Pricing of options on assets with level dependent stochastic volatility

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ABSTRACT

Many asset classes, such as interest rates, exchange rates, commodities, and equities, often exhibit a strong relationship between asset prices and asset volatilities. This paper examines an analytical model that takes into account this level dependence of volatility. We demonstrate how prices of European options under stochastic volatility can be calculated analytically via inverse Laplace transformations. We also examine a Hull-White stochastic volatility expansion. While a success of this expansion in approximate computation of option prices has already been established empirically, the question of convergence has been left unanswered. We demonstrate, in this paper, that this expansion diverges essentially for all possible stochastic volatility processes. In contrast to a majority of volatility expansion models reported in the literature, we construct expansions that explicitly show the contribution of all of the variance moments. Such complete expansions are very useful in analyzing properties of option prices, as we demonstrate by examining why empirical volatility surfaces plotted as a function of the rescaled strike can sometimes exhibit striking time invariance.

Keywords: stochastic volatility, derivatives, volatility expansion, leverage effect, Laplace transform, CEV

1. INTRODUCTION

The relationship between asset price and its volatility has been a center of attention in financial research for a long time. The pioneering work by Black (1976) was one of the first to document the fact that stock return volatility is negatively correlated with stock returns. Black (1976) and Christie (1982) argued that a stock price decline increases a firm’s leverage, which makes the stock riskier and increases its volatility. The asymmetric nature of volatility manifests itself in option prices via an implied volatility skew that suggests that market participants expect volatility to rise as prices fall. A number of alternative explanations have been proposed. Pindyck (1984), French, Schwert, and Stambaugh (1987) attributed the same effect to the time-varying risk premiums. Any increase in volatility makes stock more risky and, therefore, demands a higher premium. This mechanism would lead to an immediate stock price decline. An investigation of different factors that contribute to price level dependence of volatility is still an active area of research and many problems are still unsolved (Bekaert, Wu (2000)). The asymmetric volatility is not limited to equity markets, and has a much broader applicability. For example, Pindyck (2001) showed that changes in volatility of commodity prices directly affect the marginal value of storage. Typically, an increase in volatility leads to inventory build-ups and, thereby, can raise prices.

The above arguments inspired a number of models with volatility that depend on price level (Cox and Ross (1976), Geske (1979)). The constant elasticity of variance (CEV) model of the Cox and Ross (1976) models level dependence as a power function. These models provide only a partial solution because deterministic level dependence does not completely reflect the stochastic nature of volatility. For example, Eraker (2001) found that a pure CEV model does a poor job in capturing the time-varying volatility interest-rate data. He showed that an addition of the stochastic volatility component provides a vastly superior fit to that of the CEV model. Hull and White (1987), Stein and Stein (1991), Heston (1993) proposed analytically solvable models that have stochastic volatility. Stochastic volatility models are usually so complex that modeling asymmetric volatility was always restricted to the introduction of a simple correlation coefficient between stock price and volatility processes. The analysis of the level-dependent stochastic volatility models was limited mainly to numerical methods. Johnson and Shanno (1987), Melino and Turnbull (1990) used the Monte Carlo approach to calculate option prices for a stochastic volatility model with a power level dependence. Melino and Turnbull found that their model provides a good fit to the empirical distribution of the Canada-US exchange rate. In fact, a power law is one of the most popular functions to model level dependence. Hagan, Kumar, Lesniewski (2002) argued that using any function for level dependence, except for a power law, introduces an intrinsic “length scale” for the stock price. There are no clear financial reasons for prices to be inhomogeneous and, therefore, power law is the most natural candidate to use as a model of asymmetric volatility.
In a recent study, Hagan, Kumar, and Lesniewski (2002) used singular perturbation techniques to obtain approximate option prices for a stochastic volatility model with a power level dependence (the SABR model). The stochastic volatility in the SABR model follows drift-less geometrical Brownian motion that is correlated to stock price. The SABR model provided good fits to the implied volatility curves observed in the marketplace. The volatility elasticity parameter $\beta$ and correlation affect the volatility “smile” in similar ways. Hagan, Kumar, and Lesniewski (2002) found that it is difficult to distinguish between the two parameters, and market “smiles” can be fit equally well with any specific value of $\beta$. As a result, the authors of the SABR model suggest setting $\beta$ a priori at some “aesthetically” appealing number. Since $\beta$ and correlation duplicate each other, we feel that, for any parsimonious model, it is unnecessary to keep them both. In this paper, we set correlation to zero and use an elasticity parameter to control the leverage effect.

Being motivated by the above considerations, we investigated a model of the asset prices that has the following risk-neutral dynamics:

$$dF = \sigma F^\alpha dB_1$$  \hspace{1cm} (1)

$$d\sigma = b(\sigma)dt + a(\sigma)dB_2$$  \hspace{1cm} (2)

$$F = Se^{-rt}$$  \hspace{1cm} (3)

where two Brownian motions are uncorrelated. To be specific, we assume that elasticity parameter $\alpha \leq 1$ which is the case for equity markets. Results for option prices when $\alpha > 1$ can be obtained in a similar way, but would require a more careful consideration because of the possibility of volatility explosions. An excellent treatment of this subject is given in Lewis (2000).

