Introduction

- For many financial time series there is a tendency to volatility clustering. E.g. periods of high and low market uncertainty.
- ARCH and GARCH models is a way of modelling this feature. Specify equations for the (conditional) mean and the (conditional) variance.

Outline:
1. ARCH defined.
2. Test of no-ARCH.
3. GARCH defined.
4. Estimation.
5. Example: The Nasdaq index.
ARCH Defined

- Consider an equation for the conditional mean:
  \[ y_t = x_t' \theta + \epsilon_t, \quad t = 1, 2, \ldots, T. \]  
  (*)
  Often \( x_t \) contains lags of \( y_t \) and dummies for special features of the market.

- The ARCH(1) model also specifies an equation for the conditional variance:
  \[ \sigma_t^2 = E[\epsilon_t^2 \mid I_{t-1}] = \varpi + \alpha \epsilon_{t-1}^2. \]  
  (**) 
  To ensure that \( \sigma_t^2 \geq 0 \), we need \( \varpi \geq 0, \alpha \geq 0 \).

- If \( \epsilon_{t-1}^2 \) is high, the variance of the next shock, \( \epsilon_t \), is large.

- We condition on the information set \( I_{t-1} = \{ \epsilon_{t-1}, \epsilon_{t-2}, \epsilon_{t-3}, \ldots \} \).

- In the presence of ARCH, OLS is consistent but inefficient. 
  There exist a non-linear estimator that takes the ARCH structure into account.

- It is useful to define the surprise in the squared innovations
  \[ v_t = \epsilon_t^2 - E[\epsilon_t^2 \mid I_{t-1}] = \epsilon_t^2 - \sigma_t^2, \]
  and rewrite (**) as
  \[ \epsilon_t^2 = \varpi + \alpha \epsilon_{t-1}^2 + v_t. \]
  The squared innovation, \( \epsilon_t^2 \), follows an AR(1) process.

- The unconditional variance is given by
  \[
  E[\epsilon_t^2] = \varpi + \alpha E[\epsilon_{t-1}^2] \\
  \sigma^2 = \frac{\varpi}{1 - \alpha},
  \]
  which has a stationary solution for \( 0 \leq \alpha < 1 \).

- Generalizes to an ARCH(p) model:
  \[ \sigma_t^2 = \varpi + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \ldots + \alpha_p \epsilon_{t-p}^2. \]
Example I: Price of IBM Stock

(A) IBM stock, percent month-on-month

(B) Squared returns

(C) ACF - Returns

(D) ACF - Squared returns

Example II: Danish Stock Market Index (KFX)

(A) KFX stock index (log)

(B) Log return on KFX stock index

(C) Squared returns

(D) ACF - Squared returns
Example III: Danish NEER, 1990-2005

Test for ARCH

- Use the Breusch-Pagan LM test for heteroskedasticity.

- To test for ARCH of order $p$ consider the auxiliary regression model

  \[ \epsilon_t^2 = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 \epsilon_{t-2}^2 + \ldots + \beta_p \epsilon_{t-p}^2 + \eta_t. \]

  Under the null of no ARCH,

  \[ H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0. \]

  The hypothesis can be tested using the familiar statistic

  \[ T \cdot R^2 \rightarrow \chi^2(p). \]

- The ARCH test has also power against residual autocorrelation.
  Test for autocorrelation first.
GARCH Defined

• The popular GARCH(1,1) model is defined by

\[ \sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \]

For \( \sigma_t^2 \) to be non-negative we require the coefficients to be non-negative.

• Using the definition \( \sigma_t^2 = \epsilon_t^2 - v_t \), we get that

\[
\begin{align*}
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
\epsilon_t^2 - v_t &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 (\epsilon_{t-1}^2 - v_{t-1}) \\
\epsilon_t^2 &= \omega + (\alpha_1 + \beta_1) \epsilon_{t-1}^2 + v_t - \beta_1 v_{t-1},
\end{align*}
\]

which is an ARMA(1,1) model for the squared innovation. Stationarity requires that \( \alpha_1 + \beta_1 < 1 \).

• Generalizes to a GARCH(p,q) model:

\[ \sigma_t^2 = \omega + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2. \]

The GARCH model is equivalent to an infinite ARCH model.

ARCH in Mean

• There are many extensions and elaborations of the ARCH/GARCH model. Some are mentioned in the book.

