Note

Subadditive functions and their (pseudo-)inverses

Lars Peter Østerdal

Institute of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen, Denmark

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Abstract

The paper considers non-negative increasing functions on intervals with left endpoint closed at zero and investigates the duality between subadditivity and superadditivity via the inverse function and pseudo-inverses.

Keywords: Subadditive functions; Superadditive functions; Pseudo-inverse

1. Introduction

This paper considers non-negative increasing functions $f$ defined on intervals $I$, where $I = [0, \omega]$ for some $\omega > 0$ or $I = [0, \infty[$.

A function $f$ is subadditive if $f(x) + f(y) \geq f(x + y)$ whenever $x, y, x + y \in I$, and $f$ is superadditive if $f(x) + f(y) \leq f(x + y)$ whenever $x, y, x + y \in I$. Subadditive functions have been studied by, e.g., Hille and Phillips [5, Chapter 7] and Rosenbaum [8] and Matkowski and Świątkowski [6,7], and non-negative superadditive functions have been treated by, e.g., Bruckner [1,2] and Bruckner and Ostrow [3].

Since $f$ is subadditive if and only if $-f$ is superadditive (e.g., Rosenbaum [8]) results for non-negative superadditive functions translate directly to non-positive subadditive
functions as well. For many applications, however, the function class of interest is the class of non-negative subadditive functions.\(^1\)

There is an elementary connection between strictly increasing continuous subadditive and superadditive functions, which is likely to be folk knowledge.\(^2\) (Recall that if \(f\) is strictly increasing and continuous, then \(J = f(I)\) is an interval and there is a uniquely determined function \(f^{-1}\) on \(J\), the inverse function, such that \(f^{-1}(y) = \{x \mid f(x) = y\}\), \(f^{-1}\) is continuous and strictly increasing and \((f^{-1})^{-1} = f\).

**Proposition.** Let \(f : I \to \mathbb{R}_+\) be strictly increasing and continuous. Then \(f\) is subadditive if and only if \(f^{-1}\) is superadditive.

**Proof.** Let \(f\) be subadditive. We claim that if \(x, y, z \in I, f(x) = x', f(y) = y'\) and \(f(z) = x' + y'\) then \(x + y \leq z\). For this, assume otherwise that \(x + y > z\). Since \(y > z - x\) and \(f\) is strictly increasing we have \(f(x) + f(y) > f(x) + f(z - x) \geq f(z) = x' + y'\), a contradiction proving the claim.

Now, let \(x', y', x' + y' \in f(I)\) and \(x = f^{-1}(x'), y = f^{-1}(y')\). From our claim we have \(x + y \leq f^{-1}(x' + y')\), i.e., \(f^{-1}(x') + f^{-1}(y') \leq f^{-1}(x' + y')\), proving that \(f^{-1}\) is superadditive.

Conversely, let \(g \equiv f^{-1}\) be superadditive on \(J \equiv f(I)\). Let \(x', y', x' + y' \in g(J)\) and \(x = g^{-1}(x'), y = g^{-1}(y')\). Since \(g\) is strictly increasing and superadditive we have \(x + y \geq g^{-1}(x' + y')\) since if otherwise \((x + y < g^{-1}(x' + y'))\) we would have \(x + y \in J\) and \(g(x + y) < g(x) + g(y)\) contradiction superadditivity. Thus \(g^{-1}(x' + y') \leq x + y = g^{-1}(x') + g^{-1}(y')\), proving that \(g^{-1} = (f^{-1})^{-1} = f\) is subadditive on \(g(J) = f^{-1}(f(I)) = I.\)

The assumption that \(f\) is strictly increasing and continuous is restrictive. However, when \(f\) is discontinuous or fails to be strictly increasing, the inverse \(f^{-1}\) may not be a well-defined function. The classes of subadditive and superadditive functions may then also fail to share certain regularities. For instance, a non-negative increasing superadditive function vanish at the origin, whereas this is not necessarily the case for non-negative increasing subadditive functions.

