Iterated weak dominance and subgame dominance

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Abstract

In this paper, we consider finite normal form games satisfying transference of decisionmaker indifference. We show that any set of strategies surviving $k$ rounds of elimination of some weakly dominated strategies can be reduced to a set of strategies equivalent to the set of strategies surviving $k$ rounds of elimination of all weakly dominated strategies in every round by (at most $k$) further rounds of elimination of weakly dominated strategies. The result develops work by Gretlein [Gretlein, R., 1983. Dominance elimination procedures on finite alternative games. International Journal of Game Theory 12, 107–113]. We then consider applications and demonstrate how we may obtain a unified approach to the work by Gretlein and recent results by Ewerhart [Ewerhart, C., 2002. Iterated weak dominance in strictly competitive games of perfect information. Journal of Economic Theory 107, 474-482] and Marx and Swinkels [Marx, L.M., Swinkels, J.M., 1997. Order independence for iterated weak dominance. Games and Economic Behavior 18, 219-245].

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1. Introduction

For the class of normal form games where a finite number of players have strict preferences over a finite set of outcomes (utility vectors), Gretlein (1983) showed that the set
of outcomes resulting from iterative elimination of some weakly dominated strategies contains the set of outcomes that remains from a procedure removing all weakly dominated strategies in every step until no more strategies can be removed.

In this paper, we extend this result and show that any set of strategies surviving $k$ rounds of elimination of some weakly dominated strategies can be reduced to a set of strategies equivalent to the strategies surviving $k$ rounds of elimination of all weakly dominated strategies in every round. The reduction can be carried out by at most $k$ further rounds of elimination of some weakly dominated strategies (Theorem 1). From this we obtain a strengthened version of Gretlein’s result (Corollary 1). Moreover, the result extends to the class of games satisfying transference of decisionmaker indifference which is less restrictive than strict preferences over outcomes. This conditions says that, for given strategies, if a player obtains the same payoff from shifting to a new strategy, then all other players remain unaffected as well.

We then consider applications and demonstrate how we may obtain a unified approach to the above-mentioned result by Gretlein and recent work by Ewerhart (2000a, 2002) and Marx and Swinkels (1997).

Our first application is to two-player strictly competitive finite games of perfect information. Recently, Ewerhart (2000a) demonstrated that any chess-like game (a strictly competitive, finite game of perfect information with three outcomes) is solved by two rounds of elimination of all weakly dominated strategies. Moreover, he conjectured that the following generalization is true: Any finite, strictly competitive game of perfect information with at most $n$ outcomes is dominance solvable by $n - 1$ rounds of elimination of all weakly dominated strategies. A proof of this conjecture has now been provided by Ewerhart (2002). The proof is complicated by the fact that for an extensive form game of perfect information, after one round of elimination of all weakly dominated strategies, the surviving strategies do not necessarily represent the strategic form of any residual extensive form game. In other words, the procedure eliminating all weakly dominated strategies in every step does not correspond to any procedure removing ‘dominated’ branches from the game tree. Another difficulty is that the procedure eliminating all weakly dominated strategies in every step does not necessarily remove the largest number of strategies from the second round and onwards, compared to other less ‘greedy’ elimination procedures.

The present paper considers iterative elimination of all weakly subgame dominated strategies, a procedure that intuitively can be viewed as the removal of certain ‘dominated’ branches in every step. More precisely, after every step, the remaining strategy set is, up to some redundant strategies, the strategy set of a residual extensive form game where what we call ‘weakly dominated subgames’ have been removed. We claim that any finite, strictly competitive game of perfect information with at most $n$ outcomes is dominance solvable by $n - 1$ rounds of elimination of weakly subgame dominated strategies (Theorem 2), and give a short and intuitive proof. By combining the results (Theorems 1 and 2), we also obtain the result by Ewerhart (2002) (Corollary 2).

1 An earlier version of Ewerhart’s proof was reported in Ewerhart (2000b). A proof has also been reported in independent work by Shimoji (2001).

