Long Memory in Financial Time Series: Estimation of the Bivariate Multi-Fractal Model And Its Application For Value-at-Risk

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Abstract

Long memory (long-term dependence) seems to be as widespread in financial time series as in nature. Inspired by the long memory property, Multi-fractal processes have recently been introduced as a new tool for modeling the stylized facts in financial time series. In this paper, we attempt to construct a bivariate multi-fractal model, and implement its estimation via both GMM and likelihood approaches. For its empirical assessment, we apply the model on portfolio investment concerning VaR using time series of foreign exchange rates and bond maturity rates.

Keyword: Long memory, Bivariate Multifractal, GMM estimation, VaR.

JEL Classification: C20, G15

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1 Introduction

Traditionally, long memory (long-term dependence) has been specified in the time domain in terms of long lag autocorrelation, or in the frequency domains in terms of explosion of low frequency spectra. A large numbers of papers demonstrates the existence of long memory in financial economics. Baillie et al (1996) find long memory in the volatility of the DM-USD exchange rate; long term dependence in the German DAX is found by Lux (1996). Apart from the research on the developed financial markets, there are also dozens of papers on smaller and less developed markets; the stock market in Finland is analyzed by Tolvi. J (2003). Madhusoodanan (1998), provides evidence on the individual stocks in the Indian Stock Exchange and Golaka (2002) also gives
significant indication of long-term dependence for all time lags in Indian; Similar
evidence on the Greek financial market is given by Barkoulas and Baum (2000);
Cavalcante et al (2002) demonstrate long memory in Brazil stock market. In
addition, Chung (2002), and Zumbach (2002) also provide convincing evidence
in favor of long memory models.

Following statistical analysis like Hurst R/S test and modified R/S by Lo
(1991); as well as some econometric models such as ARFIMA (Fractional Inte-
grated Autoregressive Moving Average); FIGARCH (Fractional Integrated Gen-
eral Autoregressive Conditional Heteroscedasticity), the Multi-Fractal Model
(MF) has been recently introduced as a competitive alterative, which conceives
volatility as a hierarchical, multiplicative process with heterogeneous compo-
ents. The essential new feature of MF models is its ability of generating dif-
ferent degrees of long-term dependence in various powers of returns - a feature
colossively found in financial empirical data (Ding et al 1993). Research on
Multi-Fractal models originated from statistical physics (Mandelbrot, 1974).
Unfortunately, the models used in physics are of a combinatorial nature and
suffer from non-stationarity due to the limitation to a bounded interval and the
non-convergence of moments in the continuous-time limit. This major weakness
was overcome by introducing a iterative version of the multi-fractal model.

So far, available multi-fractal models are mostly univariate ones. However,
for many important questions in empirical literature, multi-variate settings are
preferable. In particular it is now well accepted that financial volatilities move
together over time across assets and markets. This is particularly important
when considering asset allocation, Value-at-Risk and Portfolio hedging strate-
gies. Secondly, the source of long memory in the volatility process is quite
limited, the Multivariate (bivariate) model may provide additional information
about the factors responsible for long memory.

The rest of this paper is organized as follows: Section 2 presents a review of
the Multi-fractal model in financial returns. Section 3 introduces the bivariate
multi-fractal model and three different estimation processes, for GMM, we show
the details of the analytical moments in the Appendix. Empirical works in
Section 4.

2 Review of Multifractal Model

Financial markets display some common properties with fluid turbulence. For
example, both turbulent fluctuations and financial fluctuations display intermitt-
ency at all scales. A cascade of energy flux is known to occur from the large
scale of injection to the small scale of dissipation. This cascade is typically
modeled by multiplicative cascade, which then leads to multi-fractal field.

Mandelbrot et al. (1997) are first introduced the Multi-Fractal Model, trans-
lating the approach of Mandelbrot (1974) from the pure physics area into fi-
nance. Fisher and Calvet (2004a) report advantages of Multi-Fractal model

\footnote{An exception is Calvet, et al (2004b) whose approach, however, differs from ours in various aspects.}
compared to GARCH and FIGARCH in various financial time series. Lux (2004b) provides related evidence on forecasting of future volatility generated from the Multi-Fractal model, the results demonstrating its potential advantage.

The first type of the MF proposed by Mandelbrot et al. (1997), named the Multi-Fractal Model of Assets Returns (MMAR), it assumes that returns $x(t)$ follow a compound process:

$$x(t) = B_H[\theta(t)]$$

in which an incremental fractional Brownian motion with index $H$, $B_H[·]$, is subordinate to the cumulative distribution function $\theta(t)$ of a multi-fractal measure, which was already employed by Mandelbrot (1974), when modeling the distribution of energy in turbulent dissipation.

The simplest way to create a multi-fractal measure is the “binomial multi-fractal”, constructed on a unit interval $[0,1]$ with uniform density. one proceeds as follows: Divide the interval into two of equal length. Let $m_0$ and $m_1$ be two positive numbers adding up to 1. In the step 1 that is $k = 1$, this interval is split into two equal subintervals, and the measure uniformly spreads mass equal to $m_0$ on the subinterval $[0; 0.5]$ and mass equal to $m_1$ on $[0.5; 1]$, in step 2, the set $[0; 0.5]$ is split into two subintervals, $[0; 0.25]$ and $[0.25; 0.5]$; which respectively receive a fraction measure $m_0$ and $m_1$ of the total mass $[0; 0.5]$; We apply the same procedure to the dyadic set $[0.5; 1]$, and the above procedure is then repeated ad infinitum, and iteration of this procedure generates an infinite sequence of measures.

As a minor extension of the original binomial measure one could simply dispense with the rule of always assigning $m_0$ to the left, and $m_1$ to the right, randomizing the assignment instead; or, one may uniformly split the interval into an arbitrary number $b$ larger than 2 at each stage of the cascade, and receive the fractions $m_0, m_1, \ldots, m_{b-1}$, which leads to a so-called multinomial measure. Furthermore, we can also randomize the allocations between the subintervals, taking $m_0, m_1, \ldots, m_{b-1}$ with certain probabilities, or using random numbers for $m_0$ instead of the same constant value, such as draws from a Lognormal distribution in Mandelbrot (1974, 1997).

The above mechanism is called combinatorial MF model, it is immediately obvious that one important limitation of this approach is the limited domain of any retaliations with an underlying cascade extending over $k$ steps, that is, we have exactly $2^k$ (Binary cascade) different subintervals at our disposal and, therefore, could generate only “time series” which are no longer than $2^k$. Later, this difficulty was overcome by the introduction of an iterative Markov-switching MF model in Calvet and Fisher (2001). Returns are modeled as:

$$x_t = \sigma \left( \prod_{i=1}^{k} M_t^{(i)} \right)^{1/2} \cdot u_t$$

with $u_t$ drawn from a standard Normal distribution $N(0, 1)$ and instantaneous volatility being determined by the product of $k$ volatility components or
multipliers $M_{1,t}, M_{2,t} \ldots, M_{k,t},$ \footnote{Additionally, $E[M_t]$ or $E[\sum M_t]$ equal to some arbitrary value are usually imposed for the sake of normalizing the time-varying components of volatility. Both Calvet, Fisher (2002) and Lux (2004) assume a Binomial distribution with parameters $m_0$ and $2-m_0$ (thus guaranteeing an expectation of unity for all $M_{i,t}$), and for Lognormal distribution $E[M] = 1.$} and a constant scale parameter $\sigma.$

Each volatility component is renewed at time $t$ with probability $\gamma_i$ depending on its rank within the hierarchy of multipliers or remains unchanged with probability $1 - \gamma_i.$ The transition probabilities are specified as:

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^k - 1)}$$

with parameters $\gamma_1 \in [0, 1]$ and $b \in (1, \infty).$ Estimation of this model, then, involves the parameters $\gamma_1$ and $b$ as well as those characterizing the distribution of the components $M_{i,t}.$

The main attraction of Multi-Fractal model is that it shares certain properties of asset returns: fat tails and asymptotic power-law behavior of the autocovariance function (long memory)\footnote{I.e. $\text{Cov}(|x_t|^q, |x_{t+\tau}|^q) \propto \tau^{2d(q)-1}$. However, one should note that the Markov-switching multi-fractal of eq. (2) only has “long memory over a hundred interval” Calvet and Fisher, 2001, for details.}. Furthermore, multifractality implies that different powers of the measure have different decay rates of their autocovariances. Calvet and Fisher (2002) show that this feature carries over to absolute moments of returns in the MMAR (eq.1). In this sense, other alternatives like FIGARCH or ARFIMA models belong to the catalogue of uni-fractal model, i.e. they have the same decay rate for all moments. As a rather new model in financial economics, there are various attempts at estimating the parameters of the multi-fractal model. Available options include the traditional Scaling estimator; GMM estimation by Lux (2003, 2004b) and Maximum Likelihood Estimation by Calvet and Fisher (2004a), which will be showed in the next section.