2. Results

Let \(\mathbb{R}_+^* = \mathbb{R}_+ \cup \{\infty\}\) denote the extended non-negative reals. For an increasing function \(g : I \to \mathbb{R}_+^*\), let

\[
J \equiv \begin{cases} 
\{\inf_{x \in I} g(x), \sup_{x \in I} g(x)\}, & \text{if } g \text{ is bounded,} \\
\{\inf_{x \in I} g(x), \infty\}, & \text{otherwise.}
\end{cases}
\]

Then a function \(\tilde{g}^{-1} : J \to \mathbb{R}_+^*\) is a *pseudo-inverse* to \(g\) if

\[
\sup\{x \mid g(x) < y\} \leq \tilde{g}^{-1}(y) \leq \sup\{x \mid g(x) \leq y\},
\]

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1 In economic theory for instance, \(f\) can for example represent a non-linear tariff or long-run production costs.

2 The following proposition is probably well known, but we have been unable to find literature where it has been stated and provide it here for ease of reference.
whenever \( y \in J \). As noted by Denneberg [4, p. 5], \( \tilde{g}^{-1} \) is uniquely determined except on an at most countable number of points on \( J \). With respect to Lebesgue integration (studied by Denneberg) there is no reason to distinguish between different pseudo-inverses, but in our context it is useful to consider two specific forms.

If \( g : I \to \mathbb{R}_+ \) is increasing and lower semi-continuous, then we define \( \leftarrow g : J \to \mathbb{R}_+ \) by
\[
\leftarrow g(x) = \sup \{ y \mid g(y) \leq x \}.
\]
If \( g : I \to \mathbb{R}_+^* \) is increasing and upper semi-continuous, then \( \rightarrow g : J \to \mathbb{R}_+ \) is defined by
\[
\rightarrow g(x) = \min \{ y \mid x \leq g(y) \}.
\]

Lemma 2.1. Let \( f : I \to \mathbb{R}_+ \) be increasing, then:

1. If \( f \) is lower semi-continuous, then \( \leftarrow f \) is upper semi-continuous.
2. If \( f \) is upper semi-continuous, then \( \rightarrow f \) is lower semi-continuous.

Proof. We prove (1). Assume that \( \leftarrow f \) is not upper semi-continuous, i.e., there exists \( x \) and a sequence \( \{ x_n \} \), such that \( \lim_{n \to \infty} x_n = x \) and \( \limsup x_n = \leftarrow f(x) > 0 \). Since \( \leftarrow f \) is increasing there exists a decreasing subsequence \( \{ x_n^+ \} \) of \( \{ x_n \} \) such that \( \lim_{n \to \infty} x_n^+ = x \) and \( x + \frac{1}{n} \geq x_n^+ \geq x \). By the definition of \( \leftarrow f \) and lower semi-continuity of \( f \) there exists \( y \) such that \( f(y) \leq x \) and \( f(y') > x \) for all \( y' > y \). But this contradicts that for all \( y'' \in [y, y + \varepsilon] \) we have \( f(y'') \leq x_n^+ \leq x + \frac{1}{n} \) for all \( n \), i.e., \( f(y'') \leq x \).

The proof of (2) is similar. \( \square \)

For two functions \( g \) and \( h \), we write \( g = h \) if they are defined on the same domain \( I \) and \( f(x) = g(x) \) for all \( x \in I \).

Lemma 2.2. Let \( f : I \to \mathbb{R}_+ \) be increasing, then:

1. If \( f \) is lower semi-continuous then \( f = \leftarrow f \).
2. If \( f \) is upper semi-continuous then \( f = \rightarrow f \).