2 Battigalli (1997) provides an example.
Our second application is to work by Marx and Swinkels (1997). For the class of finite normal form games satisfying the transference of decisionmaker indifference condition, Marx and Swinkels show that regardless of the order in which weakly dominated strategies are removed, any two full reductions are the same up to removal of redundant strategies and renaming of strategies. We round off by formulating this as a corollary of our main result (Corollary 3).

2. Preliminaries

We consider a finite set of players $I = \{1, 2, \ldots, m\}$. Let $S_i = \{s_i, s'_i, \ldots\}$ be a finite strategy set for player $i$. A game in normal form $N(S, u)$ consists of strategies $S = S_1 \times \cdots \times S_m$ and utility functions $u = (u_1, \ldots, u_m)$. We write $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_m$. Let $s = (s_1, \ldots, s_m) \in S$ and $u(s) = (u_1(s), \ldots, u_m(s))$.

Gretlein (1983) studied the class of games satisfying $u_i(s) = u_i(s') \Rightarrow u(s) = u(s')$ for all $i \in I$, $s, s' \in S$, formulated as strict preferences over outcomes. Throughout this paper we relax this condition and assume that $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \Rightarrow u(s_i, s_{-i}) = u(s_i', s_{-i})$ for all $i \in I$, $s_i, s_i' \in S_i, s_{-i} \in S_{-i}$. This condition has been referred to as transference of decisionmaker indifference (TDI), see Marx and Swinkels (1997) for a discussion.

A strategy $s_i \in S_i$ is weakly dominated by $s_i' \in S_i$ on $S_{-i}$ (or $S$) if $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ with at least one strict inequality. Without the latter requirement we say that $s_i'$ is at least as good as $s_i$. Let $F^1(S) = S$, and for $k \geq 1$ define $F^k(S) = F^k(S) \times \cdots \times F^k(S)$ recursively such that $F^k(S)$ is the set of strategies remaining in $F^{k-1}(S)$ after all weakly dominated strategies w.r.t. $F^{k-1}(S)$ have been removed. A set of strategies $L^k(S)$ is obtained from $S$ by iteratively eliminating some weakly dominated strategies in $k$ rounds if there is a sequence $\{L^k(S)\}_{k=0}^h$ such that $L^0(S) = S$, and for $1 \leq h \leq k$, $L^k(S) = L^1(S) \times \cdots \times L^h(S)$ is a set of strategies where for every $i$ if $s_i \notin L^h_i$ and $s_j \in L^{h-1}_i$ then $s_j$ is weakly dominated w.r.t. $L^{h-1}$. We write $L^kS = L^k(S)$ and $LS = L^1(S)$.

A game $N(S, u)$ is dominance solvable in $k$ steps if the utility functions are constant on $F^k(S)$.

We say that strategies $s_i$ and $s_i'$ are equivalent w.r.t. $S_{-i}$ if $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$. If $s_i$ and $s_i'$ are equivalent then $s_i$ is redundant to $s_i'$ and vice versa.

Let $N$ and $\hat{N}$ be normal form games with $m$ players and respective strategy sets $S$ and $\hat{S}$ and utility functions $u_i$ and $\hat{u}_i$, $i = 1, \ldots, m$. Then $\hat{N}$ is a reduction of $N$ if there exist surjective maps $f_i : S_i \rightarrow \hat{S}_i, i = 1, \ldots, m$, such that $\hat{u}_i(f_1(s_1), \ldots, f_m(s_m)) = u_i(s)$ for all $i$ and $s$. We say that $N$ and $\hat{N}$ are equivalent games, written $N \sim \hat{N}$, if they have a common reduction. In words, two games are equivalent if they are identical up to removal of redundant strategies and renaming.

