Econometrics 2 – Fall 2005

Non-Stationary Time Series, Cointegration and Spurious Regression

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Motivation: Regression with Non-Stationarity

- What happens to the properties of OLS if variables are non-stationary?

- Consider two presumably unrelated variables:

CONS	Danish private consumption in 1995 prices.
BIRD	Number of breeding cormorants (skarv) in Denmark.

  And consider a static regression model

$$\log(\text{CONS}_t) = \beta_0 + \beta_1 \cdot \log(\text{BIRD}_t) + u_t.$$

  We would expect (or hope) to get $\widehat{\beta}_1 \approx 0$ and $R^2 \approx 0$.

- Applying OLS to yearly data $1982 - 2001$ gives the result:

$$\log(\text{CONS}_t) = \underset{(80.90)}{12.145} + \underset{(6.30)}{0.095} \cdot \log(\text{BIRD}_t) + u_t,$$

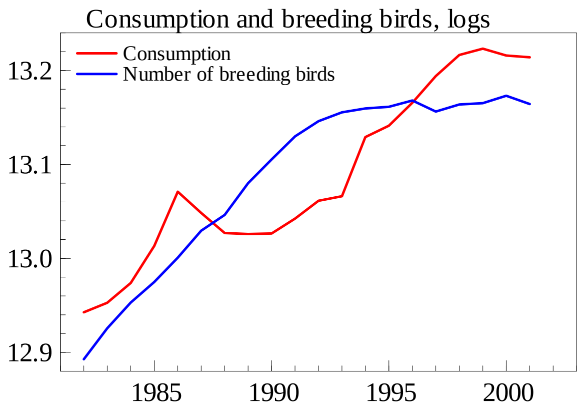
  with $R^2 = 0.688$.

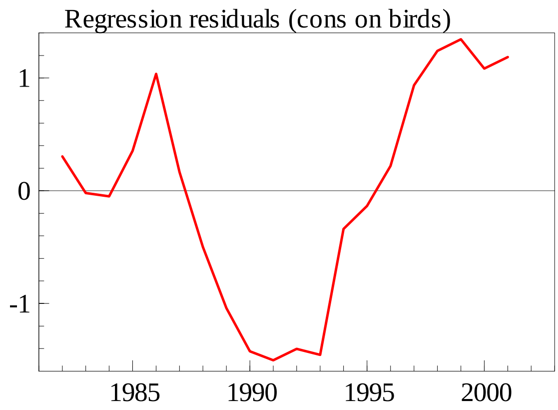
- It looks like a reasonable model. But it is complete nonsense: spurious regression.

- The variables are non-stationary.
  The residual, u_t, is non-stationary and standard results for OLS do not hold.

- In general, regression models for non-stationary variables give spurious results.
  Only exception is if the model eliminates the stochastic trends to produce stationary residuals: Cointegration.

- For non-stationary variables we should always think in terms of cointegration.
  Only look at regression output if the variables cointegrate.





Outline

Definitions and concepts:

(1) Combinations of non-stationary variables; Cointegration defined.

(2) Economic equilibrium and error correction.

Engle-Granger two-step cointegration analysis:

(3) Static regression for cointegrated time series.

(4) Residual based test for no-cointegration.

(5) Models for the dynamic adjustment.

Cointegration analysis based on dynamic models:

(6) Estimation in the unrestricted ADL or ECM model.

(7) PcGive test for no-cointegration.

What if variables do not cointegrate?

(8) Spurious regression revisited.

Cointegration Defined

- Let $X_t = \begin{pmatrix} X_{1t} & X_{2t} \end{pmatrix}'$ be two I(1) variables, i.e. they contain stochastic trends:

$$\begin{aligned} X_{1t} &= \sum\nolimits_{i=1}^{t} \epsilon_{1i} + \text{initial value} + \text{stationary process} \\ X_{2t} &= \sum\nolimits_{i=1}^{t} \epsilon_{2i} + \text{initial value} + \text{stationary process}. \end{aligned}$$

- In general, a linear combination of X_{1t} and X_{2t} will also have a random walk. Define $\beta = \begin{pmatrix} 1 & -\beta_2 \end{pmatrix}'$ and consider the linear combination:

$$\begin{aligned} Z_t &= \beta' X_t = \begin{pmatrix} 1 & -\beta_2 \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = X_{1t} - \beta_2 X_{2t} \\ &= \sum\nolimits_{i=1}^{t} \epsilon_{1i} - \beta_2 \sum\nolimits_{i=1}^{t} \epsilon_{2i} + \text{initial value} + \text{stationary process}. \end{aligned}$$

- Important exception: There exist a β, so that Z_t is stationary: We say that X_{1t} and X_{2t} co-integrate with cointegration vector β.

Remarks:

(1) Cointegration occurs if the stochastic trends in X_{1t} and X_{2t} are the same so they cancel, $\sum_{i=1}^{t} \epsilon_{1i} = \beta_1 \cdot \sum_{i=1}^{t} \epsilon_{2i}$. This is called a common trend.