Proof. Suppose that \( f \) is lower semi-continuous. By Lemma 2.1, \( \leftarrow f \) is upper semi-continuous, so \( \leftarrow f \) is well defined. It is readily verified that the domain of \( \leftarrow f \) is the same as the domain of \( f \). Now let \( x \in I \), then:
\[
\leftarrow f(x) = \min \{ y \mid x \leq f(y) \} = \min \{ y \mid x \leq \sup \{ z \mid f(z) \leq y \} \}.
\]

Since \( x \leq \sup \{ z \mid f(z) \leq f(x) \} \), \( \leftarrow f(x) \leq f(x) \). It remains to verify that if \( y' < f(x) \) then \( y' \notin \{ y \mid x \leq \sup \{ z \mid f(z) \leq y \} \} \). This follows from the fact that if \( y' < f(x) \) then \( \sup \{ z \mid f(z) \leq y' \} < x \) since \( f \) is increasing. This means that \( \leftarrow f(x) \neq f(x) \), i.e., \( \leftarrow f(x) = f(x) \).

The proof of (2) is similar. \( \square \)
Notice that Lemma 2.2 would not be true for functions $g : I \rightarrow \mathbb{R}^*_+$ taking values on the extended reals, since for the function $g^1$ on $[0, 1]$ defined by
\[
g^1(x) \equiv \begin{cases} \frac{1}{1-x}, & 0 \leq x < 1, \\ \infty, & x = 1, \end{cases}
\]
and the function $g^2$ on $[0, \infty[$ defined by
\[
g^2(x) \equiv \begin{cases} \frac{1}{1-x}, & 0 \leq x < 1, \\ \infty, & x \geq 1, \end{cases}
\]
we have $g^1 = g^2$ and the domain is $[0, \infty[$, $g^1 = g^2$ and the domain is $[0, 1]$. Hence $g^1 \neq g^2$.

**Theorem 2.3.** Let $f : I \rightarrow \mathbb{R}^*_+$ be increasing, then:

1. If $f$ is lower semi-continuous, then $f$ is subadditive if and only if $\tilde{f}$ is superadditive.
2. If $f$ is upper semi-continuous, then $f$ is superadditive if and only if $\tilde{f}$ is subadditive.

**Proof.** Suppose that $f$ is lower semi-continuous.

First let $f$ be subadditive. If $\tilde{f}$ is not superadditive then there exist $x^*, y^*, x^* + y^* \in J$ such that $\sup\{y' : f(y') \leq x^* \} + \sup\{y'' : f(y'') \leq y^* \} > \sup\{y''' : f(y''') \leq x^* + y^* \}$. But $f$ subadditive implies $\sup\{y''' : f(y''') \leq x^* + y^* \} \geq \sup\{y' + y'' : f(y') \leq x^*, f(y'') \leq y^* \} = \sup\{y' : f(y') \leq x^* \} + \sup\{y'' : f(y'') \leq y^* \}$ which is a contradiction.

Now let $\tilde{f}$ be superadditive. For arbitrary $u, v \in I$ such that $f(u) = x$ and $f(v) = y$, define $u^* = \sup\{y' : f(y') \leq x \}$ and $v^* = \sup\{y'' : f(y'') \leq y \}$. Since $\tilde{f}$ is superadditive we have $\sup\{y' : f(y') \leq x \} + \sup\{y'' : f(y'') \leq y \} = \sup\{y' + y'' : f(y') \leq x, f(y'') \leq y \} \leq \sup\{y''' : f(y''') \leq x + y \}$ for all $x, y, x + y \in J$. But then there exists $y''' \geq u^* + v^* \geq u + v$ such that $f(y''') \leq x + y$. Then $f(u + v) \leq f(u^* + v^*) \leq f(y''') \leq x + y = f(u) + f(v)$, i.e., $f$ is subadditive.