3 The Bivariant Multi-Fractal Model and its Estimation

We introduced a parsimonious Bivariate Multi-fractal model (BMF) under the hypothesis of two time series having certain amounts of joint cascade levels in both multi-fractal processes.

$$V_{q,t} = \left[ \left( \prod_{i=1}^{k} m_i^{(i)} \right) \cdot \left( \prod_{l=k+1}^{n} m_l^{(l)} \right) \right]^{1/2} \cdot u_t$$

$q = 1, 2$ refers to two time series, both having $n$ levels of their volatility cascade, and they share $k$ numbers of joint cascade levels which govern the strength of their volatility correlation. Obviously, the larger $k,$ the more correlation between them. After $k$ joint multiplications, each series has separates additional multifractal components, $u_t$ follows a standard normal distribution and both independent.
Furthermore, we restrict the specification of the transition probabilities as:

\[ \gamma_j = 2^{-(k' - j)} \]  

(5)

Each component is renewed at time \( t \) with probability \( \gamma_i \) depending on its rank within the hierarchy of multipliers and remains unchanged with probability \( 1 - \gamma_i \).

We specify the multipliers to be random draws from either a Binomial or Lognormal distribution, for the latter, we assume \(-log_2 M \sim N(\lambda, \sigma^2)\), and assign constraint \( E[M_i^{(i)}] = 0.5 \) which leads to \( \sigma^2_m = 2(\lambda - 1)/\ln 2 \), so that also in the Lognormal model we only have to estimate one parameter like in the binomial case in which we assume two draws \( m_0 \in (0, 2) \) and alternative \( m_1 = 2 - m_0 \).

Figure 1 and Figure 2 show the simulations of the bivariate multi-fractal model \((k = 4, n = 20)\) with Binomial distribution together with its ACFs. The simulation apparently shares some of the stylized facts of financial time series, namely volatility clustering and hyperbolical decay of the autocorrelation function. One also easily recognizes the correlation in the volatility of both time series.

### 3.1 Generalized Method of Moments Estimation

Historically, the first attempt at estimating the multi-fractal models is the scaling estimator. Since multifractal measures are characterized by a non-linear scaling function of moments (scaling law), through a Legendre transformation, parameter estimation is achieved by matching the empirical and hypothetical spectrum of Hölder exponents. In our proceeding bivariate MF model, we will, however, exclude the scaling estimator due to its bias and lack of asymptotic distribution theory, Lux (2003, 2004a).

Instead, we adopt a GMM (Generalized Method of Moments) by Hansen (1982) approach with analytical solutions of a set of appropriate moment conditions. In the GMM approach, the vector of parameter estimates of the model, say \( \beta \), can be obtained as:

\[ \hat{\beta} = \arg \min_{\beta \in \Theta} \bar{M}(\beta)'W\bar{M}(\beta) \]  

(6)

with \( \Theta \) the parameter vector, \( \bar{M}(\beta) \) the vector of differences between sample moments and analytical moments, and \( W \) a positive definite weighting matrix, which controls the over-identification when applying GMM. Implementing (6), one typically starts with identity matrix, then the inverse of the covariance matrix obtained from the first round estimation is used as the weighting matrix in the next step, and the procedure will continue until the estimates and weighting matrices converge. Under suitable conditions, \( \hat{\beta} \) is consistent and asymptotically converges to \( T^{1/2}(\hat{\beta} - \beta_0) \sim N(0, \Xi) \) with covariance matrix \( \Xi \).

The applicability of GMM for multi-fractal models has been discussed by Lux (2003). The approach recommended in this paper is using log differences of
absolute returns together with the pertinent analytical moment conditions, i.e. to transform the observed data \( V_t \) into \( T \)th differences of the log observations:

\[
X_{t,T} = \ln |V_t| - \ln |V_{t-T}|
\]

\[
= \left( 0.5 \sum_{i=1}^{k} \varepsilon_t^{(i)} + 0.5 \sum_{h=k+1}^{n} \varepsilon_t^{(h)} + \ln |u_t| \right) - \left( 0.5 \sum_{i=1}^{k} \varepsilon_{t-T}^{(i)} + 0.5 \sum_{h=k+1}^{n} \varepsilon_{t-T}^{(h)} + \ln |u_{t-T}| \right)
\]

\[
= 0.5 \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-T}^{(i)}) + 0.5 \sum_{h=k+1}^{n} (\varepsilon_t^{(h)} - \varepsilon_{t-T}^{(h)}) + (\ln |u_t| - \ln |u_{t-T}|)
\]

(7)

with \( \varepsilon_t^{(i)} = \ln \left( M_t^{(i)} \right) \), and in the same way to define the second time series, say \( Y_{t,T} \).

In order to exploit as much as possible information of the multi-fractal model, the moment conditions that we consider include two categories: the first set of conditions is obtained by considering some order of log-squared observations, the second set of moment conditions is derived from the absolute observations. In particular, we select moment conditions for the powers of \( X_{t,T} \) and \( Y_{t,T} \), i.e. moments of the raw observations and square observations:

\[
\text{Cov}[X_t^q, Y_t^q]; \text{Cov}[X_{t+1,t}^q, Y_{t+1,t}^q]; \text{Cov}[X_t^q, X_{t,T}^q]; \text{Cov}[Y_t^q, Y_{t,T}^q]
\]

for \( q = 1, 2 \) and \( T = 1, 5, 10, 20 \). It is straightforward to get the moments for the raw observations, but the moment calculations for the squared data seem a bit tedious. The detailed analytical moments are given in the Appendix.

We proceed by conducting several Monte Carlo experiments to explore the performance of the GMM estimation. Moment conditions for Binomial and Lognormal distribution can be found in Appendices A and B. We start with the Binomial Model (\( n = 20 \)) with certain number of joint multipliers, \( k = 4, 6, 8 \), and we choose multipliers from \( m_0 = 1.2 \) to \( 1.5 \) by \( 0.1 \) increment with sample sizes \( N_1 = 2000, N_2 = 5000, \) and \( N_3 = 10000 \). Table 1 shows the statistical result of the GMM estimator for the Binomial distribution: not only the Bias but also the FSSE (finite sample standard error) and root mean squared error show quite encouraging behavior, even in the small sample size \( N = 2000 \) and \( N = 5000 \), the average bias of the Monte Carlo estimates is moderate throughout and practically close to zero for the larger sample sizes \( N = 10000 \). It is also interesting to note that our estimates are in harmony with \( T^{\frac{3}{2}} \) consistency, and the Hansen’s \( J \) test reveals that there is almost absence of rejection of the over-identification restrictions for the numbers of moment conditions (only around 30 cases among 500 simulations are rejected), all these results can be viewed as a positive signal of the log transformation in practice. Furthermore, the very slight sensitivity of the estimates of \( m_0 \) with respect to the number of joint cascades might be viewed as a very welcome phenomenon as it implies that estimation of \( m_0 \) is hardly affected by the potential misspecification of joint cascade level \( k \).
Then, we turn to the Bivariate MF with continuous distribution \((-\log_2 M \sim N(\lambda, \sigma^2))\). Imposing constraint on \(\sigma\) we introduced in the last section, it reduces to one-parameter estimation again. In our Monte Carlo simulations reported in Table 2, we cover parameter values \(\lambda = 1.10, 1.20, 1.30\) and 1.40, and use the same numbers of joint multiplier cascade levels and the sample sizes as in the Binomial case above. As can be seen, results are not too different from those obtained with the Binomial model: Biases are moderate again, and results for \(\lambda\) are almost insensitive with respect to \(k\). Somewhat in contrast to the Binomial case, we notice a very slight deterioration of efficiency with increasing \(\lambda\), while might be due to increasing \(\lambda\) leading to increasing \(\sigma_m^2\) by their dependence (recall that \(\sigma_m^2 = 2(\lambda - 1)/\ln 2\)). All in all, the results from both the Binomial and Lognormal Monte Carlo simulation and estimation show that GMM seems to work quite well for multi-fractal process both in the discrete and in the continuous state space.