(2) You can think of an equation eliminating the random walks in X_{1t} and X_{2t}:

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t. \qquad (\dagger)$$

If u_t is I(0) (mean zero) then $\beta = \begin{pmatrix} 1 & -\beta_2 \end{pmatrix}'$ is a cointegrating vector.

(3) The cointegrating vector is only unique up to a constant factor. If $\beta' X_t \sim$I(0). Then so is $\widetilde{\beta}' X_t = b\beta' X_t$ for $b \neq 0$. We can choose a normalization

$$\beta = \begin{pmatrix} 1 \\ -\beta_2 \end{pmatrix} \quad \text{or} \quad \widetilde{\beta} = \begin{pmatrix} -\widetilde{\beta}_1 \\ 1 \end{pmatrix}.$$

This corresponds to different variables on the left hand side of (†)

(4) Cointegration is easily extended to more variables. The variables in $X_t = \begin{pmatrix} X_{1t} & X_{2t} & \cdots & X_{pt} \end{pmatrix}'$ cointegrate if

$$Z_t = \beta' X_t = X_{1t} - \beta_2 \cdot X_{2t} - ... - \beta_p \cdot X_{pt} \sim \text{I(0)}.$$

Cointegration and Economic Equilibrium

- Consider a regression model for two I(1) variables, X_{1t} and X_{2t}, given by

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t. \qquad (*)$$

The term, u_t, is interpretable as the deviation from the relation in $(*)$.

- If X_{1t} and X_{2t} cointegrate, then the deviation

$$u_t = X_{1t} - \mu - \beta_2 X_{2t}$$

is a stationary process with mean zero.
Shocks to X_{1t} and X_{2t} have permanent effects. X_{1t} and X_{2t} co-vary and $u_t \sim$I(0).
We can think of $(*)$ as defining an equilibrium between X_{1t} and X_{2t}.

- If X_{1t} and X_{2t} do not cointegrate, then the deviation u_t is I(1).
There is no natural interpretation of $(*)$ as an equilibrium relation.

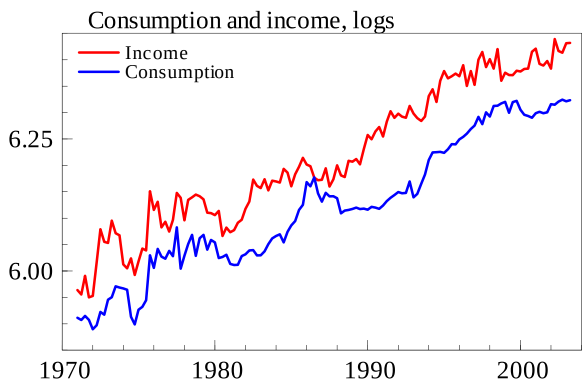
Empirical Example: Consumption and Income

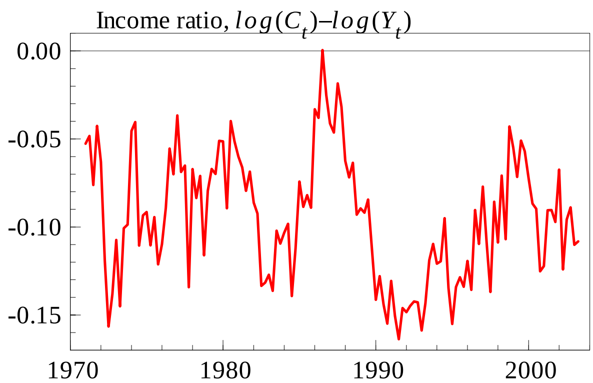
- Time series for log consumption, C_t, and log income, Y_t, are likely to be I(1).
Define a vector $X_t = \begin{pmatrix} C_t & Y_t \end{pmatrix}'$.

- Consumption and income are cointegrated with cointegration vector $\beta = \begin{pmatrix} 1 & -1 \end{pmatrix}'$ if the (log-) consumption-income ratio,

$$Z_t = \beta' X_t = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} C_t \\ Y_t \end{pmatrix} = C_t - Y_t,$$

is a stationary process. The consumption-income ratio is an equilibrium relation.





How is the Equilibrium Sustained?

- There must be forces pulling X_{1t} or X_{2t} towards the equilibrium.
- Famous representation theorem: X_{1t} and X_{2t} cointegrate if and only if there exist an error correction model for either X_{1t}, X_{2t} or both.
- As an example, let $Z_t = X_{1t} - \beta_2 X_{2t}$ be a stationary relation between I(1) variables. Then there exists a stationary ARMA model for Z_t. Assume for simplicity an AR(2):

$$Z_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \epsilon_t, \qquad \theta(1) = 1 - \theta_1 - \theta_2 > 0.$$

This is equivalent to

$$\begin{aligned}(X_{1t} - \beta_2 X_{2t}) &= \theta_1 (X_{1t-1} - \beta_2 X_{2t-1}) + \theta_2 (X_{1t-2} - \beta_2 X_{2t-2}) + \epsilon_t \\ X_{1t} &= \beta_2 X_{2t} + \theta_1 X_{1t-1} - \theta_1 \beta_2 X_{2t-1} + \theta_2 X_{1t-2} - \theta_2 \beta_2 X_{2t-2} + \epsilon_t,\end{aligned}$$

or

$$\Delta X_{1t} = \beta_2 \Delta X_{2t} + \theta_2 \beta_2 \Delta X_{2t-1} - \theta_2 \Delta X_{1t-1} - (1 - \theta_1 - \theta_2) \{X_{1t-1} - \beta_2 X_{2t-1}\} + \epsilon_t.$$

In this case we have a common-factor restriction. That is not necessarily true.