3 In this paper we restrict attention to finite pure strategy spaces.

4 This terminology is used by Ewerhart (2000b).
We collect some useful observations below.

**Lemma A.**

1. If $s_i \in L_i^k(S)$ and $s_i \notin L_i^{k+1}(S)$ then there is $s_i' \in L_i^{k+1}(S)$ such that $s_i'$ weakly dominates $s_i$ on $L_i^{k+1}(S)$.
2. If $s_i \in S_i$ and $k > 0$ then there is $s_i' \in L_i^k(S)$ such that $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in L_{-i}^k(S)$.
3. The relation $\sim$ is an equivalence relation on the class of finite normal form games, and if $N(S, u) \sim N(\bar{S}, \bar{u})$ then $N(F^k(S), u|_{F^k(S)}) \sim N(F^k(\bar{S}), \bar{u}|_{F^k(\bar{S})})$, $k > 0$.

**Proof.** For A.1, see Gretlein (1983). For A.2, Ewerhart (2000b, 2002) provides a proof for the elimination procedure $\{F^k(S)\}_{k \geq 0}$ in the two-player case, but the proof applies to any elimination procedure $\{L^k(S)\}_{k \geq 0}$ in the $m$-player case by a change of notation. For A.3, see Ewerhart (2000b) for a proof in the two-player case for $k = 1$ that easily generalizes to the $m$-player case and for any $k$. □

3. Iterated weak dominance

In the following, consider a normal form game $N(S, u)$ satisfying TDI. If $R_i \subseteq T_i$ for all $i$ we say that $R$ is in $T$. If $R$ is in $S$, $R$ induces a game $N(R, u|R)$ and we occasionally refer to $R$ as a game. For $R$ and $T$ in $S$, we write $R \sim T$ if $N(R, u|R) \sim N(T, u|_T)$. We can now formulate the main result of this section.

**Theorem 1.** For any $k$ and $L^k(S)$ there is $T \sim F^k(S)$, where $T$ is obtained from $L^k(S)$ by at most $k$ rounds of elimination of some weakly dominated strategies.

Gretlein (1983) shows (in his Theorem 2) that

$$\lim_{k \to \infty} u(F^k(S)) \leq \lim_{h \to \infty} u(L^h(S)).$$

By Theorem 1 above, we obtain the following strengthened version.

**Corollary 1.** $u(F^k(S)) \leq u(L^k(S))$ for all $k > 0$.

In order to prove Theorem 1, we state some intermediate results. Following Gretlein (1983), we write $RaT$ if (i) $R$ is in $T$, and (ii) for all $i$ and all $t_i \in T_i$ there is $r_i \in R_i$ such that $r_i$ is at least as good as $t_i$ on $T$. Moreover, $R\beta T$ if (i) $R$ is in $T$, and (ii) for all $i$ and all $t_i \in T_i$ there is $r_i \in R_i$ such that $r_i$ is equivalent to $t_i$ on $T$.

**Lemma 1.** Let $RaT$. Then for any $k \geq 0$ there are strategies $Q[k]$, such that $F^kR \sim Q[k]aF^kT$.

**Proof.** Define $R^k$ and $T^k$ recursively: $R^0 \equiv R$ and $T^0 \equiv T$. $R^{k+1} = FR^k \cap FT^k$ and $T^{k+1} = R^k \cap FT^k$. We show that if $RaT$ then $F^kR \sim R^k a F^k T$ for all $k \geq 0$, i.e. the claim holds with $Q[k] \equiv R^k$. □
Step 1. First we recall some useful results developed by Gretlein within the proofs of his Lemma 4 and Theorem 1.

1.1 If $R \beta T$ then $F_k R \beta F_k T$ for all $k$.
1.2 If $R \alpha T$ then $R_k \alpha T_k$ for all $k$.
1.3 $R_k^{k+1} \beta F_k$ and $T^{k+1}_k \beta F_k$ for all $k$.

Gretlein considers games with strict preferences over outcomes, but his results are valid for games satisfying TDI.\footnote{In fact, Gretlein uses only the TDI property implied by strict preferences over outcomes in his proofs.} Gretlein obtains 1.1, 1.2, and 1.3 in Gretlein (1983) (p. 112, at lines 3, 15, and 16, respectively).

