Christian Groth

A suggested solution to the problem set at the exam in **Advanced Macroeconomics 2** January 16, 2012

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

The rule requires that

$$\lambda i D_t + P_t(G_t - T_t) \leq \alpha P_t Y_t, \tag{*}$$

where $\lambda > 0$, $\alpha > 0$,

$$
\begin{aligned}
G_t &= \text{real government spending on goods and services in period } t, \\
T_t &= \text{real net tax revenue in period } t, \\
i &= (1+r)(1+\pi) - 1, \text{ where } r \text{ is the real interest rate, a given non-negative constant,} \\
\pi &= \frac{P_t - P_{t-1}}{P_{t-1}} = \text{the inflation rate, a given non-negative constant,}
\end{aligned}
$$

and we are told that money financing of the government deficit never occurs.

a) We have

$$GBD_t = iD_t + P_t(G_t - T_t),$$

and

$$D_{t+1} = D_t + GBD_t = (1+i)D_t + P_t(G_t - T_t).$$

[1] The solution below contains more details and more precision than can be expected at a three hours exam.

b) Yes, the deficit rule of the SGP in the EMU is a special case of (*), namely the case $\alpha = 0.03$ and $\lambda = 1$.

c) Let $b_t \equiv D_t/(P_{t-1}Y_t)$. When the "ceiling" (*) is always binding, we have

$$\begin{aligned}
b_{t+1} &\equiv \frac{D_{t+1}}{P_t Y_{t+1}} = \frac{(1+i)D_t + P_t(G_t - T_t)}{P_t(1+g_Y)Y_t} \\
&= \frac{(1+i)D_t - \lambda i D_t + \alpha P_t Y_t}{P_t(1+g_Y)Y_t} \qquad \text{(by (*) with =)} \\
&= \frac{(1+(1-\lambda)i)D_t}{P_{t-1}(1+\pi)(1+g_Y)Y_t} + \frac{\alpha}{1+g_Y} \equiv \frac{1+(1-\lambda)i}{(1+\pi)(1+g_Y)}b_t + \frac{\alpha}{1+g_Y} \\
&\equiv \gamma b_t + \frac{\alpha}{1+g_Y}. \qquad (1.1)
\end{aligned}$$

d) λ is such that $0 < 1 + (1-\lambda)i < (1+\pi)(1+g_Y)$. Hence, $0 < \gamma < 1$. The solution of the difference equation (1.1) is (cf. the hint)

$$b_t = (b_0 - b^*)\gamma^t + b^*, \qquad \text{where } b^* = \frac{\alpha}{(1-\gamma)(1+g_Y)} > 0.$$

In view of $0 < \gamma < 1$, the debt-income ratio, b_t, converges towards a constant, b^*, for $t \to \infty$. Since, in practice the "ceiling" may not always be binding, b^* is an upper bound for b_t in the long run. As long as the rule (*) is followed, there will be no "debt explosion".

e) We have

$$\frac{\partial b^*}{\partial \lambda} = \frac{\partial b^*}{\partial \gamma}\frac{\partial \gamma}{\partial \lambda} = \frac{\alpha}{(1-\gamma)^2(1+g_Y)} \cdot \frac{-i}{(1+\pi)(1+g_Y)} \leq 0,$$

since $i \geq 0$. In the "normal case", both i and D_t are positive. Then the interest expenses, iD_t, will be positive and a higher weight, λ, on them will imply less scope for debt creation in the long run. The mechanism follows from (*) with equality: $P_t(G_t - T_t) = \alpha P_t Y_t - \lambda i D_t$. That is, for a larger λ, the rule gives less scope for the primary deficit, and this puts the brake on the long-run debt-income ratio.

