Econometrics 2 – Fall 2005

Autoregressive Conditional Heteroskedasticity (ARCH)

Heino Bohn Nielsen

Introduction

- For many financial time series there is a tendency to volatility clustering. E.g. periods of high and low market uncertainty.
- ARCH and GARCH models is a way of modelling this feature. Specify equations for the (conditional) mean and the (conditional) variance.

Outline:

(1) ARCH defined.

(2) Test of no-ARCH.

(3) GARCH defined.

(4) Estimation.

(5) Example: The Nasdaq index.

ARCH Defined

- Consider an equation for the conditional mean:

$$y_t = x_t'\theta + \epsilon_t, \quad t = 1, 2, ..., T. \qquad (*)$$

Often x_t contains lags of y_t and dummies for special features of the market.

- The ARCH(1) model also specifies an equation for the conditional variance:

$$\sigma_t^2 = E[\epsilon_t^2 \mid \mathcal{I}_{t-1}] = \varpi + \alpha\epsilon_{t-1}^2. \qquad (**)$$

To ensure that $\sigma_t^2 \geq 0$, we need $\varpi \geq 0$, $\alpha \geq 0$.

- If ϵ_{t-1}^2 is high, the variance of the next shock, ϵ_t, is large.

- We condition on the information set $\mathcal{I}_{t-1} = \{\epsilon_{t-1}, \epsilon_{t-2}, \epsilon_{t-3}, ...\}$.

- In the presence of ARCH, OLS is consistent but inefficient.
There exist a non-linear estimator that takes the ARCH structure into account.

- It is useful to define the surprise in the squared innovations

$$v_t = \epsilon_t^2 - E[\epsilon_t^2 \mid \mathcal{I}_{t-1}] = \epsilon_t^2 - \sigma_t^2,$$

and rewrite $(**)$ as

$$\epsilon_t^2 = \varpi + \alpha\epsilon_{t-1}^2 + v_t.$$

The squared innovation, ϵ_t^2, follows an AR(1) process.

- The unconditional variance is given by

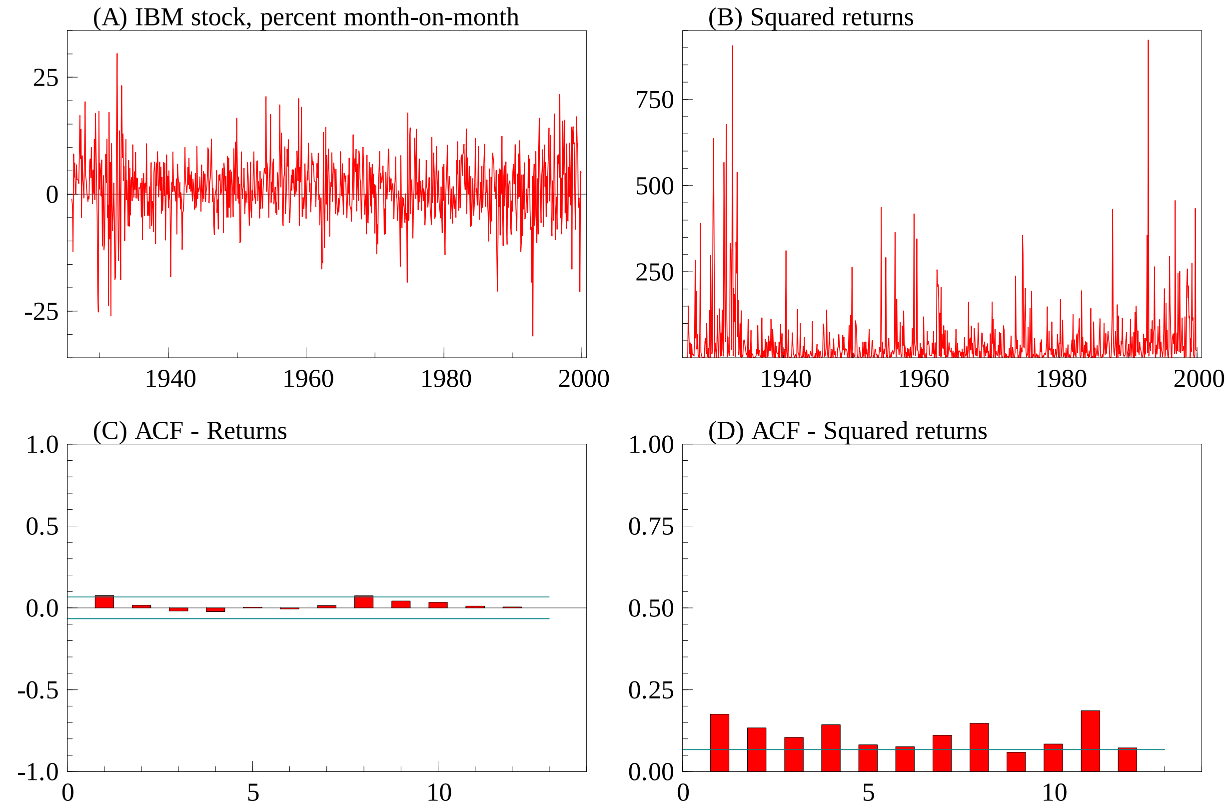
$$\begin{aligned} E[\epsilon_t^2] &= \varpi + \alpha E[\epsilon_{t-1}^2] \\ \sigma^2 &= \frac{\varpi}{1-\alpha}, \end{aligned}$$