Step 2. Let $R$, $T$ and $P$ be in $\mathcal{S}$. We claim that

2.1 If $R \beta T$ then $R \sim T$.
2.2 If $R$ is in $T$ and $R \sim T$ then $R \beta T$.
2.3 If $R \beta T \beta P$ then $R \beta P$.
2.4 If $R \alpha T \beta P$ then $R \alpha P$.

A verification of Steps 2.1 and 2.2 is left to the reader.

For Step 2.3, note that since $R$ is in $T$ and $T$ is in $P$, $R$ is in $P$. Moreover, since $R \sim T$ and $T \sim P$ then by Lemma A.3 $R \sim P$ implying by Step 2.2 that $R \beta P$.

For Step 4.4, since $T \beta P$ for any $p_i \in P$, there is $t_i \in T_i$ equivalent to $p_i$ on $P_{-i}$. Moreover, by $RaT$, there is $r_i \in R_i$ at least as good as $t_i$ on $T_{-i}$. Now, assume that $r_i$ is not at least as good as $p_i$ on $P_{-i} \setminus T_{-i}$. Let $p_{-i} \in P_{-i} \setminus T_{-i}$ be such that $u_i(r_i, p_{-i}) < u_i(t_i, p_{-i}) = u_i(p_i, p_{-i})$. Moreover, for all $j \neq i$ let $l_j \in T_j$ be a strategy equivalent to $p_j$ w.r.t. $P$, $T_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_m) \in T_{-i}$. Then $u_i(r_i, l_{-i}) = u_i(r_i, p_{-i}) < u_i(t_i, p_{-i}) = u_i(t_i, l_{-i})$, a contradiction.

Step 3. We claim that $R^k \beta F^k R$ and $T^k \beta F^k T$ for all $k \geq 0$.

We prove the first part of the claim by induction on $k$. The claim is trivial for $k = 0$. Assume that $R^k \beta F^k R$ for some $k \geq 0$. Then by Step 1.1 we have that $F_{k+1} \beta F_{k+1} R$. By Step 1.3 $R^{k+1} \beta F R^{k+1}$. Thus, by Step 2.3 we have $R^k \beta F^k R$. The proof of the second part is similar.

Step 4. By Step 1.2 $R^k \alpha T^k$ and by Step 3 $T^k \beta F^k T$ thus by Step 2.4 we have $R^k \alpha F^k T$. Hence, by Step 3 and Step 2.1, $F^k R \sim R^k \alpha F^k T$.

Lemma 2. Let $\bar{R}$ be obtained from $R$ by one round of elimination of some weakly dominated strategies, and let $\bar{T} \sim \bar{R}$. Then if $P \sim R$ there is $\bar{P}$ obtained from $P$ by one round of elimination of some weakly dominated strategies such that $\bar{P} \sim \bar{T}$.
Proof. By Lemma A.3 if $R \sim P$ and $R \sim T$ then $T \sim P$. Thus, it is sufficient to show that there is $P$ obtained from $P$ by one round of elimination of some weakly dominated strategies such that $R \sim P$.

For this, let $N(\tilde{S}, \tilde{u})$ be a common reduction of $R$ and $P$, and note that $\tilde{S}$ is equivalent to $R$ and $P$.

Let $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$ be surjective maps from $R$ and $P$, respectively to $\tilde{S}$ that gives a common reduction. Let $S_i = f_i(\tilde{R}_i)$ for all $i$, i.e. $S_i$ is the subset of strategies in $\tilde{S}_i$ which are the image of $R_i$. Thus, $S = S_1 \times \cdots \times S_m$.