More on Error-Correction

- Cointegration is a system property. Both variables could error correct, e.g.:

$$\begin{aligned}\Delta X_{1t} &= \delta_1 + \Gamma_{11} \Delta X_{1t-1} + \Gamma_{12} \Delta X_{2t-1} + \alpha_1 (X_{1t-1} - \beta_2 X_{2t-1}) + \epsilon_{1t} \\ \Delta X_{2t} &= \delta_2 + \Gamma_{21} \Delta X_{1t-1} + \Gamma_{22} \Delta X_{2t-1} + \alpha_2 (X_{1t-1} - \beta_2 X_{2t-1}) + \epsilon_{2t},\end{aligned}$$

- We may write the model as the so-called vector error correction model,

$$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \begin{pmatrix} \Delta X_{1t-1} \\ \Delta X_{2t-1} \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} (X_{1t-1} - \beta_2 X_{2t-1}) + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix},$$

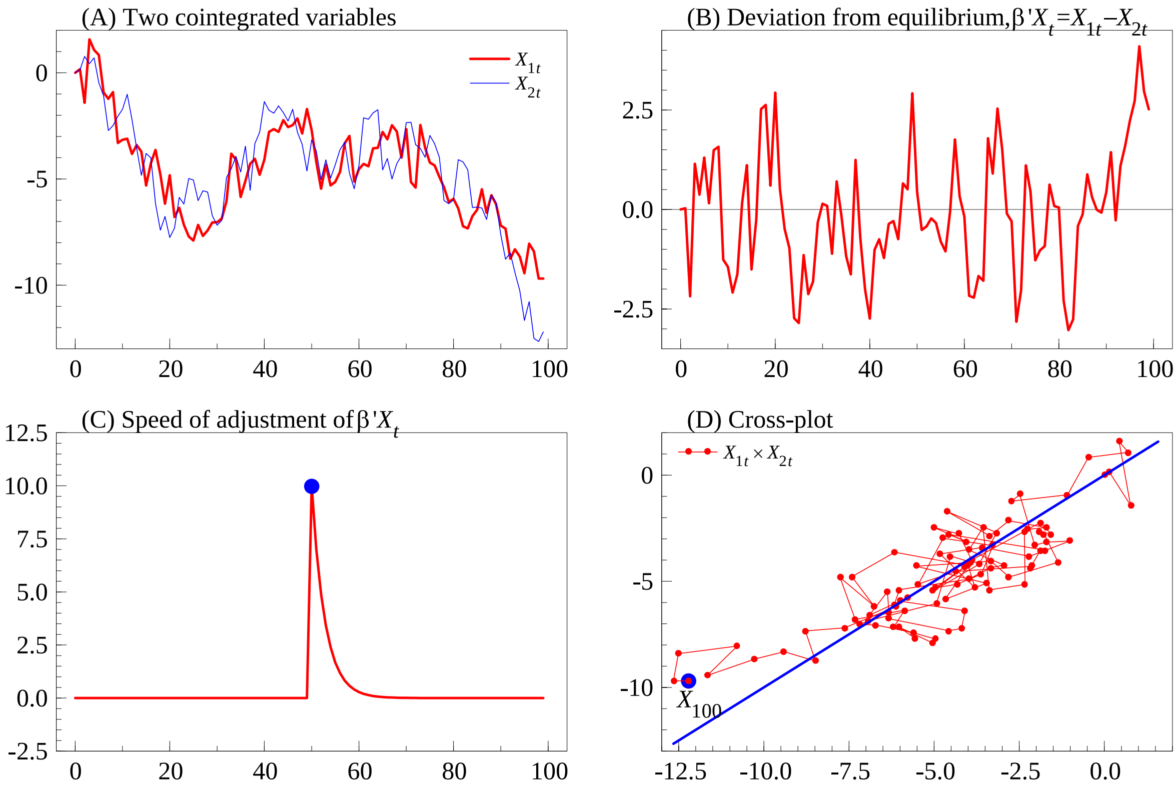
or simply

$$\Delta X_t = \delta + \Gamma \Delta X_{t-1} + \alpha \beta' X_{t-1} + \epsilon_t.$$

- Note that $\beta' X_{t-1} = X_{1t-1} - \beta_2 X_{2t-1}$ appears in both equations.
- For X_{1t} to error correct we need $\alpha_1 < 0$.
  For X_{2t} to error correct we need $\alpha_2 > 0$.

Consider a simple model for two cointegrated variables:

$$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.1 \end{pmatrix} (X_{1t-1} - X_{2t-1}) + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}.$$



OLS Regression with Cointegrated Series

- In the cointegration case there exists a β_2 so that the error term, u_t, in

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t. \qquad (**)$$

  is stationary.