which has a stationary solution for $0 \leq \alpha < 1$.

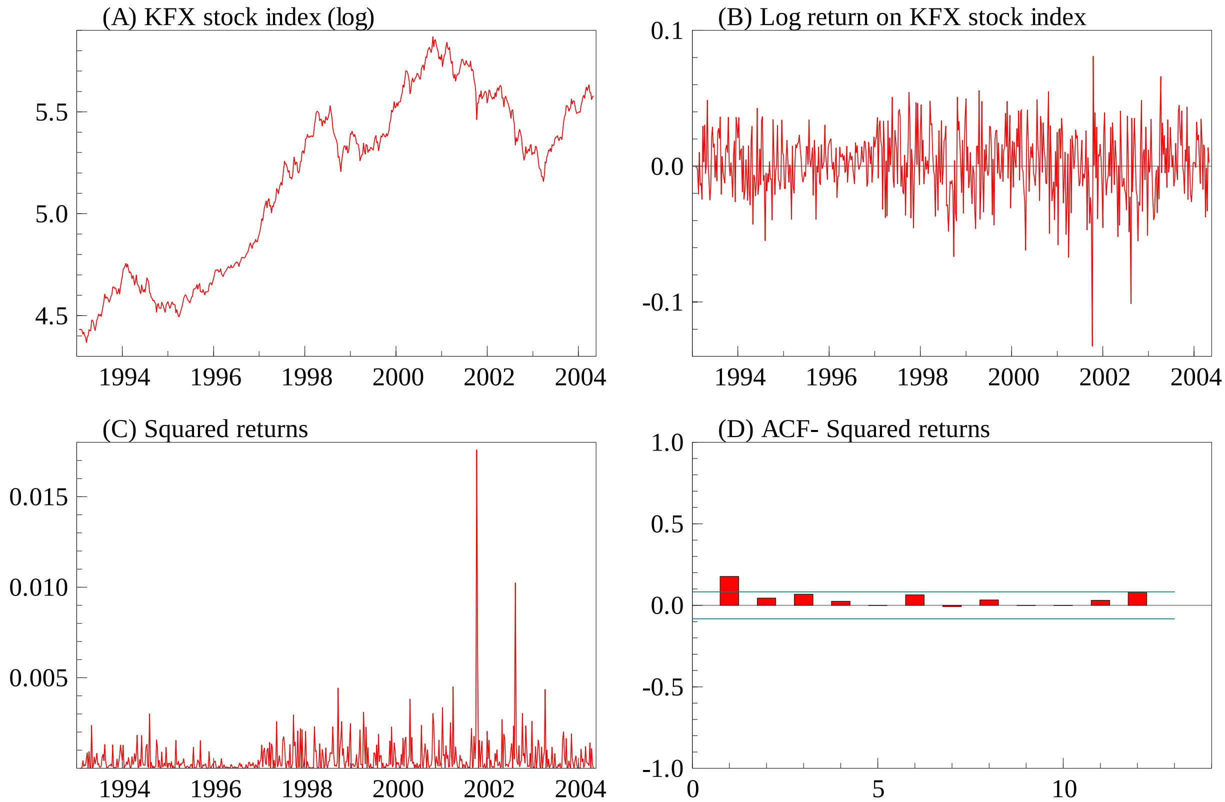
- Generalizes to an ARCH(p) model:

$$\sigma_t^2 = \varpi + \alpha_1\epsilon_{t-1}^2 + \alpha_2\epsilon_{t-2}^2 + ... + \alpha_p\epsilon_{t-p}^2.$$

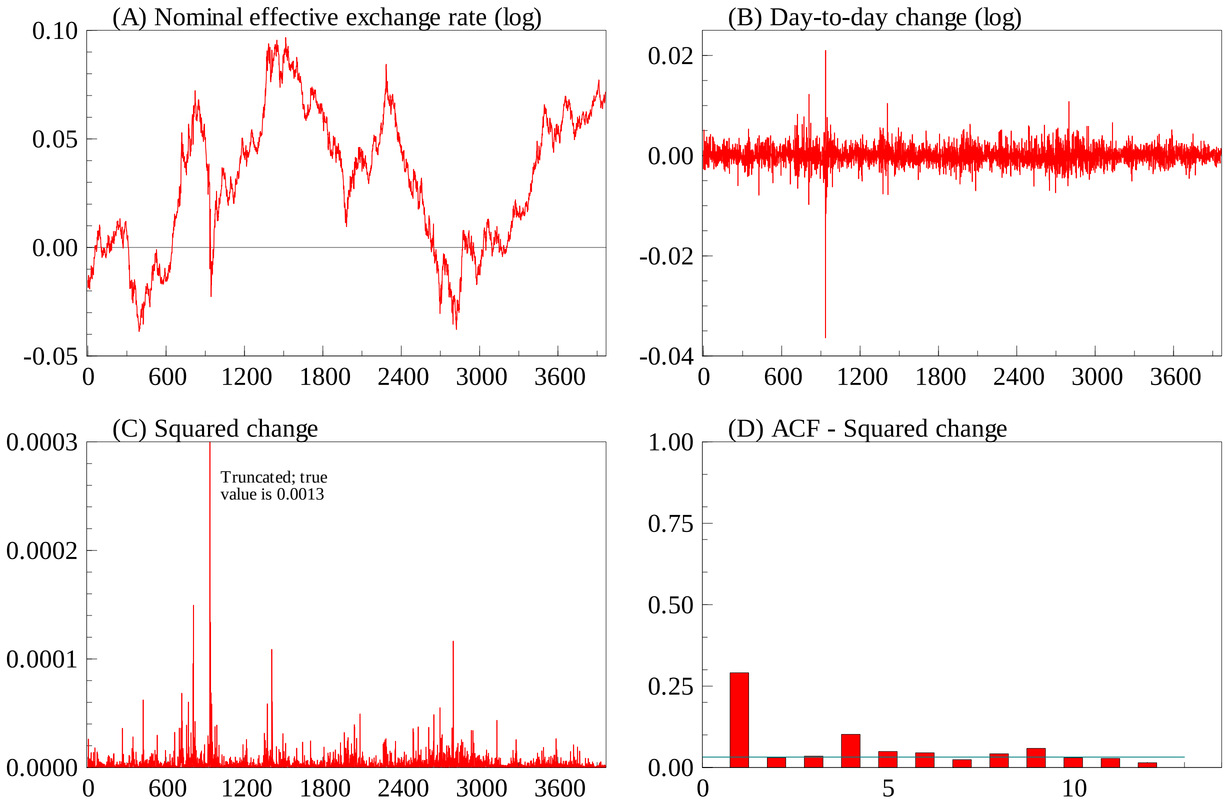
Example I: Price of IBM Stock



Example II: Danish Stock Market Index (KFX)



Example III: Danish NEER, 1990-2005



Test for ARCH

- Use the Breusch-Pagan LM test for heteroskedasticity.