2. Solution to Problem 2

The model is:

$$\dot{Y}_t = \lambda(D(Y_t, R_t, \rho) + G - Y_t), \qquad \lambda > 0,\ 0 < D_Y < 1,\ D_R < 0,\ D_\rho > 0, \quad (2.1)$$

$$\frac{M_t}{P_t} = L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0, \quad (2.2)$$

$$R_t = \frac{1}{q_t}, \quad (2.3)$$

$$\frac{1 + \dot{q}_t^e}{q_t} = r_t^e, \quad (2.4)$$

$$r_t^e \equiv i_t - \pi_t^e, \quad (2.5)$$

$$\pi_t = \pi, \quad (2.6)$$

$$i_t = \alpha_0 + \alpha_1 Y_t, \qquad \alpha_1 > 0, \quad (2.7)$$

where the exogenous constants $\rho, G,$ and π are positive. The initial values Y_0 and P_0 are historically given. In questions a) - e) it is assumed that exogenous variables and initial conditions are such that $i_t > 0$ for all $t \geq 0$.

a) The model is essentially Blanchard's dynamic IS/LM model, which describes the adjustment in the "very short run" towards a "short-run equilibrium" with respect to output and interest rates. The adjustment of output to demand takes time and during the adjustment process also demand changes (since the output level and asset prices are among its determinants). There are three financial assets: money, a long-term inflation-indexed bond, and a short-term bond.

Eq. (2.1) tells how output adjusts to demand; the parameter λ is the speed of adjustment. Output demand depends positively on current income, Y, (to reflect, e.g., that a fraction of the consumers are credit constrained) and negatively on the real long-term interest rate, since in particular investment is likely to depend negatively on this rate. And because of the substitution and wealth effects, it tends to be similar with consumption.

Eq. (2.2) expresses equilibrium in the money market. Real money demand depends positively on Y, because Y is a proxy for the number of transactions per time unit, and negatively on the short-term nominal interest rate, the opportunity cost of holding money. The asset markets are assumed to clear instantaneously.

When the "long-term real interest rate", R_t, is identified with the internal rate of return on the consol, we have

$$q_t = \int_t^\infty 1 \cdot e^{-R_t(s-t)} ds = \frac{1}{R_t}.$$

Reordering gives (2.3).

Eq. (2.4) is a no-arbitrage condition saying that the expected real rate of return on holding the consol one time unit equals the expected real short-term interest rate, r_t^e. So there is no risk premium (which is explained below). Eq. (2.5) defines the expected real interest rate, r_t^e. Eq. (2.6) states the simplifying assumption that the actual rate of inflation is a constant π. Finally, eq. (2.7) describes the central bank's counter-cyclical interest rate rule, a kind of truncated Taylor rule ("truncated" because the constant inflation term is subsumed under the constant α_0).

We now assume perfect foresight (expectations are rational and there are no stochastic elements in the model, hence no risk premium in (2.4)). We also assume that speculative bubbles never arise.

b) In view of perfect foresight, $\pi_t^e = E_t\pi_t = \pi_t = \pi > 0$ for all t, implying that $r_t^e = i_t - \pi \equiv r_t$. Similarly, $\dot{q}_t^e = E_t\dot{q}_t = \dot{q}_t$. Substituting into (2.4), we get

$$\frac{1}{q_t} + \frac{\dot{q}_t}{q_t} = R_t - \frac{\dot{R}_t}{R_t} = i_t - \pi = \alpha_0 + \alpha_1 Y_t - \pi, \tag{2.8}$$

by (2.7). By reordering,

$$\dot{R}_t = \left[R_t - (\alpha_0 + \alpha_1 Y_t - \pi)\right] R_t. \tag{2.9}$$

Combining this with (2.1), we thus have two coupled differential equations in R and Y.

The money market equilibrium condition, (2.2), has a rather subordinate role in the model. It only describes the needed adjustments of the money supply when the central bank through open-market operations continuously succeeds, as assumed here, in equating i_t to the target level, given by the right-hand side of (2.7).

c) Given $R > 0$, (2.9) implies

$$\dot{R} \gtreqless 0 \qquad \text{for} \qquad R \gtreqless \alpha_0 + \alpha_1 Y - \pi, \qquad \text{respectively.} \tag{2.10}$$

We have $\frac{\partial R}{\partial Y}|_{\dot{R}=0} = \alpha_1 > 0$. The $\dot{R} = 0$ locus is thus an upward-sloping line, named the "MP curve" in Fig. 2.1.