By Lemma A.3 if $\bar{R} \equiv \bar{g}^{-1}(\tilde{S}_i)$ for all $i$. Then $R \sim \tilde{S}_i$ since $\tilde{S}$ is a reduction of $\bar{R}$ via surjective maps $f_i|_{\bar{R}_i}$, $i = 1, \ldots, m$, and since $\bar{u}_i(f_i(s_1), \ldots, s_m(s_m)) = u_i(s)$ for all $i$ and $s \in R$ implies $\bar{u}_i|_{\bar{g}}(f_i(s_1), \ldots, s_m(s_m)) = \bar{u}_i|_{\bar{g}}(s)$ for all $i$ and $s \in \bar{R}$.

Now, let $\bar{P}_i \equiv \bar{g}_i^{-1}(\tilde{S}_i)$ for all $i$. Then $P \sim \tilde{S}_i$ since $\tilde{S}$ is a reduction of $P$ via surjective maps $g_i|_{\bar{P}_i}$, $i = 1, \ldots, m$, and since $\bar{u}_i(g_i(s_1), \ldots, s_m(s_m)) = u_i(s)$ for all $i$ and $s \in \bar{P}$ implies $\bar{u}_i|_{\bar{g}}(g_i(s_1), \ldots, s_m(s_m)) = \bar{u}_i|_{\bar{g}}(s)$ for all $i$ and $s \in \bar{P}$. Thus, $P \sim R$ by Lemma A.3.

Finally, since the strategies in $R \setminus \bar{R}$ are weakly dominated w.r.t. $R$, the strategies in $S_i \setminus \tilde{S}_i$ are weakly dominated w.r.t. $\tilde{S}$ for all $i$. Hence, the strategies in $P_1 \setminus \bar{P}_i$ are weakly dominated w.r.t. $P$ for all $i$. □

Lemma 3. For $h = 1, \ldots, k$ let $\bar{R}^h$ be obtained from $R^{h-1}$ by one round of elimination of some weakly dominated strategies, and let $R^h \sim \bar{R}^h$. Then there is $T \sim R^h$ where $T$ is obtained from $R^0$ in at most $k$ rounds of elimination of some weakly dominated strategies.

Proof. By induction on $k$. For $k = 1$, the claim holds with $T = R^h$. Assume that the claim holds for some $k \geq 1$. Now, let $\bar{R}^h$ be obtained from $R^{h-1}$ by one round of elimination of some weakly dominated strategies, and $R^h \sim \bar{R}^h$, $h = 1, \ldots, k + 1$. By the induction hypothesis, there is $T \sim \bar{R}^h$ where $T$ is obtained from $R^0$ by at most $k$ rounds of elimination of some weakly dominated strategies. Then by Lemma 2 there is $T$ obtained from $T$ by one round of elimination of some weakly dominated strategies such that $\bar{R}^h \sim \bar{R}^{h+1}$, and since $T$ is obtained from $R^h$ in at most $k + 1$ rounds of elimination of some weakly dominated strategies the claim holds for $k + 1$. □

We are then ready to proof Theorem 1.

Proof of Theorem 1. We have $F^{h+1}(S) \alpha L^h(S)$ for all $h = 1, \ldots, k$. By Lemma 1 $F^{k-h+1}L^{h+1}(S) \sim Q[h]$ where $Q[h]$ is obtained from $F^{k-h}L^h(S)$ by one round of elimination of some weakly dominated strategies. Hence, by Lemma 3 there is $T \sim F^k(S)$ where $T$ is obtained from $L^k(S)$ by at most $k$ rounds of elimination of some weakly dominated strategies. □

4. Application 1: strictly competitive games of perfect information

We now consider a finite, two-player strictly competitive extensive form game $G$ of perfect information. For definitions of standard concepts, see Ewerhart (2000a) (or, for example, Osborne and Rubinstein, 1994). Let $i, j \in \{1, 2\}$ be players, $i \neq j$. $X_i$ denotes the set of $i$’s nodes, $Z$ denotes the set of terminal nodes, and $X = X_i \cup X_j \cup Z$. Let $\omega_i : Z \rightarrow \}$
A strategy $s_i$ for player $i$ specifies a move at every node $x \in X_i$. Let $S_i$ be the set of strategies for player $i$. A pair of strategies $s = (s_1, s_2)$ uniquely determines a path $p(s) = (x_0, \ldots, x_H)$ where $x_0 = x^0$ (the initial node) and $x_H = z(s)$ is a terminal node.