2. EUROPEAN OPTIONS AS LAPLACE TRANSFORMS

We assume that volatility of a risk-neutral process can depend on the asset price level, but otherwise is uncorrelated with the asset:

$$dF = \sigma \eta(F)dB_1$$  \hspace{1cm} (4)

$$du = b(u)dt + a(u)dB_2$$  \hspace{1cm} (5)

$$u = \sigma^2$$  \hspace{1cm} (6)

$$\nu = V/t = \frac{1}{t} \int_0^t \sigma^2(\tau)d\tau$$  \hspace{1cm} (7)

Lewis (2000) has shown that a price of a European option can be represented as an integral

$$h(S,u_0,\tau) = e^{-\tau V} \int_0^\infty \phi(F,V)P(v,u_0,\tau)dv$$  \hspace{1cm} (8)

where $P$ is a probability distribution of variance and $\Phi$ function is related to a price $f$ of the same option under constant volatility:

$$f(S,u_0,\tau) = e^{-\tau \nu} \Phi(F,u_0)$$  \hspace{1cm} (9)

Therefore, the price of a European option under stochastic volatility is a mixture of constant volatility prices. Most realistic volatility processes do not have an explicit probability function $P$, but have an explicit Laplace transform that can be found via a Feynman-Kac theorem. Therefore, mixing theorems in their standard form can be used for numerical computations only. We demonstrate below that it is possible to derive analytical option prices that would not require numerical inversions of the Laplace transforms. The mean reverting volatility process of Stein and Stein (1991), the square root process used by Heston (1993), and the geometrical Brownian motion process are examples of stochastic volatility processes with known Laplace transforms.
Our argument relies on an observation that, when volatility is constant, then, in most cases, European options can be considered as Laplace transforms of simple functions with respect to variance $V$. The price of the European derivative is equal to the discounted expectation:

$$C = e^{-rt} \int_0^\infty G(S) p(S,V) dS$$  \hspace{1cm} (10)$$

where $p(S,V)$ is a probability density of the stock price at maturity and $G(S)$ is a derivative payoff function. We show later that, if stock price follows a geometric Brownian motion or a constant elasticity of variance (CEV) type process, then a stock price probability density can be represented as a Laplace transform

$$p(S,V) = \int_0^\infty e^{-V\omega} \hat{p}(S,\omega) d\omega$$  \hspace{1cm} (11)$$

If we can change the order of integration, then option price is indeed a Laplace transform:

$$C = \int_0^\infty e^{-V\omega} \left( e^{-rt} \int_0^\infty G(S) \hat{p}(S,\omega) dS \right) d\omega = \int_0^\infty e^{-V\omega} \hat{C}(\omega) d\omega$$  \hspace{1cm} (12)$$

where $\hat{C}(\omega)$ represents an inverse Laplace transformation of the option price with respect to the variance $V$. We assume that European option prices and stock price distribution densities can be computed from the corresponding constant volatility results via mixing theorems. If $\hat{C}(\omega)$ can be computed explicitly, then option prices under stochastic volatility can be computed analytically, without a numerical inversion of the volatility Laplacian:

$$C = \int_0^\infty \hat{C}(V) P(V) dV = \int_0^\infty \left( \int_0^\infty e^{-V\omega} P(V) dV \right) \hat{C}(\omega) d\omega = \int_0^\infty L[P] \hat{C}(\omega) d\omega$$  \hspace{1cm} (13)$$

Mixing theorems are not limited to the situations with zero correlation between volatility and stock price. Romano and Touzi (1997) have shown that, if there is no level dependence, then mixing theorems can be formulated for correlated volatility too. These theorems have stock prices and volatilities substituted by effective values that depend on correlation. Therefore, the inverse Laplace transform technique is not restricted to zero correlation cases and have a broader applicability too.

3. STOCHASTIC VOLATILITY WITH NO LEVEL DEPENDENCE

We start our analysis by demonstrating the application of the inverse Laplace transformation method in a simple and more familiar case of level independent volatility. Therefore, we assume that the stock price follows a geometrical Brownian motion. When volatility is deterministic, the stock prices have a lognormal distribution (Black and Sholes (1973)): 

$$f_{norm}^0(S) = \frac{1}{S \sqrt{2\pi V}} \exp \left( -\frac{(w - V/2)^2}{2V} \right)$$  \hspace{1cm} (14)$$

$$w = \log(S/S_0) - rt$$  \hspace{1cm} (15)$$

The inverse Laplace transformation of the lognormal probability density can be easily derived from the general properties of the Laplace transformation (Schiff (1999)):

$$L^{-1} \left[ \frac{e^{-aV - c/V}}{\sqrt{V}} \right] = \frac{u_a(\omega) \cos(2\sqrt{c(\omega - a)})}{\sqrt{\pi(\omega - a)}}$$  \hspace{1cm} (16)$$

where $u$ is a step function. As a result, when volatility is stochastic, the stock price distribution can be represented as an integral:
In a similar way, we can compute the inverse Laplace transformation of the call option (Appendix):

\[
L^{-1}[C(V)] = S_0 \delta(\omega) - \frac{S_0 K e^{-\xi}}{2\pi \sqrt{2}} \int_{0}^{\infty} L(\omega^2/2 + 1/8) \cos(\omega \omega) d\omega
\]  

(18)

Therefore, option prices under stochastic volatility can be expressed analytically:

\[
C = S_0 - \frac{S_0 K e^{-\xi}}{2\pi} \int_{0}^{\infty} L(\omega^2/2 + 1/8) \cos(\omega \omega) d\omega
\]  

(19)

Stein and Stein (1991), and Ball and Roma (1994) analyzed the case of level independent volatility with Fourier techniques and obtained a formula for stock price distribution. Curiously enough, they were not able to derive an analytical formula for option prices. This demonstrates that some results can be obtained more easily with inverse Laplace transformation methods than with alternative techniques.