- To test for ARCH of order p consider the auxiliary regression model

$$\epsilon_t^2 = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 \epsilon_{t-2}^2 + ... + \beta_p \epsilon_{t-p}^2 + \eta_t.$$

  Under the null of no ARCH,

$$H_0 : \beta_1 = \beta_2 = ... = \beta_p = 0.$$

  The hypothesis can be tested using the familiar statistic

$$T \cdot R^2 \rightarrow \chi^2(p).$$

- The ARCH test has also power against residual autocorrelation.
  Test for autocorrelation first.

GARCH Defined

- The popular GARCH(1,1) model is defined by

$$\sigma_t^2 = \varpi + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

  For σ_t^2 to be non-negative we require the coefficients to be non-negative.

- Using the definition $\sigma_t^2 = \epsilon_t^2 - v_t$, we get that

$$\begin{aligned} \sigma_t^2 &= \varpi + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ \epsilon_t^2 - v_t &= \varpi + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \left(\epsilon_{t-1}^2 - v_{t-1} \right) \\ \epsilon_t^2 &= \varpi + (\alpha_1 + \beta_1) \epsilon_{t-1}^2 + v_t - \beta_1 v_{t-1}, \end{aligned}$$

  which is an ARMA(1,1) model for the squared innovation.
  Stationarity requires that $\alpha_1 + \beta_1 < 1$.

- Generalizes to a GARCH(p,q) model:

$$\sigma_t^2 = \varpi + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

  The GARCH model is equivalent to an infinite ARCH model.

ARCH in Mean

- There are many extensions and elaborations of the ARCH/GARCH model.
  Some are mentioned in the book.

- An interesting extension is where the volatility, as measured by σ_t^2, affects the conditional mean, i.e.

$$\begin{aligned} y_t &= x_t'\theta + \delta \sigma_t^2 + \epsilon_t \\ \text{or} \quad y_t &= x_t'\theta + \delta \sigma_t + \epsilon_t \end{aligned}$$

- An example could be that market participants require higher average returns to compensate a higher risk.

Maximum Likelihood Estimation

- Consider the ARCH(1) case

$$\begin{aligned} y_t &= x_t'\theta + \epsilon_t \\ \sigma_t^2 &= \varpi + \alpha\epsilon_{t-1}^2. \end{aligned}$$

- Assume conditional normality:

$$\epsilon_t = y_t - x_t'\theta = \sigma_t v_t, \quad v_t \sim N(0,1).$$

- We specify the normal likelihood function as

$$L_t(\theta, \varpi, \alpha \mid y_t, x_t, \mathcal{I}_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{1}{2}\frac{\epsilon_t^2}{\sigma_t^2}\right),$$

  and maximize wrt. θ, ϖ and α.
  Note: Cannot solve the likelihood equations analytically.

- Other (typically fat-tailed distributions) can also be used.