### 3.2 Maximum Likelihood Estimation

The MF dynamics can be interpreted as a special case of a Markov-switching process with a huge state states. This makes Maximum Likelihood Estimation feasible. In our parsimonious bivariate MF model, the state spaces is finite when the multipliers follow a discrete distribution (i.e. Binomial distribution). The likelihood function can be derived by determining the exact form of each possible component in the transition matrix, and is similar to the likelihood function developed for uni-variate process by Calvet et al (2004b), but differs in so far as the transition matrix of each multifractal component contains two starting cascade level:

$$f(V_1, \cdots, V_T; \Theta) = \prod_{t=1}^{T} f(V_t|V_1, \cdots, V_{t-1})$$ (8)

$$= \prod_{t=1}^{T} \left[ \sum_{i=1}^{2^n} P(M_t = m^i|V_1, \cdots, V_{t-1}) \cdot f(V_t|M_t = m^i) \right]$$

$$= \prod_{t=1}^{T} (\Omega_{t-1}A) \cdot f(V_t|M_t = m^i).$$

With transition matrix \(A\) which has components \(A_{ij}\) equals to

$$P(M_{t+1} = m^j|M_t = m^i)$$ (9)

$$= \prod_{k=1}^{n} \left[ (1 - \gamma_k) \cdot 1\{m^i_k = m^j_k\} + \gamma_k P(M_t = m^j_k) \right]$$

Both \(M_t\) and \(m^{(i)}\) are vectors, \(M_t = (M_t^1, \cdots, M_t^k, M_t^{k+1}, \cdots, M_t^n)\), \(m^i_k\) denotes the \(k\)th component of vector \(m^i\).
The density of the innovation $V_t$ conditional on $M_t$ is:

$$f(V_t | M_t = m^i) = \frac{F_N \left[ \frac{V_t}{\prod_{i=1}^{k} M_t^{(i)} \prod_{j=k+1}^{n} M_t^{(j)}} \right]}{\prod_{i=1}^{k} M_t^{(i)} \prod_{j=k+1}^{n} M_t^{(j)}}$$

$F_N[\cdot]$ denotes the Normal density function.

The last unknown component in the likelihood function above is $\Omega$, which refers to conditional probability which is defined by $\Omega_t^i = P(M_t = m^i | V_1, \ldots, V_t)$, and due to $\sum_{i=1}^{2^n} \Omega_t^i = 1$, by Bayesian updating, we get

$$\Omega_{t+1} = \frac{f(V_t | M_t = m^i) \otimes (\Omega_t \ast A)}{\sum f(V_t | M_t = m^i) \otimes (\Omega_t \ast A)}$$

(11)

Table 3 presents the results of Monte Carlo estimation for the same design as applied for GMM, and it shows that there is not much difference between these two previous tables (apart from the ML seeming a more efficient estimator due to its extraction all the information in the data). However, applicability of the ML approach is constrained by its computational demands: First, it is not applicable for models with an infinite state space, i.e. continuous distributions of the multipliers such as Lognormal distribution we use here. Secondly, for the discrete distributions, say the Binomial case, current computational limitations make choices of cascades with a number of steps $n$ beyond 10 unfeasible because of the implied evaluation of a $2^n \times 2^n$ transition matrix in each iteration.

### 4 Value at Risk

One widely used tool to measure, gear and control market risk is Value-at-Risk (VaR), which measures the worst loss over a specified target horizon with a given statistical confidence level. In other words, it represents a quantile of an estimated profit-loss distribution. Various organizations and interest groups have recommended VaR as a portfolio risk-measurement tool.

In this section, We will apply some empirical works on the application of the Bivariate MF model for Value at risk assessment. $\tilde{x}_{t,t+h}$ is defined as the forward-looking $h$-period return at time $t$: $\tilde{x}_{t,t+h} = \sum_{i=1}^{h} x_{t+i}$. VaR at the $h$-period horizon is defined as the $\alpha$ quantile of the conditional probability distribution of $\tilde{x}_{t,t+h}$:

$$Pr \left( \tilde{x}_{t,t+h} \leq VaR_{t,t+h}^\alpha | I_t \right) = \alpha.$$  

$\otimes$ represents element by element product.
The performance of our model is assessed by computing the failure rate for the returns and the portfolio. By definition, the failure rate is the number of times returns exceed (here in absolute value) the forecasted VaR. If the model is well specified, the failure rate is expected to be as close as possible to the prespecified VaR level.

In the empirical application we consider daily data for a collection of U.S. 1 Year and 2 Year Treasury Constant Maturity Bond Rate (USTR, June 1976 - Oct.2004), and two Foreign Exchange rates: British Pound to US Dollar (BP, March 1973 - Nov. 2004), Australian Dollar to US Dollar (AUD, March 1973 - Nov. 2004), where the numbers in parentheses are the start and end period for the sample at hand and the first symbol inside the parentheses designates the short notation for the Bonds and FXs. For all time series daily observations are denoted \( p_t \), returns are defined as \( x_t = \ln(p_t) - \ln(p_{t-1}) \).

Empirical results for the time series are given in tables 4 and 5. We estimate the bivariate Lognormal model by GMM and both Maximum Likelihood and GMM estimation are employed for the bivariate Binomial model. Then we simulate the bivariate time series to calculate the cumulative returns for single time series and portfolio\(^5\). \( \text{VaR}_{t,h}(p) \) is obtained as the \((1 - p)^{th}\) empirical quantile based on the estimators.

For Foreign Exchange rates \( BP \) and \( AUD \), VaR from both GMM and ML seem well specified, except with 4 cases of a conservative VaR for AUD in table 4, and 2 cases with too risky VaR in table 5. For US Bond maturity rates, we find that VaR forecasts through GMM (table 4) are also quite successful at confidence levels of 0.1 and 0.05, but at confidence level of 0.01 are again somewhat too conservative. On the contrary, in the same confidence level of 0.01 produces quite satisfactory results for ML in table 5, but gives rise to too risky VaR for the equal weighted portfolio, which is against the investment principle, of course, excessive conservativeness also does not imply superior risk management in financial investment.

5 Conclusion

In this paper we have developed a bivariate multi-fractal model extending the univariate Markov-switching multi-fractal model, and we implemented both GMM and Maximum Likelihood estimation. For GMM, eight moments have been employed by the log transformation of observations. Our Monte Carlo experiments indicate that there is not too much difference between both method although ML is expected to be more efficient, and there is no restriction on the choice of the number of cascade levels with GMM compared to a maximum of about 10 cascade levels in ML estimation. Furthermore, empirically speaking, GMM is faster compared to the very time-consuming ML process. In the last part of this paper, we applied the model to Value at Risk assessment with time

\(^5\)Although Bivariate Binomial MF has also been implemented by GMM, we don’t present the results in the table as there is not too much deviation from the GMM for the Lognormal.

5,000 simulations are conducted for each scenario.
series of two Foreign Exchange rates and US Bond maturity rates, the results demonstrate the applicability of the bivariate multi-fractal model.