4. THE PROBABILITY DENSITY OF STOCK PRICES

We assume that volatility has a power law dependence on the stock price. The completely deterministic dependence corresponds to the CEV model of Cox and Ross (1976). The continuous part of the stock price probability distribution in the CEV model is equal to:

\[
f_0(S) = e^{-\kappa t} \frac{(Se^{-\kappa t})^{-2\beta-3/2} S_0^{1/2}}{\sigma^2 \beta^t} \exp\left(-\frac{(Se^{-\kappa t})^{-2\beta} + S_0^{-2\beta}}{2\sigma^2 \beta^t}\right) I_\lambda\left(\frac{(Se^{-\kappa t})^{-\beta} S_0^{-\beta}}{\sigma^2 \beta^t}\right)
\]  

(20)

where \(I_\lambda\) is a modified Bessel function of the first kind and \(\beta = \alpha - 1, \lambda = 1/(2\beta\beta)\). This distribution has the following inverse Laplace transform (Gradshteyn, et al. (2000)):

\[
L^{-1}\left[\frac{1}{V} \exp\left(-\frac{x+y}{2V}\right) I_\lambda\left(\frac{\sqrt{xy}}{V}\right)\right] = J_\lambda\left(\sqrt{2x\omega}\right) J_\lambda\left(\sqrt{2y\omega}\right)
\]  

(21)

Therefore, the continuous part of the stock price distribution under stochastic volatility can be expressed as:

\[
f(S) = \frac{S_0^2 S^{-2\beta-3/2} e^{(2\beta+1/2)\kappa t}}{2\lambda} \int_{0}^{\infty} L(\omega^2/2\beta^2) J_\lambda(S^{-\beta} \omega) J_\lambda(S^{-\beta} e^{\kappa t} \omega) \omega d\omega
\]  

(22)

In their influential paper, Stein and Stein (1991) obtained a similar formula for a simpler case when volatility is completely independent of stock price, and wrote that they do not know whether a tractable exact solution exists for stock price distributions generated by the CEV type stock price dependence. Our formula demonstrates that the exact solution does exist.

Cox and Ross (1976) showed that the CEV model with constant volatility has the probability of default:

\[
P_0^{\text{def}} = \frac{\Gamma(\lambda, \alpha/V)}{\Gamma(\lambda)}
\]  

(23)

\[
\alpha = \frac{S_0^{-2\beta}}{2\beta^2}
\]  

(24)

where \(\Gamma(\lambda, \alpha)\) is the complimentary Gamma distribution function. We derived the inverse Laplace transformation of this default probability (Appendix):
Therefore, the probability of default in the presence of stochastic volatility is:

\[
P_{\text{def}} = 1 - \frac{\sqrt{S_0}}{\Gamma(\lambda) 2^{\lambda-1}} \int_{0}^{\infty} L(\omega^2 \beta^2 / 2) J_{\lambda}(S_0 \beta \omega) \omega^{\lambda-1} \, d\omega
\]

This formula is important for the calculation of the option prices.

### 5. OPTION PRICES

In the risk-neutral world, the prices of the European options are equal to the discounted expectations. For example, the price of the European put option can be represented as:

\[
P = e^{-rt} \int_{0}^{s} f(S)[K - S] \, dS + P_{\text{def}} Ke^{-rt}
\]

where \( f(S) \) is the continuous part of the stock price distribution. In order to evaluate this expectation, we change the order of \( dS \) and \( d\omega \) integrations, and use the properties of the Bessel functions (Abramowitz and Stegun (1974)):

\[
P = Ke^{-rt} - \sqrt{S_0 Ke^{-rt}} 2 \lambda \int_{0}^{\infty} L(\omega^2 \beta^2 / 2) J_{\lambda}(S_0^{-\beta} \omega) J_{\lambda}(K^{-\beta} e^{-rt} \omega) \frac{d\omega}{\omega}
\]

The prices of European call options immediately follow from a put-call parity relationship. These formulas extend the work of Stein and Stein (1991) in two directions. The first advancement is that we considered the case when volatility depends on the stock price; the second advancement is that we obtained analytical expressions, not only for the stock price densities but also for the option prices.

### 6. A VARIANCE MOMENT EXPANSION FOR LEVEL DEPENDENT STOCHASTIC VOLATILITY

Schroder (1989) has shown that, if volatility is constant, then a European option price for the CEV model can be expressed in terms of the non-central chi-square cumulative distribution function \( \chi^2 \):

\[
C^{\text{call}} = S_0 - \{Ke^{-rt} \chi^2(x, b, y) + S_0 \chi^2(y, b + 2, x)\}
\]

\[
x = \frac{S_0}{(1 - \alpha)^2 \sigma^2 \tau}
\]

\[
y = \frac{(Ke^{-rt})^2(1 - \alpha)}{(1 - \alpha)^2 \sigma^2 \tau}
\]

\[
b = \frac{1}{1 - \alpha}
\]

where \( \alpha > 1 \) and \( \chi^2(x, b, y) \) is the cumulative probability that a variable with a non-central \( \chi^2 \) distribution with non-centrality parameter \( y \) and \( b \) degrees of freedom is less than \( x \). We will calculate an expansion of the stochastic CEV model in terms of variance moments using a mixing property:
The expansions of this type were introduced in the pioneering work of Hull and White (1987). One of the most thorough treatments of the volatility expansion methods is given in the outstanding book by Lewis (2000).

The complementary non-central chi-square function can be represented as a weighted average of the complementary central probability functions (Abramowitz and Stegun (1974)) or as a double series of the gamma functions (Schroder (1989)). Both representations allow negative degrees of freedom for the non-central chi-square function. These representations can be used to compute the following identities for derivatives:

\[
\mathcal{X}_x^2(x, b, y) = \frac{1}{2} \mathcal{X}_x^2(x, b - 2, y) - \frac{1}{2} \mathcal{X}_x^2(x, b, y)
\]

(35)

\[
\mathcal{X}_y^2(x, b, y) = \frac{1}{2} \mathcal{X}_y^2(x, b + 2, y) - \frac{1}{2} \mathcal{X}_y^2(x, b, y)
\]

(36)

Our calculations demonstrate that the prices for European options under stochastic volatility can be represented as following series (Appendix):

\[
C_{\text{call}}^\text{call} = S_0 - \sum_{n=0}^{\infty} \mu_n \sum_{k=-n}^{n} P_k^n (x, y) \left[ Ke^{-\nu \tau} \mathcal{X}_x^2(x, b + 2k, y) + S_0 \mathcal{X}_y^2(y, b + 2 - 2k, x) \right]
\]