If $f$ is upper semi-continuous, by Lemma 2.1 $\tilde{f}$ is lower semi-continuous. By (1) and Lemma 2.2 $\tilde{f}$ is subadditive if and only if $\tilde{f} = f$ is superadditive. □

Let $f : [0, \omega] \rightarrow \mathbb{R}^*_+$. For $x \in [0, \infty[$ we say that $x^1, \ldots, x^n$ form a $\omega$-partition if $x^1 + \cdots + x^n = x$ and $0 \leq x^i \leq \omega$ for all $i$. Let $\hat{f}$ be a function on $[0, \hat{\omega})$, $\omega < \hat{\omega}$, or $[0, \infty[$ defined as
\[
\hat{f}(x) = \inf\{f(x^1) + \cdots + f(x^n) \mid x^1, \ldots, x^n \text{ form a } \omega\text{-partition for } x \}.
\]

One may show that $\hat{f}$ is subadditive (see [1], for a proof in the case of superadditivity) and hence the maximal subadditive extension of $f$ on $[0, \hat{\omega})$.\(^3\)

A central problem is how to check whether a given function is subadditive. We verify below that Bruckner’s test for superadditivity [2] has a counterpart for increasing, non-negative subadditive functions. For this, we make use of the following lemma.

\(^3\) For a subadditive function $f : I \rightarrow \mathbb{R}^*_+$, $\hat{f} : I \rightarrow \mathbb{R}^*_+$ is a maximal subadditive extension on $I \supset I$, if $\hat{f}$ is subadditive and $f = \hat{f}$ on $I$ and if $g$ is a subadditive function on $I$ and $g = f$ on $I$ then $g \leq \hat{f}$.
Lemma 2.4. Let \( f : [0, \omega] \rightarrow \mathbb{R}^+ \) be increasing, and \( \hat{\omega} > \omega \):

1. If \( f \) is upper semi-continuous and superadditive, then the minimal superadditive extension \( \hat{f} \) on \([0, \hat{\omega}]\) is upper semi-continuous.

2. If \( f \) is lower semi-continuous and subadditive, then the maximal subadditive extension \( \hat{f} \) on \([0, \hat{\omega}]\) is lower semi-continuous.

Proof. By contradiction. Assume that \( \hat{f} \) is not upper semi-continuous at some point \( x \in [\omega, \hat{\omega}] \). Since \( \hat{f} \) is increasing there is an infinite sequence \( \{x^1, x^2, x^3, \ldots\} \) where \( x^i > x \) for all \( i \) and \( \{x^i\} \rightarrow x \) such that \( \hat{f}(x) < \lim_{i \rightarrow \infty} \hat{f}(x^i) \). By [1] \( \hat{f}(x) = \sup\{f(x^1) + \cdots + f(x^n) | x^1, \ldots, x^n \text{ forms a } \omega\text{-partition for } x\}, x \in [0, \hat{\omega}] \).

By our assumption, for each \( x^i \) there is a \( \omega\)-partition \( \{x^1_i, \ldots, x^n_i\} \) such that \( f(x^1_i) + \cdots + f(x^n_i) \geq f(x^i) + \frac{1}{2} \lim_{i \rightarrow \infty} \hat{f}(x^i) - f(x) \). Note that since \( f \) is superadditive on \([0, \omega]\) we can assume that \( n_i \leq N \) for all \( i \) and some positive integer \( N \). Since \([0, \omega]\) is compact there is a subsequence \( \{y^1, y^2, y^3, \ldots\} \subseteq \{x^1, x^2, x^3, \ldots\} \) such that \( \{x^1_i, \ldots, x^n_i\} \) forms a \( \omega\)-partition for \( y^i \), \( n_i = K \leq N \) for all \( i \), and \( \{x^1_i, \ldots, x^n_i\} \) converges pointwise to some \( \omega\)-partition \( \{x^1, \ldots, x^K\} \) for \( i \rightarrow \infty \).

We therefore have \( f(x^1) + \cdots + f(x^K) < \lim_{i \rightarrow \infty} f(x^1_i) + \cdots + f(x^n_i) \) contradicting that \( f \) is upper semi-continuous hence right-continuous on \([0, \omega]\).