References


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Appendix: Moment Conditions

Recall the model from Section 3. Let $\varepsilon_i^{(j)} = \ln(|M_i^{(j)}|)$, and we compute the first log difference:

$$X_{t,1} = \ln(|V_{1,t}|) - \ln(|V_{1,t-1}|)$$

$$= \left( \frac{1}{2} \sum_{i=1}^{k} \varepsilon_i^{(i)} + \frac{1}{2} \sum_{l=k+1}^{n} \varepsilon_t^{(l)} + \ln|u_t| \right) - \left( \frac{1}{2} \sum_{i=1}^{k} \varepsilon_{t-1}^{(i)} + \frac{1}{2} \sum_{l=k+1}^{n} \varepsilon_{t-1}^{(l)} + \ln|u_{t-1}| \right)$$

$$= \frac{1}{2} \sum_{i=1}^{k} \left( \varepsilon_i^{(i)} - \varepsilon_{t-1}^{(i)} \right) + \frac{1}{2} \sum_{l=k+1}^{n} \left( \varepsilon_t^{(l)} - \varepsilon_{t-1}^{(l)} \right) + \left( \ln|u_t| - \ln|u_{t-1}| \right)$$

$$Y_{t,1} = \ln(|V_{2,t}|) - \ln(|V_{2,t-1}|)$$

$$= \left( \frac{1}{2} \sum_{i=1}^{k} \varepsilon_i^{(i)} + \frac{1}{2} \sum_{h=k+1}^{n} \varepsilon_t^{(h)} + \ln|u_t| \right) - \left( \frac{1}{2} \sum_{i=1}^{k} \varepsilon_{t-1}^{(i)} + \frac{1}{2} \sum_{h=k+1}^{n} \varepsilon_{t-1}^{(h)} + \ln|u_{t-1}| \right)$$

$$= \frac{1}{2} \sum_{i=1}^{k} \left( \varepsilon_i^{(i)} - \varepsilon_{t-1}^{(i)} \right) + \frac{1}{2} \sum_{h=k+1}^{n} \left( \varepsilon_t^{(h)} - \varepsilon_{t-1}^{(h)} \right) + \left( \ln|u_t| - \ln|u_{t-1}| \right)$$

A Binomial case

$$\text{cov}[X_{t,1}, Y_{t,1}]$$

$$= E \left[ (X_{t,1} - E[X_{t,1}]) \cdot (Y_{t,1} - E[Y_{t,1}]) \right] = E[X_{t,1} \cdot Y_{t,1}]$$

$$= E \left\{ \left[ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{l=k+1}^{n} (\varepsilon_t^{(l)} - \varepsilon_{t-1}^{(l)}) + (\ln|u_t| - \ln|u_{t-1}|) \right] \cdot \left[ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{h=k+1}^{n} (\varepsilon_t^{(h)} - \varepsilon_{t-1}^{(h)}) + (\ln|u_t| - \ln|u_{t-1}|) \right] \right\}$$

$$= \frac{1}{4} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right]$$

We firstly consider $E[(\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)})^2]$, the only one non-zero contribution is $[\ln(m_0) - \ln(2 - m_0)]^2$, and it occurs when new draws take place in cascade level $i$ between $t$ and $t-1$, whose probability by definition is $\frac{1}{2} \frac{1}{2^{i-1}}$. Summing up we get:

$$\text{cov}[X_{t,1}, Y_{t,1}] = 0.25 \cdot [\ln(m_0) - \ln(2 - m_0)]^2 \cdot \sum_{i=1}^{k} \frac{1}{2} \frac{1}{2^{i-1}}$$
$$\text{cov}[X_{t+1,1}, Y_{t,1}]$$

$$= E \left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) + \frac{1}{2} \sum_{l=k+1}^{n} (\varepsilon_{t+1}^{(l)} - \varepsilon_{t}^{(l)}) + (ln|u_{t+1}| - ln|u_{t}|) \right\} \cdot$$

$$\left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{h=k+1}^{n} (\varepsilon_{t}^{(h)} - \varepsilon_{t-1}^{(h)}) + (ln|u_{t}| - ln|u_{t-1}|) \right\} \right)$$

$$= \frac{1}{4} E \left[ \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \cdot \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right].$$

For $$(\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)})(\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)})$$, the non-zero value only occurs in case of two changes of the multiplier from time $$t + 1$$ to time $$t - 1$$, the probability of this occurrence is $$\left(\frac{1}{2} \frac{1}{2\pi-i}\right)^2$$. So, we have the result:

$$\text{cov}[X_{t+1,1}, Y_{t,1}]$$

$$= 0.25 \cdot [2ln(m_0) \cdot ln(2 - m_0) - (ln(m_0))^2 - (ln(2 - m_0))^2] \cdot \sum_{i=1}^{k} \left(\frac{1}{2} \frac{1}{2\pi-i}\right)^2$$

Then, we look at the moment condition for one single time series:

$$\text{cov}[X_{t+1,1}, X_{t,1}]$$

$$= E \left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) + \frac{1}{2} \sum_{l=k+1}^{n} (\varepsilon_{t+1}^{(l)} - \varepsilon_{t}^{(l)}) + (ln|u_{t+1}| - ln|u_{t}|) \right\} \cdot$$

$$\left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{l=k+1}^{n} (\varepsilon_{t}^{(l)} - \varepsilon_{t-1}^{(l)}) + (ln|u_{t}| - ln|u_{t-1}|) \right\} \right)$$

$$= \frac{1}{4} E \left[ \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \cdot \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right] + \frac{1}{4} E \left[ \sum_{l=k+1}^{n} (\varepsilon_{t+1}^{(l)} - \varepsilon_{t}^{(l)}) \cdot \sum_{l=k+1}^{n} (\varepsilon_{t}^{(l)} - \varepsilon_{t-1}^{(l)}) \right]$$

$$+ E [ln|u_{t}|]^2 - E [ln|u_{t}|]^2.$$

$$(A3)$$

The first component is identical to the one of the case of $$\text{cov}[X_{t+1,1}, Y_{t,1}]$$, and the second component can be derived in the same way. Adding together we arrive at:

$$\text{cov}[X_{t+1,1}, X_{t,1}]$$

$$= 0.25 \cdot [2ln(m_0) \cdot ln(2 - m_0) - (ln(m_0))^2 - (ln(2 - m_0))^2] \cdot \sum_{i=1}^{k} \left(\frac{1}{2} \frac{1}{2\pi-i}\right)^2$$

$$+ 0.25 \cdot [2ln(m_0) \cdot ln(2 - m_0) - (ln(m_0))^2 - (ln(2 - m_0))^2] \cdot \sum_{i=k+1}^{n} \left(\frac{1}{2} \frac{1}{2\pi-i}\right)^2$$

$$+ E[ln|u_{t}|]^2 - E[ln|u_{t}|]^2.$$
By our assumption of both time series having the same number of cascade levels, the moments for the two individual time series are identical for the same length of time lags.