(37)

where \( \mu_n \) are normalized variance moments:

\[
\mu_n = \frac{(v - \bar{v})^n}{\bar{v}^n}, \quad \mu_0 = 1, \quad \mu_1 = 0.
\]

(38)

We use a bar over variables to indicate average values. \( P_k^n (x, y) \) is the \( n \)th degree polynomial:

\[
P_0^0 (x, y) = 1
\]

(39)

\[
P_k^n (x, y) = \sum_{1 \leq \lambda_1 \leq \lambda_2 \leq n} x^{\lambda_1} y^{\lambda_2} \frac{C^{\lambda_1 + \lambda_2}_{\lambda_2 - k} C^{n-1}_{\lambda_1 + \lambda_2 - 1}}{\lambda_1 ! \lambda_2 ! 2^{\lambda_1 + \lambda_2}} (-1)^{n+k+\lambda_1+\lambda_2}
\]

(40)

The last equation has a double summation over nonnegative integers \( \lambda_1 \) and \( \lambda_2 \). We adopted the following binomial coefficient convention: \( C^a_b = 0 \) if \( b<0 \) or \( b>a \). Our polynomials have an elegant symmetry:

\[
P_{-k}^n (x, y) = P_k^n (y, x)
\]

(41)

\[
\sum_{k=-n}^{n} P_k^n (x, y) = 0
\]

(42)

where the last equation is valid for \( n>0 \). Using this symmetry, it is possible to write down an interesting identity for the value of the at-the-money call option:

\[
C_{\text{call}}^\text{call} (x, b, x) = \sum_{n=0}^{\infty} \mu_n \sum_{k=-n}^{n} P_k^n (x, x) C_{\text{call}}^\text{call} (x, b + 2k, x)
\]

(43)
and a similar result for the put option. If parameter \((b+2k)\) is not positive, then the constant volatility formula 
\[ C^{\text{call}}(x, b + 2k, y) \] 
does not represent the value of a call option, and should be interpreted as an analytical continuation of a call price.

Generally, in order to compute moments, we would have to know the moment generating function of the average variance \(v\). The moment generating function for the process of Stein and Stein (1991) is derived in their paper and involves only elementary functions. The square root volatility process used by Heston (1993) also possesses well-known moments. The innovative paper of Hull and White (1987) gives the first moments for stochastic volatility that follow a geometrical Brownian motion.

The power dependence of volatility on the stock price in conjunction with the mean reverting volatility process of Stein and Stein (1991) or Heston (1993) reproduces general features of the typical volatility surfaces: the sharp “skew” or “smile” for short dated options and flattening of the volatility surface for longer dated options. The expansion assumes that volatility fluctuations are small in comparison with an average volatility. In contrast to that, the perturbation expansion of the SABR model requires not only volatility of volatility, but also volatility itself to be small and, therefore, the SABR model is restricted to small volatilities only.

7. TIME SCALING OF THE IMPLIED VOLATILITY SURFACE

Complete asymptotic expansions, such as the one that we derived for a level dependent volatility model, can be very useful in analyzing properties of option prices. In this section, we give an example for a level independent model that reveals an interesting decomposition of the option price.

Implied volatility of options differs across strike prices and terms to expiration. It was found empirically that, in order to capture regularities in the implied volatility and compare different underlying assets, it is very helpful to standardize implied volatility surfaces (Natenberg (1994), Tompkins (2001)). We demonstrate below how benefits and limitations of the standardization follow from a volatility expansion.

We can use our formula for the inverse Laplace transformation of option prices to compute analytically all terms of the variance expansion (Appendix):

\[
\frac{\partial^n C(V)}{\partial V^n} = \frac{(-1)^{n-1}}{2} S_0 N'(d_1) \sum_{k=0}^{n-1} \frac{(n-1)!}{8^k k!} V^{k-n+1/2} L_{n-k-1}^{-1/2} \left( \frac{g^2}{2V} \right)
\]

(44)

\[ g = \ln\left(\frac{S_0}{K}\right) + rt \]

(45)

where \(L\) is a generalized Laguerre polynomial, \(N'\) is a standard normal distribution function and \(d_1\) is a usual factor from the Black-Scholes formula. The very first term dominates the entire sum in the above formula. It happens because each subsequent term is multiplied by a very small factor \(\sigma^2 t / 8\). This factor is about one percent for a regular range of parameters. Therefore, a price of a call option can be naturally decomposed into two parts:

\[
C = C^S + C^V = C(\bar{V}) + \frac{1}{2} S_0 N'(d_1) \sum_{n=2}^{\infty} \mu_n \sum_{k=0}^{n-1} \frac{(-1)^{n-1}}{2n} L_{n-k-1}^{-1/2} \left( \frac{\sigma^2}{V} \right) (M^2)
\]

(46)

\[
C^S = \frac{S_0}{\sqrt{\sigma^2 t}} N'(d_1) \sqrt{\frac{\sigma^2 t}{8}} \sum_{n=2}^{\infty} \mu_n \frac{(-1)^{n-1}}{2n} L_{n-1}^{-1/2} \left( \frac{\sigma^2}{V} \right) (M^2)
\]

(47)

\[
C^V = S_0 N'(d_1) \sum_{n=2}^{\infty} \mu_n \frac{(-1)^{n-1}}{2n} \left( \frac{\sigma^2}{V} \right)^{k+1/2} L_{n-k-1}^{-1/2} \left( \frac{\sigma^2}{V} \right) (M^2)
\]

(48)

\[
M = \frac{-g}{\sqrt{2V}} \log(Ke^{-rt} / S_0)
\]

