

# Chapter 3

## The basic OLG model: Diamond

There exists two main analytical frameworks for analyzing the basic intertemporal choice, consumption versus saving, and the dynamic implications of this choice: *overlapping-generations* (OLG) models and *representative agent* models. In the first type of models the focus is on (a) the interaction between different generations alive at the same time, and (b) the never-ending entrance of new generations and thereby new decision makers. In the second type of models the household sector is modelled as consisting of a finite number of infinitely-lived dynasties. One interpretation is that the parents take the utility of their descendants into account by leaving bequests and so on forward through a chain of intergenerational links. This approach, which is also called the Ramsey approach (after the British mathematician and economist Frank Ramsey, 1903-1930), will be described in Chapter 8 (discrete time) and Chapter 10 (continuous time).

In the present chapter we introduce the OLG approach which has shown its usefulness for analysis of many issues such as: public debt, taxation of capital income, financing of social security (pensions), design of educational systems, non-neutrality of money, and the possibility of speculative bubbles. We will focus on what is known as Diamond's OLG model<sup>1</sup> after the American economist and Nobel Prize laureate Peter A. Diamond (1940-).

Among the strengths of the model are:

- The *life-cycle* aspect of human behavior is taken into account. Although the economy is infinitely-lived, the individual agents are not. During lifetime one's educational level, working capacity, income, and needs change and this is reflected in the individual labor supply and saving behavior. The aggregate implications of the life-cycle behavior of coexisting individual agents at different stages in their life is at the centre of attention.

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<sup>1</sup>Diamond (1965).

- The model takes elementary forms of *heterogeneity* in the population into account – there are “old” and “young”, there are the currently-alive people and the future generations whose preferences are not reflected in current market transactions. Questions relating to the distribution of income and wealth across generations can be studied. For example, how does the investment in fixed capital and environmental protection by current generations affect the conditions for the succeeding generations?

### 3.1 Motives for saving

Before going into the specifics of Diamond’s model, let us briefly consider what may in general motivate people to save:

- (a) The *consumption-smoothing motive for saving*. Individuals go through a life cycle where earnings typically have a hump-shaped time pattern; by saving and dissaving the individual then attempts to obtain the desired smoothing of consumption across lifetime. This is the essence of the *life-cycle saving hypothesis* put forward by Nobel laureate Franco Modigliani (1918-2003) and associates in the 1950s. This hypothesis states that consumers plan their saving and dissaving in accordance with anticipated variations in income and needs over lifetime. Because needs vary less over lifetime than income, the time profile of saving tends to be hump-shaped with some dissaving early in life (for instance if studying), positive saving during the years of peak earnings and then dissaving after retirement.
- (b) The *precautionary motive for saving*. Income as well as needs may vary due to conditions of *uncertainty*: sudden unemployment, illness, or other kinds of bad luck. By saving, the individual can obtain a buffer against such unwelcome events.

Horioka and Watanabe (1997) find that empirically, the saving motives (a) and (b) are of dominant importance (Japanese data). Yet other motives include:

- (c) Saving enables the purchase of *durable consumption goods* and owner-occupied housing as well as repayment of debt.
- (d) Saving may be motivated by the *desire to leave bequests* to heirs.
- (e) Saving may simply be motivated by the fact that financial wealth may lead to *social prestige* and economic or political *power*.

Diamond's OLG model aims at simplicity and concentrates on motive (a). In fact only one aspect of motive (a) is considered, namely the saving for retirement. People live for two periods only, as "young" they work full-time and as "old" they retire and live by their savings. The model abstracts from a possible bequest motive.

Now to the details.

## 3.2 The model framework

The flow of time is divided into successive periods of equal length, taken as the time unit. Given the two-period lifetime of (adult) individuals, the period length is understood to be very long, around, say, 30 years. The main assumptions are:

1. The number of young people in period  $t$ , denoted  $L_t$ , changes over time according to  $L_t = L_0(1 + n)^t$ ,  $t = 0, 1, 2, \dots$ , where  $n$  is a constant,  $n > -1$ . Indivisibility is ignored and so  $L_t$  is just considered a positive real number.
2. Only the young work. Each young supplies one unit of labor inelastically. The division of available time between work and leisure is thereby considered as exogenous.
3. Output is homogeneous and can be used for consumption as well as investment in physical capital. Physical capital is the only non-human asset in the economy; it is owned by the old and rented out to the firms. Output is the numeraire (unit of account) used in trading. Money (means of payment) is ignored.<sup>2</sup>
4. The economy is closed (no foreign trade).
5. Firms' technology has constant returns to scale.
6. In each period three markets are open, a market for output, a market for labor services, and a market for capital services. Perfect competition rules in all markets. Uncertainty is absent; when a decision is made, its consequences are known.
7. Agents have perfect foresight.

Assumption 7 entails the following. First, the agents are assumed to have "rational expectations" or, with a better name, "model-consistent expectations".

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<sup>2</sup>As to the disregard of money we may imagine that agents have safe electronic accounts in a fictional central bank allowing costless transfers between accounts.

This means that forecasts made by the agents coincide with the forecasts that can be calculated on the basis of the model. Second, as there are no stochastic elements in the model (no uncertainty), the forecasts are point estimates rather than probabilistic forecasts. Thereby the model-consistent expectations take the extreme form of *perfect foresight*: the agents agree in their expectations about the future evolution of the economy and ex post this future evolution fully coincides with what was expected.

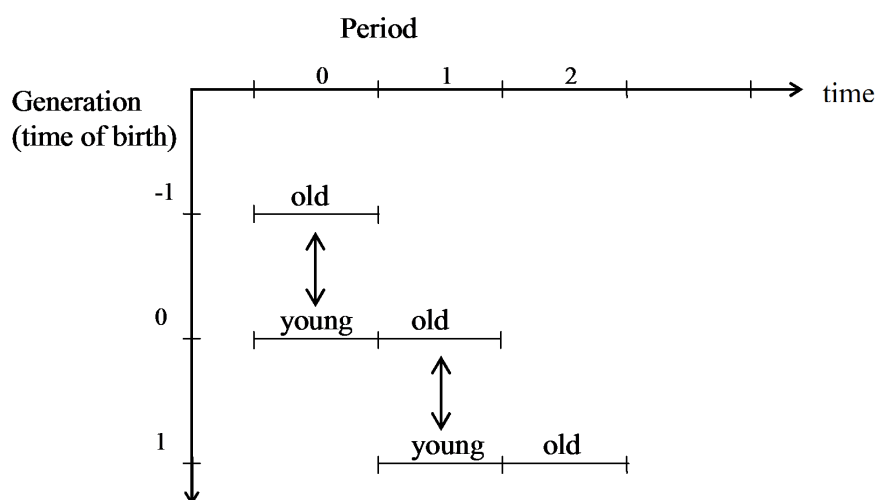


Figure 3.1: The two-period model's time structure.

Of course, this is an unrealistic assumption. The motivation is to simplify in a first approach. The results that emerge will be the outcome of economic mechanisms in isolation from expectational errors. In this sense the model constitutes a “pure” case (benchmark case).

The time structure of the model is illustrated in Fig. 3.1. In every period two generations are alive and interact with each other as indicated by the arrows. The young supply labor and earn a labor income. They consume an endogenous fraction of this income and save the remainder for retirement. Thereby the young offset the dissaving by the old, and possibly positive net investment arises in the economy. At the end of the first period the savings by the young are converted into direct ownership of capital goods. In the next period the now old owners of the capital goods rent them out to the firms. We may imagine that the firms are owned by the old, but this ownership is not visible in the equilibrium allocation because pure profits will be nil due to the combination of perfect competition and constant returns to scale.

Let the output good be the numeraire and let  $\hat{r}_t$  denote the rental rate for capital in period  $t$ ; that is,  $\hat{r}_t$  is the real price a firm has to pay at the end of

period  $t$  for the right to use one unit of someone else's physical capital through period  $t$ . So the owner of  $K_t$  units of physical capital receives a

$$\text{real (net) rate of return on capital} = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta, \quad (3.1)$$

where  $\delta$  is the rate of physical capital depreciation which is assumed constant,  $0 \leq \delta \leq 1$ .

Suppose there is also a market for loans. Assume you have lent out one unit of output from the end of period  $t - 1$  to the end of period  $t$ . If the *real interest rate* in the loan market is  $r_t$ , then, at the end of period  $t$  you should get back  $1 + r_t$  units of output. In the absence of uncertainty, equilibrium requires that capital and loans give the same rate of return,

$$\hat{r}_t - \delta = r_t. \quad (3.2)$$

This *no-arbitrage* condition indicates how the rental rate for capital and the more everyday concept, the interest rate, would be related in an equilibrium where both the market for capital services and a loan market were active. We shall see, however, that in this model no loan market will be active in an equilibrium. Nevertheless we will follow the tradition and call the right-hand side of (3.2) the *interest rate*.

Table 3.1 provides an overview of the notation. As to our timing convention, notice that any stock variable dated  $t$  indicates the amount held at the beginning of period  $t$ . That is, the capital stock accumulated by the end of period  $t - 1$  and available for production in period  $t$  is denoted  $K_t$ . We therefore write  $K_t = (1 - \delta)K_{t-1} + I_{t-1}$  and  $Y_t = F(K_t, L_t)$ , where  $F$  is an aggregate production function. In this context it is useful to think of “period  $t$ ” as running from date  $t$  to right before date  $t + 1$ . So period  $t$  is the half-open time interval  $[t, t + 1)$  on the continuous-time axis. Whereas production and consumption take place *in* period  $t$ , we imagine that all decisions are made at discrete points in time  $t = 0, 1, 2, \dots$  (“dates”). We further imagine that receipts for work and lending as well as payment for the consumption in period  $t$  occur at the end of the period. These timing conventions are common in discrete-time growth and business cycle theory;<sup>3</sup> they are convenient because they make switching between discrete and continuous time analysis fairly easy.

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<sup>3</sup>In contrast, in the *accounting* and *finance* literature, typically  $K_t$  would denote the *end-of-period- $t$*  stock that begins to yield its services *next* period.

Table 3.1. List of main variable symbols

<i>Symbol</i>	<i>Meaning</i>
$L_t$	the number of young people in period $t$
$n$	generation growth rate
$K_t$	aggregate capital available in period $t$
$c_{1t}$	consumption as young in period $t$
$c_{2t}$	consumption as old in period $t$
$w_t$	real wage in period $t$
$r_t$	real interest rate (from end of per. $t - 1$ to end of per. $t$ )
$\rho$	rate of time preference (impatience)
$\theta$	elasticity of marginal utility
$s_t$	saving of each young in period $t$
$Y_t$	aggregate output in period $t$
$C_t = c_{1t}L_t + c_{2t}L_{t-1}$	aggregate consumption in period $t$
$S_t = Y_t - C_t$	aggregate gross saving in period $t$
$\delta \in [0, 1]$	capital depreciation rate
$K_{t+1} - K_t = I_t - \delta K_t$	aggregate net investment in period $t$

### 3.3 The saving by the young

Suppose the preferences of the young can be represented by a lifetime utility function as specified in (3.3). Given  $w_t$  and  $r_{t+1}$ , the decision problem of the young in period  $t$  then is:

$$\max_{c_{1t}, c_{2t+1}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + (1 + \rho)^{-1}u(c_{2t+1}) \quad \text{s.t.} \quad (3.3)$$

$$c_{1t} + s_t = w_t \cdot 1 \quad (w_t > 0), \quad (3.4)$$

$$c_{2t+1} = (1 + r_{t+1})s_t \quad (r_{t+1} > -1), \quad (3.5)$$

$$c_{1t} \geq 0, c_{2t+1} \geq 0. \quad (3.6)$$

The interpretation of the variables is given in Table 3.1 above. We may think of the “young” as a household consisting of one adult and  $1 + n$  children whose consumption is included in  $c_{1t}$ . Note that “utility” appears at two levels. There is a *lifetime utility function*,  $U$ , and a *period utility function*,  $u$ .<sup>4</sup> The latter is assumed to be the same in both periods of life (this has no effects on the qualitative results and simplifies the exposition). The period utility function is assumed twice continuously differentiable with  $u' > 0$  and  $u'' < 0$  (positive, but diminishing marginal utility of consumption). Many popular specifications of  $u$ , e.g.,  $u(c) = \ln c$ , have the property that  $\lim_{c \rightarrow 0} u(c) = -\infty$ ; then we *define*  $u(0) = -\infty$ .

<sup>4</sup>Other names for these two functions are the *intertemporal utility function* and the *subutility function*, respectively.

The parameter  $\rho$  is called the *rate of time preference*. It acts as a utility discount *rate*, whereas  $(1+\rho)^{-1}$  is a utility discount *factor*. By definition,  $\rho > -1$ , but  $\rho > 0$  is usually assumed. We interpret  $\rho$  as reflecting the degree of *impatience* with respect to the “arrival” of utility. When preferences can be represented in this additive way, they are called *time-separable*. In principle, as seen from period  $t$  the interest rate appearing in (3.5) should be interpreted as an *expected* real interest rate. But as long as we assume perfect foresight, there is no need that our notation distinguishes between actual and expected magnitudes.

*Box 3.1. Discount rates and discount factors*

By a *discount rate* is meant an interest rate applied in the construction of a discount factor. A *discount factor* is a factor by which future benefits or costs, measured in some unit of account, are converted into present equivalents. The higher the discount rate the lower the discount factor.

One should bear in mind that a discount rate depends on what is to be discounted. In (3.3) the unit of account is “utility” and  $\rho$  acts as a *utility discount rate*. In (3.7) the unit of account is the consumption good and  $r_{t+1}$  acts as a *consumption discount rate*. If people also work as old, the right-hand side of (3.7) would read  $w_t + (1 + r_{t+1})^{-1}w_{t+1}$  and thus  $r_{t+1}$  would act as an *earnings discount rate*. This will be the same as the consumption discount rate if we think of real income measured in consumption units. But if we think of nominal income, that is, income measured in monetary units, there would be a *nominal earnings discount rate*, namely the *nominal* interest rate, which in an economy with inflation will exceed the consumption discount rate. Unfortunately, confusion of different discount rates is not rare.

In (3.5) the interest rate  $r_{t+1}$  acts as a (net) rate of return on saving.<sup>5</sup> An interest rate may also be seen as a discount rate relating to consumption over time. Indeed, by isolating  $s_t$  in (3.5) and substituting into (3.4), we may consolidate the two period budget constraints of the individual into *one* budget constraint,

$$c_{1t} + \frac{1}{1 + r_{t+1}}c_{2t+1} = w_t. \quad (3.7)$$

<sup>5</sup>While  $s_t$  in (3.4) appears as a *flow* (non-consumed income), in (3.5)  $s_t$  appears as a *stock* (the accumulated financial wealth at the end of period  $t$ ). This notation is legitimate because the magnitude of the two is the same when the time unit is the same as the period length. Indeed, the interpretation of  $s_t$  in (3.5)  $s_t = s_t \cdot \Delta t = s_t \cdot 1$  units of account.

In real life the gross payoff of individual saving may sometimes be nil (if invested in a project that completely failed). Unless otherwise indicated, it is in this book understood that an interest rate is a number exceeding  $-1$  as indicated in (3.5). Thereby the discount factor  $1/(1 + r_{t+1})$  is well-defined. In general equilibrium, the condition  $1 + r_{t+1} > 0$  is always met in the present model.

In this *intertemporal budget constraint* the interest rate appears as the discount rate entering the discount factor converting future amounts of consumption into present equivalents, cf. Box 3.1.

### Solving the saving problem

To avoid the possibility of corner solutions, we impose the No Fast Assumption

$$\lim_{c \rightarrow 0} u'(c) = \infty. \quad (\text{A1})$$

In view of the sizeable period length in the model, this is definitely plausible.

Inserting the two budget constraints into the objective function in (3.3), we get  $U(c_{1t}, c_{2t+1}) = u(w_t - s_t) + (1 + \rho)^{-1}u((1 + r_{t+1})s_t) \equiv \tilde{U}_t(s_t)$ , a function of only one decision variable,  $s_t$ . According to the non-negativity constraint on consumption in both periods, (3.6),  $s_t$  must satisfy  $0 \leq s_t \leq w_t$ . Maximizing with respect to  $s_t$  gives the first-order condition

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1 + \rho)^{-1}u'((1 + r_{t+1})s_t)(1 + r_{t+1}) = 0. \quad (\text{FOC})$$

The second derivative of  $\tilde{U}_t$  is

$$\frac{d^2\tilde{U}_t}{ds_t^2} = u''(w_t - s_t) + (1 + \rho)^{-1}u''((1 + r_{t+1})s_t)(1 + r_{t+1})^2 < 0. \quad (\text{SOC})$$

Hence there can at most be one  $s_t$  satisfying (FOC). Moreover, for a positive wage income there always exists such an  $s_t$ . Indeed:

**LEMMA 1** Let  $w_t > 0$  and suppose the No Fast Assumption (A1) applies. Then the saving problem of the young has a unique solution  $s_t = s(w_t, r_{t+1})$ . The solution is interior, i.e.,  $0 < s_t < w_t$ , and  $s_t$  satisfies (FOC).

*Proof.* Assume (A1). For any  $s \in (0, w_t)$ ,  $d\tilde{U}_t(s)/ds > -\infty$ . Now consider the endpoints  $s = 0$  and  $s = w_t$ . By (FOC) and (A1),

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{d\tilde{U}_t}{ds} &= -u'(w_t) + (1 + \rho)^{-1}(1 + r_{t+1}) \lim_{s \rightarrow 0} u'((1 + r_{t+1})s) = \infty, \\ \lim_{s \rightarrow w_t} \frac{d\tilde{U}_t}{ds} &= - \lim_{s \rightarrow w_t} u'(w_t - s) + (1 + \rho)^{-1}(1 + r_{t+1})u'((1 + r_{t+1})w_t) = -\infty. \end{aligned}$$

By continuity of  $d\tilde{U}_t/ds$  follows that there exists an  $s_t \in (0, w_t)$  such that at  $s = s_t$ ,  $d\tilde{U}_t/ds = 0$ . This is an application of the *intermediate value theorem*. It follows that (FOC) holds for this  $s_t$ . By (SOC),  $s_t$  is unique and can therefore



be written as an implicit function,  $s(w_t, r_{t+1})$ , of the exogenous variables in the problem,  $w_t$  and  $r_{t+1}$ .  $\square$

Inserting the solution for  $s_t$  into the two period budget constraints, (3.4) and (3.5), we immediately get the optimal consumption levels,  $c_{1t}$  and  $c_{2t+1}$ .

The simple optimization method we have used here is called the *substitution method*: by substitution of the constraints into the objective function an unconstrained maximization problem is obtained.<sup>6</sup>

### The consumption Euler equation

The first-order condition (FOC) can conveniently be written

$$u'(c_{1t}) = (1 + \rho)^{-1}u'(c_{2t+1})(1 + r_{t+1}). \quad (3.8)$$

This is known as an *Euler equation*, after the Swiss mathematician L. Euler (1707-1783) who was the first to study dynamic optimization problems. In the present context the condition is called a *consumption Euler equation*.

Intuitively, in an optimal plan the marginal utility cost of saving must equal the marginal utility benefit obtained by saving. The marginal utility cost of saving is the opportunity cost (in terms of current utility) of saving one more unit of account in the current period (approximately). This one unit of account is transferred to the next period with interest so as to result in  $1 + r_{t+1}$  units of account in that period. An optimal plan requires that the utility cost equals the utility benefit of instead having  $1 + r_{t+1}$  units of account in the next period. And this utility benefit is the discounted value of the extra utility that can be obtained next period through the increase in consumption by  $1 + r_{t+1}$  units compared with the situation without the saving of the marginal unit.