Then, let’s turn to the squared observations:

\[
E[X_{t,1}^2, Y_{t,1}^2]
\]

\[
= E \left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{i=k+1}^{n} \varepsilon_{t}^{(l)} + (ln|u_t| - ln|u_{t-1}|) \right\}^2 \cdot \left\{ \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) + \frac{1}{2} \sum_{h=k+1}^{n} \varepsilon_{t}^{(h)} + (ln|u_t| - ln|u_{t-1}|) \right\}^2
\]

\[
= \frac{1}{16} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^4 \right] + \frac{1}{16} E \left[ \left( \sum_{i=k+1}^{n} \varepsilon_{t}^{(l)} \right)^2 \left( \sum_{h=k+1}^{n} \varepsilon_{t}^{(h)} \right)^2 \right] + \frac{1}{16} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \left( \sum_{i=k+1}^{n} (\varepsilon_{t}^{(l)} - \varepsilon_{t-1}^{(l)}) \right)^2 \right] + \frac{1}{4} \left\{ 2E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right] \right\} \cdot (2E[ln|u_t|^2] - 2E[ln|u_t|^2])
\]

\]

By examining each component in the expression above combining with the calculations of the previous moments, it is not difficult to find the solution:

\[
E[X_{t,1}^2, Y_{t,1}^2]
\]

\[= [ln (m_0) - ln (2 - m_0)]^4 \cdot \frac{1}{16} \sum_{i=1}^{k} \frac{1}{2^{2i-1}} + [ln (m_0) - ln (2 - m_0)]^4 \cdot \frac{1}{16} \sum_{i=k+1}^{n} \frac{1}{2^{2i-1}} \sum_{i=k+1}^{n} \frac{1}{2^{2i-1}}
\]

\[+2 [ln (m_0) - ln (2 - m_0)]^4 \cdot \frac{1}{16} \sum_{i=1}^{k} \frac{1}{2^{2i-1}} \sum_{i=k+1}^{n} \frac{1}{2^{2i-1}}
\]

\[+ (E[ln|u_t|^2] - E[ln|u_t|^2]) \cdot [ln (m_0) - ln (2 - m_0)]^2 \cdot \left( \sum_{i=1}^{k} \frac{1}{2^{2i-1}} + \sum_{i=k+1}^{n} \frac{1}{2^{2i-1}} \right)
\]

\]

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\[ E[X_{t+1,1}, Y_{t,1}] \]

\[
= E \left[ \left( \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) + \frac{1}{2} \sum_{l=k+1}^{n} (\varepsilon_{t+1}^{(l)} - \varepsilon_{t}^{(l)}) + (\ln|u_{t+1}| - \ln|u_{t}|) \right) \right]^{2}.
\]

\[
= \frac{1}{16} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \right)^{2} \left( \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^{2} \right]
+ \frac{1}{16} E \left( \sum_{i=k+1}^{n} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \right)^{2},
\]

\[
+ \frac{1}{16} E \left( \sum_{i=k+1}^{n} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^{2}
+ \frac{1}{4} \left\{ 2E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \right)^{2} \right] + 2E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \right)^{2} \right] \right\} \cdot (2E[\ln|u_{t}|]^{2} - 2E[\ln|u_{t}|]^{2})
\]

\[ + 4E[\ln|u_{t}|]^{2} + 4E[\ln|u_{t}|]^{4} - 8E[\ln|u_{t}|]^{2}E[\ln|u_{t}|]^{2} \]

until now, the only unfamiliar component is the first term:

\[ E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \right)^{2} \cdot \left( \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^{2} \right], \]

there are three different forms to be distinguished:

1. \[ \left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)} \right)^{2} \left( \varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)} \right)^{2} \] which have non-zero value only if \( \varepsilon_{t+1}^{(i)} \neq \varepsilon_{t}^{(i)} \neq \varepsilon_{t-1}^{(i)} \) and this possibility is \( \left( \frac{1}{2} \right)^{2} \), combining with the non-zero expectation value,

we have \[ \left( \sum_{i=1}^{k} \left( \frac{1}{2} \right)^{2} \right) \cdot [\ln (m_{0}) - \ln (2 - m_{0})]^{4} \]

2. \[ \left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)} \right)^{2} \left( \varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)} \right)^{2} \] which are non-zero for \( i \neq j, \varepsilon_{t+1}^{(j)} \neq \varepsilon_{t}^{(j)} \) and \( \varepsilon_{t}^{(i)} \neq \varepsilon_{t-1}^{(i)} \), the probability of its occurrence is

\[ \left\{ \sum_{i=1}^{k} \left( \frac{1}{2} \right)^{2} \cdot \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2} \right)^{2} \right\} \]
finally arrive at:

$$\text{[ln}(m_0) - \text{ln}(2 - m_0)^4 \cdot \left( \sum_{i=1}^{k} \frac{1}{2^{n/2}} \sum_{j=1}^{k} \frac{1}{2^{n/2}} \right)$$

(3) form $\left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)} \right) \left( \varepsilon_{t+1}^{(j)} - \varepsilon_{t}^{(j)} \right) \left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t+1}^{(i-1)} \right)$ which for $i \neq j$ and $\varepsilon_{t+1}^{(n)} \neq \varepsilon_{t+1}^{(n)}$, $n = i, j$ are non-zero,

$$2 \left\{ \sum_{i=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \sum_{j=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \right\} \cdot [\text{ln}(m_0) - \text{ln}(2 - m_0)]^4.$$

Which implies the solution:

$$E \left[ \left( \sum_{i=1}^{k} \left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)} \right) \right)^2 \left( \sum_{i=1}^{k} \left( \varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)} \right) \right)^2 \right]$$

$$= [\text{ln}(m_0) - \text{ln}(2 - m_0)]^4 \left[ \sum_{i=1}^{k} \frac{1}{2^{n/2}} \sum_{j=1}^{k} \frac{1}{2^{n/2}} + 2 \sum_{i=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \sum_{j=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \right]$$

The other components can be solved by previous calculations, All in all, we finally arrive at:

$$E[X_{i+1,t}^2, Y_{i,t}^2]$$

$$= [\text{ln}(m_0) - \text{ln}(2 - m_0)]^4 \cdot \frac{1}{16} \left[ \sum_{i=1}^{k} \frac{1}{2^{n/2}} \sum_{j=1}^{k} \frac{1}{2^{n/2}} + 2 \sum_{i=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \sum_{j=1}^{k} \left( \frac{1}{2^{n/2}} \right)^2 \right] + [\text{ln}(m_0) - \text{ln}(2 - m_0)]^4 \cdot \frac{1}{16} \sum_{i=k+1}^{n} \frac{1}{2^{n/2}} \sum_{i=k+1}^{n} \frac{1}{2^{n/2}} + \frac{1}{8} [\text{ln}(m_0) - \text{ln}(2 - m_0)]^4 \sum_{i=1}^{k} \frac{1}{2^{n/2}} \sum_{i=k+1}^{n} \frac{1}{2^{n/2}}$$

$$+ \left(E[|u_t|^2] - E[|u_t|^2] \cdot [\text{ln}(m_0) - \text{ln}(2 - m_0)]^2 \cdot \left( \sum_{i=1}^{k} \frac{1}{2^{n/2}} + \sum_{i=k+1}^{n} \frac{1}{2^{n/2}} \right) + 4E[|u_t|^2]^2 + 4E[|u_t|^2]^4 \right)$$

(A6)
\[
E[X_{t+1,1}^2, X_{t,1}^2]
\]
\[
= E \left\{ \frac{1}{2} \sum_{i=1}^k (\varepsilon_{t+1}^{(i)} - \varepsilon_t^{(i)})^2 + \frac{1}{2} \sum_{l=k+1}^n (\varepsilon_{t+1}^{(l)} - \varepsilon_t^{(l)})^2 + (\ln|u_{1,t+1}| - \ln|u_{1,t}|) \right\}^2 
\]
\[
= \frac{1}{T^2} E \left[ \sum_{i=1}^k (\varepsilon_{t+1}^{(i)} - \varepsilon_t^{(i)})^2 \right] ^2 + \frac{1}{T^2} E \left[ \sum_{l=k+1}^n (\varepsilon_{t+1}^{(l)} - \varepsilon_t^{(l)})^2 \right] ^2 
\]
\[
+ \frac{1}{T} \left\{ 2E \left[ \sum_{i=1}^k (\varepsilon_{t+1}^{(i)} - \varepsilon_t^{(i)})^2 \right] \cdot (2E[|lnu_t|^2] - 2E[|lnu_t|^2]) \right\} 
\]
\[
+ \frac{1}{T} \cdot 4E \left[ \sum_{i=1}^k (\varepsilon_{t+1}^{(i)} - \varepsilon_t^{(i)}) \cdot \sum_{l=k+1}^n (\varepsilon_{t+1}^{(l)} - \varepsilon_t^{(l)}) \right] \cdot (E[|lnu_t|^2] - E[|lnu_t|^2]) 
\]
\[
+ \frac{1}{T} \cdot 4E \left[ \sum_{l=k+1}^n (\varepsilon_{t+1}^{(l)} - \varepsilon_t^{(l)}) \cdot \sum_{i=1}^k (\varepsilon_{t+1}^{(i)} - \varepsilon_t^{(i)}) \right] \cdot (E[|lnu_t|^2] - E[|lnu_t|^2]) 
\]
\[
+ 3E[|lnu_t|^2]^2 + E[|lnu_t|^4] - 4E[|lnu_t|^2] E[|lnu_t|]. \tag{A7}
\]