The proof of (2) is similar. \( \square \)

Let \( f : [0, \omega] \rightarrow \mathbb{R}^+ \). The functions \( f_1, f_2, \ldots, f_K \) on \([0, \omega_1], [0, \omega_2], \ldots, [0, \omega_K]\) respectively form a decomposition of \( f \), if \( \omega_1 + \cdots + \omega_K = \omega \), and

\[
 f(x) = \begin{cases} 
 f_1(x), & 0 \leq x \leq \omega_1, \\
 f_2(x - \omega_1) + f_1(\omega_1), & \omega_1 < x \leq \omega_1 + \omega_2, \\
 \vdots & \vdots \\
 f_K(x - \omega_1 - \cdots - \omega_{K-1}) + f_1(\omega_1) + \cdots + f_{K-1}(\omega_{K-1}), & \omega_1 + \cdots + \omega_{K-1} < x \leq \omega. 
\end{cases}
\]

Following Bruckner [2], if \( f_1, f_2, \ldots, f_K \) form a decomposition of \( f \), we write \( f = f_1 \wedge f_2 \wedge \cdots \wedge f_K \).

Theorem 2.5. Let \( f_1 \) and \( f_2 \) be non-negative subadditive functions defined on \([0, \omega_1]\) and \([0, \omega_2]\), respectively, and \( f = f_1 \wedge f_2 \). Let \( \hat{f}_1 \) be the maximal subadditive extension of \( f_1 \). Then \( f \) is subadditive on \([0, \omega]\) if and only if \( f \leq \hat{f}_1 \) on \([0, \omega]\).

Proof. If \( g_1 \) and \( g_2 \) are non-negative superadditive upper semi-continuous functions defined on \([0, \omega_1]\) and \([0, \omega_2]\), respectively, and \( g = g_1 \wedge g_2 \), then \( g \) is superadditive on \([0, \omega]\) if and only if \( \hat{g}_1 \leq g \) on \([0, \omega]\), where \( \hat{g}_1 \) is the minimal superadditive extension of \( g_1 \) [2, Theorem 1].

By Lemma 2.4 \( \hat{g}_1 \) is upper semi-continuous, hence by Lemma 2.1 and Theorem 2.3 \( \hat{g}_1 \) is lower semi-continuous and subadditive.

We claim that \( \hat{\hat{g}_1} \) is the maximal subadditive extension of \( \hat{g}_1 \). For this, assume that the maximal subadditive extension of \( \hat{g}_1 \), \( \hat{\hat{g}_1} \), does not coincide with \( \hat{g}_1 \). Then we must have
\( \hat{g}_1(x) \geq \hat{g}_1(x) \) for all \( x \in [\omega, \hat{\omega}] \), with at least one strict inequality. However, then (by Theorem 2.3) \( \hat{g}_1 \) is a superadditive extension of \( g_1 \) and \( \hat{g}_1(x) \leq \hat{g}_1(x) \) with at least one strict inequality—a contradiction proving the claim.

We conclude that \( f \leq \hat{f}_1 \) if and only if \( \hat{f} \geq \hat{f}_1 \) and the result follows from Bruckner’s Theorem 1. \( \square \)

By induction, it follows from Theorem 2.5 that:

**Theorem 2.6.** Let \( f_1, \ldots, f_K \) be non-negative and subadditive on \([0, \omega_1], \ldots, [0, \omega_K]\), respectively, and let \( f = f_1 \land \cdots \land f_K \). Let \( \hat{f}_k \) be the maximal subadditive extension of \( f_k \), \( k = 1, \ldots, K \). Then \( f \) is subadditive on \([0, \omega]\) if \( f_k \land \cdots \land f_K \leq \hat{f}_k \) for every \( k \).