From (2.1) we have

$$\dot{Y} \gtreqless 0 \qquad \text{for} \qquad D(Y_t, R_t, \rho) + G \gtreqless Y, \qquad \text{respectively.} \tag{2.11}$$

Hence, $\frac{\partial R}{\partial Y}|_{\dot{Y}=0} = (1 - D_Y)/D_R < 0$. The $\dot{Y} = 0$ locus is thus a downward-sloping curve, named the "IS curve" in Fig. 2.1.

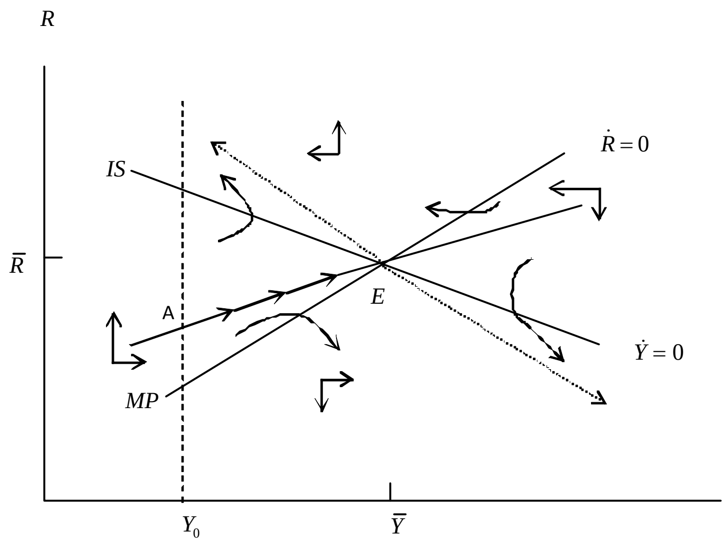

Figure 2.1:

In addition, the figure shows the direction of movement in the different regions, as described by (2.10) and (2.11). We see that the steady state point, E, with coordinates $(\bar{Y}, \bar{R})$, is a saddle point. This implies that two and only two solution paths – one from each side – converges towards E. These two saddle paths, which together make up the stable arm (with positive slope), are shown in the figure. Also the unstable arm is displayed in the figure (the negatively sloped stippled line).

We are told that parameters are such that there exists a steady state with a nominal short-term interest rate $\bar{i} > \pi > 0$. So, $\bar{R} = \alpha_0 + \alpha_1 \bar{Y} - \pi = \bar{i} - \pi > 0$.

Since Y_0 is pre-determined, at time $t = 0$ the economy must be somewhere on the vertical line $Y = Y_0$. Since speculative bubbles are by assumption ruled out, neither the explosive nor the implosive paths in Fig. 2.1 can materialize. We are then left with the path AE (along the saddle path) as the unique solution to the model. So the economy will at $t = 0$ be at point A in Fig. 2.1 and then gradually approach the steady state, E, over time, moving along the saddle path.

d) The steady-state value, $\bar{Y}$, is determined implicitly by

$$\bar{Y} = D(\bar{Y}, \bar{R}, \rho) + G = D(\bar{Y}, \alpha_0 + \alpha_1 \bar{Y} - \pi, \rho) + G.$$

We find

$$d\bar{Y} = D_Y d\bar{Y} + D_R \alpha_1 d\bar{Y} + dG.$$

So

$$\frac{\Delta \bar{Y}}{\Delta G} \approx \frac{\partial \bar{Y}}{\partial G} = \frac{1}{1 - D_Y - D_R \alpha_1} \in (0, \frac{1}{1 - D_Y}).$$