- OLS applied to $(**)$ gives consistent results, so that $\widehat{\beta}_2 \rightarrow \beta_2$ for $T \rightarrow \infty$.

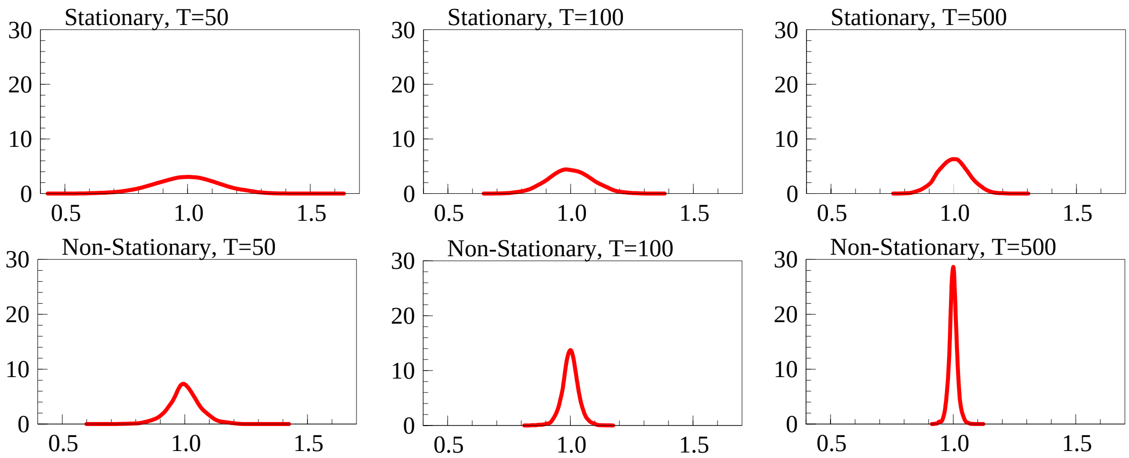
- Note that consistency is obtained even if potential dynamic terms are neglected.
  This is because the stochastic trends in X_{1t} and X_{2t} dominate.
  We can even get consistent estimates in the reverse regression

$$X_{2t} = \delta + \gamma_1 X_{1t} + v_t.$$

- Unfortunately, it turns out that $\widehat{\beta}_2$ is not asymptotically normal in general.
  The normal inferential procedures do not apply to $\widehat{\beta}_2$!
  We can use $(**)$ for estimation–not for testing.

Super-Consistency

- For stationary series, the variance of $\widehat{\beta}_1$ declines at a rate of T^{-1}.
- For cointegrated I(1) series, the variance of $\widehat{\beta}_1$ declines at a faster rate of T^{-2}.
- Intuition: If $\widehat{\beta}_1 = \beta_1$ then u_t is stationary. If $\widehat{\beta}_1 \neq \beta_1$ then the error is I(1) and will have a large variance. The 'information' on the parameter grows very fast.



Test for No-Cointegration, Known β_1

- Suppose that X_{1t} and X_{2t} are I(1).
  Also assume that $\beta = \begin{pmatrix} 1 & -\beta_2 \end{pmatrix}'$ is known.

- The series cointegrate if
$$Z_t = X_{1t} - \beta_2 X_{2t}$$
is stationary.

- This can be tested using an ADF unit root test, e.g. the test for $H_0 : \pi = 0$ in
$$\Delta Z_t = \delta + \sum_{i=1}^{k} \Delta Z_{t-i} + \pi Z_{t-1} + \eta_t.$$
The usual DF critical values apply to $t_{\pi=0}$.

- Note, that the null, $H_0 : \pi = 0$, is a unit root, i.e. no cointegration.

Test for No-Cointegration, Estimated β_1

- Engle-Granger (1987) two-step procedure.
- If $\beta = (\ 1 \quad -\beta_2\ )'$ is unknown, it can be (super-consistently) estimated in

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t. \qquad (***)$$

  $\widehat{\beta}$ is a cointegration vector if $\widehat{u}_t = X_{1t} - \widehat{\mu} - \widehat{\beta}_2 X_{2t}$ is stationary.

- This can be tested using a DF unit root test, e.g. the test for $H_0 : \pi = 0$ in