It may seem odd to attempt an intuitive interpretation this way, that is, in terms of “utility units”. The utility concept is just a convenient mathematical device used to represent the assumed *preferences*. Our interpretation is only meant as an as-if interpretation: as if utility were something concrete. An interpretation in terms of concrete *measurable quantities* goes like this. We rewrite (3.8) as

$$\frac{u'(c_{1t})}{(1 + \rho)^{-1}u'(c_{2t+1})} = 1 + r_{t+1}. \quad (3.9)$$

The left-hand side measures the *marginal rate of substitution*, MRS, of consumption as old for consumption as young, evaluated at the point  $(c_1, c_2)$ . MRS is defined as the increase in period- $t + 1$  consumption needed to compensate for a

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<sup>6</sup>Alternatively, one could use the Lagrange method.

marginal decrease in period- $t$  consumption. That is,

$$MRS_{c_2c_1} = -\frac{dc_{2t+1}}{dc_{1t}} \Big|_{U=\bar{U}} = \frac{u'(c_{1t})}{(1+\rho)^{-1}u'(c_{2t+1})}, \quad (3.10)$$

where we have used implicit differentiation in  $U(c_{1t}, c_{2t+1}) = \bar{U}$ . The right-hand side of (3.9) indicates the marginal rate of transformation, MRT, which is the rate at which saving allows an agent to shift consumption from period  $t$  to period  $t+1$  via the market. In an optimal plan MRS must equal MRT.

Even though interpretations in terms of “MRS equals MRT” are more satisfactory, we will often use “as if” interpretations like the one before. They are a convenient short-hand for the more elaborate interpretation.

By the Euler equation (3.8),

$$\rho \lesseqgtr r_{t+1} \text{ implies } u'(c_{1t}) \gtrless u'(c_{2t+1}), \text{ i.e., } c_{1t} \gtrless c_{2t+1},$$

respectively, in the optimal plan (because  $u'' < 0$ ). That is, absent uncertainty the optimal plan entails either increasing, constant or decreasing consumption over time according to whether the rate of time preference is below, equal to, or above the market interest rate, respectively. For example, when  $\rho < r_{t+1}$ , the plan is to start with relatively low consumption in order to take advantage of the relatively high rate of return on saving.

Note that there are infinitely many pairs  $(c_{1t}, c_{2t+1})$  satisfying the Euler equation (3.8). Only when requiring the two period budget constraints, (3.4) and (3.5), satisfied, do we get the unique solution  $s_t$  and thereby the unique solution for  $c_{1t}$  and  $c_{2t+1}$ .

### Properties of the saving function

The first-order condition (FOC), where the two budget constraints are inserted, determines the saving as an implicit function of the market prices faced by the young decision maker, i.e.,  $s_t = s(w_t, r_{t+1})$ .

The partial derivatives of this function can be found by applying the *implicit function theorem* on (FOC). A practical procedure is the following. We first interpret  $d\tilde{U}_t/ds_t$  in (FOC) as a function,  $f$ , of the variables involved,  $s_t$ ,  $w_t$ , and  $r_{t+1}$ , i.e.,

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}) \equiv f(s_t, w_t, r_{t+1}).$$

By (FOC),

$$f(s_t, w_t, r_{t+1}) = 0. \quad (*)$$

The implicit function theorem (see Math tools) now implies that if  $\partial f/\partial s_t \neq 0$ , then the equation (\*) defines  $s_t$  as an implicit function of  $w_t$  and  $r_{t+1}$ ,  $s_t = s(w_t, r_{t+1})$ , with partial derivatives

$$\frac{\partial s_t}{\partial w_t} = -\frac{\partial f/\partial w_t}{D} \quad \text{and} \quad \frac{\partial s_t}{\partial r_{t+1}} = -\frac{\partial f/\partial r_{t+1}}{D},$$

where  $D \equiv \partial f/\partial s_t \equiv d^2\tilde{U}_t/ds_t^2 < 0$  by (SOC). We find

$$\begin{aligned} \frac{\partial f}{\partial w_t} &= -u''(c_{1t}) > 0, \\ \frac{\partial f}{\partial r_{t+1}} &= (1 + \rho)^{-1} [u'(c_{2t+1}) + u''(c_{2t+1})s_t(1 + r_{t+1})]. \end{aligned}$$

Consequently, the partial derivatives of the saving function  $s_t = s(w_t, r_{t+1})$  are

$$s_w \equiv \frac{\partial s_t}{\partial w_t} = \frac{u''(c_{1t})}{D} > 0 \quad (\text{but} < 1), \quad (3.11)$$

$$s_r \equiv \frac{\partial s_t}{\partial r_{t+1}} = -\frac{(1 + \rho)^{-1} [u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1}]}{D}, \quad (3.12)$$

where in the last expression we have used (3.5).<sup>7</sup>

The role of  $w_t$  for saving is straightforward. Indeed, (3.11) shows that  $0 < s_w < 1$ , which implies that  $0 < \partial c_{1t}/\partial w_t < 1$  and  $0 < \partial c_{2t}/\partial w_t < 1 + r_{t+1}$ . The positive sign of these two derivatives indicate that consumption in each of the periods is a *normal* good (which certainly is plausible since we are talking about the total consumption by the individual in each period).<sup>8</sup>

The sign of  $s_r$  in (3.12) is seen to be ambiguous. This ambiguity regarding the role of  $r_{t+1}$  for saving reflects that the Slutsky substitution and income effects on consumption as young of a rise in the interest rate are of opposite signs. To

<sup>7</sup>A perhaps more straightforward procedure, not requiring full memory of the exact content of the implicit function theorem, is based on “implicit differentiation”. First, keeping  $r_{t+1}$  fixed, one calculates the total derivative w.r.t.  $w_t$  on both sides of (FOC). Next, keeping  $w_t$  fixed, one calculates the total derivative w.r.t.  $r_{t+1}$  on both sides of (FOC).

Yet another possible procedure is based on “total differentiation” in terms of *differentials*. Taking the differential w.r.t.  $s_t, w_t$ , and  $r_{t+1}$  on both sides of (FOC) gives  $-u''(c_{1t})(dw_t - ds_t) + (1 + \rho)^{-1} \cdot \{u''(c_{2t+1})[(1 + r_{t+1})ds_t + s_t dr_{t+1}] + u'(c_{2t+1})dr_{t+1}\} = 0$ . By rearranging we find the ratios  $ds_t/dw_t$  and  $ds_t/dr_{t+1}$ , which will indicate the value of the partial derivatives (3.11) and (3.12).

<sup>8</sup>Recall, a consumption good is called *normal* for given consumer preferences if the demand for it is an increasing function of the consumer’s wealth. Since in this model the consumer is born without any financial wealth, the consumer’s wealth at the end of period  $t$  is simply the present value of labor earnings through life, which here, evaluated at the beginning of period  $t$ , is  $w_t/(1 + r_t)$  as there is no labor income in the second period of life, cf. (3.7).

understand this, it is useful to keep the intertemporal budget constraint, (3.7), in mind. The *substitution effect* on  $c_{1t}$  is negative because the higher interest rate makes future consumption cheaper in terms of current consumption. And the *income effect* on  $c_{1t}$  is positive because with a higher interest rate, a given budget can buy more consumption in both periods, cf. (3.7). Generally there would be a third Slutsky effect, a *wealth effect* of a rise in the interest rate. But such an effect is ruled out in this model. This is because there is no labor income in the second period of life. Indeed, as indicated by (3.4), the human wealth of a member of generation  $t$ , evaluated at the end of period  $t$ , is simply  $w_t$ , which is independent of  $r_{t+1}$ . (In contrast, with labor income,  $w_{t+1}$ , in the second period, the human wealth would be  $w_t + w_{t+1}/(1 + r_{t+1})$ . This present discounted value of life-time earnings clearly depends negatively on  $r_{t+1}$ , and so a negative wealth effect on  $c_{1t}$  of a rise in the interest rate would arise.)

Rewriting (3.12) gives

$$s_r = \frac{(1 + \rho)^{-1} u'(c_{2t+1}) [\theta(c_{2t+1}) - 1]}{D} \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ for } \theta(c_{2t+1}) \begin{matrix} \leq \\ \geq \end{matrix} 1, \quad (3.13)$$

respectively, where  $\theta(c_{2t+1})$  is the absolute *elasticity of marginal utility* of consumption in the second period, that is,

$$\theta(c_{2t+1}) \equiv -\frac{c_{2t+1}}{u'(c_{2t+1})} u''(c_{2t+1}) \approx -\frac{\Delta u'(c_{2t+1})/u'(c_{2t+1})}{\Delta c_{2t+1}/c_{2t+1}} > 0,$$

where the approximation is valid for a “small” increase,  $\Delta c_{2t+1}$ , in  $c_{2t+1}$ . The inequalities in (3.13) show that when the absolute elasticity of marginal utility is below one, then the substitution effect on consumption as young of an increase in the interest rate dominates the income effect and saving increases. The opposite is true if the elasticity of marginal utility is above one.

The reason that  $\theta(c_{2t+1})$  has this role is that  $\theta(c_{2t+1})$  reflects how sensitive marginal utility of  $c_{2t+1}$  is to a rise in  $c_{2t+1}$ . To see the intuition, consider the case where consumption as young – and thus saving – happens to be unaffected by an increase in the interest rate. Even in this case, consumption as old,  $c_{2t+1}$ , is automatically increased (in view of the higher income as old through the higher rate of return on the unchanged saving); and the marginal utility of  $c_{2t+1}$  is thus decreased in response to a higher interest rate. The point is that this outcome can only be optimal if the elasticity of marginal utility of  $c_{2t+1}$  is of “medium” size. A very high absolute elasticity of marginal utility of  $c_{2t+1}$  would result in a sharp decline in marginal utility – so sharp that not much would be lost by dampening the automatic rise in  $c_{2t+1}$  and instead increase  $c_{1t}$ , thus reducing saving. On the other hand, a very low elasticity of marginal utility of  $c_{2t+1}$  would result in only a small decline in marginal utility – so small that it is beneficial to take advantage

of the higher rate of return and save *more*, thus accepting a first-period utility loss brought about by a lower  $c_{1t}$ .

We see from (3.12) that an absolute elasticity of marginal utility equal to exactly one is the case leading to the interest rate being *neutral* vis-a-vis the saving of the young. What is the intuition behind this? Neutrality vis-a-vis the saving of the young of a rise in the interest rate requires that  $c_{1t}$  remains unchanged since  $c_{1t} = w_t - s_t$ . In turn this requires that the marginal utility,  $u'(c_{2t+1})$ , on the right-hand side of (3.8) falls by the same percentage as  $1 + r_{t+1}$  rises. At the same time, the budget (3.5) as old tells us that  $c_{2t+1}$  has to rise by the same percentage as  $1 + r_{t+1}$  if  $s_t$  remains unchanged. Altogether we thus need that  $u'(c_{2t+1})$  falls by the same percentage as  $c_{2t+1}$  rises. But this requires that the absolute elasticity of  $u'(c_{2t+1})$  with respect to  $c_{2t+1}$  is exactly one.

The elasticity of marginal utility, also called the marginal utility flexibility, will generally depend on the level of consumption, as implicit in the notation  $\theta(c_{2t+1})$ . There exists a popular special case, however, where the elasticity of marginal utility is constant.

**EXAMPLE 1** *The CRRA utility function.* If we impose the requirement that  $u(c)$  should have an absolute elasticity of marginal utility of consumption equal to a constant  $\theta > 0$ , then one can show (see Appendix A) that the utility function must, up to a positive linear transformation, be of the CRRA form:

$$u(c) = \begin{cases} \frac{c^{1-\theta} - 1}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases}, \quad (3.14)$$

It may seem odd that when  $\theta \neq 1$ , we subtract the constant  $1/(1 - \theta)$  from  $c^{1-\theta}/(1 - \theta)$ . Adding or subtracting a constant from a utility function does not affect the marginal rate of substitution and consequently not behavior. So we could do without this constant, but its occurrence in (3.14) has two formal advantages. One is that in contrast to  $c^{1-\theta}/(1 - \theta)$ , the expression  $(c^{1-\theta} - 1)/(1 - \theta)$  can be interpreted as valid even for  $\theta = 1$ , namely as identical to  $\ln c$ . This is because  $(c^{1-\theta} - 1)/(1 - \theta) \rightarrow \ln c$  for  $\theta \rightarrow 1$  (by L'Hôpital's rule for "0/0"). Another advantage is that the kinship between the different members, indexed by  $\theta$ , of the CRRA family becomes more transparent. Indeed, by defining  $u(c)$  as in (3.14), all graphs of  $u(c)$  will go through the same point as the log function, namely  $(1, 0)$ , cf. Fig. 3.2. The equation (3.14) thus displays the CRRA utility function in *normalized form*.

The higher is  $\theta$ , the more "curvature" does the corresponding curve in Fig. 3.2 have. In turn, more "curvature" reflects a higher incentive to smooth consumption across time. The reason is that a large curvature means that the marginal utility will drop sharply if consumption rises and will increase sharply if consumption falls. Consequently, not much utility is lost by lowering consumption when it

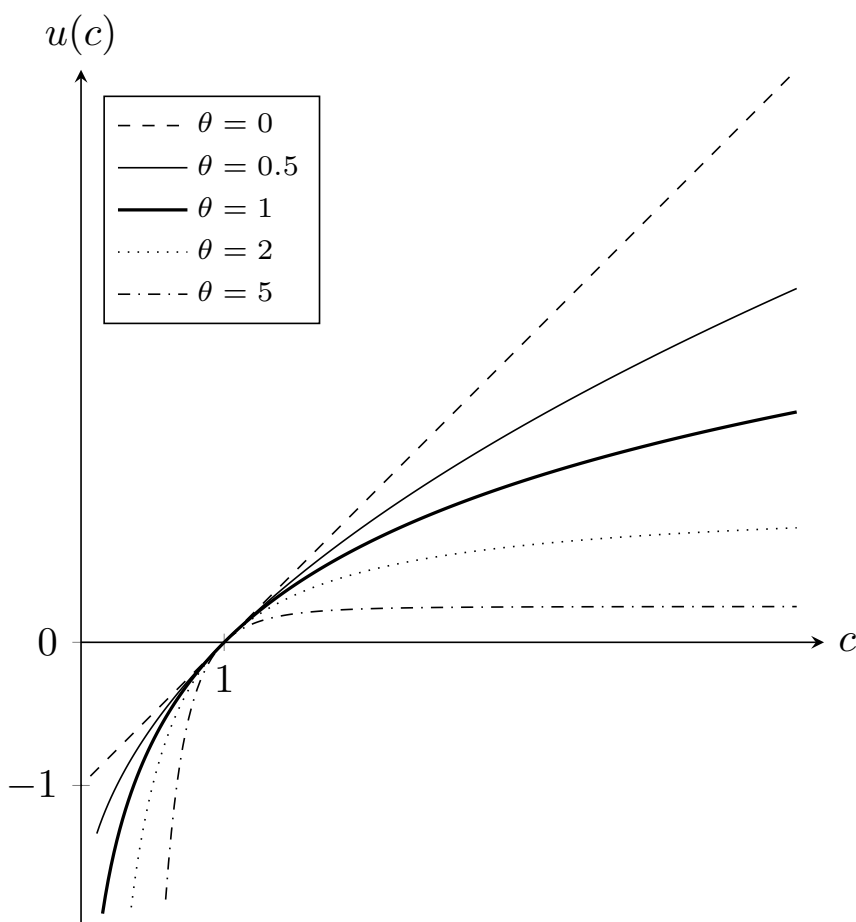


Figure 3.2: The CRRA family of utility functions.

is relatively high but there is a lot of utility to be gained by raising it when it is relatively low. So the curvature  $\theta$  indicates the degree of *aversion towards variation in consumption*. Or we may say that  $\theta$  indicates the strength of the *preference for consumption smoothing*.<sup>9</sup>  $\square$

Suppose the period utility is of CRRA form as given in (3.14). (FOC) then yields an explicit solution for the saving of the young:

$$s_t = \frac{1}{1 + (1 + \rho)\left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{\theta-1}{\theta}}} w_t. \quad (3.15)$$

We see that the signs of  $\partial s_t / \partial w_t$  and  $\partial s_t / \partial r_{t+1}$  shown in (3.11) and (3.13), re-

<sup>9</sup>The name CRRA is a shorthand for *Constant Relative Risk Aversion* and comes from the theory of behavior under uncertainty. Also in that theory does the CRRA function constitute an important benchmark case. And  $\theta$  is in that context called the *degree of relative risk aversion*.

spectively, are confirmed. Moreover, the saving of the young is in this special case proportional to income with a factor of proportionality that depends on the interest rate (as long as  $\theta \neq 1$ ). But in the general case the saving-income ratio depends also on the income level.

A major part of the attempts at empirically estimating  $\theta$  suggests that  $\theta > 1$ . Based on U.S. data, Hall (1988) provides estimates above 5, while Attanasio and Weber (1993) suggest  $1.25 \leq \theta \leq 3.33$ . For Japanese data Okubo (2011) suggests  $2.5 \leq \theta \leq 5.0$ . As these studies relate to much shorter time intervals than the implicit time horizon of about  $2 \times 30$  years in the Diamond model, we should be cautious. But if the estimates *were* valid also to that model, we should expect the income effect on current consumption of an increase in the interest rate to dominate the substitution effect, thus implying  $s_r < 0$  *as long as there is no wealth effect* of a rise in the interest rate.

When the elasticity of marginal utility of consumption is a constant,  $\theta$ , its inverse,  $1/\theta$ , equals the *elasticity of intertemporal substitution* in consumption. This concept refers to the willingness to substitute consumption over time when the interest rate changes. Under certain conditions the elasticity of intertemporal substitution reflects the elasticity of the ratio  $c_{2t+1}/c_{1t}$  with respect to  $1 + r_{t+1}$  when we move along a given indifference curve. The next subsection, which can be omitted in a first reading, goes more into detail with the concept.

### Digression: The elasticity of intertemporal substitution\*

Consider a two-period consumption problem like the one above. Fig. 3.3 depicts a particular indifference curve,  $u(c_1) + (1 + \rho)^{-1}u(c_2) = \bar{U}$ . At a given point,  $(c_1, c_2)$ , on the curve, the marginal rate of substitution of period-2 consumption for period-1 consumption,  $MRS$ , is given by

$$MRS = - \frac{dc_2}{dc_1} \Big|_{U=\bar{U}} ,$$

that is,  $MRS$  at the point  $(c_1, c_2)$  is the absolute value of the slope of the tangent to the indifference curve at that point.<sup>10</sup> Under the “normal” assumption of “strictly convex preferences” (as for instance in the Diamond model),  $MRS$  is rising along the curve when  $c_1$  decreases (and thereby  $c_2$  increases). Conversely, we can let  $MRS$  be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio  $c_2/c_1$ , as a function of  $MRS$ . If we raise  $MRS$  along the indifference curve, the corresponding value of the ratio  $c_2/c_1$  will also rise.

<sup>10</sup>When the meaning is clear from the context, to save notation we just write  $MRS$  instead of the more precise  $MRS_{c_2c_1}$ .

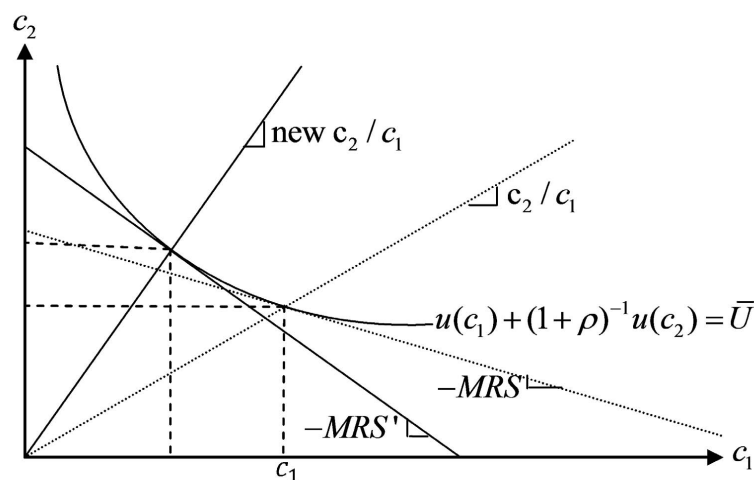


Figure 3.3: Substitution of period 2-consumption for period 1-consumption as  $MRS$  increases to  $MRS'$ .

The *elasticity of intertemporal substitution in consumption* at a given point is defined as the elasticity of the ratio  $c_2/c_1$  with respect to the marginal rate of substitution of  $c_2$  for  $c_1$ , when we move along the indifference curve through the point  $(c_1, c_2)$ . Letting the elasticity with respect to  $x$  of a differentiable function  $f(x)$  be denoted  $El_x f(x)$ , the elasticity of intertemporal substitution in consumption can be written

$$El_{MRS} \frac{c_2}{c_1} = \frac{MRS}{c_2/c_1} \frac{d(c_2/c_1)}{dMRS} \Big|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta MRS}{MRS}},$$

where the approximation is valid for a “small” increase,  $\Delta MRS$ , in  $MRS$ .