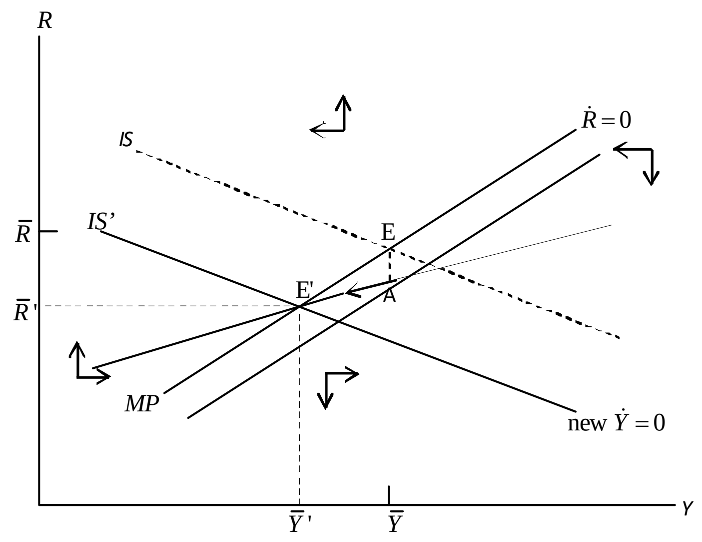

Figure 2.2:

The spending multiplier is positive but smaller than it would be with a constant nominal interest rate. This is in accordance with the idea of a counter-cyclical monetary policy.

e) The effect of the downward shift in the level of confidence, ρ, is shown in Fig. 2.2. When ρ shifts down, the MP curve is not affected, but the IS curve is shifted downward. Hence, the long-term interest rate immediately jumps down to R_A, reflecting that the market value of the consol jumps up in the expectation of lower short-term interest rates as a result of the recession. This is where the given formula,

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds}, \tag{2.12}$$

is useful. This formula, valid in the absence of bubbles, indicates that the long-term interest rate is a kind of average of the expected future short-term rates.

The mechanism behind the jump is as follows. The lower level of confidence implies lower output demand. This triggers an expectation of decreasing Y and therefore also an expectation of decreasing i and r in view of the (truncated) Taylor rule. The implication is, by (2.12), a higher q, hence a lower R already immediately after time t_1, as illustrated in Fig. 2.2. As time proceeds and the economy gets closer to the expected low future values of r, these lower values gradually become dominating in the determination of R. Hence, R gradually decreases toward its new steady-state value, the same as that for r.

Time profiles showing the evolution of i_t, r_t, R_t, and Y_t for $t \geq 0$ are given in Fig. 2.3. After t_1, output Y and the short-term rates, i and r, gradually decrease toward their new steady state values. These new steady-state values of the three interest rates are denoted

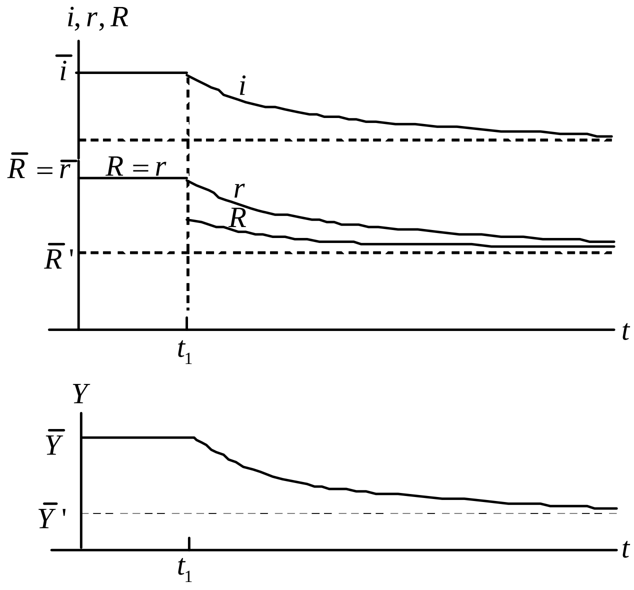

Figure 2.3:

$\bar{i}'$, $\bar{r}'$, and $\bar{R}'$, respectively.

f) Among the positive and useful aspects of the model are:

1) It includes dynamics and endogenous forward-looking expectations, thereby allowing endogenous changes in the yield curve.