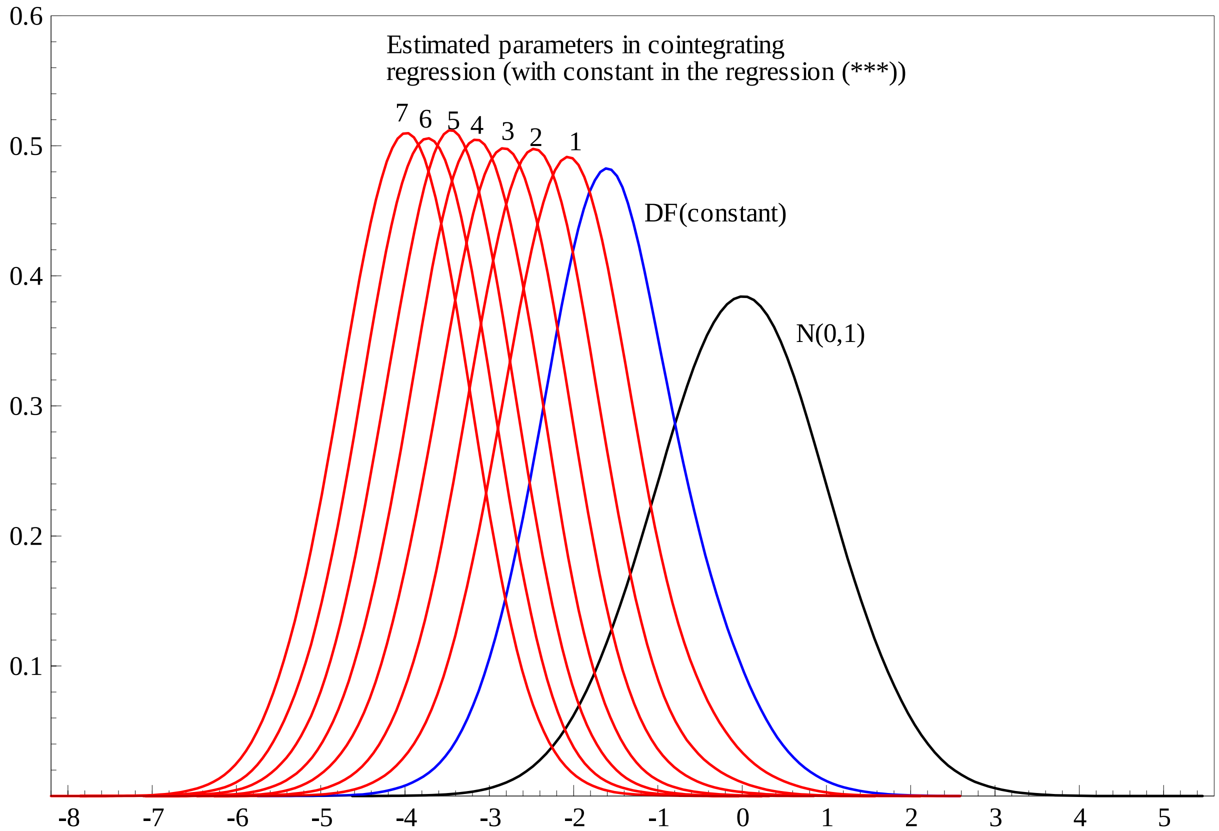
$$\Delta \widehat{u}_t = \sum_{i=1}^{k} \Delta \widehat{u}_{t-i} + \pi \widehat{u}_{t-1} + \eta_t.$$

  Remarks:

  (1) The residual $\widehat{u}_t$ has mean zero. No deterministic terms in DF regression.

  (2) The critical value for $t_{\pi=0}$ still depends on the deterministic regressors in $(***)$.

  (3) The fact that $\widehat{\beta}_1$ is estimated also changes the critical values.
  OLS minimizes the variance of $\widehat{u}_t$. Look 'as stationary as possible'.
  Critical value depends on the number of regressors.



- Critical values for the Dickey-Fuller test for no-cointegration are given by:

Case 1: A constant term in $(***)$.

Number of estimated parameters	Significance level		
	1%	5%	10%
0	-3.43	-2.86	-2.57
1	-3.90	-3.34	-3.04
2	-4.29	-3.74	-3.45
3	-4.64	-4.10	-3.81
4	-4.96	-4.42	-4.13

Case 2: A constant and a trend in $(***)$.

Number of estimated parameters	Significance level		
	1%	5%	10%
0	-3.96	-3.41	-3.13
1	-4.32	-3.78	-3.50
2	-4.66	-4.12	-3.84
3	-4.97	-4.43	-4.15
4	-5.25	-4.72	-4.43

Dynamic Modelling

- Given the estimated cointegrating vector we can define the error correction term

$$\text{ecm}_t = \widehat{u}_t = X_{1t} - \widehat{\mu} - \widehat{\beta}_2 X_{2t},$$

which is, per definition, a stationary stochastic variable.

- Since $\widehat{\beta}_2$ converges to β_2 very fast we can treat it as a fixed regressor and formulate an error correction model conditional on ecm_{t-1}, i.e.

$$\Delta X_{1t} = \delta + \lambda_1 \Delta X_{1t-1} + \kappa_0 \Delta X_{2t} + \kappa_1 \Delta X_{2t-1} - \alpha \cdot \text{ecm}_{t-1} + \epsilon_t,$$

where $\alpha > 0$ is consistent with error-correction.

- Given cointegration, all terms are stationary, and normal inference applies to δ, λ, κ_0, κ_1, and α.

Outline of an Engle-Granger Analysis

(1) Test individual variables, e.g. X_{1t} and X_{2t}, for unit roots.

(2) Run the static cointegrating regression

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t.$$

Note that the $t-$ratios cannot be used for inference.

(3) Test for no-cointegration by testing for a unit root in the residuals, $\widehat{u}_t$.