The strategic form $N(G)$ of $G$ is the normal form game with strategy sets $S_i$ and utility functions $u_i(s) = \omega_0(z(s)), i = 1, 2$.

It is well known (Moulin, 1979; Gretlein, 1982, 1983) that $G$ is dominance solvable in a finite number of steps, and that the outcome is equal to the backward induction outcome. The value $v(x) = (v_1(x), v_2(x))$ of $x \in X$ is the backward induction outcome of $G(x)$.

Now, let $x \in X \setminus x^0$, and let player $i$ be the player called to move at node $y$ immediately preceding $x$. Then $G(x)$ is a weakly dominated subgame if

$$v_i(y) \geq \max_{z \in Z(x)} \omega_i(z) \quad \text{and} \quad v_i(y) > v_i(x).$$

Thus, a proper subgame is weakly dominated if the highest possible outcome within the subgame is not higher than, and the value of the subgame is lower than, the value of the subgame arising from the anterior node (for the player called to move at the anterior node leading to the subgame). We then say that a strategy $s_i \in S_i$ is weakly subgame dominated on $S_j$ if there exists $s_j \in S_j$, such that $p(s_i, s_j)$ reaches a weakly dominated subgame. Let $E^0(S) = S$, and let $E^k(S) = E^k_1(S) \times E^k_2(S)$ be the set of strategies not weakly subgame dominated on $E^{k-1}(S) = E^{k-1}_1(S) \times E^{k-1}_2(S)$. $G$ is subgame dominance solvable in $k$ steps if the outcomes in $E^k(S)$ are constant.

For any strictly competitive game $G$ with associated strategies $S$, weak subgame dominance is stronger than weak dominance since a weakly subgame dominated strategy $s_i$ is weakly dominated (for example, by a strategy $s'_i$ which consists of a maxmin strategy at subgames beginning at nodes leading to dominated subgames and equal to $s_i$ elsewhere).

On the other hand, a weakly dominated strategy does not necessarily lead the outcome path to a weakly dominated subgame.

With the definitions in place, we may then find an upper bound for the number of steps necessary to solve a game of perfect information, removing all dominated subgames at each step.

**Theorem 2.** Let $G$ be a finite, strictly competitive game of perfect information with at most $n$ outcomes. Then $N(G)$ is subgame dominance solvable in $n - 1$ steps.

We proceed with the following two lemmas.

**Lemma 4.** Let $G$ be a finite, strictly competitive game of perfect information with strategy set $S$. Let $x \in X$ and consider the subgame $G(x)$. Assume that $v_i(x) = \max_{z \in Z(x)} \omega_i(z)$ for a

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*For extensive form games with strict preferences over outcomes, any subgame perfect equilibrium outcome yields the backward induction outcome (cf. e.g. Osborne and Rubinstein, 1994, Chapter 6).*
player $i$. Let $s \in S$ be a strategy leading the path through $x$ and with $o_i(z(s)) < v_i(x)$. Then $s_i$ is a weakly subgame dominated strategy.

**Proof.** Let $p(s) = (x_0, \ldots, x_H)$ be the path determined by $s$. Let $\tilde{h} = \max\{h \mid v_i(x_h) = v_i(x), 0 \leq h \leq H\}$. As $v_i(x) > v_i(x_{\tilde{h} + 1})$, player $i$ is called to move at node $x_{\tilde{h}}$. Moreover, $v_i(x_{\tilde{h}}) \geq \max_{z \in Z(x_{\tilde{h} + 1})} o_i(z)$. Thus, $G(x_{\tilde{h} + 1})$ is a weakly dominated subgame and $s_i$ is a weakly subgame dominated strategy. □

**Lemma 5.** Let $G$ be a finite, strictly competitive game of perfect information with strategy set $S$. Assume that $v_i(x^0) < \max_{z \in Z} o_i(z)$ for a player $i$, and let $s \in S$ be a strategy where $o_i(z(s)) = \max_{z \in Z} o_i(z)$. Then $s$ is removed after two rounds of elimination of weakly subgame dominated strategies.