A more concrete understanding is obtained when we take into account that in the consumer’s optimal plan,  $MRS$  equals the ratio of the discounted prices of good 1 and good 2, that is, the ratio  $1/(1/(1+r))$  given in (3.7). Indeed, from (3.10) and (3.9), omitting the time indices, we have

$$MRS = -\frac{dc_2}{dc_1} \Big|_{U=\bar{U}} = \frac{u'(c_1)}{(1+\rho)^{-1}u'(c_2)} = 1+r \equiv R. \quad (3.16)$$

Letting  $\sigma(c_1, c_2)$  denote the elasticity of intertemporal substitution, evaluated at the point  $(c_1, c_2)$ , we then have

$$\sigma(c_1, c_2) = \frac{R}{c_2/c_1} \frac{d(c_2/c_1)}{dR} \Big|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta R}{R}}. \quad (3.17)$$



Consequently, the elasticity of intertemporal substitution can here be interpreted as the approximate percentage increase in the consumption ratio,  $c_2/c_1$ , triggered by a one percentage increase in the inverse price ratio, holding the utility level unchanged.<sup>11</sup>

Given  $u(c)$ , we let  $\theta(c)$  be the absolute elasticity of marginal utility of consumption, i.e.,  $\theta(c) \equiv -cu''(c)/u'(c)$ . As shown in Appendix B, we then find the elasticity of intertemporal substitution to be

$$\sigma(c_1, c_2) = \frac{c_2 + Rc_1}{c_2\theta(c_1) + Rc_1\theta(c_2)}. \quad (3.18)$$

We see that if  $u(c)$  belongs to the CRRA class and thereby  $\theta(c_1) = \theta(c_2) = \theta$ , then  $\sigma(c_1, c_2) = 1/\theta$ . In this case (as well as whenever  $c_1 = c_2$ ) the elasticity of marginal utility and the elasticity of intertemporal substitution are simply the inverse of each other.

## 3.4 Production

Output is homogeneous and can be used for consumption as well as investment in physical capital. The capital stock is thereby just accumulated non-consumed output. We may imagine a “corn economy” where output is corn, part of which is eaten (flour) while the remainder is accumulated as capital (seed corn).

The specification of technology and production conditions follows the simple competitive one-sector setup discussed in Chapter 2. Although the Diamond model is a long-run model, we shall in this chapter for simplicity ignore technological change.

### The representative firm

There is a representative firm with a neoclassical production function and constant returns to scale (CRS). Omitting the time argument  $t$  when not needed for clarity, we have

$$Y = F(K, L) = LF(k, 1) \equiv Lf(k), \quad f' > 0, f'' < 0, \quad (3.19)$$

where  $Y$  is output (GNP) per period,  $K$  is capital input,  $L$  is labor input, and  $k \equiv K/L$  is the capital-labor ratio. The derived function,  $f$ , is the production

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<sup>11</sup>This characterization is equivalent to saying that the elasticity of substitution between two consumption goods indicates the approximate percentage *decrease* in the ratio of the chosen quantities of the goods (when moving along a given indifference curve) induced by a one-percentage *increase* in the *corresponding* price ratio.

function in intensive form. Capital installation and other adjustment costs are ignored. With  $\hat{r}$  denoting the rental rate for capital, profit is  $\Pi \equiv F(K, L) - \hat{r}K - wL$ . The firm maximizes  $\Pi$  under perfect competition. This gives, first,  $\partial\Pi/\partial K = F_K(K, L) - \hat{r} = 0$ , that is,

$$F_K(K, L) = \frac{\partial[Lf(k)]}{\partial K} = f'(k) = \hat{r}. \quad (3.20)$$

Second,  $\partial\Pi/\partial L = F_L(K, L) - w = 0$ , that is,

$$F_L(K, L) = \frac{\partial[Lf(k)]}{\partial L} = f(k) - kf'(k) = w. \quad (3.21)$$

We may interpret these two conditions as saying that the firm will in every period use capital and labor up to the point where the marginal productivities of  $K$  and  $L$ , respectively, given the chosen input of the other factor, are equal to the respective factor prices from the market. Such an intuitive formulation does not take us far, however. Indeed, because of CRS there may be infinitely many pairs  $(K, L)$ , if any, that satisfy (3.20) and (3.21). What we can definitely ascertain is that in view of  $f'' < 0$ , a  $k > 0$  satisfying (3.20) will be *unique*.<sup>12</sup> Let us call it the *desired capital-labor ratio* and recognize that at this stage the separate factor inputs,  $K$  and  $L$ , are indeterminate. While (3.20) and (3.21) are just first-order conditions for the profit maximizing representative firm, to get further we have to appeal to *equilibrium* in the factor markets.

### Factor prices in equilibrium

Let the aggregate demand for capital services and labor services be denoted  $K^d$  and  $L^d$ , respectively. Clearing in factor markets in period  $t$  implies

$$K_t^d = K_t, \quad (3.22)$$

$$L_t^d = L_t = L_0(1+n)^t, \quad (3.23)$$

where  $K_t$  is the aggregate supply of capital services and  $L_t$  the aggregate supply of labor services. As was called attention to in Chapter 1, unless otherwise specified it is understood that the rate of utilization of each production factor is constant over time and normalized to one. So the quantity  $K_t$  will at one and the same time measure both the capital input, a flow, and the available capital stock. Similarly,

<sup>12</sup>It might seem that  $k$  is overdetermined because we have two equations, (3.20) and (3.21), but only one unknown. This reminds us that for *arbitrary* factor prices,  $\hat{r}$  and  $w$ , there will generally *not* exist a  $k$  satisfying both (3.20) and (3.21). But in equilibrium the factor prices faced by the firm are not arbitrary. They are equilibrium prices, i.e., they are adjusted so that (3.20) and (3.21) become consistent.

the quantity  $L_t$  will at one and the same time measure both the labor input, a flow, and the size of the labor force as a stock (= the number of young people).

The aggregate input demands,  $K^d$  and  $L^d$ , are linked through the desired capital-labor ratio,  $k^d$ . In equilibrium we have  $K_t^d/L_t^d = k_t^d = K_t/L_t \equiv k_t$ , by (3.22) and (3.23). The  $k$  in (3.20) and (3.21) can thereby be identified with the ratio of the stock supplies,  $k_t \equiv K_t/L_t > 0$ , which is a predetermined variable. Interpreted this way, (3.20) and (3.21) *determine* the equilibrium factor prices  $\hat{r}_t$  and  $w_t$  in each period. In view of the no-arbitrage condition (3.2), the real interest rate satisfies  $r_t = \hat{r}_t - \delta$ , where  $\delta$  is the capital depreciation rate,  $0 \leq \delta \leq 1$ . So in equilibrium we end up with

$$r_t = f'(k_t) - \delta \equiv r(k_t) \quad (r'(k_t) = f''(k_t) < 0), \quad (3.24)$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \quad (w'(k_t) = -k_t f''(k_t) > 0), \quad (3.25)$$

where causality is from the right to the left in the two equations. In line with our general perception of perfect competition, cf. Section 2.4 of Chapter 2, it is understood that the factor prices,  $\hat{r}_t$  and  $w_t$ , adjust quickly to the market-clearing levels.

*Technical Remark.* In these formulas it is understood that  $L > 0$ , but we may allow  $K = 0$ , i.e.,  $k = 0$ . In case  $f'(0)$  is not immediately well-defined, we interpret  $f'(0)$  as  $\lim_{k \rightarrow 0^+} f'(k)$  if this limit exists. If it does not, it must be because we are in a situation where  $\lim_{k \rightarrow 0^+} f'(k) = \infty$ , since  $f''(k) < 0$  (an example is the Cobb-Douglas function,  $f(k) = Ak^\alpha$ ,  $0 < \alpha < 1$ , where  $\lim_{k \rightarrow 0^+} f'(k) = \lim_{k \rightarrow 0^+} A\alpha k^{\alpha-1} = +\infty$ ). In this situation we simply include  $+\infty$  in the range of  $r(k)$  and define  $r(0) \cdot 0 \equiv \lim_{k \rightarrow 0^+} (f'(k) - \delta)k = 0$ , where the last equality comes from the general property of a neoclassical CRS production function that  $\lim_{k \rightarrow 0^+} k f'(k) = 0$ , cf. (2.18) of Chapter 2. Letting  $r(0) \cdot 0 = 0$  also fits well with intuition since, when  $k = 0$ , nobody receives capital income anyway. Note that since  $\delta \in [0, 1]$ ,  $r(k) > -1$  for all  $k \geq 0$ . What about  $w(0)$ ? We interpret  $w(0)$  as  $\lim_{k \rightarrow 0} w(k)$ . From (2.18) of Chapter 2 we have that  $\lim_{k \rightarrow 0^+} w(k) = f(0) \equiv F(0, 1) \geq 0$ . If capital is essential,  $F(0, 1) = 0$ . Otherwise,  $F(0, 1) > 0$ . Finally, since  $w' > 0$ , we have, for  $k > 0$ ,  $w(k) > 0$  as also noted in Chapter 2.  $\square$

To fix ideas we have assumed that households (here the old) own the physical capital and rent it out to the firms. In view of perfect competition and constant returns to scale, pure profit is nil in equilibrium. As long as the model ignores uncertainty and capital installation costs, the results will be unaffected if instead we let the firms themselves own the physical capital and finance capital investment by issuing bonds and shares. These bonds and shares would then be accumulated by the households and constitute their financial wealth instead of the capital goods themselves. The equilibrium rate of return,  $r_t$ , would be the same.

## 3.5 The dynamic path of the economy

As in microeconomic general equilibrium theory, it is important to distinguish between the set of technically feasible allocations and an allocation brought about, within this set, by a specific economic institution (the rules of the game). The economic institution assumed by the Diamond model is the private-ownership perfect-competition market institution.

We shall in the next subsections introduce three different concepts concerning allocations over time in this economy. The three concepts are: *technically feasible paths*, *temporary equilibrium*, and *equilibrium path*. These concepts are mutually related in the sense that there is a whole *set* of technically feasible paths, *within which* there may exist a unique equilibrium path, which in turn is a sequence of states that have certain properties, including the temporary equilibrium property.

### 3.5.1 Technically feasible paths

When we speak of technically feasible paths, the focus is merely upon what is feasible from the point of view of the given technology as such and available initial resources. That is, we disregard the agents' preferences, their choices given the constraints, their interactions in markets, the market forces etc.

The technology is represented by (3.19) and there are two exogenous resources, the labor force,  $L_t = L_0(1+n)^t$ , and the initial capital stock,  $K_0$ . From national income accounting aggregate consumption can be written  $C_t \equiv Y_t - S_t = F(K_t, L_t) - S_t$ , where  $S_t$  denotes aggregate gross saving, and where we have inserted (3.19). In a closed economy aggregate gross saving equals (ex post) aggregate gross investment,  $K_{t+1} - K_t + \delta K_t$ . So

$$C_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t). \quad (3.26)$$

Let  $c_t$  denote aggregate consumption per unit of labor in period  $t$ , i.e.,

$$c_t \equiv \frac{C_t}{L_t} = \frac{c_{1t}L_t + c_{2t}L_{t-1}}{L_t} = c_{1t} + \frac{c_{2t}}{1+n}.$$

Combining this with (3.26) and using the definitions of  $k$  and  $f(k)$ , we obtain the dynamic resource constraint of the economy:

$$c_{1t} + \frac{c_{2t}}{1+n} = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \quad (3.27)$$

**DEFINITION 1** Let  $\bar{k}_0 \geq 0$  be the historically given initial ratio of available capital and labor. Let the path  $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$  have nonnegative  $k_t$ ,  $c_{1t}$ , and  $c_{2t}$

for all  $t = 0, 1, 2, \dots$ . The path is called *technically feasible* if it has  $k_0 = \bar{k}_0$  and satisfies (3.27) for all  $t = 0, 1, 2, \dots$ .

The next subsections consider how, for given household preferences, the private-ownership market institution with profit-maximizing firms under perfect competition generates a *selection* within the set of technically feasible paths. A member of this selection (which may but need not have just one member) is called an *equilibrium path*. It constitutes a sequence of states with certain properties, one of which is the temporary equilibrium property.

### 3.5.2 A temporary equilibrium

Standing in a given period, it is natural to think of next period's interest rate as an *expected* interest rate that provisionally can deviate from the ex post realized one. We let  $r_{t+1}^e > -1$  denote the expected real interest rate of period  $t + 1$  as seen from period  $t$ .

Essentially, by a temporary equilibrium in period  $t$  is meant a state where for a *given*  $r_{t+1}^e$ , all markets clear in the period. There are three markets, namely two factor markets and a market for produced goods. We have already described the two factor markets. In the market for produced goods the representative firm supplies the amount  $Y_t^s = F(K_t^d, L_t^d)$  in period  $t$ . The demand side in this market has two components, consumption,  $C_t$ , and *gross* investment,  $I_t$ . Equilibrium in the goods market requires that demand equals supply, i.e.,

$$C_t + I_t = c_{1t}L_t + c_{2t}L_{t-1} + I_t = Y_t^s = F(K_t^d, L_t^d), \quad (3.28)$$

where consumption by the young and old,  $c_{1t}$  and  $c_{2t}$ , respectively, were determined in Section 3.

By definition, aggregate gross investment equals aggregate net investment,  $I_t^N$ , plus capital depreciation, i.e.,

$$I_t = I_t^N + \delta K_t \equiv I_{1t}^N + I_{2t}^N + \delta K_t \equiv S_{1t}^N + S_{2t}^N + \delta K_t = s_t L_t + (-K_t) + \delta K_t. \quad (3.29)$$

The first equality follows from the definition of net investment and the assumption that capital depreciation equals  $\delta K_t$ . Next comes an identity reflecting that aggregate net investment is the sum of net investment by the young and net investment by the old. In turn, saving in this model is directly an act of acquiring capital goods. So the net investment by the young,  $I_{1t}^N$ , and the old,  $I_{2t}^N$ , are identical to their net saving,  $S_{1t}^N$  and  $S_{2t}^N$ , respectively. As we have shown, the net saving by the young in the model equals  $s_t L_t$ . And the net saving by the old is negative and equals  $-K_t$ . Indeed, because they have no bequest motive, the old consume all they have and leave nothing as bequests. Hence, the young

in any period enter the period with no non-human wealth. Consequently, any non-human wealth existing at the beginning of a period must belong to the old in that period and be the result of their saving as young in the previous period. As  $K_t$  constitutes the aggregate non-human wealth in our closed economy at the beginning of period  $t$ , we therefore have

$$s_{t-1}L_{t-1} = K_t. \quad (3.30)$$

Recalling that the net saving of any group is by definition the same as the increase in its non-human wealth, the net saving of the old in period  $t$  is  $-K_t$ . Aggregate net saving in the economy is thus  $s_tL_t + (-K_t)$ , and (3.29) is thereby explained.

**DEFINITION 2** Let a given period  $t$  have capital stock  $K_t \geq 0$ , labor supply  $L_t > 0$ , and hence capital-labor ratio  $k_t = K_t/L_t$ . Let the expected real interest rate be given as  $r_{t+1}^e > -1$ . And let the functions  $s(w_t, r_{t+1}^e)$ ,  $w(k_t)$ , and  $r(k_t)$  be defined as in Lemma 1, (3.25), and (3.24), respectively. Then a *temporary equilibrium* in period  $t$  is a state  $(k_t, c_{1t}, c_{2t}, w_t, r_t)$  of the economy such that (3.22), (3.23), (3.28), and (3.29) hold (i.e., all markets clear) for  $c_{1t} = w_t - s_t$ ,  $c_{2t} = (k_t + r(k_t)k_t)(1 + n)$ , where  $w_t = w(k_t) > 0$  and  $s_t = s(w_t, r_{t+1}^e)$ .

The reason for the requirement  $w_t > 0$  in the definition is that if  $w_t = 0$ , people would have nothing to live on as young and nothing to save from for retirement. The system would not be economically viable in this case. With regard to the equation for  $c_{2t}$  in the definition, note that (3.30) gives  $s_{t-1} = K_t/L_{t-1} = (K_t/L_t)(L_t/L_{t-1}) = k_t(1 + n)$ , which is the wealth of each old at the beginning of period  $t$ . Substituting into  $c_{2t} = (1 + r_t)s_{t-1}$ , we get  $c_{2t} = (1 + r_t)k_t(1 + n)$ , which can also be written  $c_{2t} = (k_t + r_t k_t)(1 + n)$ . This last way of writing  $c_{2t}$  has the advantage of being applicable even if  $k_t = 0$ , cf. Technical Remark in Section 3.4. The remaining conditions for a temporary equilibrium are self-explanatory.

**PROPOSITION 1** Suppose the No Fast Assumption (A1) applies. Consider a given period  $t$  with a given  $k_t \geq 0$ . Then for any  $r_{t+1}^e > -1$ ,

- (i) if  $k_t > 0$ , there exists a temporary equilibrium,  $(k_t, c_{1t}, c_{2t}, w_t, r_t)$ , and  $c_{1t}$  and  $c_{2t}$  are positive;
- (ii) if  $k_t = 0$ , a temporary equilibrium exists if and only if capital is not essential; in that case,  $w_t = w(k_t) = w(0) = f(0) > 0$  and  $c_{1t}$  and  $s_t$  are positive (while  $c_{2t} = 0$ );
- (iii) whenever a temporary equilibrium exists, it is unique.

*Proof.* We begin with (iii). That there is at most one temporary equilibrium is immediately obvious since  $w_t$  and  $r_t$  are functions of the given  $k_t$ :  $w_t = w(k_t)$  and  $r_t = r(k_t)$ . And given  $w_t, r_t$ , and  $r_{t+1}^e$ ,  $c_{1t}$  and  $c_{2t}$  are uniquely determined.

(i) Let  $k_t > 0$ . Then, by (3.25),  $w(k_t) > 0$ . We claim that the state  $(k_t, c_{1t}, c_{2t}, w_t, r_t)$ , with  $w_t = w(k_t)$ ,  $r_t = r(k_t)$ ,  $c_{1t} = w(k_t) - s(w(k_t), r_{t+1}^e)$ , and  $c_{2t} = (1 + r(k_t))k_t(1 + n)$ , is a temporary equilibrium. Indeed, Section 3.4 showed that the factor prices  $w_t = w(k_t)$  and  $r_t = r(k_t)$  are consistent with clearing in the factor markets in period  $t$ . Given that these markets clear (by price adjustment), it follows by Walras' law (see Appendix C) that also the third market, the goods market, clears in period  $t$ . So all criteria in Definition 2 are satisfied. That  $c_{1t} > 0$  follows from  $w(k_t) > 0$  and the No Fast Assumption (A1), in view of Lemma 1. That  $c_{2t} > 0$  follows from  $c_{2t} = (1 + r(k_t))k_t(1 + n)$  when  $k_t > 0$ , since  $r(k_t) > -1$  always.

(ii) Let  $k_t = 0$ . Suppose  $f(0) > 0$ . Then, by Technical Remark in Section 3.4,  $w_t = w(0) = f(0) > 0$  and  $c_{1t} = w_t - s(w_t, r_{t+1}^e)$  is well-defined, positive, and less than  $w_t$ , in view of Lemma 1; so  $s_t = s(w_t, r_{t+1}^e) > 0$ . The old in period 0 will starve since  $c_{2t} = (0 + 0)(1 + n)$ , in view of  $r(0) \cdot 0 = 0$ , cf. Technical Remark in Section 3.4. Even though this is a bad situation for the old, it is consistent with the criteria in Definition 2. On the other hand, if  $f(0) = 0$ , we get  $w_t = f(0) = 0$ , which violates one of the criteria in Definition 2.  $\square$

Point (ii) of the proposition says that a temporary equilibrium *may* exist even in a period where  $k = 0$ . The old in this period will starve and not survive. But if capital is not essential, the young get positive labor income out of which they will save a part for their old age and be able to maintain life also next period which will be endowed with positive capital. Then, by our assumptions the economy is viable forever.<sup>13</sup>

Generally, the term “equilibrium” is used to denote a state of “rest”, often just “temporary rest”. The temporary equilibrium in the present model is an example of a state of “temporary rest” in the following sense: (a) the agents optimize, given their expectations and the constraints they face; and (b) the aggregate demands and supplies in the given period are mutually consistent, i.e., markets clear. The qualification “temporary” is motivated by two features. First, in the next period the conditioning circumstances may be different, possibly as a direct consequence of the currently chosen actions. Second, the given expectations may turn out wrong.

### 3.5.3 An equilibrium path

The concept of an equilibrium path, also called an intertemporal equilibrium, requires more conditions satisfied. The concept refers to a sequence of temporary equilibria such that *expectations* of the agents are *fulfilled* in every period:

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<sup>13</sup>For simplicity, the model ignores that in practice a certain minimum per capita consumption level (the subsistence minimum) is needed for viability.

**DEFINITION 3** An *equilibrium path* is a technically feasible path  $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$  such that for  $t = 0, 1, 2, \dots$ , the state  $(k_t, c_{1t}, c_{2t}, w_t, r_t)$  is a temporary equilibrium with  $r_{t+1}^e = r(k_{t+1})$ .

To characterize such a path, we forward (3.30) one period and rearrange so as to get

$$K_{t+1} = s_t L_t. \quad (3.31)$$

Since  $K_{t+1} \equiv k_{t+1} L_{t+1} = k_{t+1} L_t (1+n)$ , this can be written

$$k_{t+1} = \frac{s(w(k_t), r(k_{t+1}))}{1+n}, \quad (3.32)$$

using that  $s_t = s(w_t, r_{t+1}^e)$ ,  $w_t = w(k_t)$ , and  $r_{t+1}^e = r_{t+1} = r(k_{t+1})$  in a sequence of temporary equilibria with fulfilled expectations. Equation (3.32) is a first-order difference equation, known as the *fundamental difference equation* or the *law of motion* of the Diamond model.

**PROPOSITION 2** Suppose the No Fast Assumption (A1) applies. Then,

- (i) for any  $k_0 > 0$  there exists at least one equilibrium path;
- (ii) if  $k_0 = 0$ , an equilibrium path exists if and only if  $f(0) > 0$  (i.e., capital not essential);
- (iii) in any case, an equilibrium path has a positive real wage in all periods and positive capital in all periods except possibly the first;
- (iv) an equilibrium path satisfies the first-order difference equation (3.32).

*Proof.* (i) and (ii): see Appendix D. (iii) For a given  $t$ , let  $k_t \geq 0$ . Then, since an equilibrium path is a sequence of temporary equilibria, we have, from Proposition 1,  $w_t = w(k_t) > 0$  and  $s_t = s(w(k_t), r_{t+1}^e)$ , where  $r_{t+1}^e = r(k_{t+1})$ . Hence, by Lemma 1,  $s(w(k_t), r_{t+1}^e) > 0$ , which implies  $k_{t+1} > 0$ , in view of (3.32). This shows that only for  $t = 0$  is  $k_t = 0$  possible along an equilibrium path. (iv) This was shown in the text above.  $\square$

The formal proofs of point (i) and (ii) of the proposition are quite technical and placed in the appendix. But the graphs in the ensuing figures 3.4-3.7 provide an intuitive verification. The “only if” part of point (ii) reflects the not very surprising fact that *if* capital were an essential production factor, no capital “now” would imply no income “now”, hence no saving and investment and thus no capital in the next period and so on. On the other hand, the “if” part of point (ii) says that when capital is not essential, an equilibrium path can set off even from an initial period with no capital. Then point (iii) adds that an equilibrium path will have positive capital in all subsequent periods. Finally, as to point (iv), note that the fundamental difference equation, (3.32), rests on equation (3.31). Recall from the previous subsection that the economic logic behind this



key equation is that since capital is the only non-human asset in the economy and the young are born without any inheritance, the aggregate capital stock at the beginning of period  $t + 1$  *must* be owned by the old generation in that period. It must thereby equal the aggregate saving these people had in the previous period where they were young.

### The transition diagram

To be able to further characterize equilibrium paths, we construct a transition diagram in the  $(k_t, k_{t+1})$  plane. The *transition curve* is defined as the set of points  $(k_t, k_{t+1})$  satisfying (3.32). Its form and position depends on the households' preferences and the firms' technology. Fig. 3.4 shows one *possible*, but far from necessary configuration of this curve. A complicating circumstance is that the equation (3.32) has  $k_{t+1}$  on both sides. Sometimes we are able to solve the equation explicitly for  $k_{t+1}$  as a function of  $k_t$ , but sometimes we can do so only implicitly. What is even worse is that there are cases where  $k_{t+1}$  is not unique for a given  $k_t$ . We will proceed step by step.

First, what can we say about the *slope* of the transition curve? In general, a point on the transition curve has the property that at least in a small neighborhood of this point, the equation (3.32) will define  $k_{t+1}$  as an implicit function of  $k_t$ .<sup>14</sup> Taking the total derivative with respect to  $k_t$  on both sides of (3.32), we get

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+n} \left( s_w w'(k_t) + s_r r'(k_{t+1}) \frac{dk_{t+1}}{dk_t} \right). \quad (3.33)$$

By ordering, the slope of the transition curve within this small neighborhood can be written

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w(w(k_t), r(k_{t+1})) w'(k_t)}{1+n - s_r(w(k_t), r(k_{t+1})) r'(k_{t+1})}, \quad (3.34)$$

when the denominator,

$$\mathcal{D}(k_t, k_{t+1}) \equiv 1+n - s_r(w(k_t), r(k_{t+1})) r'(k_{t+1}),$$

differs from nil.

In view of  $s_w > 0$  and  $w'(k_t) = -k_t f''(k_t) > 0$ , the numerator in (3.34) is always positive and we have

$$\frac{dk_{t+1}}{dk_t} \geq 0 \text{ for } s_r(w(k_t), r(k_{t+1})) \geq \frac{1+n}{r'(k_{t+1})},$$

respectively, since  $r'(k_{t+1}) = f''(k_{t+1}) < 0$ .

<sup>14</sup>An exception occurs if the denominator in (3.34) below vanishes.

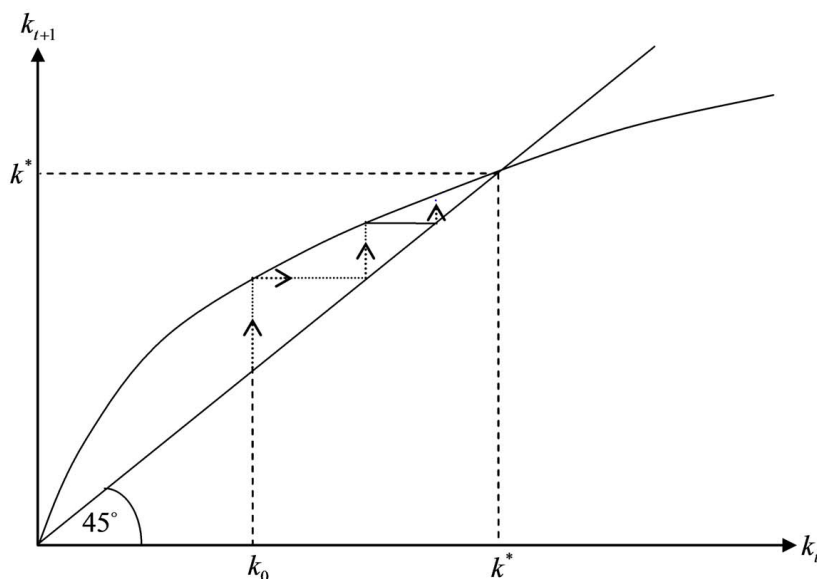


Figure 3.4: Transition curve and the resulting dynamics in the log-utility Cobb-Douglas case, cf. Example 2.

It follows that the transition curve is universally upward-sloping if and only if  $s_r(w(k_t), r(k_{t+1})) > (1+n)/r'(k_{t+1})$  everywhere along the transition curve. The intuition behind this becomes visible by rewriting (3.34) in terms of small changes in  $k_t$  and  $k_{t+1}$ . Since  $\Delta k_{t+1}/\Delta k_t \approx dk_{t+1}/dk_t$  for  $\Delta k_t$  “small”, (3.34) implies

$$[1+n - s_r(\cdot) r'(k_{t+1})] \Delta k_{t+1} \approx s_w(\cdot) w'(k_t) \Delta k_t. \quad (*)$$

Let  $\Delta k_t > 0$ . This rise in  $k_t$  will always raise wage income and, via the resulting rise in  $s_t$ , raise  $k_{t+1}$ , everything else equal. Everything else is *not* equal, however, since a rise in  $k_{t+1}$  implies a fall in the rate of interest. There are four cases to consider:

Case 1:  $s_r(\cdot) = 0$ . Then there is no feedback effect from the fall in the rate of interest. So the tendency to a rise in  $k_{t+1}$  is neither offset nor fortified.

Case 2:  $s_r(\cdot) > 0$ . Then the tendency to a rise in  $k_{t+1}$  will be partly offset through the *dampening* effect on saving resulting from the fall in the interest rate. This negative feedback can not fully offset the tendency to a rise in  $k_{t+1}$ . The reason is that the negative feedback on the saving of the young will only be there *if* the interest rate falls in the first place. We cannot in a period have both a *fall* in the interest rate triggering lower saving *and* a *rise* in the interest rate (via a lower  $k_{t+1}$ ) *because* of the lower saving. So a *sufficient* condition for a universally upward-sloping transition curve is that the saving of the young is a non-decreasing function of the interest rate.

Case 3:  $(1+n)/r'(k_{t+1}) < s_r(\cdot) < 0$ . Then the tendency to a rise in  $k_{t+1}$  will be fortified through the *stimulating* effect on saving resulting from the fall in the interest rate.

Case 4:  $s_r(\cdot) < (1+n)/r'(k_{t+1}) < 0$ . Then the expression in brackets on the left-hand side of (\*) is negative and requires therefore that  $\Delta k_{t+1} < 0$  in order to comply with the positive right-hand side. This is a situation where self-fulfilling expectations operate, a case to which we return. We shall explore this case in the next sub-section.

Another feature of the transition curve is the following:

LEMMA 2 (*the transition curve is nowhere flat*) For all  $k_t > 0$  such that the denominator,  $\mathcal{D}(k_t, k_{t+1})$ , in (3.34) differs from nil, we have  $dk_{t+1}/dk_t \neq 0$ .

*Proof.* Since  $s_w > 0$  and  $w'(k_t) > 0$  always, the numerator in (3.34) is always positive.  $\square$

The implication is that no part of the transition curve can be horizontal.<sup>15</sup>

When the transition curve crosses the 45° degree line for some  $k_t > 0$ , as in the example in Fig. 3.4, we have a steady state at this  $k_t$ . Formally:

DEFINITION 4 An equilibrium path  $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$  is in a *steady state* with capital-labor ratio  $k^* > 0$  if the fundamental difference equation, (3.32), is satisfied with  $k_t$  as well as  $k_{t+1}$  replaced by  $k^*$ .

This exemplifies the notion of a steady state as a stationary point in a dynamic process. Some economists use the term “dynamic equilibrium” instead of “steady state”. As in this book the term “equilibrium” refers to situations where the constraints and decided actions of the market participants are mutually compatible, an economy can be in “equilibrium” without being in a steady state. A steady state is seen as a *special* sequence of temporary equilibria with fulfilled expectations, namely one with the property that the endogenous variable, here  $k$ , entering the fundamental difference equation does not change over time.

EXAMPLE 2 (the log utility Cobb-Douglas case) Let  $u(c) = \ln c$  and  $Y = AK^\alpha L^{1-\alpha}$ , where  $A > 0$  and  $0 < \alpha < 1$ . Since  $u(c) = \ln c$  is the case  $\theta = 1$  in Example 1, by (3.15) we have  $s_r = 0$ . Indeed, with logarithmic utility the substitution and income effects on  $s_t$  of a rise in the interest rate offset each other; and, as discussed above, in the Diamond model there can be no wealth effect of a rise in  $r_{t+1}$ . Further, the equation (3.32) reduces to a transition *function*,

$$k_{t+1} = \frac{(1-\alpha)Ak_t^\alpha}{(1+n)(2+\rho)}. \quad (3.35)$$

<sup>15</sup>This would not generally hold if the utility function were not time-separable.

The associated transition curve is shown in Fig. 3.4 and there is for  $k_0 > 0$  both a unique equilibrium path and a unique steady state with capital-labor ratio

$$k^* = \left( \frac{(1 - \alpha)A}{(2 + \rho)(1 + n)} \right)^{1/(1-\alpha)} > 0.$$

At  $k_t = k^*$ , the slope of the transition curve is necessarily less than one. The dynamics therefore lead to convergence to the steady state as illustrated in the figure.<sup>16</sup> In the steady state the interest rate is  $r^* = f'(k^*) - \delta = \alpha(1 + n)(2 + \rho)/(1 - \alpha) - \delta$ . Note that a higher  $n$  results in a lower  $k^*$ , hence a higher  $r^*$ .  $\square$

Because the Cobb-Douglas production function implies that capital is essential, (3.35) implies  $k_{t+1} = 0$  if  $k_t = 0$ . The state  $k_{t+1} = k_t = 0$  is thus a stationary point of the difference equation (3.35) considered in isolation. This state is not, however, an equilibrium path as defined above (not a steady state of an *economic* system since there is no production). We may call it a *trivial* steady state in contrast to the economically viable steady state  $k_{t+1} = k_t = k^* > 0$  which is then called a *non-trivial* steady state.

Theoretically, there may be more than one (non-trivial) steady state. Non-existence of a steady state is also possible. But before considering these possibilities, the next subsection (which may be skipped in a first reading) addresses an even more defiant feature which is that for a given  $k_0$  there may exist more than one equilibrium path.

### The possibility of multiple equilibrium paths\*

It turns out that a *backward-bending* transition curve like that in Fig. 3.5 is possible within the model. Not only are there two steady states but for  $k_t \in (\underline{k}, \bar{k})$  there are *three temporary equilibria* with self-fulfilling expectations. That is, for a given  $k_t$  in this interval, there are three different values of  $k_{t+1}$  that are consistent with self-fulfilling expectations. Exercise 3.3 at the end of the chapter documents this possibility by way of a numerical example.

The theoretical possibility of multiple equilibria with self-fulfilling expectations requires that there is at least one interval on the horizontal axis where a section of the transition curve has negative slope. Let us see if we can get an intuitive understanding of why in this situation multiple equilibria can arise. Consider the specific configuration in Fig. 3.5 where  $k'$ ,  $k''$ , and  $k'''$  are the possible values for the capital-labor ratio next period when  $k_t \in (\underline{k}, \bar{k})$ . In a small neighborhood of the point P associated with the intermediate value,  $k''$ , the slope of the transition curve is negative. In Fig. 3.5 a relevant neighborhood is indicated

<sup>16</sup>A formal proof can be based on the mean value theorem.

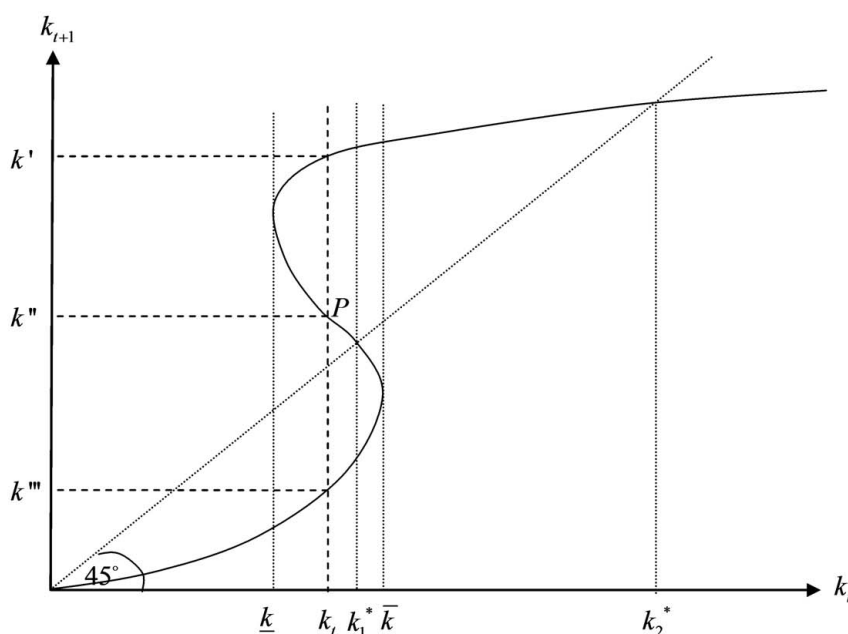


Figure 3.5: A backward-bending transition curve leads to multiple temporary equilibria with self-fulfilling expectations.

by the rectangle R. Within this rectangle the fundamental difference equation (3.32) does indeed define  $k_{t+1}$  as an implicit function of  $k_t$ , the graph of which goes through the point P and has negative slope. The only points in Fig. 3.5 that have no such neighborhood are the two points where the transition curve has vertical tangent, that is, the two points with abscissas  $\underline{k}$  and  $\bar{k}$ , respectively.

Now, as we saw above, the negative slope requires not only that in this neighborhood  $s_r(w_t, r(k_{t+1})) < 0$ , but that the stricter condition  $s_r(w_t, r(k_{t+1})) < (1+n)/f''(k'')$  holds (we take  $w_t$  as given since  $k_t$  is given and  $w_t = w(k_t)$ ). That the point P with coordinates  $(k_t, k'')$  is on the transition curve indicates that, given  $w_t = w(k_t)$  and an expected interest rate  $r_{t+1}^e = r(k'')$ , the induced saving by the young,  $s(w_t, r(k''))$ , will be such that  $k_{t+1} = k''$ , that is, the expectation is fulfilled. The fact that also the point  $(k_t, k')$ , where  $k' > k''$ , is on transition curve indicates that also a lower interest rate,  $r(k')$ , can be self-fulfilling. By this is meant that *if* an interest rate at the level  $r(k')$  is expected, then this expectation induces *more* saving by the young, just enough more to make  $k_{t+1} = k' > k''$ , thus confirming the expectation of the lower interest rate level  $r(k')$ . What makes this possible is exactly the negative dependency of  $s_t$  on  $r_{t+1}^e$ . The fact that also the point  $(k_t, k''')$ , where  $k''' < k''$ , is on the transition curve can be similarly interpreted. It is exactly  $s_r < 0$  that makes it possible that *less* saving by the

young than at P can be induced by an expected *higher* interest rate,  $r(k''')$ , than at P.

Recognizing the ambiguity arising from the possibility of multiple equilibrium paths, we face an additional ambiguity, known as the “expectational coordination problem”. The model presupposes that all the young *agree* in their expectations. Only then will one of the three mentioned temporary equilibria appear. But the model is silent about how the needed coordination of expectations is brought about, and if it is, why this coordination ends up in one rather than another of the three possible equilibria with self-fulfilling expectations. Each single young is isolated in the market and will not know what the others will expect. The market mechanism by itself provides no coordination of expectations.

As it stands, the model consequently cannot determine how the economy will evolve in the present situation with a backward-bending transition curve. Since the topic is complicated, we will here take an ad-hoc approach – we will circumvent the indeterminacy problem.<sup>17</sup> There are at least three ways to try to rule out the possibility of multiple equilibrium paths. One way is to discard the assumption of perfect foresight. Instead, some kind of adaptive expectations may be assumed, for example in the form of *myopic foresight*, also called *static expectations*. This means that the expectation formed by the agents in the current period about the value of a variable next period is that it will stay the same as in the current period. So here the assumption would be that the young have the expectation  $r_{t+1}^e = r_t$ . Then, given  $k_0 > 0$ , a *unique* sequence of temporary equilibria  $\{(k_t, c_{1t}, c_{2t}, w_t, r_t)\}_{t=0}^{\infty}$  is generated by the model. *Oscillations* in the sense of repetitive movements up and down of  $k_t$  are possible. Even *chaotic* trajectories are possible (see Exercise 3.6).

Outside steady state the agents will experience that their expectations are systematically wrong. And the assumption of myopic foresight rules out that learning occurs. This may be too simplistic, although it *can* be argued that human beings to a certain extent have a psychological disposition to myopic foresight.

Another approach to the indeterminacy problem could be motivated by the general observation that sometimes the possibility of multiple equilibria in a model arises because of a “rough” time structure imposed on the model in question. In the present case, each period in the Diamond model corresponds to half of an adult person’s lifetime. And in the first period of life there is no capital income, in the second there is no labor income. This coarse notion of time may

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<sup>17</sup>Yet the fact that multiple self-fulfilling equilibrium paths are in several contexts theoretically possible is certainly of interest and plays an important role in certain business cycle theories of booms and busts. We shall have a little more to say about this in part VI of this book.

artificially generate a multiplicity of equilibria or, with myopic foresight, oscillations. An expanded model where people live many periods may “smooth” the responses of the system to the events impinging on it. Indeed, with working life stretching over more than one period, wealth effects of changes in the interest rate arise, thereby reducing the likelihood of a backward-bending transition curve. In Chapter 12 we shall see an example of an overlapping-generations model in continuous time where the indeterminacy problem never arises.

For now, our approach will be to stay with the rough time structure of the Diamond model because of its analytical convenience and then make the best of it by imposing conditions on the utility function, the production function, and/or parameter values so as to rule out multiple equilibria. We stay with the assumption of perfect foresight, but assume that circumstances are such that multiple equilibria with self-fulfilling expectations do not arise. Fortunately, the “circumstances” needed for this in the present model are not defying empirical plausibility.

### Conditions for uniqueness of the equilibrium path

Sufficient for the equilibrium path to be unique is that preferences and technology in combination are such that the slope of the transition curve is everywhere positive. Hence we impose the Positive Slope Assumption that

$$s_r(w(k_t), r(k_{t+1})) > \frac{1+n}{f''(k_{t+1})} \quad (\text{A2})$$

everywhere along an equilibrium path. This condition is of course always satisfied when  $s_r \geq 0$  (reflecting an elasticity of marginal utility of consumption not above one) and *can* be satisfied even if  $s_r < 0$  (as long as  $s_r$  is “small” in absolute value). Essentially, (A2) is an assumption that the income effect on consumption as young of a rise in the interest rate does not dominate the substitution effect “too much”.

Unfortunately, when stated as in (A2), this condition is not as informative as we might wish. a condition like (A2) is not in itself very informative. This is because it is expressed in terms of an *endogenous* variable,  $k_{t+1}$ , for given  $k_t$ . A model assumption should preferably be stated in terms of what is *given*, also called the “primitives” of the model; in this model the “primitives” comprise the given preferences, demography, technology, and market form. We can state *sufficient* conditions, however, in terms of the “primitives”, such that (A2) is ensured. Here we state two such sufficient conditions, both involving a CRRA period utility function with parameter  $\theta$  as defined in (3.14):

- (a) If  $0 < \theta \leq 1$ , then (A2) holds for all  $k_t > 0$  along an equilibrium path.

- (b) If the production function is of CES-type,<sup>18</sup> i.e.,  $f(k) = A(\alpha k^\gamma + 1 - \alpha)^{1/\gamma}$ ,  $A > 0$ ,  $0 < \alpha < 1$ ,  $-\infty < \gamma < 1$ , then (A2) holds along an equilibrium path even for  $\theta > 1$ , if the elasticity of substitution between capital and labor,  $1/(1 - \gamma)$ , is not too small, i.e., if

$$\frac{1}{1 - \gamma} > \frac{1 - 1/\theta}{1 + (1 + \rho)^{-1/\theta}(1 + f'(k) - \delta)^{(1-\theta)/\theta}} \quad (3.36)$$

for all  $k > 0$ . In turn, sufficient for this is that  $(1 - \gamma)^{-1} > 1 - \theta^{-1}$ .

That (a) is sufficient for (A2) is immediately visible in (3.15). The sufficiency of (b) is proved in Appendix D. The elasticity of substitution between capital and labor is a concept analogue to the elasticity of intertemporal substitution in consumption. It is a measure of the sensitivity of the chosen  $k = K/L$  with respect to the relative factor price. The next chapter goes more into detail with the concept and shows, among other things, that the Cobb-Douglas production function corresponds to  $\gamma = 0$ . So the Cobb-Douglas production function will satisfy the inequality  $(1 - \gamma)^{-1} > 1 - \theta^{-1}$  (since  $\theta > 0$ ), hence also the inequality (3.36).

With these or other sufficient conditions in the back of our mind we shall now proceed imposing the Positive Slope Assumption (A2). To summarize:

**PROPOSITION 3** (*uniqueness of an equilibrium path*) Suppose the No Fast and Positive Slope assumptions, (A1) and (A2), apply. Then:

- (i) if  $k_0 > 0$ , there exists a unique equilibrium path;
- (ii) if  $k_0 = 0$ , an equilibrium path exists if and only if  $f(0) > 0$  (i.e., capital not essential).

When the conditions of Proposition 3 hold, the fundamental difference equation, (3.32), of the model defines  $k_{t+1}$  as an implicit function of  $k_t$ ,

$$k_{t+1} = \varphi(k_t),$$

for all  $k_t > 0$ , where  $\varphi(k_t)$  is called a *transition function*. The derivative of this implicit function is given by (3.34) with  $k_{t+1}$  on the right-hand side replaced by  $\varphi(k_t)$ , i.e.,

$$\varphi'(k_t) = \frac{s_w(w(k_t), r(\varphi(k_t))) w'(k_t)}{1 + n - s_r(w(k_t), r(\varphi(k_t))) r'(\varphi(k_t))} > 0. \quad (3.37)$$

The positivity for all  $k_t > 0$  is due to (A2). Example 2 above leads to a transition function.

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<sup>18</sup>CES stands for Constant Elasticity of Substitution. The CES production function was briefly considered in Section 2.1 and is considered in detail in Chapter 4.



Having determined the evolution of  $k_t$ , we have in fact determined the evolution of “everything” in the economy: the factor prices  $w(k_t)$  and  $r(k_t)$ , the saving of the young  $s_t = s(w(k_t), r(k_{t+1}))$ , and the consumption by both the young and the old. The mechanism behind the evolution of the economy is the Walrasian (or Classical) mechanism where prices, here  $w_t$  and  $r_t$ , always adjust so as to generate market clearing as if there were a Walrasian auctioneer and where expectations always adjust so as to be model consistent.

### Existence and stability of a steady state?

Possibly the equilibrium path converges to a steady state. To address this issue, we examine the possible configurations of the transition curve in more detail. In addition to being positively sloped everywhere, the transition curve will always, for  $k_t > 0$ , be situated strictly below the solid curve,  $k_{t+1} = w(k_t)/(1+n)$ , shown in Fig. 3.6. In turn, the latter curve is always, for  $k_t > 0$ , strictly below the stippled curve,  $k_{t+1} = f(k_t)/(1+n)$ , in the figure. To be precise:

LEMMA 3 (*ceiling*) Suppose the No Fast Assumption (A1) applies. Along an equilibrium path, whenever  $k_t > 0$ ,

$$0 < k_{t+1} < \frac{w(k_t)}{1+n} < \frac{f(k_t)}{1+n}, \quad t = 0, 1, \dots \quad (*)$$

*Proof.* From (iii) of Proposition 2, an equilibrium path has  $w_t = w(k_t) > 0$  and  $k_{t+1} > 0$  for  $t = 0, 1, 2, \dots$ . Thus,

$$0 < k_{t+1} = \frac{s_t}{1+n} < \frac{w_t}{1+n} = \frac{w(k_t)}{1+n} = \frac{f(k_t) - f'(k_t)k_t}{1+n} < \frac{f(k_t)}{1+n},$$

where the first equality comes from (3.32), the second inequality from Lemma 1 in Section 3.3, and the last inequality from the fact that  $f'(k_t)k_t > 0$  when  $k_t > 0$ . This proves (\*).  $\square$

We will call the graph  $(k_t, w(k_t)/(1+n))$  in Fig. 3.6 a *ceiling*. It acts as a ceiling on  $k_{t+1}$  simply because the saving of the young cannot exceed the income of the young,  $w(k_t)$ . The stippled graph,  $(k_t, f(k_t)/(1+n))$ , in Fig. 3.6 may be called the *roof* (“everything of interest” occurs below it). While the ceiling is the key concept in the proof of Proposition 4 below, the roof is a more straightforward construct since it is directly given by the production function and is always strictly concave. The roof is always above the ceiling and so it appears as a convenient first “enclosure” of the transition curve. Let us therefore start with a characterization of the roof:

LEMMA 4 The roof,  $\mathcal{R}(k) \equiv f(k)/(1+n)$ , has positive slope everywhere, crosses the 45° line for at most one  $k > 0$  and can only do that from above. A necessary

and sufficient condition for the *roof* to be above the  $45^\circ$  line for small  $k$  is that either  $\lim_{k \rightarrow 0} f'(k) > 1 + n$  or  $f(0) > 0$  (capital not essential).

*Proof.* Since  $f' > 0$ , the roof has positive slope. Since  $f'' < 0$ , it can only cross the  $45^\circ$  line once and only from above. If and only if  $\lim_{k \rightarrow 0} f'(k) > 1 + n$ , then for small  $k_t$ , the roof is steeper than the  $45^\circ$  line. Obviously, if  $f(0) > 0$ , then close to the origin, the roof will be above the  $45^\circ$  line.  $\square$

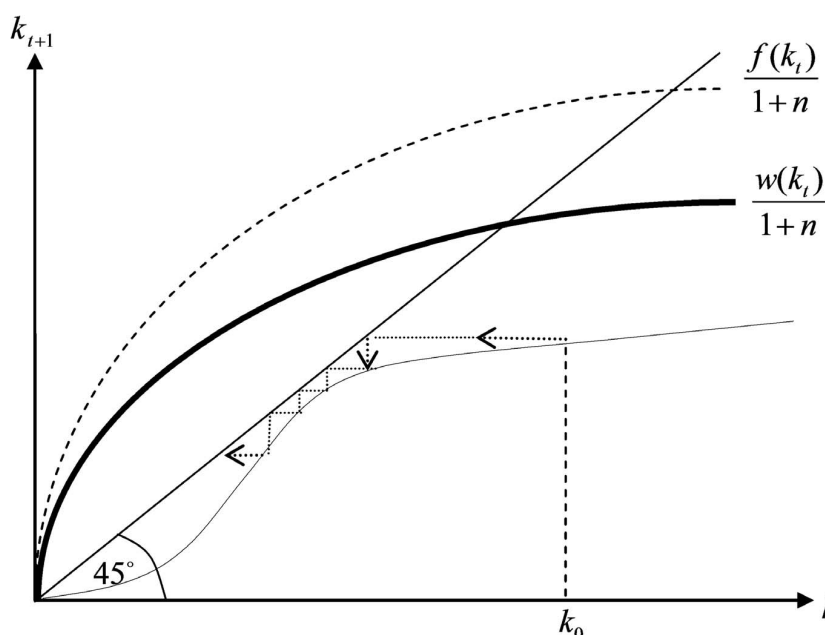


Figure 3.6: A case where both the roof and the ceiling cross the  $45^\circ$  line, but the transition curve does not (no steady state exists).

The ceiling is generally a more complex construct. It can have convex sections and for instance cross the  $45^\circ$  line at more than one point if at all. While the roof can be above the  $45^\circ$  line for *all*  $k_t > 0$ , the ceiling cannot. Indeed, (ii) of the next lemma implies that if for small  $k_t$  the ceiling is above the  $45^\circ$  line, the ceiling will necessarily cross the  $45^\circ$  line at least once for larger  $k_t$ .

**LEMMA 5** Given  $w(k) = f(k) - f'(k)k$  for all  $k \geq 0$ , where  $f(k)$  satisfies  $f(0) \geq 0$ ,  $f' > 0$ ,  $f'' < 0$ , the following holds:

- (i)  $\lim_{k \rightarrow \infty} w(k)/k = 0$ ;
- (ii) the ceiling,  $\mathcal{C}(k) \equiv w(k)/(1+n)$ , is positive and has positive slope for all  $k > 0$ ; moreover, there exists  $\bar{k} > 0$  such that  $\mathcal{C}(k) < k$  for all  $k > \bar{k}$ .

*Proof.* (i) In view of  $f(0) \geq 0$  combined with  $f'' < 0$ , we have  $w(k) > 0$  for all  $k > 0$ . Hence,  $\lim_{k \rightarrow \infty} w(k)/k \geq 0$  if this limit exists. Consider an arbitrary

$k_1 > 0$ . We have  $f'(k_1) > 0$ . For all  $k > k_1$ , it holds that  $0 < f'(k) < f'(k_1)$ , in view of  $f' > 0$  and  $f'' < 0$ , respectively. Hence,  $\lim_{k \rightarrow \infty} f'(k)$  exists and

$$0 \leq \lim_{k \rightarrow \infty} f'(k) < f'(k_1). \quad (3.38)$$

We have

$$\lim_{k \rightarrow \infty} \frac{w(k)}{k} = \lim_{k \rightarrow \infty} \frac{f(k)}{k} - \lim_{k \rightarrow \infty} f'(k). \quad (3.39)$$

There are two cases to consider. *Case 1:*  $f(k)$  has an upper bound. Then,  $\lim_{k \rightarrow \infty} f(k)/k = 0$  so that  $\lim_{k \rightarrow \infty} w(k)/k = -\lim_{k \rightarrow \infty} f'(k) = 0$ , by (3.39) and (3.38), as  $w(k)/k > 0$  for all  $k > 0$ . *Case 2:*  $\lim_{k \rightarrow \infty} f(k) = \infty$ . Then, by L'Hôpital's rule for " $\infty/\infty$ ",  $\lim_{k \rightarrow \infty} (f(k)/k) = \lim_{k \rightarrow \infty} f'(k)$  so that (3.39) implies  $\lim_{k \rightarrow \infty} w(k)/k = 0$ .

(ii) As  $n > -1$  and  $w(k) > 0$  for all  $k > 0$ ,  $\mathcal{C}(k) > 0$  for all  $k > 0$ . From  $w'(k) = -kf''(k) > 0$  follows  $\mathcal{C}'(k) = -kf''(k)/(1+n) > 0$  for all  $k > 0$ ; that is, the ceiling has positive slope everywhere. For  $k > 0$ , the inequality  $\mathcal{C}(k) < k$  is equivalent to  $w(k)/k < 1+n$ . By (i) follows that for all  $\varepsilon > 0$ , there exists  $k_\varepsilon > 0$  such that  $w(k)/k < \varepsilon$  for all  $k > k_\varepsilon$ . Now, letting  $\varepsilon = 1+n$  and  $\bar{k} = k_\varepsilon$  proves that there exists  $\bar{k} > 0$  such that  $w(k)/k < 1+n$  for all  $k > \bar{k}$ .  $\square$

A necessary condition for existence of a (non-trivial) steady state is that the roof is above the  $45^0$  line for small  $k_t$ . But this is not sufficient for also the transition curve to be above the  $45^0$  line for small  $k_t$ . Fig. 3.6 illustrates this. Here the transition curve is in fact everywhere below the  $45^0$  line. In this case no steady state exists and the dynamics imply convergence towards the "catastrophic" point  $(0,0)$ . Given the rate of population growth, the saving of the young is not sufficient to avoid famine in the long run. This outcome will occur if the technology implies so low productivity that even when all income of the young were saved, we would have  $k_{t+1} < k_t$  for all  $k_t > 0$ , cf. Exercise 3.2. The Malthusian mechanism will be at work and bring down  $n$  (outside the model). This exemplifies that even a trivial steady state (the point  $(0,0)$ ) may be of interest in so far as it may be the point the economy is heading to (though never reaching it).

To help existence of a steady state we will impose the condition that either capital is not essential or preferences and technology fit together in such a way that the slope of the transition curve is larger than one for small  $k_t$ . That is, we assume that either

$$\begin{aligned} \text{(i)} \quad & f(0) > 0 \quad \text{or} \\ \text{(ii)} \quad & \lim_{k \rightarrow 0} \varphi'(k) > 1, \end{aligned} \quad (\text{A3})$$

where  $\varphi'(k)$  is implicitly given in (3.37). Whether condition (i) of (A3) holds in a given situation can be directly checked from the production function. If it does

not, we should check condition (ii). But this condition is less amenable because the transition function  $\varphi$  is not one of the “primitives” of the model. There exist cases, though, where we can find an explicit transition function and try out whether (ii) holds (like in Example 2 above). But generally we can not. Then we have to resort to *sufficient* conditions for (ii) of (A3), expressed in terms of the “primitives”. For example, if the period utility function belongs to the CRRA class and the production function is Cobb-Douglas at least for small  $k$ , then (ii) of (A3) holds (see Appendix E). Anyway, as (i) and (ii) of (A3) can be interpreted as reflecting two different kinds of “early steepness” of the transition curve, we shall call (A3) the Early Steepness Assumption.<sup>19</sup>

Before stating the proposition aimed at, we need a definition of the concept of asymptotic stability.

**DEFINITION 5** Consider a first-order autonomous difference equation  $x_{t+1} = g(x_t)$ ,  $t = 0, 1, 2, \dots$ . A steady state  $x^* > 0$  is (locally) *asymptotically stable* if there exists  $\varepsilon > 0$  such that  $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$  implies that  $x_t \rightarrow x^*$  for  $t \rightarrow \infty$ . A steady state  $x^* > 0$  is *globally asymptotically stable* if for all feasible  $x_0 > 0$ , it holds that  $x_t \rightarrow x^*$  for  $t \rightarrow \infty$ .

Applying this definition on our difference equation  $k_{t+1} = \varphi(k_t)$ , we have:

**PROPOSITION 4** (*existence and stability of a steady state*) Assume that the No Fast Assumption (A1) and the Positive Slope assumption (A2) apply as well as the Early Steepness Assumption (A3). Then there exists at least one steady state  $k_1^* > 0$  that is locally asymptotically stable. If  $k_t$  does not converge to  $k_1^*$ ,  $k_t$  converges to another steady state. Oscillations do not occur.

*Proof.* By (A1), Lemma 3 applies. From Proposition 2 we know that if (i) of (A3) holds, then  $k_{t+1} = s_t/(1+n) > 0$  even for  $k_t = 0$ . Alternatively, (ii) of (A3) is enough to ensure that the transition curve lies above the  $45^\circ$  line for small  $k_t$ . According to (ii) of Lemma 5, for large  $k_t$  the ceiling is below the  $45^\circ$  line. Being below the ceiling, cf. Lemma 3, the transition curve must therefore cross the  $45^\circ$  line at least once. Let  $k_1^*$  denote the smallest  $k_t$  at which it crosses. Then  $k_1^* > 0$  is a steady state with the property  $0 < \varphi'(k_1^*) < 1$ . By graphical inspection we see that this steady state is asymptotically stable. If it is the only (non-trivial) steady state, it is globally asymptotically stable. Otherwise, if  $k_t$  does not converge to  $k_1^*$ ,  $k_t$  converges to one of the other steady states. Indeed, divergence is ruled out since, by Lemma 5, there exists  $\bar{k} > 0$  such that  $w(k)/(1+n) < k$  for all  $k > \bar{k}$  (Fig. 3.7 illustrates). For oscillations to come about there must exist a steady state,  $k^{**}$ , with  $\varphi'(k^{**}) < 0$ , but this is impossible in view of (A2).  $\square$

<sup>19</sup>In (i) of (A3), the “steepness” is rather a “hop” at  $k = 0$  if we imagine  $k$  approaching nil from below.

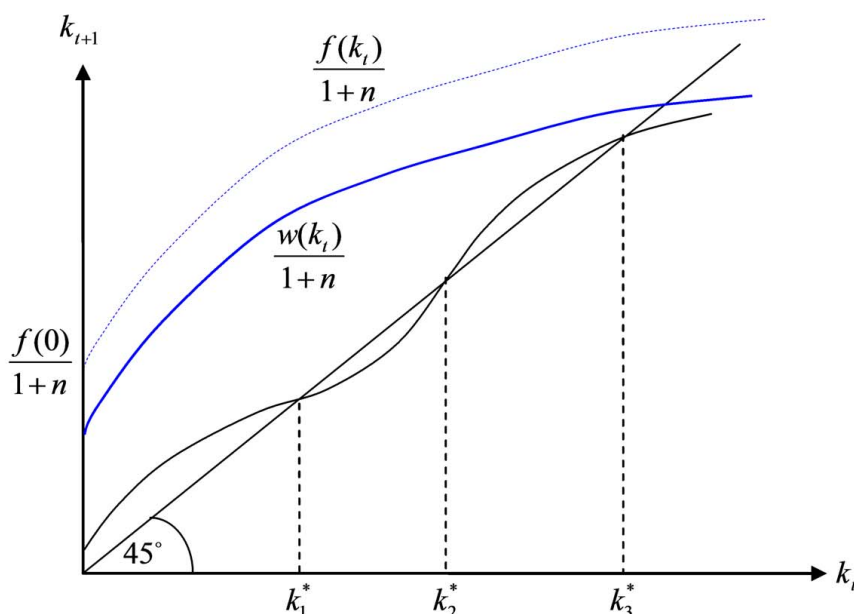


Figure 3.7: A case of multiple steady states (and capital being not essential).

From Proposition 4 we conclude that, given  $k_0$ , the assumptions (A1) - (A3) ensure existence and uniqueness of an equilibrium path; moreover, the equilibrium path converges towards *some* steady state. Thus with these assumptions, for any  $k_0 > 0$ , sooner or later the system settles down at some steady state  $k^* > 0$ . For the factor prices we therefore have

$$\begin{aligned} r_t &= f'(k_t) - \delta \rightarrow f'(k^*) - \delta \equiv r^*, \quad \text{and} \\ w_t &= f(k_t) - k_t f'(k_t) \rightarrow f(k^*) - k^* f'(k^*) \equiv w^*, \end{aligned}$$

for  $t \rightarrow \infty$ . But there may be more than one steady state and therefore only *local* stability is guaranteed. This can be shown by examples, where the utility function, the production function, and parameters are specified in accordance with the assumptions (A1) - (A3) (see Exercise 3.5 and ...).

Fig. 3.7 illustrates such a case (with  $f(0) > 0$  so that capital is not essential). Moving West-East in the figure, the first steady state,  $k_1^*$ , is stable, the second,  $k_2^*$ , unstable, and the third,  $k_3^*$ , stable. In which of the two stable steady states the economy ends up depends on the initial capital-labor ratio,  $k_0$ . The lower steady state,  $k_1^*$ , is known as a *poverty trap*. If  $0 < k_0 < k_2^*$ , the economy is caught in the trap and converges to the low steady state. But with high enough  $k_0$  ( $k_0 > k_2^*$ ), perhaps obtained by foreign aid, the economy avoids the trap and converges to the high steady state. Looking back at Fig. 3.6, we can interpret that figure's scenario as exhibiting an *inescapable* poverty trap.

It turns out that CRRA utility combined with a Cobb-Douglas production function ensures both that (A1) - (A3) hold and that a *unique* (non-trivial) steady state exists. So in this case *global* asymptotic stability of the steady state is ensured.<sup>20</sup> Example 2 and Fig. 3.4 above display a special case of this, the case  $\theta = 1$ .

This is of course a convenient case for the analyst. A Diamond model satisfying assumptions (A1) - (A3) *and* featuring a unique steady state is called a *well-behaved* Diamond model.

We end this section with the question: Is it possible that aggregate consumption, along an equilibrium path, for some periods exceeds aggregate income? We shall see that this is indeed the case in this model if  $K_0$  (wealth of the old in the initial period) is large enough. Indeed, from national accounting we have:

$$\begin{aligned} C_{10} + C_{20} &= F(K_0, L_0) - I_0 > F(K_0, L_0) \Leftrightarrow I_0 < 0 \\ &\Leftrightarrow K_1 < (1 - \delta)K_0 \Leftrightarrow K_0 - K_1 > \delta K_0. \end{aligned}$$

So aggregate consumption in period 0 being greater than aggregate income is equivalent to a fall in the capital stock from period 0 to period 1 greater than the capital depreciation in period 0. Consider the log utility Cobb-Douglas case in Fig. 3.4 and suppose  $\delta < 1$  and  $L_t = L_0 = 1$ , i.e.,  $n = 0$ . Then  $k_t = K_t$  for all  $t$  and by (3.35),  $K_{t+1} = \frac{(1-\alpha)A}{2+\rho} K_t^\alpha$ . Thus  $K_1 < (1 - \delta)K_0$  for

$$K_0 > \left( \frac{(1 - \alpha)A}{(2 + \rho)(1 - \delta)} \right)^{1/(1-\alpha)}.$$

As initial  $K$  is arbitrary, this situation is possible. When it occurs, it reflects that the financial wealth of the old is so large that their consumption (recall they consume all their financial wealth as well as the interest on this wealth) exceeds what is left of current aggregate production after subtracting the amount consumed by the young. So aggregate gross investment in the economy will be negative. Of course this is only feasible if capital goods can be “eaten” or at least be immediately (without further resources) converted into consumption goods. As it stands, the model has implicitly assumed this to be the case. And this is in line with the general setup since the output good is assumed homogeneous and can either be consumed or piled up as capital.

We now turn to efficiency problems.

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<sup>20</sup>See last section of Appendix E.

### 3.6 The golden rule and dynamic inefficiency

An economy described by the Diamond model has the property that even though there is perfect competition and no externalities, the outcome brought about by the market mechanism may not be Pareto optimal.<sup>21</sup> Indeed, the economy may *overaccumulate* forever and thus suffer from a distinctive form of production inefficiency.

A key element in understanding the concept of overaccumulation is the concept of a *golden-rule capital-labor ratio*. Overaccumulation occurs when aggregate saving maintains a capital-labor ratio above the golden-rule value forever. Let us consider these concepts in detail.

In the present section generally the period length is arbitrary except when we relate to the Diamond model and the period length therefore is half of adult lifetime.

#### The golden-rule capital-labor ratio

The golden rule is a principle that relates to technically feasible paths. The principle does not depend on the market form.

Consider the economy-wide resource constraint  $C_t = Y_t - S_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t)$ , where we assume that  $F$  is neoclassical with CRS. Accordingly, aggregate consumption per unit of labor can be written

$$c_t \equiv \frac{C_t}{L_t} = \frac{F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t)}{L_t} = f(k_t) + (1 - \delta)k_t - (1 + n)k_{t+1}, \quad (3.40)$$

where  $k$  is the capital-labor ratio  $K/L$ . Note that  $C_t$  will generally be greater than the workers' consumption. One should simply think of  $C_t$  as the flow of produced consumption goods in the economy and  $c_t$  as this flow divided by aggregate employment, including the labor that in period  $t$  produces investment goods. How the consumption goods are distributed to different members of society is not our concern here.

**DEFINITION 6** By the *golden-rule capital-labor ratio*,  $k_{GR}$ , is meant that value of the capital-labor ratio  $k$ , which results in the highest possible sustainable level of consumption per unit of labor.

Sustainability requires replicability forever. We therefore consider a steady

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<sup>21</sup>Recall that a *Pareto optimal* path is a technically feasible path with the property that no other technically feasible path will make at least one individual better off without making someone else worse off. A technically feasible path which is not Pareto optimal is called *Pareto inferior*.

state. In a steady state  $k_{t+1} = k_t = k$  so that (3.40) simplifies to

$$c = f(k) - (\delta + n)k \equiv c(k). \quad (3.41)$$

Maximizing gives the first-order condition

$$c'(k) = f'(k) - (\delta + n) = 0. \quad (3.42)$$

In view of  $c''(k) = f''(k) < 0$ , the condition (3.42) is both necessary and sufficient for an interior maximum. Let us assume that  $\delta + n > 0$  and that  $f$  satisfies the condition

$$\lim_{k \rightarrow \infty} f'(k) < \delta + n < \lim_{k \rightarrow 0} f'(k).$$

Then (3.42) has a solution in  $k$ , and it is unique because  $c''(k) < 0$ . The solution is called  $k_{GR}$  so that

$$f'(k_{GR}) - \delta = n.$$

That is:

**PROPOSITION 5 (the golden rule)** The highest sustainable consumption level per unit of labor in society is obtained when in steady state the net marginal productivity of capital equals the growth rate of the economy.

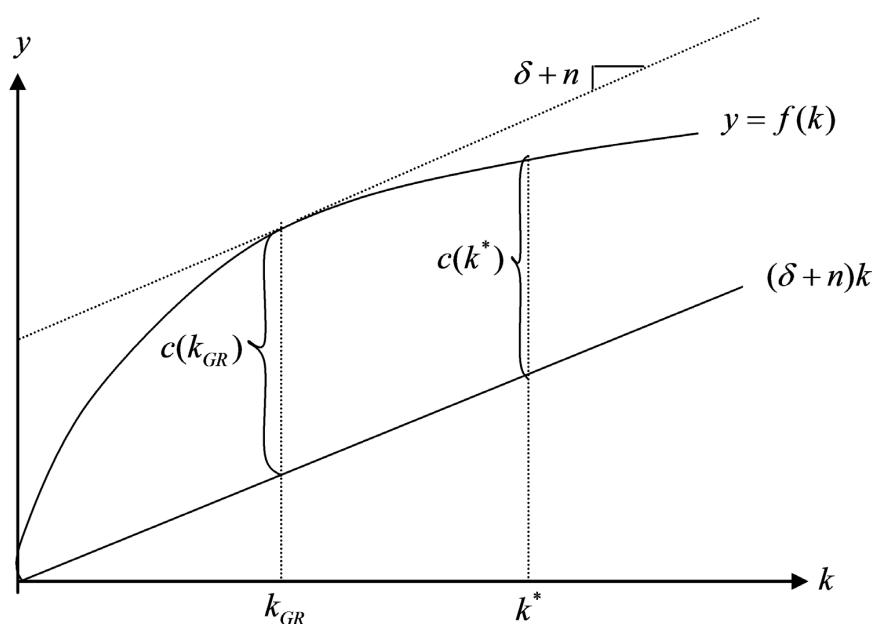


Figure 3.8: A steady state with overaccumulation.

It follows that if a society aims at the highest sustainable level of consumption and initially has  $k_0 < k_{GR}$ , society should increase its capital-labor ratio up to



the point where the extra output obtainable by a further small increase is exactly offset by the extra gross investment needed to maintain the capital-labor ratio at that level. The intuition is visible from (3.41). The golden-rule capital-labor ratio,  $k_{GR}$ , strikes the right balance in the trade-off between high output per unit of labor and a not too high investment requirement. Although a steady state with  $k > k_{GR}$  would imply higher output per unit of labor, it would also imply that a large part of that output is set aside for investment (namely the amount  $(\delta + n)k$  per unit of labor) to counterbalance capital depreciation and growth in the labor force; without this investment the high capital-labor ratio  $k^*$  would not be maintained. With  $k > k_{GR}$  this feature would dominate the first effect so that consumption per unit of labor ends up low. Fig. 3.8 illustrates.

The name golden rule hints at the golden rule from the Bible: “Do unto others as you would have them to do unto you.” We imagine that God asks the newly born generation: “What capital-labor ratio would you prefer to be presented with, given that you must hand over the same capital-labor ratio to the next generation?” The appropriate answer is: the golden-rule capital-labor ratio.

### The possibility of overaccumulation in a competitive market economy

The equilibrium path in the Diamond model with perfect competition implies an interest rate  $r^* = f'(k^*) - \delta$  in a steady state. As an implication,

$$r^* \begin{matrix} \geq \\ \leq \end{matrix} n \Leftrightarrow f'(k^*) - \delta \begin{matrix} \geq \\ \leq \end{matrix} n \Leftrightarrow k^* \begin{matrix} \leq \\ \geq \end{matrix} k_{GR}, \text{ respectively,}$$

in view of  $f'' < 0$ . Hence, a long-run interest rate below the growth rate of the economy indicates that  $k^* > k_{GR}$ . This amounts to a Pareto-inferior state of affairs. Indeed, everyone can be made better off if by a coordinated reduction of saving and investment,  $k$  is reduced. A formal demonstration of this is given in connection with Proposition 6 in the next subsection. Here we give an account in more intuitive terms.

Consider Fig. 3.8. Let  $k$  be gradually reduced to the level  $k_{GR}$  by refraining from investment in period  $t_0$  and forward until this level is reached. When this happens, let  $k$  be maintained at the level  $k_{GR}$  forever by providing for the needed investment per young,  $(\delta + n)k_{GR}$ . Then there would be higher aggregate consumption in period  $t_0$  and every future period. Both the immediate reduction of saving and a resulting lower capital-labor ratio to be maintained contribute to this result. There is thus scope for both young and old to consume more in every future period.

In the Diamond model a simple policy implementing such a Pareto improvement in the case where  $k^* > k_{GR}$  (i.e.,  $r^* < n$ ) is to incur a lump-sum tax on the young, the revenue of which is immediately transferred lump sum to the old,

hence, fully consumed. Suppose this amounts to a transfer of one good from each young to the old. Since there are  $1 + n$  young people for each old person, every old receives in this way  $1 + n$  goods in the same period. Let this transfer be repeated every future period. By decreasing their saving by one unit, the young can maintain unchanged consumption in their youth, and when becoming old, they receive  $1 + n$  goods from the next period's young and so on. In effect, the "return" on the tax payment by the young is  $1 + n$  next period. This is more than the  $1 + r^*$  that could be obtained via the market through own saving.<sup>22</sup>

A proof that  $k^* > k_{GR}$  is indeed theoretically possible in the Diamond model can be based on the log utility-Cobb-Douglas case from Example 2 in Section 3.5.3. As indicated by the formula for  $r^*$  in that example, the outcome  $r^* < n$ , which is equivalent to  $k^* > k_{GR}$ , can always be obtained by making the parameter  $\alpha \in (0, 1)$  in the Cobb-Douglas function small enough. The intuition is that a small  $\alpha$  implies a high  $1 - \alpha$ , that is, a high wage income  $wL = (1 - \alpha)K^\alpha L^{-\alpha} \cdot L = (1 - \alpha)Y$ ; this leads to high saving by the young, since  $s_w > 0$ . The result is a high  $k_{t+1}$  which generates a high real wage also next period and may in this manner be sustained forever.

An intuitive understanding of the fact that the perfectly competitive market mechanism may thus lead to overaccumulation, can be based on the following argument. Assume, first, that  $s_r < 0$ . In this case, if the young in period  $t$  expects the rate of return on their saving to end up small (less than  $n$ ), the decided saving will be large in order to provide for consumption after retirement. But the aggregate result of this behavior is a high  $k_{t+1}$  and therefore a low  $f'(k_{t+1})$ . In this way the expectation of a low  $r_{t+1}$  is confirmed by the actual events. The young persons each do the best they can as atomistic individuals, taking the market conditions as given. Yet the aggregate outcome is an equilibrium with overaccumulation, hence a Pareto-inferior outcome.

Looking at the issue more closely, we see that  $s_r < 0$  is not crucial for this outcome. Suppose  $s_r = 0$  (the log utility case) and that in the current period,  $k_t$  is, for some historical reason, at least temporarily considerably above  $k_{GR}$ . Thus, current wages are high, hence,  $s_t$  is relatively high (there is in this case no offsetting effect on  $s_t$  from the relatively low expected  $r_{t+1}$ ). Again, the aggregate result is a high  $k_{t+1}$  and thus the expectation is confirmed. Consequently, the situation in the next period is the same and so on. By continuity, even if  $s_r > 0$ , the argument goes through as long as  $s_r$  is not too large.

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<sup>22</sup>In this model with no utility of leisure, a tax on wage income, or a mandatory pay-as-you-go pension contribution (see Chapter 5), would act like a lump-sum tax on the young.

The described tax-transfers policy will affect the equilibrium interest rate negatively. By choosing an appropriate size of the tax this policy, combined with competitive markets, will under certain conditions (see Chapter 5.1) bring the economy to the golden-rule steady state where overaccumulation has ceased and  $r^* = n$ .

### Dynamic inefficiency and the double infinity

Another name for the overaccumulation phenomenon is *dynamic inefficiency*.

**DEFINITION 7** A technically feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  with the property that there does not exist another technically feasible path with higher  $c_t$  in some periods without smaller  $c_t$  in other periods is called *dynamically efficient*. A technically feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  which is not dynamically efficient is called *dynamically inefficient*.

**PROPOSITION 6** A technically feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  with the property that for  $t \rightarrow \infty$ ,  $k_t \rightarrow k^* > k_{GR}$ , is dynamically inefficient.

*Proof.* Let  $k^* > k_{GR}$ . Then there exists an  $\varepsilon > 0$  such that  $k \in (k^* - 2\varepsilon, k^* + 2\varepsilon)$  implies  $f'(k) - \delta < n$  since  $f'' < 0$ . By concavity of  $f$ ,

$$f(k) - f(k - \varepsilon) \leq f'(k - \varepsilon)\varepsilon. \quad (3.43)$$

Consider a technically feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  with  $k_t \rightarrow k^*$  for  $t \rightarrow \infty$  (the reference path). Then there exists a  $t_0$  such that for  $t \geq t_0$ ,  $k_t \in (k^* - \varepsilon, k^* + \varepsilon)$ ,  $f'(k_t) - \delta < n$  and  $f'(k_t - \varepsilon) - \delta < n$ . Consider an alternative feasible path  $\{(\hat{c}_t, \hat{k}_t)\}_{t=0}^{\infty}$ , where a) for  $t = t_0$  consumption is increased relative to the reference path such that  $\hat{k}_{t_0+1} = k_{t_0} - \varepsilon$ ; and b) for all  $t > t_0$ , consumption is such that  $\hat{k}_{t+1} = k_t - \varepsilon$ . We now show that after period  $t_0$ ,  $\hat{c}_t > c_t$ . Indeed, for all  $t > t_0$ , by (3.40),

$$\begin{aligned} \hat{c}_t &= f(\hat{k}_t) + (1 - \delta)\hat{k}_t - (1 + n)\hat{k}_{t+1} \\ &= f(k_t - \varepsilon) + (1 - \delta)(k_t - \varepsilon) - (1 + n)(k_{t+1} - \varepsilon) \\ &\geq f(k_t) - f'(k_t - \varepsilon)\varepsilon + (1 - \delta)(k_t - \varepsilon) - (1 + n)(k_{t+1} - \varepsilon) \quad (\text{by (3.43)}) \\ &> f(k_t) - (\delta + n)\varepsilon + (1 - \delta)k_t - (1 + n)k_{t+1} + (\delta + n)\varepsilon \\ &= f(k_t) + (1 - \delta)k_t - (1 + n)k_{t+1} = c_t, \end{aligned}$$

by (3.40).  $\square$

Moreover, it can be shown<sup>23</sup> that:

**PROPOSITION 7** A technically feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  such that for  $t \rightarrow \infty$ ,  $k_t \rightarrow k^* \leq k_{GR}$ , is dynamically efficient.

Accordingly, a steady state with  $k^* < k_{GR}$  is never dynamically inefficient. This is because increasing  $k$  from this level always has its price in terms of a decrease in *current* consumption; and at the same time decreasing  $k$  from this

<sup>23</sup>See Cass (1972).

level always has its price in terms of lost *future* consumption. But a steady state with  $k^* > k_{GR}$  is always dynamically inefficient. Intuitively, staying forever with  $k = k^* > k_{GR}$ , implies that society *never* enjoys its great capacity for producing consumption goods.

The fact that  $k^* > k_{GR}$  – and therefore dynamic inefficiency – cannot be ruled out might seem to contradict the First Welfare Theorem from the microeconomic theory of general equilibrium. This is the theorem saying that under certain conditions (essentially that increasing returns to scale are absent, markets are competitive, no goods are of public good character, and there are no externalities, then market equilibria are Pareto optimal. In fact, however, the First Welfare Theorem also presupposes a finite number of periods or, if the number of periods is infinite, then a finite number of agents. In contrast, in the OLG model there is a *double infinity*: an infinite number of periods *and* agents. Hence, the First Welfare Theorem breaks down. Indeed, the case  $r^* < n$ , i.e.,  $k^* > k_{GR}$ , can arise under *laissez-faire*. Then, as we have seen, everyone can be made better off by a coordinated intervention by some social arrangement (a government for instance) such that  $k$  is reduced.

The essence of the matter is that the double infinity opens up for technically feasible reallocations which are definitely beneficial when  $r^* < n$  and which a central authority can accomplish but the market can not. That *nobody* need loose by the described kind of redistribution is due to the double infinity: the economy goes on forever and there is no last generation. Nonetheless, some kind of centralized *coordination* is required to accomplish a solution.

There is an analogy in “Gamow’s bed problem”: There are an infinite number of inns along the road, each with one bed. On a certain rainy night all innkeepers have committed their beds. A late guest comes to the first inn and asks for a bed. “Sorry, full up!” But the minister of welfare hears about it and suggests that from each inn one incumbent guest moves down the road one inn.<sup>24</sup>

Whether the theoretical possibility of overaccumulation should be a matter of practical concern is an empirical question about the relative size of rates of return and economic growth. To answer the question meaningfully, we need an extension of the criterion for overaccumulation so that the presence of technological progress and rising per capita consumption in the long run can be taken into account. This is one of the topics of the next chapter. At any rate, we can already here reveal that there exists no indication that overaccumulation has ever been an actual problem in industrialized market economies.

A final remark before concluding. Proposition 5 about the golden rule can be generalized to the case where instead of one there are  $n$  different capital goods in

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<sup>24</sup>George Gamow (1904-1968) was a Russian physicist. The problem is also known as *Hilbert’s hotel problem*, after the German mathematician David Hilbert (1862-1943).

the economy. Essentially the generalization says that assuming CRS-neoclassical production functions with  $n$  different capital goods as inputs, one consumption good, no technological change, and perfectly competitive markets, a steady state in which per-unit-of labor consumption is maximized has interest rate equal to the growth rate of the labor force when technological progress is ignored (see, e.g., Mas-Colell, 1989).

### 3.7 Concluding remarks

(Unfinished)

In several respects the conclusions we get from OLG models are different than those from other neoclassical models, in particular representative agent models (to be studied later). In OLG models the aggregate quantities are the outcome of the interplay of finite-lived agents at different stages in their life cycle. The turnover in the population plays a crucial role. In this way the OLG approach lays bare the possibility of coordination failure on a grand scale. In contrast, in a representative agent model, aggregate quantities are just a multiple of the actions of the representative household.

Regarding analytical tractability, the complexity implied by having in every period two different coexisting generations is in some respects more than compensated by the fact that the finite time horizon of the households make the *dynamics* of the model *one-dimensional*: we end up with a first-order difference equation in the capital-labor ratio,  $k_t$ , in the economy. In contrast, the dynamics of the basic representative agent model (Chapter 8 and 10) is two-dimensional (owing to the assumed infinite horizon of the households considered as dynasties).

*Miscellaneous notes:*

OLG gives theoretical insights concerning macroeconomic implications of life cycle behavior, allows heterogeneity, provides training in seeing the economy as consisting of a heterogeneous population where the *distribution* of agent characteristics matters for the aggregate outcome.

Farmer (1993), p. 125, notes that OLG models are difficult to apply and for this reason much empirical work in applied general equilibrium theory has regrettably instead taken the representative agent approach.

Outlook: Rational speculative bubbles in general equilibrium, cf. Chapter ?.

### 3.8 Literature notes

1. The Nobel Laureate Paul A. Samuelson (1915-2009) is one of the pioneers of OLG models. Building on the French economist and Nobel laureate Maurice

Allais (1911-2010), a famous article by Samuelson, from 1958, is concerned with a missing market problem. Imagine a two-period OLG economy where, as in the Diamond model, only the young have an income (by Samuelson simplifying considered an exogenous endowment of consumption goods from heaven). Contrary to the Diamond model, however, there is no capital. Also other potential stores of value are absent. Then, in the laissez-faire market economy the old have to starve because they can no longer work and had no possibility of saving - transferring income - as young.

The allocation of resources in the economy is Pareto-inferior. Indeed, if each member of the young generation hands over to the old generation one unit of consumption, and this is next period repeated by the new young generation and so on in the future, everyone will be better off. Since for every old there are  $1 + n$  young, the implied rate of return would be  $n$ , the population growth rate. Such transfers do not arise under laissez-faire. A kind of social contract is required. As Samuelson pointed out, a government could in period 0 issue paper notes, “money”, and transfer these notes to the members of the old generation who would then use them to buy goods from the young. Provided the young believed the notes to be valuable in the next period, they would accept them in exchange for some of their goods in order to use them in the next period for buying from the new young generation etc.

We have here an example of how a social institution can solve a coordination problem.<sup>25</sup>

2. Diamond (1965) extended Samuelson’s contribution by adding capital accumulation. Because of its antecedents Diamonds OLG model is sometimes called the Samuelson-Diamond model or the Allais-Samuelson-Diamond model. In our exposition we have drawn upon clarifications by Galor and Ryder (1989) and de la Croix and Michel (2002). The last mentioned contribution is an extensive exploration of discrete-time OLG models and their applications. An advanced and thorough treatment from a microeconomic general equilibrium perspective is contained in Bewley (2007).

3. The *life-cycle saving hypothesis* was put forward by Franco Modigliani (1918-2003) and associates in the 1950s. See for example Modigliani and Brumberg (1954). Numerous extensions of the framework, relating to the motives (b) - (e) in the list of Section 3.1, see for instance de la Croix and Michel (2002).

4. A review of the empirics of life-cycle behavior and attempts at refining life-cycle models are given in Browning and Crossley (2001).

5. Regarding the dynamic efficiency issue, both the propositions 6 and 7 were shown in a stronger form by the American economist David Cass (1937-2008).

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<sup>25</sup>To just give a flavor of Samuelson’s contribution we have here ignored several aspects, including that Samuelson assumed three periods of life.

Cass established the *general* necessary and sufficient condition for a feasible path  $\{(c_t, k_t)\}_{t=0}^{\infty}$  to be dynamically efficient (Cass 1972). Our propositions 6 and 7 are more restrictive in that they are limited to paths that converge. Partly intuitive expositions of the deeper aspects of the theory are given by Shell (1971) and Burmeister (1980).

6. Diamond has also contributed to other fields of economics, including search theory for labor markets. In 2010 Diamond, together with Dale Mortensen and Christopher Pissarides, was awarded the Nobel price in economics.

From here very incomplete:

The two-period structure of Diamonds OLG model leaves little room for considering, e.g., education and dissaving in the early years of life. This kind of issues is taken up in three-period extensions of the Diamond model, see de la Croix and Michell (2002).

Multiple equilibria, self-fulfilling expectations, optimism and pessimism..

Dynamic inefficiency, see also Burmeister (1980).

Bewley 1977, 1980.

Two-sector OLG: Galor (1992). Galor's book on difference equations.

On the golden rule in a general setup, see Mas-Colell (1989).

## 3.9 Appendix

### A. On CRRA utility

**Derivation of the CRRA function** Consider a utility function  $u(c)$ , defined for all  $c > 0$  and satisfying  $u'(c) > 0$ ,  $u''(c) < 0$ . Let the absolute value of the elasticity of marginal utility be denoted  $\theta(c)$ , that is,  $\theta(c) \equiv -cu''(c)/u'(c) > 0$ . We claim that if  $\theta(c)$  is a positive constant,  $\theta$ , then, up to a positive linear transformation,  $u(c)$  must be of the form

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1, \end{cases} \quad (*)$$

i.e., of CRRA form.

*Proof.* Suppose  $\theta(c) = \theta > 0$ . Then,  $u''(c)/u'(c) = -\theta/c$ . By integration,  $\ln u'(c) = -\theta \ln c + A$ , where  $A$  is an arbitrary constant. Take the antilogarithm function on both sides to get  $u'(c) = e^A e^{-\theta \ln c} = e^A c^{-\theta}$ . By integration we get

$$u(c) = \begin{cases} e^A \frac{c^{1-\theta}}{1-\theta} + B, & \text{when } \theta \neq 1, \\ e^A \ln c + B, & \text{when } \theta = 1, \end{cases}$$

where  $B$  is an arbitrary constant. This proves the claim. Letting  $A = B = 0$ , we get (\*).  $\square$

When we want to make the kinship between the members of the “CRRA family” transparent, we maintain  $A = 0$  and for  $\theta = 1$  also  $B = 0$ , whereas for  $\theta \neq 1$  we set  $B = -1/(1 - \theta)$ . In this way we achieve that all members of the CRRA family will be represented by curves going through the same point as the log function, namely the point  $(1, 0)$ , cf. Fig. 3.2. For a particular  $\theta > 0, \theta \neq 1$ , we have  $u(c) = (c^{1-\theta} - 1)/(1 - \theta)$ , which makes up the CRRA utility function in normalized form. Given  $\theta$ , the transformation to normalized form is of no consequence for the economic behavior since adding or subtracting a constant does not affect marginal rates of substitution.

**The domain of the CRRA function** We want to extend the domain to include  $c = 0$ . If  $\theta \geq 1$ , the CRRA function, whether in the form  $u(c) = (c^{1-\theta} - 1)/(1 - \theta)$  or in the form (\*), is defined only for  $c > 0$ , not for  $c = 0$ . This is because for  $c \rightarrow 0$  we get  $u(c) \rightarrow -\infty$ . In this case we simply define  $u(0) = -\infty$ . This will create no problems since the CRRA function anyway has the property that  $u'(c) \rightarrow \infty$ , when  $c \rightarrow 0$  (whether  $\theta$  is larger or smaller than one). The marginal utility thus becomes very large as  $c$  becomes very small, that is, the No Fast Assumption is satisfied. This will ensure that the chosen  $c$  is strictly positive whenever there is a positive budget. So throughout this book we define the domain of the CRRA function to be  $[0, \infty)$ .

**The range of the CRRA function** Considering the CRRA function  $u(c) \equiv (c^{1-\theta} - 1)/(1 - \theta)$  for  $c \in [0, \infty)$ , we have:

$$\begin{aligned} \text{for } 0 < \theta < 1, \text{ the range of } u(c) \text{ is } &[-(1 - \theta)^{-1}, \infty), \\ \text{for } \theta = 1, \text{ the range of } u(c) \text{ is } &[-\infty, \infty), \\ \text{for } \theta > 1, \text{ the range of } u(c) \text{ is } &[-\infty, -(1 - \theta)^{-1}). \end{aligned}$$

Thus, in the latter case  $u(c)$  is bounded from above and so allows asymptotic “saturation” to occur.

## B. Deriving the elasticity of intertemporal substitution in consumption

Referring to Section 3.3, we here show that the definition of  $\sigma(c_1, c_2)$  in (3.17) gives the result (3.18). Let  $x \equiv c_2/c_1$  and  $\beta \equiv (1 + \rho)^{-1}$ . Then the first-order condition (3.16) and the equation describing the considered indifference curve constitute a system of two equations

$$\begin{aligned} u'(c_1) &= \beta u'(xc_1)R, \\ u(c_1) + \beta u(xc_1) &= \bar{U}. \end{aligned}$$



For a fixed utility level  $U = \bar{U}$  these equations define  $c_1$  and  $x$  as implicit functions of  $R$ ,  $c_1 = c(R)$  and  $x = x(R)$ . We calculate the total derivative with respect to  $R$  in both equations and get, after ordering,

$$\begin{aligned} & [u''(c_1) - \beta R u''(xc_1)x] c'(R) - \beta R u''(xc_1)c_1 x'(R) \\ & = \beta u'(xc_1), \end{aligned} \quad (3.44)$$

$$[u'(c_1) + \beta u'(xc_1)x] c'(R) = -\beta u'(xc_1)c_1 x'(R). \quad (3.45)$$

Substituting  $c'(R)$  from (3.45) into (3.44) and ordering now yields

$$- \left[ x \frac{c_1 u''(c_1)}{u'(c_1)} + R \frac{xc_1 u''(xc_1)}{u'(xc_1)} \right] \frac{R}{x} x'(R) = x + R.$$

Since  $-cu''(c)/u'(c) \equiv \theta(c)$ , this can be written

$$\frac{R}{x} x'(R) = \frac{x + R}{x\theta(c_1) + R\theta(xc_1)}.$$

Finally, in view of  $xc_1 = c_2$  and the definition of  $\sigma(c_1, c_2)$ , this gives (3.18).

### C. Walras' law

In the proof of Proposition 1 we referred to Walras' law. Here is how Walras' law works in each period in a model like this. We consider period  $t$ , but for simplicity we skip the time index  $t$  on the variables. There are three markets, a market for capital services, a market for labor services, and a market for output goods. Suppose a "Walrasian auctioneer" calls out the price vector  $(\hat{r}, w, 1)$ , where  $\hat{r} > 0$  and  $w > 0$ , and asks all agents, i.e., the young, the old, and the representative firm, to declare their supplies and demands.

The supplies of capital and labor are by assumption inelastic and equal to  $K$  units of capital services and  $L$  units of labor services. But the demand for capital and labor services depends on the announced  $\hat{r}$  and  $w$ . Let the potential pure profit of the representative firm be denoted  $\Pi$ . If  $\hat{r}$  and  $w$  are so that  $\Pi < 0$ , the firm declares  $K^d = 0$  and  $L^d = 0$ . If on the other hand at the announced  $\hat{r}$  and  $w$ ,  $\Pi = 0$  (as when  $\hat{r} = r(k) + \delta$  and  $w = w(k)$ ), the desired capital-labor ratio is given as  $k^d = f'^{-1}(\hat{r})$  from (3.20), but the firm is indifferent with respect to the absolute level of the factor inputs. In this situation the auctioneer tells the firm to declare  $L^d = L$  (recall  $L$  is the given labor supply) and  $K^d = k^d L^d$  which is certainly acceptable for the firm. Finally, if  $\Pi > 0$ , the firm is tempted to declare infinite factor demands, but to avoid that, the auctioneer imposes the rule that the maximum allowed demands for capital and labor are  $2K$  and  $2L$ , respectively.

Within these constraints the factor demands will be uniquely determined by  $\hat{r}$  and  $w$  and we have

$$\Pi = \Pi(\hat{r}, w, 1) = F(K^d, L^d) - \hat{r}K^d - wL^d. \quad (3.46)$$

The owners of both the capital stock  $K$  and the representative firm must be those who saved in the previous period, namely the currently old. These elderly will together declare the consumption  $c_2L_{-1} = (1 + \hat{r} - \delta)K + \Pi$  and the net investment  $-K$  (which amounts to disinvestment). The young will declare the consumption  $c_1L = wL - s(w, r_{+1}^e)L$  and the net investment  $sL = s(w, r_{+1}^e)L$ . So aggregate declared consumption will be  $C = (1 + \hat{r} - \delta)K + \Pi + wL - s(w, r_{+1}^e)L$  and aggregate net investment  $I - \delta K = s(w, r_{+1}^e)L - K$ . It follows that  $C + I = wL + \hat{r}K + \Pi$ . The aggregate declared supply of output is  $Y^s = F(K^d, L^d)$ . The values of excess demands in the three markets now add to

$$\begin{aligned} Z(\hat{r}, w, 1) &\equiv w(L^d - L) + \hat{r}(K^d - K) + C + I - Y^s \\ &= wL^d - wL + \hat{r}K^d - \hat{r}K + wL + \hat{r}K + \Pi - F(K^d, L^d) \\ &= wL^d + \hat{r}K^d + \Pi - F(K^d, L^d) = 0, \end{aligned}$$

by (3.46).

This is a manifestation of Walras' law for each period: *whatever the announced price vector for the period is, the aggregate value of excess demands in the period is zero*. The reason is the following. When each household satisfies its budget constraint and each firm pays out its ex ante profit,<sup>26</sup> then the economy as a whole has to satisfy an aggregate budget constraint for the period considered.

The budget constraints, demands, and supplies operating in this thought experiment (and in Walras' law in general) are the *Walrasian* budget constraints, demands, and supplies. Outside equilibrium these are somewhat artificial constructs. A Walrasian budget constraint is based on the assumption that the desired actions can be realized. This assumption will be wrong unless  $\hat{r}$  and  $w$  are already at their equilibrium levels. But the assumption that desired actions can be realized is never falsified because the thought experiment does not allow trades to take place outside Walrasian equilibrium. Similarly, the Walrasian consumption demand by the worker is rather hypothetical outside equilibrium. This demand is based on the income the worker *would* get if fully employed at the announced real wage, not on the actual employment (or unemployment) at that real wage.

These ambiguities notwithstanding, the important message of Walras' law goes through, namely that when two of the three markets clear (in the sense of the Walrasian excess demands being nil), so does the third.

<sup>26</sup>By ex ante profit is meant the hypothetical profit calculated on the basis of firms' desired supply evaluated at the announced price vector,  $(\hat{r}, w, 1)$ .

### D. Proof of (i) and (ii) of Proposition 2

For convenience we repeat the fundamental difference equation characterizing an equilibrium path:

$$k_{t+1} = \frac{s(w(k_t), r(k_{t+1}))}{1+n},$$

where  $w(k) \equiv f(k) - f'(k)k > 0$  for all  $k > 0$  and  $r(k) \equiv f'(k) - \delta > -1$  for all  $k \geq 0$ . The key to the proof of Proposition 2 about existence of an equilibrium path is the following lemma.

LEMMA D1 Suppose the No Fast Assumption (A1) applies and let  $w > 0$  and  $n > -1$  be given. Then the equation

$$\frac{s(w, r(k))}{k} = 1+n. \quad (3.47)$$

has at least one solution  $k > 0$ .

*Proof.* Note that  $1+n > 0$ . From Lemma 1 in Section 3.3 follows that for all possible values of  $r(k)$ ,  $0 < s(w, r(k)) < w$ . Hence, for any  $k > 0$ ,

$$0 < \frac{s(w, r(k))}{k} < \frac{w}{k}.$$

Letting  $k \rightarrow \infty$  we then have  $s(w, r(k))/k \rightarrow 0$  since  $s(w, r(k))/k$  is squeezed between 0 and 0 (as indicated in the two graphs in Fig. 3.9).

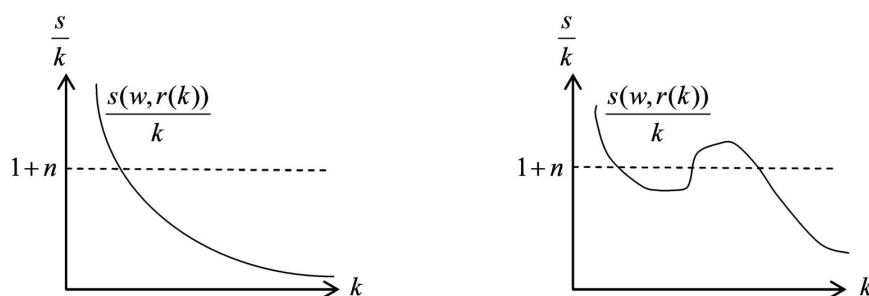


Figure 3.9: Existence of a solution to equation (3.47).

Next we consider  $k \rightarrow 0$ . There are two cases.

*Case 1:*  $\lim_{k \rightarrow 0} s(w, r(k)) > 0$ .<sup>27</sup> Then obviously  $\lim_{k \rightarrow 0} s(w, r(k))/k = \infty$ .

<sup>27</sup>If the limit does not exist, the proof applies to the *limit inferior* of  $s(w, r(k))$  for  $k \rightarrow 0$ . The limit inferior for  $i \rightarrow \infty$  of a sequence  $\{x_i\}_{i=0}^{\infty}$  is defined as  $\lim_{i \rightarrow \infty} \inf \{x_j | j = i, i+1, \dots\}$ , where  $\inf$  of a set  $S_i = \{x_j | j = i, i+1, \dots\}$  is defined as the greatest lower bound for  $S_i$ .

Case 2:  $\lim_{k \rightarrow 0} s(w, r(k)) = 0$ .<sup>28</sup> In this case we have

$$\lim_{k \rightarrow 0} r(k) = \infty. \quad (3.48)$$

Indeed, since  $f'(k)$  rises monotonically as  $k \rightarrow 0$ , the only alternative would be that  $\lim_{k \rightarrow 0} r(k)$  exists and is  $< \infty$ ; then, by Lemma 1 in Section 3.3, we would be in case 1 rather than case 2. By the second-period budget constraint, with  $r = r(k)$ , consumption as old is  $c_2 = s(w, r(k))(1 + r(k)) \equiv c(w, k) > 0$  so that

$$\frac{s(w, r(k))}{k} = \frac{c(w, k)}{[1 + r(k)]k}.$$

The right-hand side of this equation goes to  $\infty$  for  $k \rightarrow 0$  since  $\lim_{k \rightarrow 0} [1 + r(k)]k = 0$  by Technical Remark in Section 3.4 and  $\lim_{k \rightarrow 0} c(w, k) = \infty$ ; this latter fact follows from the first-order condition (3.8), which can be written

$$0 \leq u'(c(w, k)) = (1 + \rho) \frac{u'(w - s(w, r(k)))}{1 + r(k)} \leq (1 + \rho) \frac{u'(w)}{1 + r(k)}.$$

Taking limits on both sides gives

$$\lim_{k \rightarrow 0} u'(c(w, k)) = (1 + \rho) \lim_{k \rightarrow 0} \frac{u'(w - s(w, r(k)))}{1 + r(k)} = (1 + \rho) \lim_{k \rightarrow 0} \frac{u'(w)}{1 + r(k)} = 0,$$

where the second equality comes from the fact that we are in case 2 and the third comes from (3.48). But since  $u'(c) > 0$  and  $u''(c) < 0$  for all  $c > 0$ ,  $\lim_{k \rightarrow 0} u'(c(w, k)) = 0$  requires  $\lim_{k \rightarrow 0} c(w, k) = \infty$ , as was to be shown.

In both Case 1 and Case 2 we thus have that  $k \rightarrow 0$  implies  $s(w, r(k))/k \rightarrow \infty$ . Since  $s(w, r(k))/k$  is a continuous function of  $k$ , there must be at least one  $k > 0$  such that (3.47) holds (as illustrated by the two graphs in Fig. 3.14).  $\square$

Now, to prove (i) of Proposition 2, consider an arbitrary  $k_t > 0$ . We have  $w(k_t) > 0$ . In (3.47), let  $w = w(k_t)$ . By Lemma C1, (3.47) has a solution  $k > 0$ . Set  $k_{t+1} = k$ . Starting with  $t = 0$ , from a given  $k_0 > 0$  we thus find a  $k_1 > 0$  and letting  $t = 1$ , from the now given  $k_1$  we find a  $k_2$  and so on. The resulting infinite sequence  $\{k_t\}_{t=0}^{\infty}$  is an equilibrium path. In this way we have proved existence of an equilibrium path if  $k_0 > 0$ . Thereby (i) of Proposition 2 is proved.

But what if  $k_0 = 0$ ? Then, if  $f(0) = 0$ , no temporary equilibrium is possible in period 0, in view of (ii) of Proposition 1; hence there can be no equilibrium path. Suppose  $f(0) > 0$ . Then  $w(k_0) = w(0) = f(0) > 0$ , as explained in Technical Remark in Section 3.4. Let  $w$  in equation (3.47) be equal to  $f(0)$ . By Lemma C1 this equation has a solution  $k > 0$ . Set  $k_1 = k$ . Letting period 1 be the new initial period, we are back in the case with initial capital positive. This proves (ii) of Proposition 2.

<sup>28</sup>If the limit does not exist, the proof applies to the *limit inferior* of  $s(w, r(k))$  for  $k \rightarrow 0$ .

### E. Sufficient conditions for certain properties of the transition curve

**Positive slope everywhere** For convenience we repeat here the condition (3.36):

$$\frac{1}{1-\gamma} > \frac{1-\sigma}{1+(1+\rho)^{-\sigma}(1+f'(k)-\delta)^{\sigma-1}}, \quad (*)$$

where we have substituted  $\sigma \equiv 1/\theta$ . In Section 3.5.3 we claimed that in the CRRA-CES case this condition is sufficient for the transition curve to be positively sloped everywhere. We here prove the claim.

Consider an arbitrary  $k_t > 0$  and let  $w \equiv w(k_t) > 0$ . Knowing that  $w'(k_t) > 0$  for all  $k_t > 0$ , we can regard  $k_{t+1}$  as directly linked to  $w$ . With  $k$  representing  $k_{t+1}$ ,  $k$  must satisfy the equation  $k = s(w, r(k))/(1+n)$ . A sufficient condition for this equation to implicitly define  $k$  as an increasing function of  $w$  is also a sufficient condition for the transition curve to be positively sloped for all  $k_t > 0$ .

When  $u(c)$  belongs to the CRRA class, by (3.15) with  $\sigma \equiv 1/\theta$ , we have  $s(w, r(k)) = [1 + (1+\rho)^\sigma(1+r(k))^{1-\sigma}]^{-1}w$ . The equation  $k = s(w, r(k))/(1+n)$  then implies

$$\frac{w}{1+n} = k [1 + (1+\rho)^\sigma R(k)^{1-\sigma}] \equiv h(k), \quad (3.49)$$

where  $R(k) \equiv 1 + r(k) \equiv 1 + f'(k) - \delta > 0$  for all  $k > 0$ . It remains to provide a sufficient condition for obtaining  $h'(k) > 0$  for all  $k > 0$ . We have

$$h'(k) = 1 + (1+\rho)^\sigma R(k)^{1-\sigma} [1 - (1-\sigma)\eta(k)], \quad (3.50)$$

since  $\eta(k) \equiv -kR'(k)/R(k) > 0$ , the sign being due to  $R'(k) = f''(k) < 0$ . So  $h'(k) > 0$  if and only if  $1 - (1-\sigma)\eta(k) > -(1+\rho)^{-\sigma}R(k)^{\sigma-1}$ , a condition equivalent to

$$\frac{1}{\eta(k)} > \frac{1-\sigma}{1+(1+\rho)^{-\sigma}R(k)^{\sigma-1}}. \quad (3.51)$$

To make this condition more concrete, consider the CES production function

$$f(k) = A(\alpha k^\gamma + 1 - \alpha), \quad A > 0, 0 < \alpha < 1, \gamma < 1. \quad (3.52)$$

Then  $f'(k) = \alpha A^\gamma (f(k)/k)^{1-\gamma}$  and defining  $\pi(k) \equiv f'(k)k/f(k)$  we find

$$\eta(k) = (1-\gamma) \frac{(1-\pi(k))f'(k)}{1-\delta+f'(k)} \leq (1-\gamma)(1-\pi(k)) < 1-\gamma, \quad (3.53)$$

where the first inequality is due to  $0 \leq \delta \leq 1$  and the second to  $0 < \pi(k) < 1$ , which is an implication of strict concavity of  $f$  combined with  $f(0) \geq 0$ . Thus,  $\eta(k)^{-1} > (1-\gamma)^{-1}$  so that if (\*) holds for all  $k > 0$ , then so does (3.51), i.e.,  $h'(k) > 0$  for all  $k > 0$ . We have hereby shown that (\*) is sufficient for the transition curve to be positively sloped everywhere.

**Transition curve steep for  $k$  small** Here we specialize further and consider the CRRA-Cobb-Douglas case:  $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ ,  $\theta > 0$ , and  $f(k) = Ak^\alpha$ ,  $A > 0$ ,  $0 < \alpha < 1$ . In the prelude to Proposition 4 in Section 3.5 it was claimed that if this combined utility and technology condition holds at least for small  $k$ , then (ii) of (A3) is satisfied. We now show this.

Letting  $\gamma \rightarrow 0$  in (3.52) gives the Cobb-Douglas function  $f(k) = Ak^\alpha$  (this is proved in the appendix to Chapter 4). With  $\gamma = 0$ , clearly  $(1 - \gamma)^{-1} = 1 > 1 - \sigma$ , where  $\sigma \equiv \theta^{-1} > 0$ . This inequality implies that (\*) above holds and so the transition curve is positively sloped everywhere. As an implication there is a transition function,  $\varphi$ , such that  $k_{t+1} = \varphi(k_t)$ ,  $\varphi'(k_t) > 0$ . Moreover, since  $f(0) = 0$ , we have, by Lemma 5,  $\lim_{k_t \rightarrow 0} \varphi(k_t) = 0$ .

Given the imposed CRRA utility, the fundamental difference equation of the model is

$$k_{t+1} = \frac{w(k_t)}{(1+n)[1+(1+\rho)^\sigma R(k_{t+1})^{1-\sigma}]} \quad (3.54)$$

or, equivalently,

$$h(k_{t+1}) = \frac{w(k_t)}{1+n},$$

where  $h(k_{t+t})$  is defined as in (3.49). By implicit differentiation we find  $h'(k_{t+1})\varphi'(k_t) = w'(k_t)/(1+n)$ , i.e.,

$$\varphi'(k_t) = \frac{w'(k_t)}{(1+n)h'(k_{t+1})} > 0.$$

If  $k^* > 0$  is a steady-state value of  $k_t$ , (3.54) implies

$$1 + (1+\rho)^\sigma R(k^*)^{1-\sigma} = \frac{w(k^*)}{(1+n)k^*}, \quad (3.55)$$

and the slope of the transition curve at the steady state will be

$$\varphi'(k^*) = \frac{w'(k^*)}{(1+n)h'(k^*)} > 0. \quad (3.56)$$

If we can show that such a  $k^* > 0$  exists, is unique, and implies  $\varphi'(k^*) < 1$ , then the transition curve crosses the 45° line from above, and so (ii) of (A3) follows in view of  $\lim_{k_t \rightarrow 0} \varphi(k_t) = 0$ .

Defining  $x(k) \equiv f(k)/k = Ak^{\alpha-1}$ , where  $x'(k) = (\alpha-1)Ak^{\alpha-2} < 0$ , and using that  $f(k) = Ak^\alpha$ , we have  $R(k) = 1 + \alpha x(k) - \delta$  and  $w(k)/k = (1-\alpha)x(k)$ . Hence, (3.55) can be written

$$1 + (1+\rho)^\sigma (1 + \alpha x^* - \delta)^{1-\sigma} = \frac{1-\alpha}{1+n} x^*, \quad (3.57)$$

where  $x^* = x(k^*)$ . It is easy to show graphically that this equation has a unique solution  $x^* > 0$  whether  $\sigma < 1$ ,  $\sigma = 1$ , or  $\sigma > 1$ . Then  $k^* = (x^*/A)^{1/(\alpha-1)} > 0$  is also unique.

By (3.50) and (3.57),

$$\begin{aligned} h'(k^*) &= 1 + \left(\frac{1-\alpha}{1+n}x^* - 1\right) [1 - (1-\sigma)\eta(k^*)] > 1 + \left(\frac{1-\alpha}{1+n}x^* - 1\right)(1 - \eta(k^*)) \\ &\geq 1 + \left(\frac{1-\alpha}{1+n}x^* - 1\right)\alpha, \end{aligned}$$

where the first inequality is due to  $\sigma > 0$  and the second to the fact that  $\eta(k) \leq 1 - \alpha$  in view of (3.53) with  $\gamma = 0$  and  $\pi(k) = \alpha$ . Substituting this together with  $w'(k^*) = (1 - \alpha)\alpha x^*$  into (3.56) gives

$$0 < \varphi'(k^*) < \frac{\alpha x^*}{1 + n + \alpha x^*} < 1, \quad (3.58)$$

as was to be shown.

**The CRRA-Cobb-Douglas case is well-behaved** For the case of CRRA utility and Cobb-Douglas technology with CRS, existence and uniqueness of a steady state has just been proved. Asymptotic stability follows from (3.58). So the CRRA-Cobb-Douglas case is well-behaved.

## 3.10 Exercises

**3.1** The dynamic accounting relation for a closed economy is

$$K_{t+1} = K_t + S_t^N \quad (*)$$

where  $K_t$  is the aggregate capital stock and  $S_t^N$  is aggregate net saving. In the Diamond model, let  $S_{1t}$  be aggregate net saving of the young in period  $t$  and  $S_{2t}$  aggregate net saving of the old in the same period. On the basis of (\*) give a direct proof that the link between two successive periods takes the form  $k_{t+1} = s_t/(1+n)$ , where  $s_t$  is the saving of each young,  $n$  is the population growth rate, and  $k_{t+1}$  is the capital/labor ratio at the beginning of period  $t + 1$ . *Hint:* by definition, the increase in financial wealth is the same as net saving (ignoring gifts).

**3.2** Suppose the production function in Diamond's OLG model is  $Y = A(\alpha K^\gamma + (1-\alpha)L^\gamma)^{1/\gamma}$ ,  $A > 0$ ,  $0 < \alpha < 1$ ,  $\gamma < 0$ , and  $A\alpha^{1/\gamma} < 1+n$ . a) Given  $k \equiv K/L$ , find the equilibrium real wage,  $w(k)$ . b) Show that  $w(k) < (1+n)k$  for all  $k > 0$ . *Hint:*

consider the roof. c) Comment on the implication for the long-run evolution of the economy. *Hint:* consider the ceiling.

**3.3** (*multiple temporary equilibria with self-fulfilling expectations*) Fig. 3.10 shows the transition curve for a Diamond OLG model with  $u(c) = c^{1-\theta}/(1-\theta)$ ,  $\theta = 8$ ,  $\rho = 0.4$ ,  $n = 0.2$ ,  $\delta = 0.6$ ,  $f(k) = A(bk^p + 1 - b)^{1/p}$ ,  $A = 7$ ,  $b = 0.33$ ,  $p = -0.4$ .

- Let  $t = 0$ . For a given  $k_0$  slightly below 1, how many temporary equilibria with self-fulfilling expectations are there?
- Suppose the young in period 0 expect the real interest rate on their saving to be relatively low. Describe by words the resulting equilibrium path in this case. Comment (what is the economic intuition behind the path?).
- In the first sentence under b), replace “low” by “high”. How is the answer to b) affected? What kind of difficulty arises?

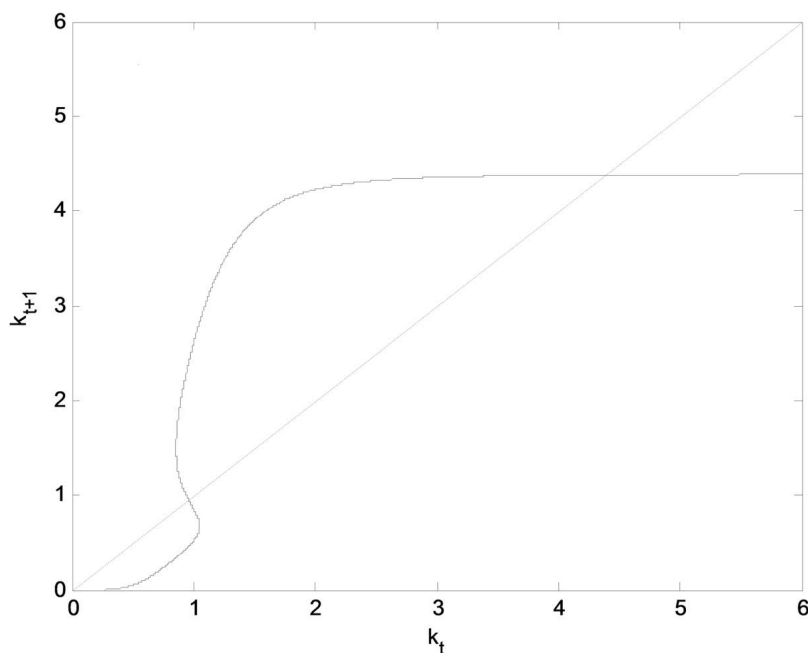


Figure 3.10: Transition curve for Diamond’s OLG model in the case described in Exercise 3.3.

**3.4** (*plotting the transition curve by MATLAB*) This exercise requires computation on a computer. You may use *MATLAB OLG program*.<sup>29</sup>

<sup>29</sup>Made by Marc P. B. Klemp and available at the address:



- a) Enter the model specification from Exercise 3.3 and plot the transition curve.
- b) Plot examples for two other values of the substitution parameter:  $p = -1.0$  and  $p = 0.5$ . Comment.
- c) Find the approximate largest lower bound for  $p$  such that higher values of  $p$  eliminates multiple equilibria.
- d) In continuation of c), what is the corresponding elasticity of factor substitution,  $\psi$ ? *Hint:* as shown in §4.4, the formula is  $\psi = 1/(1 - p)$ .
- e) The empirical evidence for industrialized countries suggests that  $0.4 < \psi < 1.0$ . Is your  $\psi$  from d) empirically realistic? Comment.

**3.5** (*one stable and one unstable steady state*) Consider the following Diamond model:  $u(c) = \ln c$ ,  $\rho = 2.3$ ,  $n = 2.097$ ,  $\delta = 1.0$ ,  $f(k) = A(bk^p + 1 - b)^{1/p}$ ,  $A = 20$ ,  $b = 0.5$ ,  $p = -1.0$ .

- a) Plot the transition curve of the model. *Hint:* you may use either a program like *MATLAB OLG Program* (available on the course website) or first a little algebra and then Excel (or similar simple software).
- b) Comment on the result you get. Will there exist a poverty trap? Why or why not?
- c) At the stable steady state calculate numerically the output-capital ratio, the aggregate saving-income ratio, the real interest rate, and the capital income share of gross national income.
- d) Briefly discuss how your results in c) comply with your knowledge of corresponding empirical magnitudes in industrialized Western countries?
- e) There is one feature which this model, as a long-run model, ought to incorporate, but does not. Extend the model, taking this feature into account, and write down the fundamental difference equation for the extended model in algebraic form.
- f) Plot the new transition curve. *Hint:* given the model specification, this should be straightforward if you use Excel (or similar); and if you use *MATLAB OLG Program*, note that by a simple “trick” you can transform your new model into the “old” form.

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<http://www.econ.ku.dk/okocg/Computation/>.

- g) The current version of the MATLAB OLG Program is not adapted to this question. So at least here you need another approach, for instance based on a little algebra and then Excel (or similar simple software). Given  $k_0 = 10$ , calculate numerically the time path of  $k_t$  and plot the *time profile* of  $k_t$ , i.e., the graph  $(t, k_t)$  in the  $tk$ -plane. Next, do the same for  $k_0 = 1$ . Comment.

**3.6** (*dynamics under myopic foresight*)

(incomplete) Show the possibility of a chaotic trajectory.

**3.7** Given the period utility function is CRRA, derive the saving function of the young in Diamond's OLG model. *Hint*: substitute the period budget constraints into the Euler equation.

**3.8** *Short questions* a) A steady-state capital-labor ratio can be in the “dynamically efficient” region or in the “dynamically inefficient” region. How are the two mentioned regions defined? b) Give a simple characterization of the two regions. c) The First Welfare Theorem states that, given certain conditions, any competitive equilibrium ( $\equiv$  Walrasian equilibrium) is Pareto optimal. Give a list of circumstances that each tend to obstruct Pareto optimality of a competitive equilibrium.

**3.9** Consider a Diamond OLG model for a closed economy. Let the utility discount rate be denoted  $\rho$  and let the period utility function be specified as  $u(c) = \ln c$ .

- a) Derive the saving function of the young. Comment.
- b) Let the aggregate production function be a neoclassical production function with CRS and ignore technological progress. Let  $L_t$  denote the number of young in period  $t$ . Derive the fundamental difference equation of the model.

From now, assume that the production function is  $Y = \alpha L + \beta KL/(K + L)$ , where  $\alpha > 0$  and  $\beta > 0$  (as in Problem 2.4).

- c) Draw a transition diagram illustrating the dynamics of the economy. Make sure that you draw the diagram so as to exhibit consistency with the production function.
- d) Given the above information, can we be sure that there exists a unique and globally asymptotically stable steady state? Why or why not?
- e) Suppose the economy is in a steady state up to and including period  $t_0 > 0$ . Then, at the shift from period  $t_0$  to period  $t_0 + 1$ , a negative technology shock occurs such that the technology level in period  $t_0 + 1$  is below that of period  $t_0$ . Illustrate by a transition diagram the evolution of the economy from period  $t_0$  onward. Comment.

- f) Let  $k \equiv K/L$ . In the  $(t, \ln k)$  plane, draw a graph of  $\ln k_t$  such that the qualitative features of the time path of  $\ln k$  before and after the shock, including the long run, are exhibited.
- g) How, if at all, is the real interest rate in the long run affected by the shock?
- h) How, if at all, is the real wage in the long run affected by the shock?
- i) How, if at all, is the labor income share of national income in the long run affected by the shock?
- j) Explain by words the economic intuition behind your results in h) and i).

**3.10**