(4) If cointegration is not rejected estimate a dynamic (ECM) model like

$$\Delta X_{1t} = \delta + \lambda_1 \Delta X_{1t-1} + \kappa_0 \Delta X_{2t} + \kappa_1 \Delta X_{2t-1} - \alpha \widehat{u}_{t-1} + \epsilon_t.$$

All terms are stationary. Remaining inference is standard.

Empirical Example: Danish Interest Rates

- Consider two Danish interest rates:

r_t : Money market interest rate
b_t : Bond Yield

for the period $t = 1972:1 - 2003:2$.

- Test for unit roots in r_t and b_t (5% critical value is -2.89):

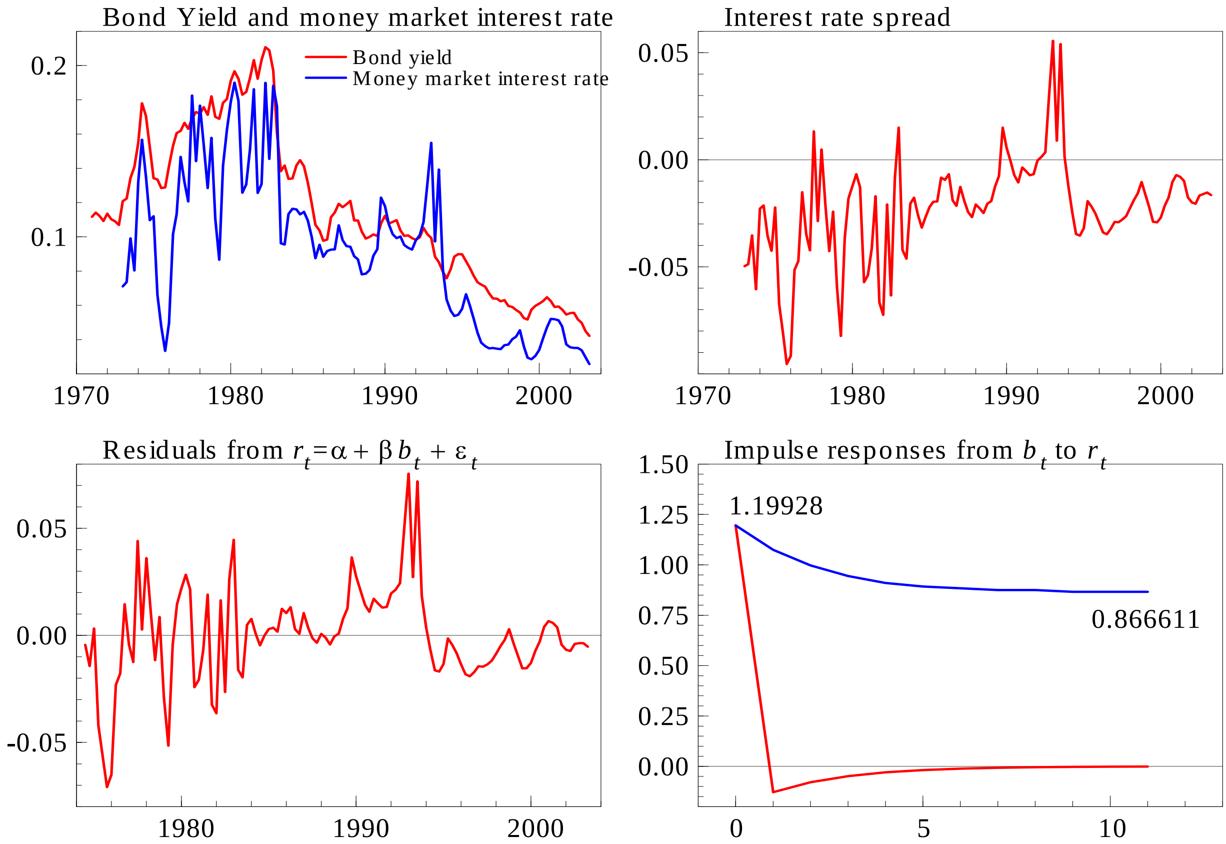
$$\widehat{\Delta r}_t = \underset{(1.35)}{0.00638118} - \underset{(-1.39)}{0.126209} \cdot \Delta r_{t-1} - \underset{(-2.70)}{0.234330} \cdot \Delta r_{t-4} - \underset{(-1.80)}{0.0826987} \cdot r_{t-1}$$

$$\widehat{\Delta b}_t = \underset{(0.658)}{0.00116558} + \underset{(4.67)}{0.395115} \cdot \Delta b_{t-1} - \underset{(-0.909)}{0.0128941} \cdot b_{t-1}$$

- We cannot reject unit roots. Test if $s_t = r_t - b_t$ is I(1) (5% crit. value is -2.89):

$$\widehat{\Delta s}_t = \underset{(-3.71)}{-0.00848594} + \underset{(2.56)}{0.207606} \cdot \Delta s_{t-3} - \underset{(-5.35)}{0.379449} \cdot s_{t-1}.$$

It is easily rejected that b_t and r_t are not cointegrating.