**Proof.** Let $p(s) = (x_0, \ldots, x_H)$, and $\tilde{h} = \max\{h \mid v_i(x_h) \neq o_i(z(s)), 0 \leq h \leq H\}$. As $v_i(x) < v_i(x_{\tilde{h} + 1})$, player $j$ is called to move at $x_{\tilde{h}}$. In the subgame $G(x_{\tilde{h} + 1})$, player $i$ has a strategy that ensures the highest possible outcome within this subgame, that is $v_i(x_{\tilde{h} + 1}) = \max_{z \in Z(x_{\tilde{h} + 1})} o_i(z)$. From Lemma 4, after one round of elimination of weakly subgame dominated strategies all remaining strategy pairs reaching the subgame $G(x_{\tilde{h} + 1})$ yield outcome $\max_{z \in Z(x_{\tilde{h} + 1})} o_i(z)$ to player $i$. If $s \in E^1(S)$ then, since $v_j(x_{\tilde{h}}) > \max_{z \in Z(x_{\tilde{h} + 1})} o_j(z), s_j$ is a weakly subgame dominated strategy on $E^1(S)$. □

Note that if strategy $s_j$ is weakly subgame dominated and hence reaches a weakly subgame dominated subgame $G(x)$ for some $s_j$, then all other strategies $s_j'$ leading play to $G(x)$ for some $s_j'$ are also weakly subgame dominated. Therefore, if a strategy points to a weakly dominated subgame then it must either be a weakly subgame dominated strategy or every weakly dominated subgame this strategy points to cannot be reached by any possible strategy of the opponent and the strategy is therefore equivalent to some strategy that does not point to weakly dominated subgames. Thus, the strategies surviving elimination of all weakly subgame dominated strategies are equivalent to the strategy set of a residual extensive form game where all weakly dominated subgames have been removed. We may now complete the proof of Theorem 2.

**Proof of Theorem 2.** Apply Lemma 5 $\min\{v_j(x^0) - 1, n - v_j(x^0)\}$ times, first on $G$ and then sequentially on the residual games where all weakly dominated subgames have been removed in each step. Then apply Lemma 4 once (with $x = x^0$) if necessary. By Lemma A.3 we then obtain that $G$ is subgame dominance solvable in at most $n - 1$ steps. □

**Corollary 2.** Any strictly competitive, finite game of perfect information with $n$ outcomes can be solved by $n - 1$ rounds of elimination of weakly dominated strategies.

5. Application II: order independence

As another application, we may observe that a corollary of Theorem 1 is the result on order independence by Marx and Swinkels (1997, 2000) for pure strategies. A reduction $L^k(S)$ is full if there are no weakly dominated strategies in $L^k(S)$. For a game sat-
isfying TDI, Marx and Swinkels demonstrate that any two full reductions are equivalent (cf. Marx and Swinkels, 1997, Corollary 1, p. 230).

By Theorem 1, if \( L^k(S) \) is a full reduction then it must be equivalent to a full reduction obtained by removing all weakly dominated strategies in every round. Thus, we have:

**Corollary 3.** Let \( L^k(S) \) and \( L^h(S) \) be full reductions, \( k, h > 0 \). Then \( L^k(S) \sim L^h(S) \).

**Proof.** If \( L^k(S) \) is a full reduction then by Theorem 1 it is equivalent to \( F^k(S) \) and \( F^k(S) \) is a full reduction. Similarly, if \( L^h(S) \) is a full reduction then it is equivalent to \( F^h(S) \) and \( F^h(S) \) is a full reduction. Since \( F^k(S) = F^h(S) \) and since \( \sim \) is transitive we have \( L^k(S) \sim L^h(S) \). \( \square \)

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