(49)

where the second part of the option price has only a minor contribution. The dominant part has one other remarkable property - it depends only on the rescaled strike \(M\) and does not have any other trivial time dependence. If we neglect the second part of the option price and if variance moments are not time dependent, then an implied volatility surface...
plotted as a function of the rescaled strike $M$ is completely time invariant. This conclusion follows from the fact that implied volatility can be found from a Taylor series:

$$C = C(V^{\text{imp}}) = C(V) + \sum_{n=1}^{\infty} \frac{(V^{\text{imp}} - \bar{V})^n}{n!} \frac{\partial^n C(V)}{\partial V^n}$$

(50)

and, therefore, implied volatility is a solution of the following equation:

$$\mu^{\text{imp}} = \sum_{n=2}^{\infty} \left( \mu_n - \mu^{\text{imp}}_n \right) \frac{(-1)^{n-1}}{n} L_{n-1}^{-1/2} (M^2)$$

(51)

where

$$\mu^{\text{imp}} = \frac{\sigma^{2}_{\text{imp}} - \sigma^2}{\sigma^2}$$

(52)

The above analysis suggests that one of the best ways of comparing shapes of implied volatility surfaces of different underlying securities is to plot the ratio $\sigma_{\text{imp}} / \sigma_{\text{atm}}$ as a function of the rescaled strike $M$. The at-the-money volatility $\sigma_{\text{atm}}$ can be used instead of unknown average volatility since they are usually very close. Any time scaling violation beyond the level that is consistent with the omission of the small $C^V$ part is a direct evidence of the changes in the distribution of the stochastic volatility or an indication that the volatility process depends on the asset price process. The dependence on the asset price can be estimated from the asymmetry of the volatility “smile” since our expansion depends only on $M^2$ and therefore strike symmetric.

8. DIVERGENCE OF THE HULL-WHITE VOLATILITY EXPANSION

In this section we demonstrate that Hull-White (1987) volatility expansion is divergent essentially for all stochastic volatility processes. Consider a Taylor expansion of a Black-Scholes (1973) price for European call with a strike $K$:

$$C(V) = S_0 \Phi(g / \sqrt{V} + \sqrt{V} / 2) - Ke^{-r} \Phi(g / \sqrt{V} - \sqrt{V} / 2)$$

(53)

$$C(V) = C(\bar{V}) + \sum_{n=1}^{\infty} (V - \bar{V})^n \frac{\partial^n C(\bar{V})}{\partial V^n}$$

(54)

where $\Phi$ is a normal cumulative distribution function. $C(V)$ can be considered as a function of the complex variable $V$. A radius of convergence of a Taylor expansion is determined by a location of a nearest singularity in a complex plain (Fisher (1999)). A Black-Scholes call price has a nearest singularity at $V=0$ which is due to the $\sqrt{V}$ factors. Therefore, the radius of convergence is equal to $\bar{V}$. As a result, this Taylor expansion converges for $0 < V < 2\bar{V}$ and diverges for all $V > 2\bar{V}$ where we restricted volatility $V$ to real values. We recheck divergence explicitly in the Appendix.

Consider a Hull-White expansion for options under stochastic volatility:

$$C = C(\bar{V}) + \int_{0}^{\infty} \left[ \sum_{n=2}^{\infty} \frac{(V - \bar{V})^n}{n!} \frac{\partial^n C(\bar{V})}{\partial V^n} \right] P(V) dV$$

(55)

where $P(V)$ is a probability of the integrated volatility $V$. An integral in the Hull-White expansion is by definition a Riemann sum:

$$\sum_{k} \left[ \sum_{n=2}^{\infty} \frac{(V_k - \bar{V})^n}{n!} \frac{\partial^n C(\bar{V})}{\partial V^n} \right] P(V_k) \Delta V_k$$

(56)
as \( \max \Delta V_k \to 0 \). Therefore, if probability distribution \( P \) is different from zero for some \( V_k > 2\tilde{V} \) then the corresponding Taylor sum in the square brackets will give a divergent contribution to the total Riemann sum. The Hull-White expansion is convergent only if integrated volatility distribution has an artificial cutoff that does not allow it to go above \( 2\tilde{V} \) (cutoff can be violated only for a zero measure set of points). The only reasonable example of a situation where such a cutoff exists naturally is when volatility is constant, but, in this case, any stochastic volatility expansion is unnecessary.

Despite the divergence, the volatility expansions can be very accurate at numerical computations of option prices if properly truncated. We can split an expansion into two parts, where the first part involves integration that does not go beyond the radius of convergence of the series:

\[
C = C(\tilde{V}) + \int_0^{2\tilde{V}} C(V)P(V)dV + \int_{2\tilde{V}}^{\infty} C(V)P(V)dV
\]

If expanded, the first part is convergent and gives an accurate representation of the option price, apart from a very small contribution from a volatility tail \( V > 2\tilde{V} \). In contrast to that, the expansion of the second part does not give a correct representation of the second integral. If volatility of volatility is small, then only high order volatility moments are sensitive to the tail of the volatility distribution. Therefore, the second part starts to contribute only later in the series and, at that moment, the Hull-White expansion series has to be truncated. An expansion that we derived for level dependent volatility has similar properties.

9. PADE APPROXIMANTS

The variance moment expansions discussed in this paper are divergent asymptotic expansions. They arise from power series taken beyond the radius of convergence. An approximation by rational functions, such as Pade approximants, can be a better alternative to a Taylor series in such circumstances. Pade approximants very often maintain accuracy far outside the radius of convergence of the series. Suppose that we are given a power series that represents some function:

\[
f(z) = \sum_{n=0}^{\infty} c_n z^n
\]

A Pade approximant is a ratio of two polynomials

\[
p_M^L(z) = \frac{a_0 + a_1 z + \ldots + a_L z^L}{b_0 + b_1 z + \ldots + b_M z^M}
\]

which has a Taylor expansion which agrees with \( f(z) \) as far as possible (Baker and Graves-Morris (1981)). For example, assume that volatility follows a geometrical Brownian motion without a drift (Hull and White (1987)) and expand volatility moments in terms of a small parameter \( \xi = \xi^2 t \) where \( \xi \) is volatility of volatility:

\[
C = C^{(0)} + \frac{(\sigma^2)^2}{2!} \left( \frac{\xi^2}{3} + \frac{\xi^4}{12} \right) C^{(2)} + \frac{(\sigma^2)^3}{3!} \left( \frac{2}{5} \xi^2 \right) C^{(3)}
\]