The first and second term are the same as the first one in the case \(E[X_{t+1,1}^2, Y_{t,1}^2]\), and the rest are our familiars. Adding together, we have the result:

\[17\]
B Lognormal case

\[ E[X_{t+1,1}^2, X_{t,1}^2] \]
\[
= [\ln (m_0) - \ln (2 - m_0)]^4 \cdot \frac{1}{16} \left[ \sum_{i=1}^{k} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 + 2 \sum_{i=1}^{k} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 \right]
\]
\[
+ [\ln (m_0) - \ln (2 - m_0)]^4 \cdot \frac{1}{16} \left[ \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 \right] + \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2
\]
\[
+ [\ln (m_0) - \ln (2 - m_0)]^4 \cdot \frac{1}{16} \left[ \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 \right] + \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2
\]
\[
+ \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 + \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 \right] \cdot (E[\ln |u_t|^2] - E[\ln |u_t|^2])
\]
\[
+ 0.25 \left[ 2\ln (m_0) \ln (2 - m_0) - (\ln (m_0))^2 - (\ln (2 - m_0))^2 \right] \cdot \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2 \cdot (E[\ln |u_t|^2] - E[\ln |u_t|^2])
\]
\[
+ 3E[\ln |u_t|^2] + E[\ln |u_t|^2] - 4E[\ln |u_t|^2]E[\ln |u_t|].
\]

(A8)

B Lognormal case

\[ \text{cov}[X_{t,1}, Y_{t,1}] = E[(X_{t,1} - E[X_{t,1}]) \cdot (Y_{t,1} - E[Y_{t,1}]) = E[X_{t,1} \cdot Y_{t,1}] \]
\[
= \frac{1}{2} E \left[ \left( \sum_{i=1}^{k} \xi _{(t)} - \xi _{(t-1)} \right)^2 \right] + E[\ln |u_{1,t}| \cdot \ln |u_{2,t}|]
\]
\[
= 0.5\sigma_x^2 \sum_{i=1}^{k} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2
\]

(B1)

Because the non-zero outcomes occur when \( \xi _{(t)} \neq \xi _{(t-1)} \), which implies:

\[
(\xi _{(t)} - \xi _{(t-1)})^2 = 2(E[\xi _{(t)}^2] - E[\xi _{(t)}^2]) = 2\sigma_x^2
\]

\[ \text{cov}[X_{t+1,1}, Y_{t,1}] = \frac{1}{4} E \left[ \sum_{i=1}^{k} \xi _{(t+1)} - \xi _{(t)} \cdot \sum_{i=1}^{k} \xi _{(t)} - \xi _{(t-1)} \right]
\]
\[
= -0.25\sigma_x^2 \sum_{i=1}^{k} \left( \frac{1}{2^{\frac{1}{2}} - 1} \right)^2
\]

(B2)
Because the non-zero outcomes occur when \( \varepsilon_{t+1}^{(i)} \neq \varepsilon_{t}^{(i)} \neq \varepsilon_{t-1}^{(i)} \), which implies:

\[
(\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \cdot (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) = E[\varepsilon_{t}^{(i)}]^2 - E[(\varepsilon_{t}^{(i)})^2] = -\sigma_{\varepsilon}^2
\]

\[
cov[X_{t+1}, X_t]
= \frac{1}{T} E \left[ \sum_{i=1}^{k} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \cdot \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right] + \frac{1}{T^2} E \left[ \sum_{i=k+1}^{n} (\varepsilon_{t+1}^{(i)} - \varepsilon_{t}^{(i)}) \cdot \sum_{i=k+1}^{n} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right]
+ E[\ln|u_t|]^2 - E[\ln|u_t|^2]
= -0.25\sigma_{\varepsilon}^2 \left[ \sum_{i=1}^{k} \left( \frac{1}{2\pi} \right)^2 \sum_{i=k+1}^{n} \left( \frac{1}{2\pi} \right)^2 \right] + E[\ln|u_t|]^2 - E[\ln|u_t|^2]
\]

\[
E[X_{t+1}, Y_{t}]
= \frac{1}{T} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^4 \right] + \frac{1}{T^2} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \left( \sum_{i=k+1}^{n} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right]
+ \frac{1}{T^2} E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \left( \sum_{i=k+1}^{n} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right]
+ \frac{1}{4} \left\{ 2E \left[ \left( \sum_{i=1}^{k} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right] + 2E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon_{t}^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^2 \right] \right\} \cdot (2E[\ln|u_t|^2] - 2E[\ln|u_t|^2])
+ 4E[\ln|u_t|^2]^2 + 4E[\ln|u_t|]^4 - 8E[\ln|u_t|^2]E[\ln|u_t|]^2
\]

\[
= 0.75\sigma_{\varepsilon}^4 \sum_{i=1}^{k} \left( \frac{1}{2\pi} \right)^4 + 0.25\sigma_{\varepsilon}^4 \sum_{i=k+1}^{n} \left( \frac{1}{2\pi} \right)^4 + 0.25\sigma_{\varepsilon}^4 \sum_{i=1}^{k} \left( \frac{1}{2\pi} \right)^4 + 0.25\sigma_{\varepsilon}^4 \sum_{i=k+1}^{n} \left( \frac{1}{2\pi} \right)^4
+ 2\sigma_{\varepsilon}^2 \left( E[\ln|u_t|^2] - E[\ln|u_t|^2] \right) \cdot \left( \sum_{i=1}^{k} \left( \frac{1}{2\pi} \right)^4 + \sum_{i=k+1}^{n} \left( \frac{1}{2\pi} \right)^4 \right)
+ 4E[\ln|u_t|^2]^2 + 4E[\ln|u_t|]^4 - 8E[\ln|u_t|^2]E[\ln|u_t|]^2
\]

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For the first term $E \left[ \left( \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^4 \right]$, let’s begin with $E \left[ (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)})^4 \right]$, the non-zero value implies:

$$E \left[ (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)})^4 \right] = 2E[\varepsilon_t^{(i)}]^4 + 6E[(\varepsilon_t^{(i)})^2]^2 - 8E[(\varepsilon_t^{(i)})^3]E[\varepsilon_t^{(i)}] = 12\sigma_t^4.$$

This occurs with probability $2^{1-}\sigma_t^4$. Then we have the solution:

$$E \left[ \left( \sum_{i=1}^{k} (\varepsilon_t^{(i)} - \varepsilon_{t-1}^{(i)}) \right)^4 \right] = 12\sigma_t^4 \cdot \sum_{i=1}^{k} \frac{1}{2^{k-i}}.$$
(2) \((ε^{(i)} - ε^{(j)} )^2 \), does not equal zero for \(i \neq j\), \(ε^{(j)}_{t+1} \neq ε^{(j)}_{t}\) and \(ε^{(i)}_{t} \neq ε^{(i)}_{t-1}\), since \(E \left( (ε^{(j)}_{t+1} - ε^{(j)}_{t} )^2 (ε^{(i)}_{t} - ε^{(i)}_{t-1} )^2 \right) = 4E[(ε^{(i)}_{t} )^2] - 8E[(ε^{(i)}_{t} )^2]E[ε^{(i)}_{t} ]^2 + 4E[ε^{(i)}_{t} ]^4 = 4σ^4\), together with the probability, this overall contribution yields:

\[
\left[ \sum_{j=1}^{k} \left( \frac{1}{2^{(t−j)}} \right) \right] · 4σ^4
\]

(3) \((ε^{(i)} - ε^{(j)} ) (ε^{(i)} - ε^{(i)} ) (ε^{(j)} - ε^{(j)} ) (ε^{(i)} - ε^{(i)} )\), which for \(i \neq j\) and \(ε^{(n)}_{t+1} \neq ε^{(n)}_{t-1}, n = i, j\) are non-zero, since \(E[(ε^{(i)}_{t+1} - ε^{(i)}_{t} ) (ε^{(i)}_{t} - ε^{(i)}_{t-1} ) (ε^{(j)}_{t+1} - ε^{(j)}_{t} ) (ε^{(j)}_{t} - ε^{(j)}_{t-1} ) = 4σ^4\), we obtain a contribution \(2 \left[ \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 \right] · σ^4\)

Combining those three cases, we have the result:

\[
E \left[ \left( \sum_{i=1}^{k} (ε^{(i)}_{t+1} - ε^{(i)}_{t}) \right) ^2 \left( \sum_{i=1}^{k} (ε^{(i)}_{t} - ε^{(i)}_{t-1}) \right) ^2 \right]
\]

\[
= 6σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 + 4σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^{(t−1)}} \right) + 2σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2
\]

\[
E[X^2_{t+1, j}, Y^2_{t, j}]
\]

\[
= \frac{1}{n} \left[ 6σ^4 \cdot n \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 + 4σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^{(t−1)}} \right) + 2σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^{(t−1)}} \right) ^2 \right]
\]

\[
+ 0.25σ^4 \cdot \sum_{i=k+1}^{n} \left( \frac{1}{2^{(t−1)}} \right) \sum_{h=k+1}^{n} \left( \frac{1}{2^{(t−1)}} \right)
\]

\[
+ 0.25σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) \sum_{h=k+1}^{n} \left( \frac{1}{2^{(t−1)}} \right) + 0.25σ^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) \sum_{l=k+1}^{n} \left( \frac{1}{2^{(t−1)}} \right)
\]

\[
+ 2σ^2 \cdot (E[ln|u_t|^2] - E[ln|u_t|^2]) \cdot \left( \sum_{i=1}^{k} \left( \frac{1}{2^{(t−1)}} \right) + \sum_{i=k+1}^{n} \left( \frac{1}{2^{(t−1)}} \right) \right)
\]

\[
\]

(B5)
\begin{equation}
E[X_{t+1,1}^2, X_{t,1}^2]
= \frac{1}{16} E \left[ \left( \sum_{i=1}^{k} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 + \left( \sum_{i=1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] + \frac{1}{16} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right]
\end{equation}

\begin{align*}
&\frac{1}{16} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] + \frac{1}{16} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right]
\end{align*}

\begin{align*}
&+ \frac{1}{4} \left\{ 2E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] + 2E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] \right\} \cdot (2E[ln|u_t|^2] - 2E[ln|u_t|^2])
\end{align*}

\begin{align*}
&+ 4 \cdot \frac{1}{16} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] \cdot (E[ln|u_t|^2] - E[ln|u_t|^2])
\end{align*}

\begin{align*}
&+ 4 \cdot \frac{1}{4} E \left[ \left( \sum_{i=k+1}^{n} (\varepsilon(i)_{t+1} - \varepsilon(i)_t) \right)^2 \right] \cdot (E[ln|u_t|^2] - E[ln|u_t|^2])
\end{align*}

\begin{align*}
&+ 3E[ln|u_t|^2]^2 + E[ln|u_t|^4] - 4E[ln|u_t|^3]E[ln|u_t]]
\end{align*}

\begin{align*}
&= \frac{1}{16} \left[ 6\sigma^4 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^n} \right)^2 + 4\sigma^4 \cdot \sum_{i=1}^{k} \frac{1}{2^n} \cdot \sum_{j=1, j \neq i}^{k} \frac{1}{2^n} \cdot \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^n} \right)^2 \right]
\end{align*}

\begin{align*}
&+ \frac{1}{16} \left[ 6\sigma^4 \cdot \sum_{i=1}^{k} \frac{1}{2^n} \cdot \sum_{j=1, j \neq i}^{k} \frac{1}{2^n} \cdot \sum_{j=1, j \neq i}^{k} \frac{1}{2^n} \cdot \sum_{j=1, j \neq i}^{k} \left( \frac{1}{2^n} \right)^2 \right]
\end{align*}

\begin{align*}
&+ 0.25\sigma^4 \sum_{i=1}^{k} \frac{1}{2^n} \cdot \sum_{i=1}^{k} \frac{1}{2^n} \cdot \sum_{i=1}^{k} \frac{1}{2^n} \cdot \sum_{i=1}^{k} \left( \frac{1}{2^n} \right)^2 
\end{align*}

\begin{align*}
&+ 2\sigma^2 \left( \sum_{i=1}^{k} \frac{1}{2^n} \right) \cdot \left( E[ln|u_t|^2] - E[ln|u_t|^2] \right)
\end{align*}

\begin{align*}
&+ 0.25\sigma^4 \sum_{i=1}^{k} \left( \frac{1}{2^n} \right)^2 \cdot \sum_{i=1}^{k} \left( \frac{1}{2^n} \right)^2
\end{align*}

\begin{align*}
&- \sigma^2 \cdot \left( \sum_{i=1}^{k} \frac{1}{2^n} \right)^2 \cdot \left( E[ln|u_t|^2] - E[ln|u_t|^2] \right)
\end{align*}

\begin{align*}
&+ 3E[ln|u_t|^2]^2 + E[ln|u_t|^4] - 4E[ln|u_t|^3]E[ln|u_t]]
\end{align*}

\text{(B6)}

Because the first term is identical with the first one of case \( E[X_{t+1,1}^2, Y_{t,1}^2] \).
Figure 1: Simulation of the Bivariate Binomial Multi-Fractal Model.
Figure 2: ACF for the Simulation of the Bivariate Binomial Multi-Fractal Model above.
Table 1: GMM Estimation of the Bivariate MF Binomial Model

<table>
<thead>
<tr>
<th></th>
<th>( k = 4 )</th>
<th>( k = 6 )</th>
<th>( k = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Bias )</td>
<td>( FSSE )</td>
<td>( RMSE )</td>
</tr>
<tr>
<td>( m_0 = 1.20 )</td>
<td>\n1 \n2 \n3</td>
<td>0.095 \n0.081 \n0.051</td>
<td>0.089 \n0.135 \n0.097</td>
</tr>
<tr>
<td>( m_0 = 1.30 )</td>
<td>\n1 \n2 \n3</td>
<td>0.082 \n0.043 \n0.019</td>
<td>0.101 \n0.092 \n0.069</td>
</tr>
<tr>
<td>( m_0 = 1.40 )</td>
<td>\n1 \n2 \n3</td>
<td>0.058 \n0.014 \n0.005</td>
<td>0.109 \n0.045 \n0.028</td>
</tr>
<tr>
<td>( m_0 = 1.50 )</td>
<td>\n1 \n2 \n3</td>
<td>0.032 \n0.008 \n0.004</td>
<td>0.057 \n0.027 \n0.018</td>
</tr>
</tbody>
</table>

Note: All simulations are based on the bivariate Multi-fractal process with the number of whole cascade levels equal to 20, and eight moment conditions as in the Appendix are used. Sample lengths are \( N_1 = 2,000 \), \( N_2 = 5,000 \) and \( N_3 = 10,000 \). Bias denotes the distance between the given and estimated parameter value (here we take absolute values), FSSE and RMSE denote the finite sample standard error and root mean squared error, respectively. For each scenario, 500 Monte Carlo simulations have been carried out.
### Table 2: GMM Estimation of the Bivariate MF Lognormal Model

<table>
<thead>
<tr>
<th>m₀ = 1.20</th>
<th>k = 4</th>
<th>Bias</th>
<th>FSSE</th>
<th>RMSE</th>
<th>k = 6</th>
<th>Bias</th>
<th>FSSE</th>
<th>RMSE</th>
<th>k = 8</th>
<th>Bias</th>
<th>FSSE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>N₁</td>
<td>0.020</td>
<td>0.042</td>
<td>0.047</td>
<td>0.022</td>
<td>0.041</td>
<td>0.046</td>
<td>0.021</td>
<td>0.040</td>
<td>0.045</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₂</td>
<td>0.007</td>
<td>0.026</td>
<td>0.027</td>
<td>0.007</td>
<td>0.025</td>
<td>0.027</td>
<td>0.007</td>
<td>0.026</td>
<td>0.028</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₃</td>
<td>0.003</td>
<td>0.018</td>
<td>0.017</td>
<td>0.002</td>
<td>0.018</td>
<td>0.017</td>
<td>0.002</td>
<td>0.018</td>
<td>0.018</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m₀ = 1.30</td>
<td>N₁</td>
<td>0.023</td>
<td>0.048</td>
<td>0.053</td>
<td>0.025</td>
<td>0.044</td>
<td>0.050</td>
<td>0.023</td>
<td>0.041</td>
<td>0.048</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₂</td>
<td>0.008</td>
<td>0.026</td>
<td>0.027</td>
<td>0.007</td>
<td>0.028</td>
<td>0.028</td>
<td>0.008</td>
<td>0.027</td>
<td>0.028</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₃</td>
<td>0.003</td>
<td>0.019</td>
<td>0.019</td>
<td>0.003</td>
<td>0.019</td>
<td>0.020</td>
<td>0.002</td>
<td>0.012</td>
<td>0.018</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m₀ = 1.40</td>
<td>N₁</td>
<td>0.029</td>
<td>0.051</td>
<td>0.059</td>
<td>0.027</td>
<td>0.048</td>
<td>0.054</td>
<td>0.025</td>
<td>0.048</td>
<td>0.054</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₂</td>
<td>0.009</td>
<td>0.029</td>
<td>0.031</td>
<td>0.008</td>
<td>0.029</td>
<td>0.030</td>
<td>0.007</td>
<td>0.028</td>
<td>0.030</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₃</td>
<td>0.002</td>
<td>0.020</td>
<td>0.021</td>
<td>0.003</td>
<td>0.020</td>
<td>0.021</td>
<td>0.002</td>
<td>0.020</td>
<td>0.020</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>m₀ = 1.50</td>
<td>N₁</td>
<td>0.030</td>
<td>0.055</td>
<td>0.056</td>
<td>0.030</td>
<td>0.053</td>
<td>0.061</td>
<td>0.027</td>
<td>0.052</td>
<td>0.060</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₂</td>
<td>0.008</td>
<td>0.032</td>
<td>0.033</td>
<td>0.007</td>
<td>0.031</td>
<td>0.031</td>
<td>0.009</td>
<td>0.031</td>
<td>0.033</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N₃</td>
<td>0.003</td>
<td>0.023</td>
<td>0.023</td>
<td>0.002</td>
<td>0.020</td>
<td>0.022</td>
<td>0.003</td>
<td>0.021</td>
<td>0.021</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: All simulations are based on the bivariate Multi-fractal process with the number of whole cascade levels equal to 20, and eight moment conditions as in the Appendix are used. Sample lengths are N₁ = 2,000, N₂ = 5,000 and N₃ = 10,000. Bias denotes the distance between the given and estimated parameter value (here we take absolute values), FSSE and RMSE denote the finite sample standard error and root mean squared error, respectively. For each scenario, 500 Monte Carlo simulations have been carried out.
Table 3: Maximum exact Likelihood Estimation of the Bivariate MF Binomial Model

<table>
<thead>
<tr>
<th></th>
<th>$k = 2$</th>
<th></th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_1$</td>
<td>$N_2$</td>
<td>$N_3$</td>
</tr>
<tr>
<td>$m_0 = 1.20$</td>
<td>Bias</td>
<td>FSSE</td>
<td>RMSE</td>
</tr>
<tr>
<td>$N_1$</td>
<td>0.0002</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0.0001</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$N_3$</td>
<td>0.0002</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$m_0 = 1.30$</td>
<td>Bias</td>
<td>FSSE</td>
<td>RMSE</td>
</tr>
<tr>
<td>$N_1$</td>
<td>0.0003</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0.0003</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$N_3$</td>
<td>0.0002</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>$m_0 = 1.40$</td>
<td>Bias</td>
<td>FSSE</td>
<td>RMSE</td>
</tr>
<tr>
<td>$N_1$</td>
<td>0.0004</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0.0003</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$N_3$</td>
<td>0.0002</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>$m_0 = 1.50$</td>
<td>Bias</td>
<td>FSSE</td>
<td>RMSE</td>
</tr>
<tr>
<td>$N_1$</td>
<td>0.0004</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>$N_2$</td>
<td>0</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$N_3$</td>
<td>0.0001</td>
<td>0.003</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Note: Simulations are based on the Bivariate Multi-Fractal process with the number of whole cascade levels 8, which is almost close to the limit of computational feasibility. Sample lengths are $N_1 = 2,000$, $N_2 = 5,000$ and $N_3 = 10,000$. Bias denotes the distance between the given and estimated parameter value (here we take absolute values), FSSE and RMSE denote the finite sample standard error and root mean squared error, respectively. For each scenario, 500 Monte Carlo simulations have been carried out.
Table 4: Failure rates for multi-period Value at Risk forecasts

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Two days horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BP</td>
<td>AUD</td>
<td>EW</td>
</tr>
<tr>
<td><strong>FXs</strong></td>
<td>p = 10%</td>
<td>0.093</td>
<td>0.080*</td>
</tr>
<tr>
<td></td>
<td>p = 5%</td>
<td>0.044</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>p = 1%</td>
<td>0.008</td>
<td>0.005*</td>
</tr>
<tr>
<td></td>
<td><strong>UST R1</strong></td>
<td><strong>UST R2</strong></td>
<td>EW</td>
</tr>
<tr>
<td><strong>Bonds</strong></td>
<td>p = 10%</td>
<td>0.104</td>
<td>0.117+</td>
</tr>
<tr>
<td></td>
<td>p = 5%</td>
<td>0.042</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>p = 1%</td>
<td>0.004*</td>
<td>0.006</td>
</tr>
</tbody>
</table>

Note: This table shows the failure rate (ratio of the observations above the VaR). FXs are Foreign Exchange rate of British Pound (BP) and Australian Dollar (AUD) to US Dollar, Bonds are the US 1-Year and 2-Year Treasury Constant Maturity Rate (UST R1, USTR2 respectively). Half of data for GMM estimation, which is based on the BMF Lognormal model using eight moment conditions as in the Appendix, half out-sample data for VaR forecasting and failure rate calculation. EW denotes Equal-Weight portfolio. + and * denote too risky and too conservative VaR, respectively.
Table 5: Failure rates for multi-period Value at Risk forecasts

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Two days horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BP</td>
<td>AUD</td>
<td>EW</td>
</tr>
<tr>
<td><strong>FXs</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p = 10%</td>
<td>0.114</td>
<td>0.110</td>
<td>0.121+</td>
</tr>
<tr>
<td>p = 5%</td>
<td>0.058</td>
<td>0.051</td>
<td>0.054</td>
</tr>
<tr>
<td>p = 1%</td>
<td>0.008</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td><strong>Bonds</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p = 10%</td>
<td>0.089</td>
<td>0.095</td>
<td>0.113</td>
</tr>
<tr>
<td>p = 5%</td>
<td>0.045</td>
<td>0.053</td>
<td>0.062</td>
</tr>
<tr>
<td>p = 1%</td>
<td>0.010</td>
<td>0.012</td>
<td>0.018*</td>
</tr>
</tbody>
</table>

Note: This table shows the failure rate (ratio of the observations above the VaR). FXs are Foreign Exchange rate of British Pound (BP) and Australian Dollar (AUD) to US Dollar, Bonds are the US 1-Year and 2-Year Treasury Constant Maturity Rate (USTR1, USTR2 respectively). Half of data for ML estimation, which is based on the BMF binomial model, half out-sample data for VaR forecasting and failure rate calculation. EW denotes Equal-Weight portfolio. + and * denote too risky and too conservative VaR, respectively.