**Proof.** \( K = 1 \) is trivial, and \( K = 2 \) is Theorem 2.5. For \( K \geq 3 \), it remains to verify that if \( f_k \land \cdots \land f_K \) is subadditive then \( f_{k-1} \land \cdots \land f_K \leq \hat{f}_{k-1} \) is subadditive. But that follows from Theorem 2.5. \( \square \)

If \( \{ f_n \} \) is a pointwise convergent sequence\(^4\) of strictly increasing continuous functions, then \( \{ f_n^{-1} \} \) is not necessarily pointwise convergent as the following example shows.

**Example 2.1.** Let \( f : [0, 100] \rightarrow \mathbb{R}_+ \) be the piecewise affine (subadditive) function spanned by the points \( \{(0, 0), (1, 10), (3, 10), (100, 12)\} \). Let \( \{ f_n \} \) be the sequence of (subadditive) functions on \([0, 100]\) where \( f_n \) is the function spanned by the points \( \{(0, 0), (1, 10 - 2\alpha_n), (3, 10 + 2\beta_n), (100, 12)\} \), where \( \alpha_n = \frac{1}{2}\alpha_{n-1} \) if \( n \) is odd, \( \alpha_n = \alpha_{n-1} \) otherwise (\( \alpha_0 \equiv 1 \)) and \( \beta_n = \frac{1}{2}\beta_{n-1} \) if \( n \) is even \( \beta_n = \beta_{n-1} \) otherwise (\( \beta_0 \equiv 1 \)). Then \( \{ f_n \} \) converges pointwise to \( f \), but \( \{ f_n^{-1} \} \) is not pointwise convergent since \( f_n^{-1}(10) = 2 \) for \( n \) even and \( f_n^{-1}(10) = \frac{5}{3} \) for \( n \) odd.

Theorem 2.7 below shows that any increasing subadditive (superadditive) function can be approximated by a bijective subadditive (superadditive) function pointwise. The class of strictly increasing continuous subadditive (superadditive) functions is in this particular sense dense in the class of increasing subadditive (superadditive) functions, and one may note that Theorem 2.7 together with the Proposition constitute an alternative proof of Theorem 2.3.

**Theorem 2.7.** Let \( f : I \rightarrow \mathbb{R}_+ \) be increasing:

\[ (1) \text{ } f \text{ is subadditive if and only if there exists a pointwise convergent sequence of strictly increasing continuous subadditive functions } \{ f_n \} \text{ where } \lim_{n \rightarrow \infty} \{ f_n \} = f \text{ and } \lim_{n \rightarrow \infty} \{ f_n^{-1} \} = \bar{f}. \]

\(^4\) An infinite sequence of functions \( \{ f_n \} \) converges pointwise to \( f \) if for all \( x \in I \) and all \( \varepsilon > 0 \) there exists \( N \) such that \( |f_n(x) - f(x)| \leq \varepsilon \) for all \( n \geq N \).
(2) $f$ is superadditive if and only if there exists a pointwise convergent sequence of strictly increasing continuous superadditive functions $\{f_n\}$ where $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} f_n^{-1} = \hat{f}$.

Proof. We prove (1). It is readily verified that if $\{f_n\}$ is a pointwise convergent sequence of subadditive (superadditive) functions, $\lim_{n \to \infty} f_n = f$, then $f$ is subadditive (superadditive) (see, e.g., [8, Theorem 1.3.2]; [3, Theorem 2]) and the “if” part of (1) follows.

For the “only if” part of (1), let $f$ be increasing and subadditive. We construct a function $f^*_n$ in a two step procedure. Roughly speaking, first the discontinuities are smoothened out (Step 1) and then flat segments are eliminated (Step 2).

Step 1: It is readily verified that if $f$ is discontinuous then $\lim_{x \to 0^+} f(x) > 0$. If $f$ is discontinuous define

$$\tilde{f}_n(x) = \begin{cases} nxf\left(\frac{1}{n}\right), & 0 \leq x \leq \frac{1}{n}, \\ f(x), & \frac{1}{n} < x, \end{cases}$$

and notice that $\{\tilde{f}_n\}$ converges pointwise to $f$. If $f$ is continuous define $\hat{f}_n = f$. Let $\hat{f}_n$ be the function on $I$ defined by

$$\hat{f}_n(x) = \inf\{ \tilde{f}_n(x^1) + \cdots + \tilde{f}_n(x^{s_n}) \},$$

where $0 \leq x^i \leq x$, $x^1 + \cdots + x^{s_n}$, $s_n \geq 1$. It is easily verified that $\hat{f}_n$ is subadditive, and notice that $\hat{f}_n$ is continuous. Further, $\hat{f}_n$ converges pointwise to $f$ since for all $x \in I$ then $\hat{f}_n(x) = \hat{f}_n(x)$ for all $n$ sufficiently large.

Step 2: We say that $s$ is a flat segment for $\hat{f}_n$ if $\hat{f}_n(x') = \hat{f}_n(x'') = s$ for some $x' \neq x''$. Let $S = \{s^{-1}, s^0, s^1, \ldots\}$ denote the increasingly ordered set of flat segments. Define $\bar{x}(s^i) = \min\{x \mid \hat{f}_n(x) = s^i\}$ and $\hat{x}(s^i) = \sup\{x \mid \hat{f}_n(x) = s^i\}$.

Now, since $\hat{f}_n$ is strictly increasing on $[0, \bar{x}(s^0)]$ the maximal subadditive extension of $\hat{f}_n$’s restriction to $[0, \bar{x}(s^0)]$ is strictly increasing. Hence by Theorem 2.5 there is some

$$z_n^0 = \left[ \bar{x}(s^0), \min\left\{ \bar{x}(s^0) + \frac{1}{n^2}, \hat{x}(s^1) \right\} \right],$$

with the convention $z_n^0 = \infty$ if $\bar{x}(s^0) = \infty$ and if $s^0$ is the largest flat segment then replace $\min(\bar{x}(s^0) + \frac{1}{n^2}, \hat{x}(s^1))$ with $\hat{x}(s^1) + \frac{1}{n^2}$, and there is a subadditive strictly increasing function $\hat{f}_n^{z_0}$ where

$$\hat{f}_n^{z_0}(x) = \begin{cases} \hat{f}_n(\bar{x}(s^0)) + c_n^0 x, & \bar{x}(s^0) \leq x \leq z_n^0, \\ \hat{f}_n(x), & \text{otherwise}, \end{cases}$$

for some $c_n^0 > 0$. Repeating this procedure for each flat segment in a way such that each flat segment will selected at some point (for example, continuing with $s = s^1, s^{-1}, s^2, s^{-2}, \ldots$), in the limit we obtain a strictly increasing continuous subadditive function $f^*_n$ on $I$. Furthermore, it is readily verified that $\{f^*_n\}$ converges pointwise to $\hat{f}_n$ hence also to $f$.

It remains to verify that $\lim_{n \to \infty} (f^*_n)^{-1} = \hat{f}$ (pointwise). If $z$ is a flat segment of $f$, then $\lim_{n \to \infty} (f^*_n)^{-1}(z) = \sup\{y \mid f(y) = z\} = \sup\{y \mid f(y) \leq z\} = \hat{f}(z)$. If $z'$ is not a flat
segment, and furthermore \[ f(x) < z' \text{ or } f(x) > z' \] for all \( x \in I \) then \( \lim_{n \to \infty} (f_n^+)^{-1}(z') = \sup\{ y \mid f(y) \leq z' \} = \overleftarrow{f}(z') \). Finally if \( z'' \) is a continuous and strictly increasing point for a pseudo-inverse of \( f \), then clearly \( (f_n^+)^{-1}(z'') = \{ y \mid f(y) = z'' \} = \sup\{ y \mid f(y) \leq z'' \} = \overleftarrow{f}(z'') \).

The proof of (2) is similar. \( \square \)

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**References**