- Instead of assuming $\beta_1 = 1$ we could estimate the coefficient

Modelling IMM by OLS (using PR0312.in7)
The estimation sample is: 1974 (3) to 2003 (2)

	Coefficient	Std.Error	t-value	t-prob	Part.R^2
Constant	-0.00468506	0.005545	-0.845	0.400	0.0062
IBZ	0.845524	0.04495	18.8	0.000	0.7563

sigma	0.0224339	RSS	0.0573738644
R^2	0.756314	F(1,114) =	353.8 [0.000]**
log-likelihood	276.885	DW	0.82
no. of observations	116	no. of parameters	2
mean(IMM)	0.0919727	var(IMM)	0.00202967

- We could test for a unit root in the residuals (5% crit. value is -3.34):

$$\Delta\widehat{\epsilon}_t = \underset{(2.95)}{0.230210} \cdot \Delta\widehat{\epsilon}_{t-3} - \underset{(-6.77)}{0.499443} \cdot \widehat{\epsilon}_{t-1}.$$

Again we reject no-cointegration.

- Finally we could estimate the error correction models based on the spread:

$$\widehat{\Delta r_t} = \underset{(-3.23)}{-0.00774026} + \underset{(4.55)}{1.17725} \cdot \Delta b_t - \underset{(5.22)}{0.406456} \cdot (r_{t-1} - b_{t-1})$$

$$\widehat{\Delta b_t} = \underset{(-2.11)}{-0.00181602} + \underset{(4.16)}{0.438970} \cdot \Delta b_{t-1} - \underset{(-2.01)}{0.0673997} \cdot \Delta r_t - \underset{(2.22)}{0.0638286} \cdot (r_{t-1} - b_{t-1})$$

Note that the short-rate, r_t, error corrects, while the bond-yield, b_t, does not.

Estimation of β In the ADL/ECM

- The estimator of β_2 from a static regression is super-consistent...but
  (1) $\widehat{\beta}_2$ is often biased (due to ignored dynamics).
  (2) Hypotheses on β_2 cannot be tested.

- An alternative estimator is based on an unrestricted ADL model, e.g.

$$X_{1t} = \delta + \theta_1 X_{1t-1} + \theta_2 X_{1t-2} + \phi_0 X_{2t} + \phi_1 X_{2t-1} + \phi_2 X_{2t-2} + \epsilon_t,$$

where ϵ_t is IID. This is equivalent to an error correction model:

$$\Delta X_{1t} = \delta + \lambda_1 \Delta X_{1t-1} + \kappa_0 \Delta X_{2t} + \kappa_1 \Delta X_{2t-1} + \gamma_1 X_{1t-1} + \gamma_2 X_{2t-1} + \epsilon_t.$$

An estimate of β_2 can be found from the long-run solutions:

$$\widehat{\beta}_2 = \frac{-\widehat{\gamma}_2}{\widehat{\gamma}_1} = \frac{\widehat{\phi}_0 + \widehat{\phi}_1 + \widehat{\phi}_2}{1 - \widehat{\theta}_1 - \widehat{\theta}_2}.$$

- The main advantage is that the analysis is undertaken in a well-specified model.
  The approach is optimal if only X_{1t} error corrects.
  Inference on $\widehat{\beta}_2$ is possible.

Testing for No-Cointegration

- Due to representation theorem, the null hypothesis of no-cointegration corresponds to the null of no-error-correction. Several tests have been designed in this spirit.
- The most convenient is the so-called PcGive test for no-cointegration.
- Consider the unrestricted ADL or ECM:

$$\Delta X_{1t} = \delta + \lambda_1 \Delta X_{1t-1} + \kappa_0 \Delta X_{2t} + \kappa_1 \Delta X_{2t-1} + \gamma_1 X_{1t-1} + \gamma_2 X_{2t-1} + \epsilon_t. \quad (\#)$$

  Test the hypothesis

$$H_0 : \gamma_1 = 0$$

  against the cointegration alternative, $\gamma_1 < 0$.
- This is basically a unit root test (not a $N(0,1)$). The distribution of the $t-$ratio,

$$t_{\gamma_1=0} = \frac{\widehat{\gamma}_1}{\mathsf{SE}(\widehat{\gamma}_1)},$$

  depends on the deterministic terms and the number of regressors in $(\#)$.

- Critical values for the PcGive test for no-cointegration are given by:

Case 1: A constant term in $(\#)$.

Number of variables in X_t	Significance level		
	1%	5%	10%
2	-3.79	-3.21	-2.91
3	-4.09	-3.51	-3.19
4	-4.36	-3.76	-3.44
5	-4.59	-3.99	-3.66

Case 2: A constant and a trend in $(\#)$.

Number of variables in X_t	Significance level		
	1%	5%	10%
2	-4.25	-3.69	-3.39
3	-4.50	-3.93	-3.62
4	-4.72	-4.14	-3.83
5	-4.93	-4.34	-4.03

Outline of a (One-Step) Cointegration Analysis

(1) Test individual variables, e.g. X_{1t} and X_{2t}, for unit roots.