Empirical Example

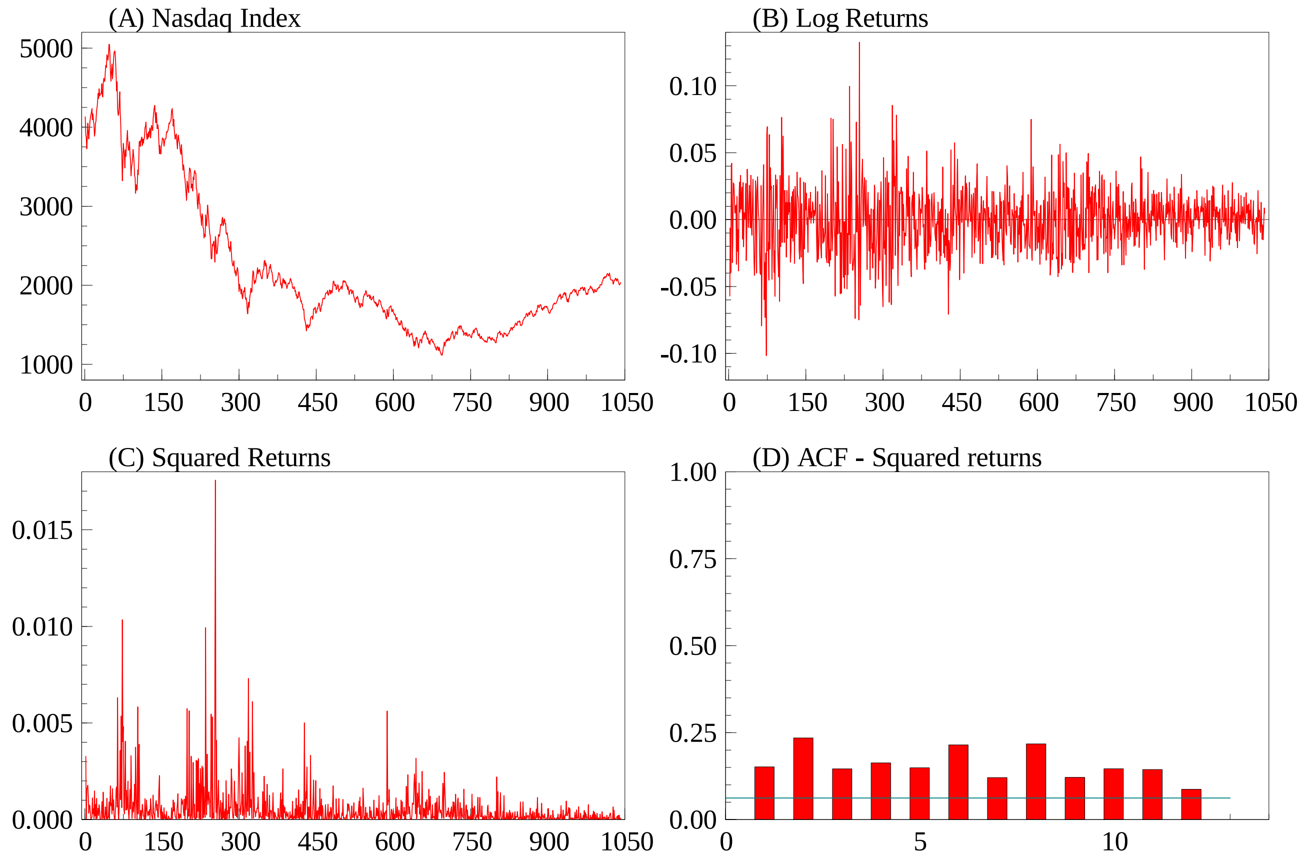
- Daily data for the Nasdaq index 31/1-2000 till 26/2-2004, 1042 observations.
  We consider the log returns

$$y_t = \log(\textsf{NASDAQ}_t) - \log(\textsf{NASDAQ}_{t-1}).$$

- We want to estimate a GARCH(1,1) model based on an AR(1), i.e.

$$\begin{aligned} y_t &= \theta_0 + \theta_1 y_{t-1} + \epsilon_t \\ \sigma_t^2 &= \varpi + \alpha_1\epsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 \end{aligned}$$

  with normal innovations.

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EQ( 1) Modelling DlogNasdaq by OLS (using Nasdaq.in7)
       The estimation sample is: 3 to 1042

                  Coefficient  Std.Error  t-value  t-prob Part.R^2
DlogNasdaq_1      -0.00243025    0.03096  -0.0785   0.937   0.0000
Constant         -0.000628696  0.0007396   -0.850   0.395   0.0007

ARCH coefficients:
  Lag Coefficient  Std.Error
    1    0.080823    0.03109
    2      0.1732    0.03106
    3    0.068651    0.03144
    4    0.087759    0.03104
    5    0.080797    0.03105
RSS = 0.00118881  sigma = 0.00107537

Testing for error ARCH from lags 1 to 5
ARCH 1-5 test:    F(5,1028)=   19.666 [0.0000]**
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VOL( 1) Modelling DlogNasdaq by restricted GARCH(1,1) (Nasdaq.in7)
        The estimation sample is: 3 to 1042