This second order expansion can be represented by a Pade approximant \( p_1 \):

\[
p_1 = \frac{a_0 + a_1 \xi}{1 + b_1 \xi}
\]

After solving for unknown coefficients \( a \) and \( b \), we find that
The Pade approximant should be very close to the initial series for small values of volatility of volatility and does not diverge as fast as the original series when volatility of volatility becomes high. Pade approximant $p_1^L$ is equivalent to Aitken’s $\Delta^2$ method of convergence acceleration (Aitken(1926), Baker and Graves-Morris(1981)). Aitken’s method accelerates convergence by finding out and summing the geometrically convergent part.

10. SUMMARY

The relationship between asset price and volatility has been a key topic in financial research. Since Black’s (1976) pioneering work, “leverage effects” have been well established for a broad range of asset classes. In this paper, we investigated a simple analytical model that has the ability to capture this asymmetric nature of volatility. We demonstrated that European options may be viewed as Laplace transforms of elementary functions and showed how this observation can be combined with mixing theorems to derive analytical option prices under stochastic volatility. This paper also examines the analytical properties of Hull-White’s stochastic volatility expansion and demonstrates that expansions of this type are divergent. We argued that such expansions may be used for numerical computations despite their divergence. We derived a stochastic variance expansion for a model with a power law asset price dependence of volatility and calculated contributions of all variance moments. Such complete expansions can be very useful, as we demonstrated by considering the time-scaling of volatility surfaces. Finally, we showed how an approximation by rational functions, such as Pade approximants, may be used as an alternative to truncated volatility power series.

11. APPENDIX

11.1. The inverse Laplace transformation of the Black-Scholes call and put prices

The Black-Scholes option prices are expressed via the normal cumulative distribution function $\Phi$:

$$C(V) = S_0 \Phi_+(V) - Ke^{-rt} \Phi_-(V)$$

$$\Phi_+(V) = \Phi(g / \sqrt{V} \pm \sqrt{V} / 2)$$

$$g = \ln(S_0 / K) + rt$$

where $C(V)$ is the price of a call option. First, we derive the inverse Laplace transformation of the normal cumulative distribution function $\Phi_\pm(V)$. We begin by considering its derivative:

$$\phi_\pm(V) = \frac{\partial \Phi_\pm(V)}{\partial V} = \frac{e^{\mp g/2-V/8}}{\sqrt{2\pi}} \left[ -\frac{ge^{-g^2/(2V)}}{2V^{3/2}} \pm \frac{e^{-g^2/(2V)}}{4\sqrt{V}} \right]$$

$$L^{-1}[\phi_\pm(V)] = \frac{e^{\mp g/2}}{4\pi \sqrt{2}} \frac{u_{i/8}(\omega)}{\sqrt{i/8}} \left[ 2g \sin(\sqrt{2g^2(\omega-1/8)} \pm \cos(\sqrt{2g^2(\omega-1/8)}) \right]$$

where $u_a(\omega)$ is a step function that equals one for $\omega \geq a$ and equals zero otherwise. The last equation is a consequence of the well-known identities (Schiff (1999)). The transformation of a function is related to a transformation of its derivative:
\[ \Phi_\pm (V) = c_\pm - \int_{V}^{\infty} \phi_\pm (V_1) dV_1 \quad (68) \]

\[
L^{-1} [\Phi_\pm (V)] = \delta_\pm (\omega) - \frac{L^{-1} [\phi_\pm (V)]}{\omega} = \\
= \delta_\pm (\omega) - \frac{e^{\frac{x}{g/2}}}{4\pi \sqrt{2}} \frac{u_{1/8}(\omega)}{\omega_{1/8}(\omega - 1/8)} \left\{ 2g \sin(\sqrt{2g^2(\omega-1/8) + \cos(\sqrt{2g^2(\omega-1/8)})}) \right\} \quad (69)
\]

where \( \delta_\pm (\omega) \) is equal to Dirac delta function for a plus sign and zero for a minus sign. The constant \( c_\pm \) is equal to one for a plus sign and a zero for a minus sign. The inverse Laplace transformation of the call and put prices follows directly from the last equation.

### 11.2. The inverse Laplace transformation of the default probability in the CEV model

\[
P_{\text{def}} (V) = \Gamma^{-1} (\lambda) \Gamma (\lambda, \alpha / V) = \Gamma^{-1} (\lambda) \int_{\frac{\alpha}{V}}^{\infty} e^{-x} x^{\lambda-1} dx \quad (70)
\]

It is easier to work with gamma function derivatives:

\[
P_{\text{def}} (V) = 1 - \int_{V}^{\infty} \phi(x) dx \quad (71)
\]

\[
\phi(V) = \frac{\partial P_{\text{def}} (V)}{\partial V} = \frac{\alpha^2 e^{-\alpha/V}}{\Gamma (\lambda) V^{\lambda+1}} \quad (72)
\]

This is a standard Laplace transformation, which is available in Abramowitz and Stegun (1974):

\[
L^{-1} [\phi(V)] = \frac{(\alpha \omega)^{\lambda/2} J_{\lambda} (2\sqrt{\alpha \omega})}{\Gamma (\lambda)} \quad (73)
\]

\[
L^{-1} [P_{\text{def}} (V)] = \delta (\omega) - \frac{L^{-1} [\phi]}{\omega} = \delta (\omega) - \frac{\alpha^{\lambda/2} \omega^{\lambda/2-1} J_{\lambda} (2\sqrt{\alpha \omega})}{\Gamma (\lambda)} \quad (74)
\]