2) Although in a rough way, it takes into account the differences in speed of adjustment of asset prices, the level of output, and the general output price level. Empirically, these differences seem important for dynamics in the short run.

Among the weaker aspects of the model are:

1) The model is somewhat ad hoc. The role of unintended positive and negative inventory investment is not incorporated.

2) The forward-looking expectations and the dynamics in the model are "truncated" in the following sense. It is not taken into account that realism requires the "short-run equilibrium" (the steady state in the model) to be just a temporary state in a continuing process where medium-term mechanisms, for example in the form of a Phillips curve, are operative.

3) As it stands, the model ignores the zero lower bound for the nominal interest rate.

3. Solution to Problem 3

The model is:

$$\dot{\pi}_t = \delta(Y_t - Y^*), \qquad \delta > 0, Y^* > 0, \qquad \pi_0 \text{ given}, \tag{3.1}$$

$$Y_t = \gamma - \beta r_t^e, \qquad \gamma > 0, \beta > 0, \tag{3.2}$$

$$r_t^e \equiv i_t - \pi_t^e, \tag{3.3}$$

$$i_t = \max\left[0, \hat{r} + \hat{\pi} + \alpha_1(Y_t - Y^*) + \alpha_2(\pi_t - \hat{\pi})\right], \qquad \hat{r} \equiv \frac{\bar{\gamma} - Y^*}{\beta} > 0, \hat{\pi} > 0, \alpha_1 > 0, \alpha_2 > 1, \tag{3.4}$$

together with an assumption of perfect foresight, i.e.,

$$\pi_t^e = \pi_t. \tag{3.5}$$

a) Eq. (3.1) is a Phillips curve saying that inflation is rising or falling according to whether output is above or below a certain level, Y^*; when output equals Y^*, inflation remains unchanged. Thus, Y^* is the NAIRU level of output. Including a Phillips curve of some form (at both the theoretical and empirical level several "forms" are on hand) is appropriate, given the "short-to-medium-run" perspective of the model.

Eq. (3.4) says that as long as the zero lower bound on the nominal interest rate is not binding, monetary policy follows a Taylor rule. Through open market operations the central bank adjusts the nominal short-term interest rate so as to increase or decrease the real interest rate depending on whether a dampening or stimulation of aggregate demand is called for. Everything else equal, if actual output is above the NAIRU level or actual inflation is above the target level, $\hat{\pi}$, the target for i_t will be higher in order to obtain a higher $r_t^e = i_t - \pi_t^e = i_t - \pi_t \equiv r_t$ (note that $\alpha_2 > 1$ so that a rise in π_t results in a larger rise in i_t).

When $\pi_t = \hat{\pi}$ and $Y_t = Y^*$, the rule implies $i_t = \hat{r} + \hat{\pi}$, so that $r_t = i_t - \pi_t = \hat{r} + \hat{\pi} - \hat{\pi} = \hat{r}$. So $\hat{r}$ is the real interest rate resulting from the Taylor rule when $\pi_t = \hat{\pi}$ and $Y_t = Y^*$ (in which case π remains constant according to (3.1)).

In an economic crisis, however, the target nominal interest rate implied by the Taylor rule may be negative. Then the zero lower bound in (3.4) becomes binding, and the actual nominal interest rate, i_t, stays at the zero level (further increases in money supply can not bring i below 0 because agents would prefer holding cash at zero interest rather than short-term bonds at negative interest).