                      Coefficient  Std.Error  robust-SE  t-value  t-prob
DlogNasdaq_1  Y         -0.0121906     0.03246    0.03105   -0.393   0.695
Constant      X       0.000595005  0.0005788  0.0006317    0.942   0.346
alpha_0       H     3.30480e-006 2.113e-006 1.872e-006     1.77   0.078
alpha_1       H         0.0746032    0.01588    0.01610     4.63   0.000
beta_1        H          0.919901    0.01578    0.01423     64.7   0.000

log-likelihood       2523.8308  HMSE                    2.16818
mean(h_t)          0.000576232  var(h_t)           1.90665e-007
no. of observations       1040  no. of parameters             5
AIC.T               -5037.6616  AIC                 -4.84390539
mean(DlogNasdaq) -0.000627028  var(DlogNasdaq)    0.000567281
alpha(1)+beta(1)      0.994504  alpha_i+beta_i>=0, alpha(1)+beta(1)<1
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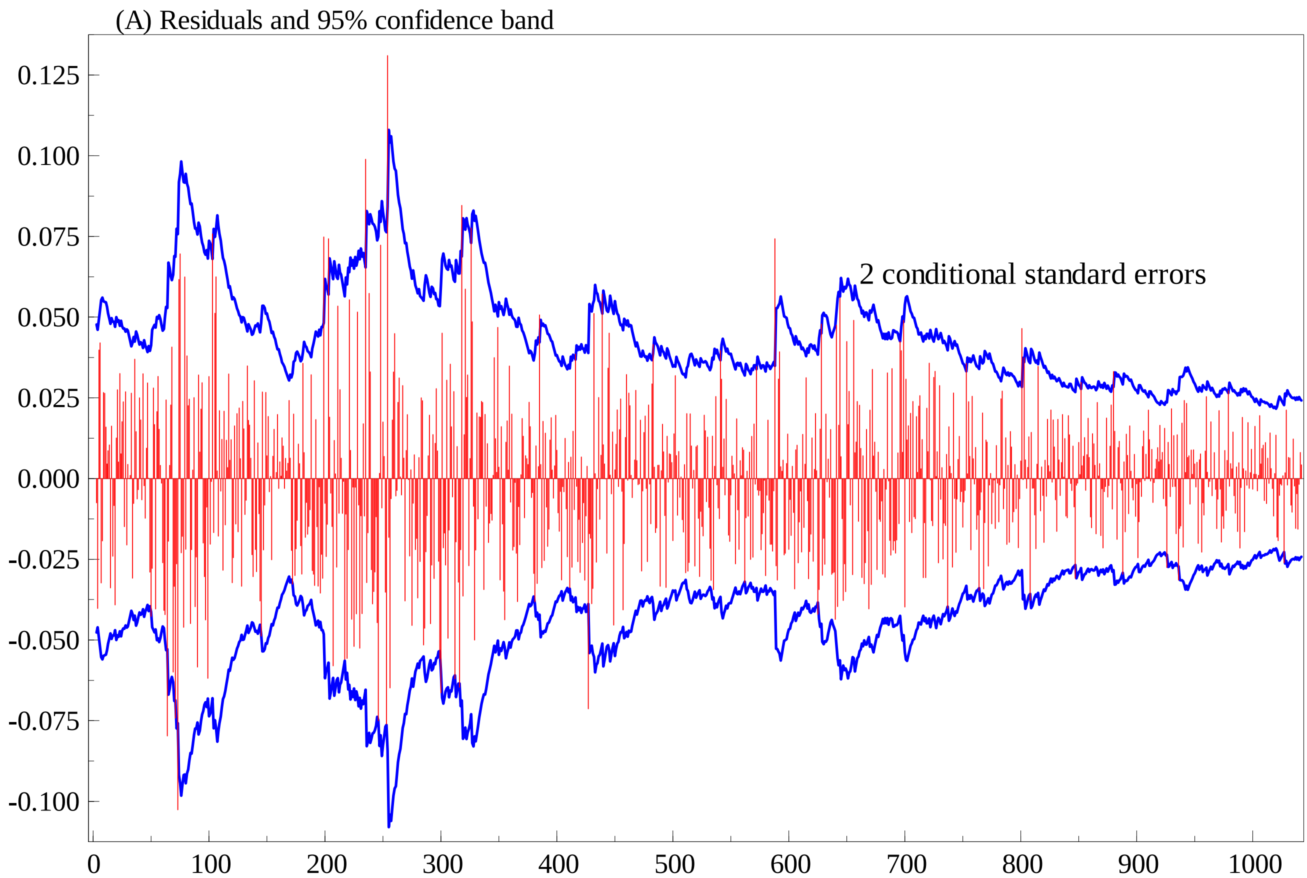

