Advanced Macroeconomics. Exercises

30.12.2008. Christian Groth

Solutions to Problem V.2

V.2

a) We have $R_t = F(K_t, L_t) - w_t L_t - I_t = F(K_t, L_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \Pi_t + r_t K_t - \dot{K}_t$.

b) From the hint we see that $\int_0^T R_t e^{-rt} dt = \int_0^T \Pi_t e^{-rt} dt - K_T e^{-rT} + K_0$. It follows that

$$\begin{aligned} V_0 &= \lim_{T\to\infty} \left(\int_0^T \Pi_t e^{-rt} dt - K_T e^{-rT} + K_0 \right) = \int_0^\infty \Pi_t e^{-rt} dt - \lim_{T\to\infty} K_T e^{-rT} + K_0 \\ &= \int_0^\infty \Pi_t e^{-rt} dt + K_0 \qquad \text{(from (A1), see below)} \\ &= \int_0^\infty \Pi(K_t, L_t) e^{-rt} dt + K_0. \qquad (\text{since } \Pi_t \equiv \Pi(K_t, L_t)) \end{aligned}$$

In line 3 we have assumed circumstances are such that the firm is not motivated to let its capital stock ultimately grow at a rate as high as the interest rate. That is, we have imposed the end condition

$$\lim_{T\to\infty} K_T e^{-rT} = 0. \tag{A1}$$

Comment: Given K_0,[1] we have that V_0 is maximized if and only if K_t and L_t are at each t chosen such that Π_t is maximized. Thus, when there are no capital adjustment costs, the firm's intertemporal value maximization problem can be reduced to a series of static profit maximization problems.

c) With capital adjustments costs the problem is:

$$\begin{aligned} \max V_0 &= \int_0^\infty (F(K_t, L_t) - w_t L_t - I_t - C(I_t)) e^{-rt} dt \quad \text{s.t.} \\ L_t &\geq 0, \; I_t \text{ free}, \\ \dot{K}_t &= I_t - \delta K_t, \\ K_t &\geq 0, \; \text{for all } t \geq 0. \end{aligned}$$

[1] Note that the historically given K_0 is no more given than the firm might wish its physical capital to immediately jump to a lower or higher level. In the first case the firm would immediately sell some of its machines, and in the last case it would immediately buy more machines than it already has.

d) The current-value Hamiltonian is: $H = F(K, L) - wL - I - C(I) + q(I - \delta K)$, where q is the adjoint variable (the co-state variable). The first-order conditions are:

$$\begin{aligned}
\frac{\partial H}{\partial L} &= F_L(K, L) - w = 0 \Rightarrow F_L(K, L) = w, && \text{(FOC1)} \\
\frac{\partial H}{\partial I} &= -1 - C'(I) + q = 0 \Rightarrow 1 + C'(I) = q, && \text{(FOC2)} \\
\frac{\partial H}{\partial K} &= F_K(K, L) - q\delta = -\dot{q} + rq && \text{(FOC3)}
\end{aligned}$$

and the transversality condition is

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0. \qquad \text{(TVC)}$$

Comment: q_t can be interpreted as the shadow price (the marginal value to the firm) of installed capital at time t.

e) We have $Y = F(K, L) = LF(k, 1) \equiv Lf(k)$, where $k \equiv K/L$ and $f' = F_K(K, L) > 0$, $f'' < 0$. (FOC1) can be written

$$F_L(K, L) = \frac{\partial\,[Lf(k)]}{\partial L} = f(k) + f'(k)\frac{-K}{L^2}L = f(k) - f'(k)k \equiv \varphi(k) = w.$$

We assume w is such that this equation has a solution $k > 0$. Since $\varphi'(k) = -kf''(k) > 0$, we have that $\varphi(\cdot)$ can be inverted. Thus,

$$k_t = \varphi^{-1}(w_t) \equiv k(w_t) \qquad \text{where} \qquad k'(w_t) = \frac{1}{\varphi'(k(w_t))} = -\frac{1}{k(w_t)f''(k(w_t))} > 0.$$

From now, $w_t = w$ for all $t \geq 0$. Hence, $k = k(w) \equiv \bar{k}$, which is a positive constant.

f) From (FOC3) we now have the linear differential equation

$$\dot{q}_t = (r + \delta)q_t - f'(\bar{k}),$$

which has the solution

$$q_t = (q_0 - q^*)e^{(r+\delta)t} + q^*, \qquad \text{where} \quad q^* = \frac{f'(\bar{k})}{r + \delta} > 0. \qquad (1)$$

Substituting this into (TVC) gives

$$\begin{aligned}
\lim_{t\to\infty}(q_0 - q^*)e^{\delta t}K_t + q^* \lim_{t\to\infty} e^{-rt}K_t &= 0 \quad \text{so that} \\
\lim_{t\to\infty}(q_0 - q^*)e^{\delta t}K_t &= 0, \qquad \text{by (A1).} \qquad \text{(TVC')}
\end{aligned}$$

Notice the assumption made in the paragraph right before question f), namely that for K below some positive threshold value, unit production costs are prohibitively large (that is, there is a minimum size of the stock of machines). Due to this threshold value for K, along an optimal plan of an active firm we can not have $\lim_{t\to\infty} K_t = 0$. Hence, (TVC') implies $q_0 = q^*$ so that, by (1),

$$q_t = q^* \quad \text{for all} \quad t \geq 0.$$

Thus, "marginal q" is a constant.

g) From (FOC2) follows

$$\begin{aligned} C'(I) &= q^* - 1, \quad \text{from which we get} \\ I &= C'^{-1}(q^* - 1) \equiv m(q^*) = m(\frac{f'(\bar{k})}{r+\delta}) \equiv I^*. \end{aligned}$$

Since $C'(0) = 0$ and $C''(I) > 0$, we have the investment rule: gross investment should be positive if (and only if) $q^* > 1$, that is, if $f'(\bar{k}) > r + \delta$, that is, if $f'(\bar{k}) - \delta > r$. Not surprising! At the same time, in spite of $f'(\bar{k}) > r + \delta$, gross investment should *not* be infinite. This is due to the strictly convex installation costs.

h) $\dot{K}_t = I^* - \delta K_t \gtreqless 0 \quad$ for $\quad K_t \lesseqgtr I^*/\delta \equiv K^*$. Thus, for $t \to \infty$

$$K_t \to K^* = \frac{1}{\delta} m(\frac{f'(\bar{k})}{r+\delta}).$$

i) If optimal net investment is denoted I_t^n, we have

$$I_t^n = I^* - \delta K_t = \delta(I^*/\delta - K_t) = \delta(K^* - K_t).$$

Comment: This result has traditionally been called the "capital adjustment principle" in that K^* is the "desired capital stock" and net investment is positive (negative) as long as the actual capital stock is below (above) the desired capital stock. The adjustment takes time due to the convex adjustment costs. Notice also that the desired capital stock and gross investment are here *stable* (decreasing) functions of the real interest rate. It is otherwise with *net investment*. Given r, I_t^n can be anything (it can be positive, negative, or zero). The point is that I_t^n depends as much on K_t as on r. A relevant net investment function is therefore $I_t^n = I^n(r, K_t)$, $I_r^n < 0$, $I_K^n < 0$, not $I^n(r)$. In a short-run model where changes in K_t are negligible, it may, however, be legitimate to simplify notation by writing $I_t^n \approx \tilde{I}^n(r) \approx I^n(r, K_t)$, $\tilde{I}^{n\prime}(r) \approx I_r^n < 0$.

j) Now $Y = K^{\alpha}L^{1-\alpha}, 0 < \alpha < 1$, and $C(I) = \frac{1}{2}\beta I^2, \beta > 0$. Thus, $f(k) = k^{\alpha}$ and $\varphi(k) = (1-\alpha)k^{\alpha}$ from which follows

$$\begin{aligned}
\bar{k} &= (\frac{w}{1-\alpha})^{1/\alpha}, \quad \text{hence,} \\
f'(\bar{k}) &= \alpha\bar{k}^{\alpha-1} = \alpha(\frac{w}{1-\alpha})^{(\alpha-1)/\alpha}, \qquad \text{so that} \\
q^* &= \frac{\alpha(\frac{w}{1-\alpha})^{(\alpha-1)/\alpha}}{r+\delta}.
\end{aligned}$$

Further, since $C'(I) = \beta I = q - 1 \Rightarrow I = (q-1)/\beta$,

$$I^* = \frac{1}{\beta}(\frac{\alpha(\frac{w}{1-\alpha})^{(\alpha-1)/\alpha}}{r+\delta} - 1).$$

Finally,

$$K^* = \frac{1}{\delta\beta}(\frac{\alpha(\frac{w}{1-\alpha})^{(\alpha-1)/\alpha}}{r+\delta} - 1).$$

k) A realistic aspect of this model is that its point of departure is that *firms* (not utility maximizing households) are the active agents in relation to most capital investment in society. Further, it seems realistic that firms try to maximize their value. And that firms' behavior is forward-looking.

Some weaknesses of the present model version are: 1) the adjustment costs should probably be a decreasing function of the level of K (this can easily be taken into account, however, cf. Chapter 10); 2) the q-theory of investment is only partly supported by the data; 3) perfect competition is assumed, but some kind of imperfect competition would be a better description of the situation for firms in the manufacturing and service sectors.

—