• An interesting extension is where the volatility, as measured by \( \sigma_t^2 \), affects the conditional mean, i.e.

\[
\begin{align*}
y_t &= x_t' \theta + \delta \sigma_t^2 + \epsilon_t \\
or \quad y_t &= x_t' \theta + \delta \sigma_t + \epsilon_t
\end{align*}
\]

• An example could be that market participants require higher average returns to compensate a higher risk.
Maximum Likelihood Estimation

- Consider the ARCH(1) case
  \[ y_t = x_t' \theta + \epsilon_t \]
  \[ \sigma_t^2 = \varpi + \alpha \epsilon_{t-1}^2. \]

- Assume conditional normality:
  \[ \epsilon_t = y_t - x_t' \theta = \sigma_t v_t, \quad v_t \sim N(0, 1). \]

- We specify the normal likelihood function as
  \[
  L_t(\theta, \varpi, \alpha \mid y_t, x_t, I_{t-1}) = \frac{1}{\sqrt{2\pi \sigma_t^2}} \exp \left( -\frac{1}{2} \frac{\epsilon_t^2}{\sigma_t^2} \right),
  \]
  and maximize wrt. \( \theta, \varpi \) and \( \alpha \).
  Note: Cannot solve the likelihood equations analytically.

- Other (typically fat-tailed distributions) can also be used.

Empirical Example

- Daily data for the Nasdaq index 31/1-2000 till 26/2-2004, 1042 observations.
  We consider the log returns
  \[ y_t = \log(\text{NASDAQ}_t) - \log(\text{NASDAQ}_{t-1}). \]

- We want to estimate a GARCH(1,1) model based on an AR(1), i.e.
  \[ y_t = \theta_0 + \theta_1 y_{t-1} + \epsilon_t \]
  \[ \sigma_t^2 = \varpi + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \]
  with normal innovations.
EQ( 1) Modelling DlogNasdaq by OLS (using Nasdaq.in7)

The estimation sample is: 3 to 1042

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
<th>Part.R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>DlogNasdaq_1</td>
<td>-0.00243025</td>
<td>-0.0785</td>
<td>0.937</td>
<td>0.0000</td>
</tr>
<tr>
<td>Constant</td>
<td>-0.000628696</td>
<td>-0.850</td>
<td>0.395</td>
<td>0.0007</td>
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</table>

ARCH coefficients:

<table>
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<tr>
<th>Lag</th>
<th>Coefficient</th>
<th>Std.Error</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.080823</td>
<td>0.03109</td>
</tr>
<tr>
<td>2</td>
<td>0.1732</td>
<td>0.03106</td>
</tr>
<tr>
<td>3</td>
<td>0.068651</td>
<td>0.03144</td>
</tr>
<tr>
<td>4</td>
<td>0.087759</td>
<td>0.03104</td>
</tr>
<tr>
<td>5</td>
<td>0.080797</td>
<td>0.03105</td>
</tr>
</tbody>
</table>

RSS = 0.00118881 sigma = 0.00107537

Testing for error ARCH from lags 1 to 5
ARCH 1-5 test: F(5, 1028) = 19.666 [0.0000]**
VOL (1) Modelling DlogNasdaq by restricted GARCH(1,1) (Nasdaq.in7)

The estimation sample is: 3 to 1042

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>robust-SE</th>
<th>t-value</th>
<th>t-prob</th>
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<tbody>
<tr>
<td>DlogNasdaq_1</td>
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<td>0.03246</td>
<td>0.03105</td>
<td>-0.393</td>
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<tr>
<td>alpha_0</td>
<td>3.30480e-006</td>
<td>2.113e-006</td>
<td>1.872e-006</td>
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<td>alpha_1</td>
<td>0.0746032</td>
<td>0.01588</td>
<td>0.01610</td>
<td>4.63</td>
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<td>beta_1</td>
<td>0.919901</td>
<td>0.01578</td>
<td>0.01423</td>
<td>64.7</td>
</tr>
</tbody>
</table>

log-likelihood 2523.8308
HMSE 2.16818

mean(h_t) 0.000576232
var(h_t) 1.90665e-007

no. of observations 1040
no. of parameters 5

AIC.T -5037.6616
AIC -4.84390539

mean(DlogNasdaq) -0.000627028
var(DlogNasdaq) 0.000567281

alpha(1)+beta(1) 0.994504
alpha_i+beta_i>=0, alpha(1)+beta(1)<1

(A) Residuals and 95% confidence band

2 conditional standard errors
(A) Forecast of conditional mean

(B) Forecast of conditional variance