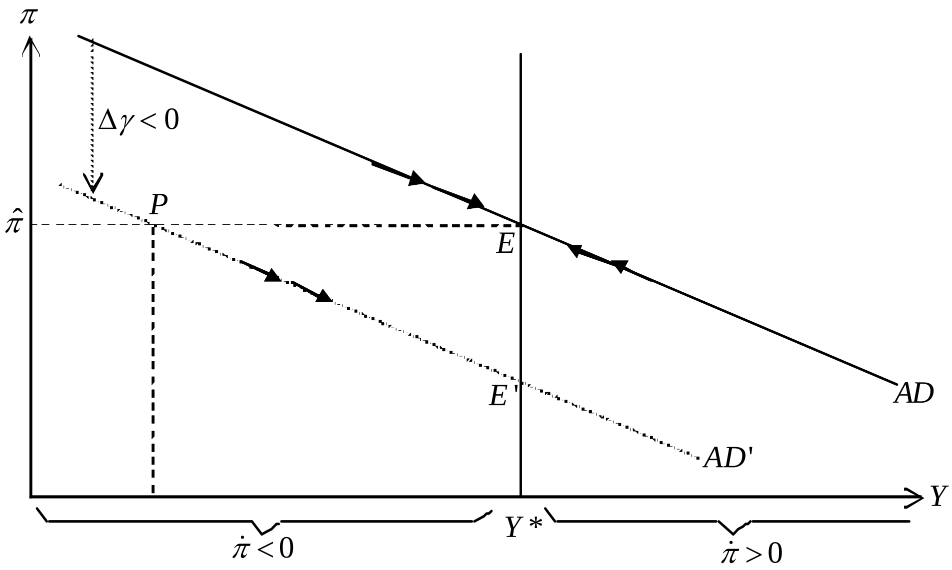

Figure 3.1:

b) We have

$$\begin{aligned} Y_t &= \gamma - \beta r_t^e = \gamma - \beta(i_t - \pi_t^e) = \gamma - \beta(i_t - \pi_t) \qquad \text{(perfect foresight)} \\ &= \gamma - \beta(\hat{r} + \hat{\pi} + \alpha_1(Y_t - Y^*) + \alpha_2(\pi_t - \hat{\pi}) - \pi_t), \qquad \text{(zero lower bound not binding)} \end{aligned}$$

so that

$$(1 + \beta\alpha_1)Y_t = \gamma - \beta(\hat{r} + \hat{\pi} - \alpha_1 Y^* - \alpha_2\hat{\pi}) - \beta(\alpha_2 - 1)\pi_t,$$

implying

$$Y_t = \theta_0(\gamma) - \theta_1\pi_t, \tag{AD}$$

where

$$\begin{aligned} \theta_0(\gamma) &= \frac{\gamma - \beta\left(\hat{r} + \hat{\pi} - \alpha_1 Y^* - \alpha_2\hat{\pi}\right)}{1 + \beta\alpha_1}, \\ \theta_1 &= \frac{\beta(\alpha_2 - 1)}{1 + \beta\alpha_1} > 0, \end{aligned}$$

as was to be shown.

c) When γ takes its "normal" value $\bar{\gamma} > Y^*$, the AD curve reads

$$Y_t = \theta_0(\bar{\gamma}) - \theta_1\pi_t. \tag{$\overline{\text{AD}}$}$$

Fig. 3.1 illustrates in the (Y, π) plane this "normal" AD curve of the economy. As long as $\gamma = \bar{\gamma}$, the AD curve is fixed and the economy must be at some point on this curve (line), depending on the current rate of inflation. Over time the economy moves along