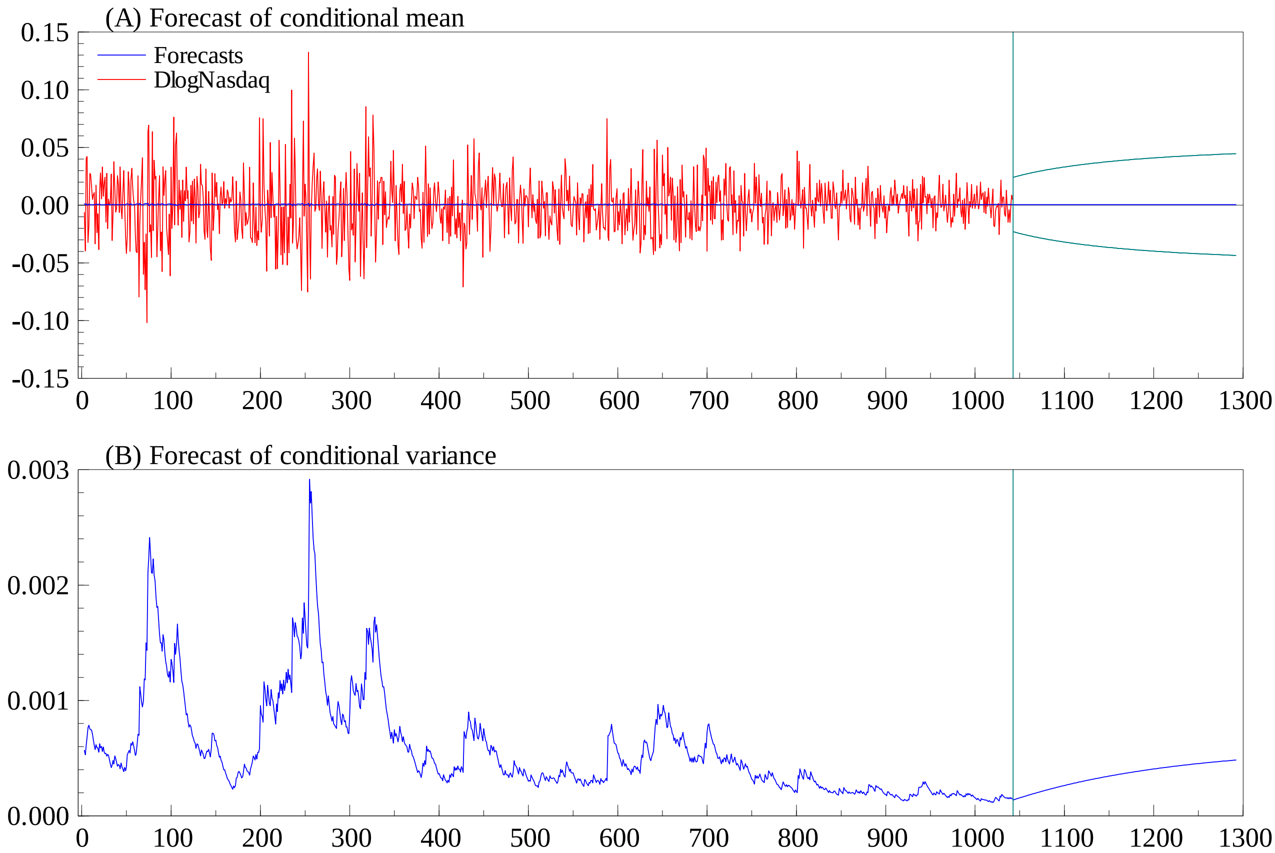

(A) Forecast of conditional mean
Forecasts
DlogNasdaq
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
(B) Forecast of conditional variance
0.003
0.002
0.001
0.000