(2) Estimate an ADL model

$$\Delta X_{1t} = \delta + \lambda_1 \Delta X_{1t-1} + \kappa_0 \Delta X_{2t} + \kappa_1 \Delta X_{2t-1} + \gamma_1 X_{1t-1} + \gamma_2 X_{2t-1} + \epsilon_t.$$

(3) Test for no-cointegration with $t_{\gamma_1=0}$.
If cointegration is found, the cointegrating relation is the long-run solution.

(4) Derive the long-run solution

$$X_{1t} = \widehat{\mu} + \widehat{\beta}_2 X_{2t}.$$

Inference on β_2 is standard (under some conditions).

Empirical Example: Interest Rates Revisited

Estimation based on a ADL model. The significant terms are:

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Modelling IMM by OLS (using PR0312.in7)
The estimation sample is: 1973 (4) to 2003 (2)
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	Coefficient	Std.Error	t-value	t-prob	Part.R^2
IMM_1	0.615152	0.07909	7.78	0.000	0.3447
Constant	-0.00250456	0.004573	-0.548	0.585	0.0026
IBZ	1.19928	0.2347	5.11	0.000	0.1851
IBZ_1	-0.865763	0.2648	-3.27	0.001	0.0851

sigma	0.0182398	RSS	0.0382594939
R^2	0.841437	F(3,115) =	203.4 [0.000]**
log-likelihood	309.674	DW	2.16
no. of observations	119	no. of parameters	4
mean(IMM)	0.092754	var(IMM)	0.00202764

The long-run solution is given in PcGive as

```
Solved static long-run equation for IMM
                     Coefficient  Std.Error  t-value  t-prob
Constant             -0.00650791    0.01184   -0.550   0.584
IBZ                     0.866611    0.09491     9.13   0.000
Long-run sigma = 0.0473948
```

Here the $t-$values can be used for testing! β_2 is not significantly different from unity.
The test for no-cointegration is given by (critical value -3.69):

```
PcGive Unit-root t-test: -4.8661
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The impulse responses $\partial X_{1t}/\partial X_{2t}, \quad \partial X_{1t}/\partial X_{2t-1}, ...$ and the cumulated $\sum \partial X_{1t}/\partial X_{2t-i}$ can be graphed.

Spurious Regression Revisited

- Recall that cointegration is a special case where all stochastic trends cancel.
  From an empirical point of view this an exception.

- What happens if the variables do not cointegrate?

- Assume that X_{1t} and X_{2t} are two totally unrelated I(1) variables.
  Then we would like the static regression

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t, \qquad (\$)$$

  to reveal that $\beta_2 = 0$ and $R^2 = 0$.

- This turns out not to be the case!
  The standard regression output will indicate a relation between X_{1t} and X_{2t}.
  This is called a spurious regression or nonsense regression result.

- With non-stationary data we always have to think in terms of cointegration.

Simulation: Stationary Case

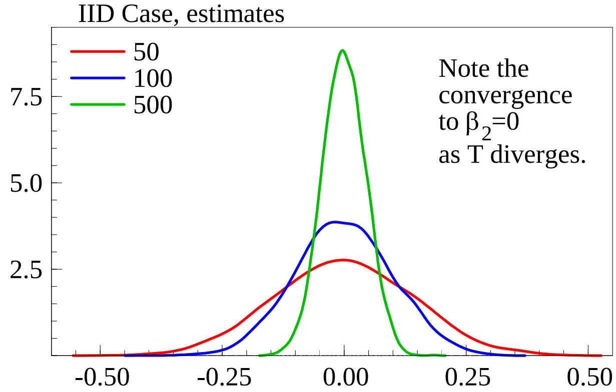
- Consider first two independent IID variables:

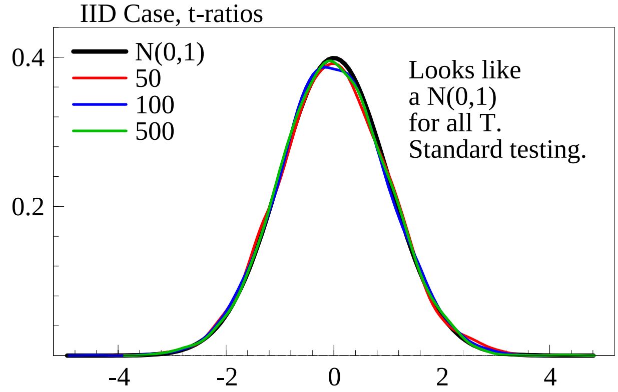
$$\begin{matrix} X_{1t} = \epsilon_{1t} \\ X_{2t} = \epsilon_{2t} \end{matrix} \quad \text{where} \quad \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

for $T = 50, 100, 500.$

- Here, we get standard results for the regression model

$$X_{1t} = \mu + \beta_2 X_{2t} + u_t.$$





Simulation: I(1) Spurious Regression

- Now consider two independent random walks

$$\begin{matrix} X_{1t} = X_{1t-1} + \epsilon_{1t} \\ X_{2t} = X_{2t-1} + \epsilon_{2t} \end{matrix} \quad \text{where} \quad \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

for $T = 50, 100, 500.$

- Under the null hypothesis, $\beta_2 = 0$, the residual is I(1). The condition for consistency is not fulfilled.