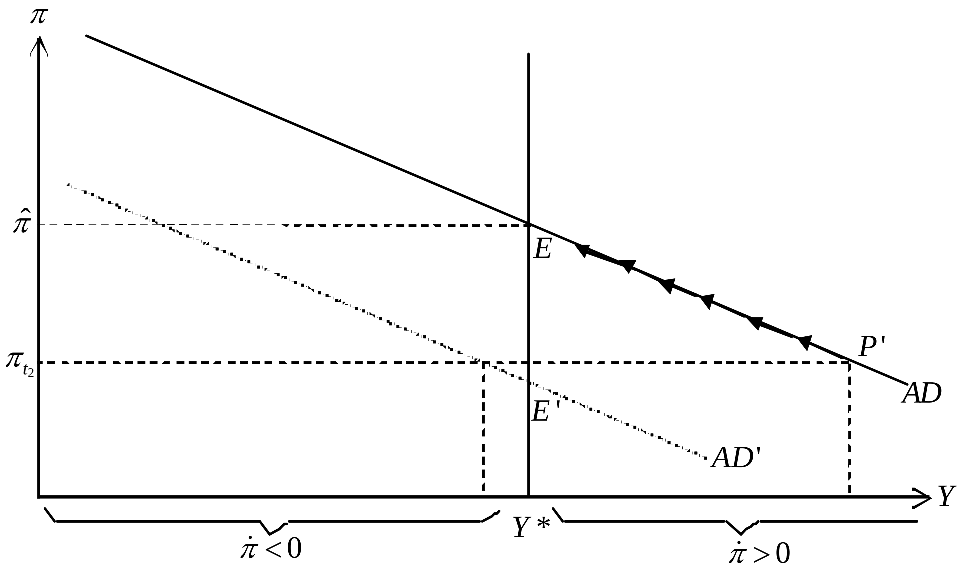

Figure 3.2:

the AD curve until steady state at the point E is "reached". If $\pi_0 > \hat{\pi}$, monetary policy will be tight so as to uphold a high interest rate in order to dampen aggregate demand and production, thereby gradually bringing π down via the Phillips curve.[2]

d) The unexpected adverse demand shock shifts the AD curve down in Fig. 3.1 to the new position, indicated by AD' in the figure. Immediately after the shock, the economy is in recession at the point P in Fig. 3.1. This triggers a decrease of the interest rate which stimulates aggregate demand and production. Thereby demand and output gradually increases while inflation continues to fall as long as $Y < Y^*$. That is, over time the economy travels down South-East along the AD' curve towards the new potential steady state, E'.

e) The restoration of confidence at time t_2 stimulates aggregate demand, thereby shifting the AD curve back to its original position, cf. Fig. 3.2. The high level of demand triggers a boom where Y jumps up to a level equal to the abscissas of the point P' in the figure. Then inflation begins to rise through the Phillips curve and monetary policy gradually dampens demand and output via the Taylor rule. Over time the economy moves North-West along the AD curve towards the old steady state point E.

It is certainly OK to use just one figure to illustrate all these dynamics (if done with clarity) instead of having two figures, as here.

[2]The suggested alternative illustration in the $(\pi, \dot{\pi})$ plane is shown in the appendix.

f) The case where the lower bound in (3.4) is *binding.* Immediately after the adverse demand shock, aggregate demand and output is

$$Y_{t_1} = \gamma' - \beta(0 - \hat{\pi}) < Y^*,$$

since a binding lower bound reflects a nominal interest rate equal to nil. Through the Phillips curve the recession, or possibly depression, triggers a falling inflation rate. This raises the real interest rate, as the nominal interest rate is already at its lower bound (the liquidity trap). This pulls aggregate demand and output further down, thus sustaining the tendency for the inflation rate to fall, thereby raising the real interest rate further - a *vicious spiral* is unfolding.

4. Solution to Problem 4

a) False. The "only if" part is not generally valid since we may have $r \leq g_Y$, in which case fiscal sustainability (a non-exploding debt-income ratio) is possible without the No-Ponzi-Game condition of the government being satisfied. And even the "if" part is not generally valid since we may have

$$1 + g_Y < \lim_{t\to\infty} B_{t+1}/B_t < 1 + r,$$

where B is real government net debt. The second inequality indicates that the No-Ponzi-Game condition holds, while the first inequality implies that the debt-income ratio explodes.

b) A life annuity contract is called *actuarially fair* if it offers the investor the same expected rate of return as a safe bond. On the one hand the life annuity contract pays a higher rate of return as long as the investor is alive; on the other hand the total investment is nullified at death. So a life annuity contract makes it possible to convert potential wealth after death to higher consumption while still alive. This is attractive for people with no or only little bequest motive.

c) False. Zero gross investment is induced by $q = 1$. But in a steady state the value of q must equal what is needed to maintain gross investment at the level required to maintain the steady state. This level of gross investment is generally positive because at least capital depreciation should be covered. So $q > 1$ is required to induce the needed gross investment. Moreover, with growth in the labor force and technology, steady state requires constancy of the capital intensity, $K/(TL)$, and then an even higher q is required to induce the gross investment needed to maintain steady state.

