquidity premia to compensate traders for bearing liquidity costs. In our model, illiquidity is a cost, but it is not a priced factor. The reason is that illiquidity makes buyers restrict their demand to buy and sellers restrict their supply to sell, and these individual effects offset each other exactly with respect to the market-clearing price and the expected return, respectively. Intuitively, as can be seen from Figure 2, buyers ($i = 1$) demand a price discount, and sellers ($i = 2$) demand a price premium, and thus both sides of the market would like to be compensated for illiquidity, but nobody is willing to grant that compensation.\(^8\)

Although our results challenge the existence of liquidity premia in illiquid markets, illiquidity and expected return are positively related in the model nevertheless. If the number of traders or trader’s risk tolerance decreases, or if the risk of the asset increases, then both illiquidity and expected return increase. Consequently, high illiquidity coincides with high expected return, but not due to any causal relationship between the two variables, but only because the risk-aversion based price impact and the risk premium of the expected return are similarly determined by the same exogenous variables. This result exemplifies the critique of Berk (1995) in the context of illiquidity, i.e., empirical findings of liquidity premia in expected returns may be solely due to mis-specifications of risk premia rather than to the pricing of illiquidity, per se.

---

\(^7\)The positive externality and the feedback effect of liquidity provision are also crucial elements in the analysis of concentration versus fragmentation of trade over time by Admati and Pfleiderer (1988) and across markets by Pagano (1989).

\(^8\)Formally, however, traders’ strategies are price-quantity functions, not prices.
5 Conclusion

This paper studies the effects of illiquidity on asset prices and traders’ portfolios when illiquidity is derived endogenously from the underlying market-microstructure process. We find that illiquidity is a cost for traders, which gives rise to allocation inefficiencies, but it is not reflected in the expected return on the risky asset. This result is different from the existing literature on exogenous illiquidity, which argues that illiquid assets yield liquidity premia to compensate traders for bearing liquidity costs.

There are several limitations to the analysis. First, illiquidity results from imperfect competition only. Other frictions that impede trading processes, e.g. asymmetric information or search costs, are not examined. Second, there is no liquidity risk in our model. Although traders submit their orders simultaneously and thus cannot observe their individual supply curves before making their trading decisions, the common-knowledge assumption allows traders to infer their supply curves in equilibrium. Third, our model of one-shot trading does not permit to analyse the impact of future illiquidity on current prices. Finally, the assumptions made for analytical tractability, notably identical constant absolute risk aversion for all traders, normally distributed payoffs, and linearity of strategies, restrict the generality of our results.

Given these limitations, our model may be seen rather as counter-example to the common notion of compensatory liquidity premia than as general argument on their nonexistence. Extending the model for additional frictions and to a more general setting would thus be promising in developing a more comprehensive microstructure foundation to the role of market liquidity in asset pricing.

Appendix

Continuation of Proof of Lemma 1:

Here, the last two cases from Lemma 1, where trader \(i\) trades against a vertical supply curve, which offers the same quantity at every price, are considered in more detail.

In Case 2, the supply curve for \(i\) is given by \(\beta^h = 0\) for all \(h \neq i\) and \(\sum_{h \neq i} \alpha^h = 0\). If trader \(i\) submits a limit order \(\beta^i > 0\), then there exists a unique market-clearing price, \(p' = \frac{\alpha^i}{\beta^i}\). At this price, \(\Theta^i(p') = 0\) applies, such that \(i\) does not trade. (There is trade between other traders, however, if \(\alpha^h \neq 0\) for at least two \(h \neq i\).) If trader \(i\) submits a market order \(\alpha^i \neq 0, \beta^i = 0\), then there does not exist a market-clearing price, and if \(i\)
employs the passive strategy $\alpha^i = 0, \beta^i = 0$, then every price clears the market. Since the trading rules exclude trade if there does not exist a unique market-clearing price, there is no trade at all. Consequently, in Case 2 trader $i$ remains with his initial endowment for each of his feasible orders, and thus is indifferent between all strategies.

In Case 3, the supply curve for $i$ is given by $\beta^h = 0$ for all $h \neq i$ and $\sum_{h \neq i} \alpha^h > 0 (< 0)$. If trader $i$ employs any strategy $\beta^i = 0$, then either no price clears the market (if $\alpha^i \neq - \sum_{h \neq i} \alpha^h$), or every price clears the market (if $\alpha^i = - \sum_{h \neq i} \alpha^h$), so that no trade takes place, and $i$ sticks to his initial endowment. If trader $i$ submits a limit order, $\beta^i \neq 0$, then there exists a unique market-clearing price, and $i$ sells (buys) $\left| \sum_{h \neq i} \alpha^h \right|$ shares. For trading against a vertical supply curve, trader $i$ can force the price for a sale (purchase) to $+\infty$ ($-\infty$) by setting $\frac{\alpha^i}{\beta^i} = \infty (-\infty)$, which gives higher expected utility than holding on to his initial endowment. It is easy to see that Case 3 does not yield an equilibrium; thus, we skip further specifications of the individual optimum for this case.

References