### 11.3. The derivation of the variance expansion

We are interested in the derivative of the non-central chi-square function

\[
\frac{\partial^n \chi^2 (x, b, y)}{\partial V^n} \quad \text{when both x and y are inversely proportional to V.}
\]

The nth derivative of the composition \( h(V) = h(x(V), y(V)) \) is given by the multivariate Faa di Bruno formula (Constantine and Savits (1996)):

\[
\frac{\partial^n h(V)}{\partial V^n} = n! \sum_{1 \leq \lambda_1 + \lambda_2 \leq n} \frac{\partial^{\lambda_1+\lambda_2} h(x, y)}{\partial x^{\lambda_1} \partial y^{\lambda_2}} \sum_{p(n, \lambda_1, \lambda_2)} \prod_{j=1}^{n} \sum_{k_1,j} \frac{(x_j)^{k_{1,j}} (y_j)^{k_{2,j}}}{k_{1,j}! k_{2,j}!} \quad (75)
\]

where
\[ p(n, \lambda_1, \lambda_2) = \{(k_{1,1}, \ldots, k_{1,n}; k_{2,1}, \ldots, k_{2,n}) : k_{i,j} \geq 0, \sum_{j=1}^{n} k_{i,j} = \lambda_i, \sum_{j=1}^{n} j(k_{1,j} + k_{2,j}) = n\} \quad (76) \]

\[ \lambda_i \in \mathbb{N}_0; \quad i = (1,2) \quad (77) \]

with \( \mathbb{N}_0 \) the set of non-negative integers. The formula is valid for all \( n > 0 \).

\[ x_j = \frac{\partial^j x}{\partial V^j} = \frac{(-1)^j j! x}{V^j} \quad (78) \]

\[ y_j = \frac{\partial^j y}{\partial V^j} = \frac{(-1)^j j! y}{V^j} \quad (79) \]

\[ \frac{\partial^n h(V)}{\partial V^n} = \frac{(-1)^n n!}{V^n} \sum_{1 \leq \lambda_1 + \lambda_2 \leq n} \frac{\partial^{\lambda_1 + \lambda_2} h(x, y)}{\partial x^{\lambda_1} \partial y^{\lambda_2}} x^{\lambda_1} y^{\lambda_2} \sum_{p(n, \lambda_1, \lambda_2)} \prod_{j=1}^{n} \frac{1}{k_{1,j}!k_{2,j}!} \quad (80) \]

Using Corollary 3.3 of Constantine and Savits (1996), it is possible to show that

\[ \sum_{p(n, \lambda_1, \lambda_2)} \prod_{j=1}^{n} \frac{1}{k_{1,j}!k_{2,j}!} = \frac{|s^+(n, \lambda_1, \lambda_2)|}{\lambda_1! \lambda_2!} \quad (81) \]

\[ s^+(n, \lambda_1, \lambda_2) = \{(m_1, \ldots, m_{\lambda_1}; m_2, \ldots, m_{\lambda_2}) : m_j \in \mathbb{N}, \sum_{j=1}^{2} \lambda_j m_j = n\} \quad (82) \]

where \( \mathbb{N}_0 \) is the set of positive integers. \( |S| \) denotes the cardinality of the set \( S \):

\[ |s^+(n, \lambda_1, \lambda_2)| = C_{\lambda_1 + \lambda_2 - 1}^{n-1} \quad (83) \]

where \( C_b^a \) is a binomial coefficient. From the properties of the chi-square function:

\[ \frac{\partial^{\lambda_1 + \lambda_2} \chi^2(x, b, y)}{\partial x^{\lambda_1} \partial y^{\lambda_2}} = \sum_{k=-\lambda_1}^{\lambda_2} \frac{(-1)^{\lambda_1 + \lambda_2}}{2^{\lambda_1 + \lambda_2}} C_{\lambda_2-k}^{\lambda_1+\lambda_2} \chi^2(x, b + 2k, y) \quad (84) \]

Therefore,

\[ \frac{\partial^n \chi^2(x, b, y)}{\partial V^n} = n!V^{-n} \sum_{k=-n}^{n} P^n_k(x, y) \chi^2(x, b + 2k, y) \quad (85) \]

where polynomials \( P \) are defined in the main text of the paper. The first few polynomials are written below explicitly:

\[ P_1^1(x,y) = \frac{1}{2}\{x, -(x + y), y\} \quad (86) \]

\[ P_2^2(x,y) = \frac{1}{8}\{x^2, -2x(y + x - 2), x^2 + y^2 + 4xy - 4(x + y), -2y(x + y - 2), y^2\} \quad (87) \]

It can be shown that the polynomials satisfy the recurrence relation:

\[ 2(n + 1)P_{n+1}^n(x, y) = (x + y - 2n)P_n^n(x, y) - xP_{n+1}^n(x, y) - yP_{n-1}^n(x, y) - 2\frac{\partial P_n^n(x, y)}{\partial x} - 2y\frac{\partial P_n^n(x, y)}{\partial y} \quad (88) \]

11.4. The low frequency contribution to option prices
Sometimes, the integrals that are used to compute option prices involve functions unbounded at \( \omega = 0 \). The numerical calculations of the integrals can be sped up if the contribution of the small vicinity near zero is computed analytically.

\[
I^\varepsilon = \sqrt{S_0 K e^{-rT}} \int_0^\varepsilon L(\omega^2 \beta^2 / 2) J_\lambda(S_0^{-\beta} \omega) J_\lambda(K^{-\beta} e^{\nu \beta / \omega}) \frac{d\omega}{\omega} = \frac{S_0 K e^{-rT} (\varepsilon / 2)^{2\lambda}}{\Gamma^2(\lambda + 1)}
\]

\[
P^\varepsilon_{\text{def}} = \frac{\sqrt{S_0}}{\Gamma(\lambda) 2^{\lambda-1}} \int_0^\varepsilon L(\omega^2 \beta^2 / 2) J_\lambda(S_0^{-\beta} \omega) \omega^{-1} d\omega = \frac{S_0 (\varepsilon / 2)^{2\lambda}}{\Gamma^2(\lambda + 1)}
\]

We used the fact that the Laplace transformation of the volatility distribution is close to one when \( \omega \) approaches zero.

11.5. Black-Scholes variance derivatives

In this section we demonstrate how the inverse Laplace transformation method can be used to compute all partial derivatives of the Black-Scholes price with respect to the variance \( \nu \).

\[
\frac{\partial^n C(V)}{\partial \nu^n} = \int_0^\varepsilon (-\omega)^n e^{-\nu \omega} L^{-1}[C(V)] d\omega = (-1)^{n-1} \sqrt{S_0 K} \frac{e^{-\nu^2 / 2 - \nu^2 / 8}}{2 \pi \sqrt{g^2}} \int_0^{\infty} \left( \frac{x^2}{2 g^2} + \frac{1}{8} \right)^{\frac{n-1}{2}} \left( 1 - \frac{V x^2}{2 g^2} \right) \cos(x) dx
\]

The integral in the last formula is standard and can be found in Gradshteyn et al. (2000). The final answer can be expressed via Laguerre polynomials as shown in the main text.

11.6. Convergence of the implied volatility expansion

In this section we explicitly check that the implied volatility expansion

\[
C = C(V^{\text{imp}}) = C(\overline{V}) + \sum_{n=1}^{\infty} \frac{(V^{\text{imp}} - \overline{V})^n}{n!} \frac{\partial^n C(\overline{V})}{\partial \nu^n}
\]

is convergent if \( V^{\text{imp}} < 2 \overline{V} \) and is divergent if \( V^{\text{imp}} > 2 \overline{V} \). We assume that \( \overline{V} > 0 \).

\[
\frac{\partial^n C(\overline{V})}{\partial \nu^n} = \frac{(-1)^{n-1} n!}{2} S_0 N'(d_1) \sum_{k=0}^{n-1} \frac{\overline{V}}{n+k+1/2} L_{n-k-1}^{1/2}(M^2)
\]

The asymptotic expansion of the Laguerre polynomials:

\[
L_n^{1/2}(x) = \frac{1}{\sqrt{\pi n}} \exp(x / 2) \cos(2\sqrt{n}x) + O(n^{-1})
\]

Therefore, the Laguerre polynomial is a bounded function of \( n \) and there is a constant \( b(x) \) such that for \( n \geq 0 \):

\[
L_n^{1/2}(x) < b(x)
\]

\[
\left| \frac{\partial^n C(\overline{V})}{\partial \nu^n} \right| < \frac{(n-1)!}{2 \sqrt{V}^{n-1/2}} S_0 |N'(d_1)| b(M^2) \sum_{k=0}^{n-1} \frac{\overline{V}^k}{n+k+1/2} \frac{1}{2 \overline{V}^{n-k-1/2}} S_0 |N'(d_1)| b(M^2) e^{V/8}
\]
Therefore, the sufficient condition for convergence of the expansion is \( V^{\text{imp}} < 2\bar{V} \).

Assume that \( V^{\text{imp}} > 2\bar{V} \). For simplicity, we will demonstrate divergence of the at-the-money options only.

\[
\left| \frac{\partial^n C(\bar{V})}{\partial \bar{V}^n} \right| \geq \frac{(n-1)!}{2\bar{V}^{n-1/2}} S_0 |N'(d_1)| L_{n-1/2}^{-1/2}(0)
\]

There exist a constant \( a_i \) and an integer \( N \) such that for all integers \( n>N \):

\[
L_{n-1/2}^{-1/2}(0) = \frac{1}{\sqrt{n}} + O(n) > \frac{a_i}{\sqrt{n}}
\]

\[
\left| \frac{(\bar{V} - V^{\text{imp}})^n}{n!} \frac{\partial^n C(\bar{V})}{\partial \bar{V}^n} \right| > \frac{a_i}{2n^{3/2}} \left| \frac{(V^{\text{imp}} - \bar{V})^n}{\bar{V}^n} \right| S_0 |N'(d_1)| \sqrt{\bar{V}}
\]

This series is divergent if \( V^{\text{imp}} > 2\bar{V} \).

11.7. Time scaling of an implied volatility surface. Integral representation

If volatility is completely independent of the asset price, then the price of a European call option can be decomposed into two parts:

\[ C = \hat{C}^S + \hat{C}^V \]

\[ \hat{C}^S = C(\bar{V}) + \int_0^\infty \frac{\hat{L} (\omega^2 / 2) - e^{-\bar{V} \omega^2 / 2}}{\omega^2 / 2} \cos(\hat{M} \omega) d\omega \]

\[ \hat{C}^V = \left[ \int_0^\infty \frac{\hat{L} (\omega^2 / 2 + t / 8) - e^{-\bar{V} (\omega^2 / 2 + t / 8)}}{(\omega^2 / 2 + t / 8)} - \frac{\tilde{L} (\omega^2 / 2) - e^{-\bar{V} \omega^2 / 2}}{\omega^2 / 2} \right] \cos(\hat{M} \omega) d\omega \]

\[ \hat{M} = \ln(K e^{-\gamma t} / S_0) / \sqrt{t} \]

\[ \tilde{L} (\omega) = L(\omega / t) / t \]

where \( \hat{L} \) is a Laplace transform of the average variance \( \bar{V} \). The first part of the option price supports time scaling, while the second part violates it. The second part usually has only a minor contribution. The functions involved in the calculation of \( \hat{C}^V \) change considerably only if their arguments have a change of the order of \( 1/\sqrt{\bar{V}} \). The \( t/8 \) factor is small compared to this scale and, therefore, can be approximately omitted, which would result in the disappearance of the part that violates time scaling.

REFERENCES