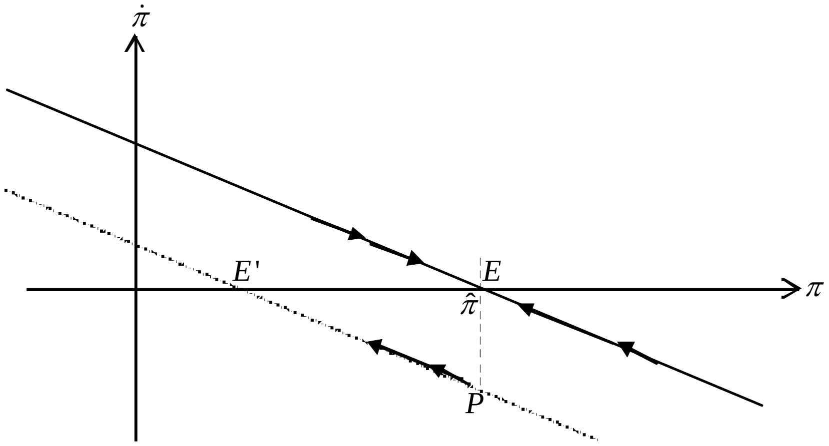

Figure 5.1:

5. Appendix: Illustrations in the $(\pi, \dot{\pi})$ plane (Problem 3)

As suggested in Problem 3, the illustrations to the questions c), d), and e) can be constructed in an alternative way.

c) After substitution of the AD curve into (3.1), we have

$$\dot{\pi}_t = \delta(\theta_0(\bar{\gamma}) - \theta_1 \pi_t - Y^*). \tag{5.1}$$

The graph of this relationship is shown in Fig. 5.1 as the downward-sloping solid line in the figure. As long as $\gamma = \bar{\gamma}$, the economy must be at some point on this line, and if $\pi < \hat{\pi}$, π will be growing towards $\hat{\pi}$, while if $\pi > \hat{\pi}$, π will be falling towards $\hat{\pi}$. Over time the economy moves along the line (5.1) until steady state at the point E is "reached".

d) Also the recession with resulting dynamics is illustrated in Fig. 5.1. The adverse demand shock shifts the line (5.1) to a new lower level in the figure. The position of the economy shifts from the point E to the point P. Hereafter, due to the recession there is a gradual fall in inflation.

e) The restoration of confidence at time t_2 with resulting dynamics is illustrated in Fig. 5.2. The favorable restoration of confidence shifts the line (5.1) back to its old, higher level and this triggers a gradual rise in inflation until the economy is back in the old steady state, E.

—

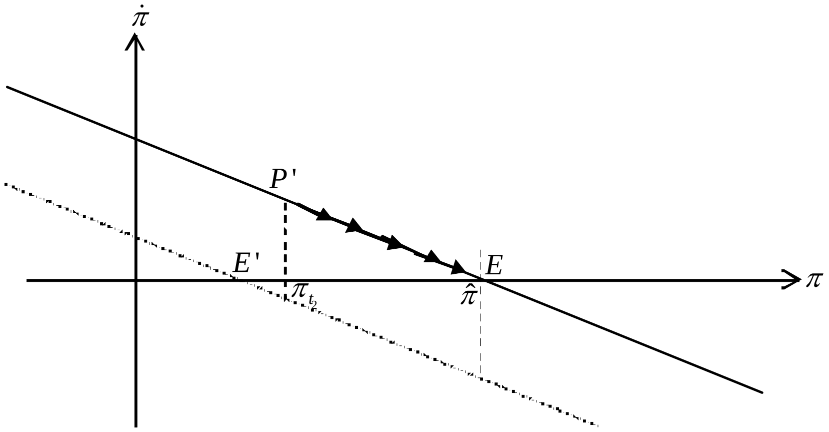

Figure 5.2: