Christian Groth

A suggested solution to the problem set at the exam in Advanced Macroeconomics January 11, 2016

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

a) We have

$$GBD_t = rB_t + G_t - T_t, \tag{GBD}$$

and

$$B_{t+1} = (1+r)B_t + G_t - T_t. \tag{DGBC}$$

As an implication,

$$B_{t+1} = B_t + GBD_t. \tag{1.1}$$

The debt-income ratio $b_t \equiv B_t/Y_t$ changes over time according to

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{(1+r)B_t + G_t - T_t}{(1+g_Y)Y_t} = \frac{1+r}{1+g_Y}b_t + \frac{\gamma-\tau}{1+g_Y}. \tag{1.2}$$

b) We use the hint, from which follows that the solution to this linear difference equation is

$$b_t = (b_0 - b^*)\left(\frac{1+r}{1+g_Y}\right)^t + b^*, \quad \text{where} \quad b^* = \frac{\tau-\gamma}{r-g_Y}, \tag{1.3}$$

and $b_0 \equiv B_0/Y_0 > 0$ is historically given.

[1] The solution below contains more details and more precision than can be expected at a three hours exam.

Only if the current fiscal policy, (γ, τ), implies a non-exploding debt-income ratio, is the policy sustainable. Since $r > g_Y$, this requires that τ and γ are such that $b^* \geq b_0$. That is, the requirement is that

$$\tau - \gamma \geq (r - g_Y)b_0.$$

Hence, the minimum value of the tax-income ratio needed for fiscal sustainability is

$$\hat{\tau} = \gamma + (r - g_Y)b_0. \tag{1.4}$$

c) If $\tau < \hat{\tau}$, then $\tau - \gamma < (r - g_Y)b_0$, which implies $b^* < b_0$. In this case we see that since $(1+r)/(1+g_Y) > 1$, the debt-income ratio, b_t, in (1.3) exhibits geometric growth and goes to $+\infty$ for $t \to \infty$. This process will not continue for long, because the government will soon be unable to find buyers for the newly issued debt. As a consequence the government will not be able to meet its financial commitments, and the private sector can see that default is under way. This means that fiscal policy is *not sustainable*.

From now on $\tau = \hat{\tau}$. We further assume that b_0 is "high". A global financial and economic crisis breaks out and also hits SOE. As an implication, $\hat{\tau}$ is only slightly below the tax-income ratio, $\bar{\tau}$, that under the current circumstances maximizes (net) tax revenue (because on top of supply effects of taxes come contractive demand effects). Suppose also that the actual spending-income ratio, γ, is only slightly above the level $\bar{\gamma}$ which is the minimum politically tolerable spending-income ratio of SOE. The world market interest rate is still r.

d) Yes, under these circumstances "fundamentals" (the tax- and spending-income ratios, the growth-corrected interest rate, and the debt-income ratio) of SOE are in accordance with fiscal sustainability in the sense that if the circumstances persist, the debt-income ratio will stay constant forever. As $\hat{\tau} - \gamma > \bar{\tau} - \bar{\gamma}$, there is even scope for a small rise in the primary surplus (as a share of income), which would be necessary should the world market interest rate rise.

e) Yes, the "high" debt-income ratio of SOE may under the given circumstances trigger a government debt default through the mechanism of self-fulfilling rational expectations. The reason is the following:

The actual primary surplus, s, can at most equal $\bar{s} \equiv \bar{\tau} - \bar{\gamma}$. Initially,

$$s = \hat{\tau} - \gamma = (r - g_Y)b_0 \leq \bar{s} \equiv \bar{\tau} - \bar{\gamma}, \tag{1.5}$$

with $b_0 > 0$. Define $\bar{r}$ to be the value of r satisfying

$$(\bar{r} - g_Y)b_0 = \bar{s}, \text{ i.e., } \bar{r} = \frac{\bar{s}}{b_0} + g_Y. \tag{1.6}$$

Thereby $\bar{r}$ is the maximum level of the interest rate consistent with absence of an explosive debt-income ratio.

Now, investors are aware that a rise in the actual interest rate, r, can always happen and that if it does, a situation with $r > \bar{r}$ is looming, in particular if the country has high debt. Indeed, the larger is b_0, the lower is the critical interest rate, $\bar{r}$, as witnessed by (1.6). The worrying scenario is that the fear of default triggers a risk premium, and if the resulting level of the interest rate ends up above the critical interest rate, $\bar{r}$, runaway debt dynamics will take off and debt default follow. So financial investors' fear can be a self-fulfilling prophesy.

The basic problem is here a liquidity problem rather than a solvency problem. In a country with its own currency, the central bank could step in and act as a "lender of last resort", buying government bonds in the open market by printing money. The common knowledge of this possibility could prevent a "bad" self-fulfilling expectations equilibrium to unfold and thereby prevent government default.

2. Solution to Problem 2

a) The problem is:

$$\begin{aligned}
\max V_0 &= \int_0^\infty (F(K_t, L_t) - w_t L_t - I_t - G(I_t))e^{-rt}dt \quad \text{s.t.} \\
L_t &\geq 0, \; I_t \text{ free}, & (2.1) \\
\dot{K}_t &= I_t - \delta K_t, & (2.2) \\
K_t &\geq 0, \; \text{for all } t \geq 0. & (2.3)
\end{aligned}$$

b) The current-value Hamiltonian is: $H = F(K, L) - wL - I - G(I) + q(I - \delta K)$, where q is the adjoint variable. The first-order conditions are:

$$\begin{aligned}
\frac{\partial H}{\partial L} &= F_L(K, L) - w = 0 \Rightarrow F_L(K, L) = w, & \text{(FOC1)} \\
\frac{\partial H}{\partial I} &= -1 - G'(I) + q = 0 \Rightarrow 1 + G'(I) = q, & \text{(FOC2)} \\
\frac{\partial H}{\partial K} &= F_K(K, L) - q\delta = -\dot{q} + rq & \text{(FOC3)}
\end{aligned}$$

and the transversality condition is

$$\lim_{t \to \infty} K_t q_t e^{-rt} = 0. \quad \text{(TVC)}$$

c) From (FOC2), which says $MC = MB$, we see that q_t can be interpreted as the shadow price (the value to the firm of the marginal unit) of installed capital at time t along the optimal path.

d) In view of CRS, we have $Y = F(K, L) = LF(k, 1) \equiv Lf(k)$, where $k \equiv K/L$. We have $f'(k) = F_K(K, L) > 0, f'' < 0$. By (FOC1) follows:

$$F_L(K, L) = \frac{\partial\,[Lf(k)]}{\partial L} = f(k) + f'(k)\frac{-K}{L^2}L = f(k) - f'(k)k \equiv \varphi(k) = w > 0. \tag{2.4}$$

As F satisfies the Inada conditions, this equation has a solution $k > 0$ for any $w > 0$. Since $\varphi'(k) = -kf''(k) \neq 0$, $\varphi(\cdot)$ can be inverted. Thus,

$$k_t = \varphi^{-1}(w_t) \equiv k(w_t), \qquad \text{where} \quad k'(w_t) = \frac{1}{\varphi'(k(w_t))} = -\frac{1}{k(w_t)f''(k(w_t))} > 0. \tag{2.5}$$

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From now on, $w_t = w > 0$ for all $t \geq 0$. Hence,

$$k_t = k(w) \equiv \bar{k} > 0 \quad \text{for all} \quad t \geq 0. \tag{2.6}$$

e) (FOC2) implies $G'(I) = q - 1$, from which we get

$$I = G'^{-1}(q - 1) \equiv \mathcal{M}(q), \quad \mathcal{M}'(q) = 1/G''(\mathcal{M}(q)) > 0. \tag{2.7}$$

In addition, $\mathcal{M}(1) = 0$ since $G'(I) = 0$ implies $I = 0$.

(FOC3) implies

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, L_t) = (r + \delta)q_t - f'(\bar{k}). \tag{*}$$

f) In addition to (*) we need a differential equation for K where in addition to K only q enters. This is obtained by substituting (2.7) into (2.2) to get

$$\dot{K}_t = \mathcal{M}(q_t) - \delta K_t, \qquad K_0 > 0 \text{ given.} \tag{**}$$

Since $\mathcal{M}' > 0$,

$$\dot{K} \gtreqless 0 \text{ for } \mathcal{M}(q) \gtreqless \delta K, \quad \text{respectively.} \tag{2.8}$$

We see that the $\dot{K} = 0$ locus, satisfying the equation $\mathcal{M}(q) = \delta K$, has the properties: 1) it is upward-sloping in the (K, q) plane, as indicated in Fig. 2.1; it goes through the point (0, 1) since, if $K = 0$, then $\mathcal{M}(q) = 0$ which requires $q = 1$.

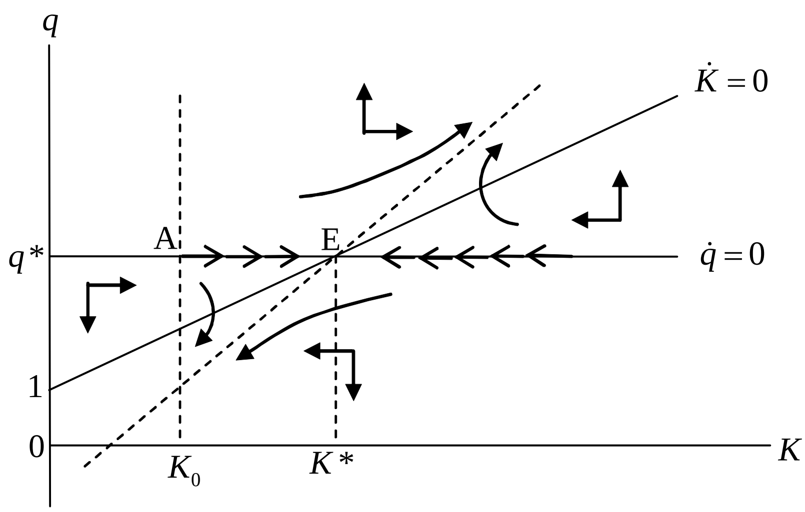

Figure 2.1:

From (*) follows

$$\dot{q}_t \gtreqqless 0 \text{ for } q_t \gtreqqless \frac{f'(\bar{k})}{r+\delta} \equiv q^*, \quad \text{respectively,} \tag{2.9}$$

The $\dot{q} = 0$ locus is thus horizontal, cf. Fig. 2.1. We are told that a steady state with $K > 0$ exists. Then, since in steady state $I = M(q^*) = \delta K^* > 0$, we have $q^* > 1$, as indicated in Fig. 2.1. The steady-state point in the figure is denoted E, and q^* is its ordinate, while its abscissa is denoted K^*. The associated labor input is $L^* = K^*/\bar{k}$. The direction of movement in the different regions of the phase diagram follows by (2.8) and (2.9), and is indicated by arrows in Fig. 2.1. The arrows taken together show that the steady state is a *saddle point*. As $r > 0$, the steady state satisfies the transversality condition, (TVC). The saddle path coincides with the $\dot{q} = 0$ locus!

The system has one predetermined variable, K, and one jump variable, q, the saddle path is not parallel to the jump-variable axis, and the diverging paths can be ruled out (they can be shown to violate the transversality condition). Hence the steady state is *saddle-point stable.*

As the saddle path coincides with the $\dot{q} = 0$ locus which is horizontal, along the optimal path, we have $q_t = q^*$ for all $t \geq 0$. So optimal gross investment is

$$I_t = \mathcal{M}(q^*) = \mathcal{M}\left(\frac{f'(\bar{k})}{r+\delta}\right) = \delta K^* \quad \text{for all } t \geq 0, \tag{2.10}$$

and thus constant. For the given $K_0 > 0$, which happens to be less than K^*, the optimal initial position is at the point A in Fig. 2.1. Optimal initial gross investment is thus $I_0 = \delta K^* > \delta K_0$ (in this case) so that there is positive net investment, and K will rise

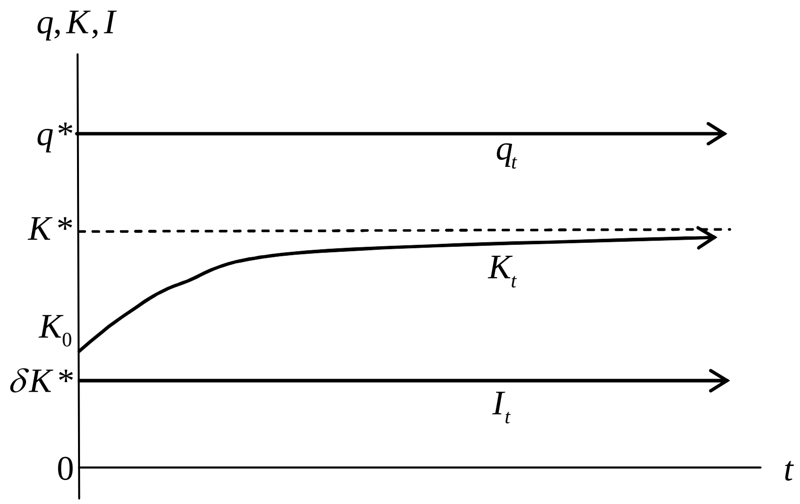

Figure 2.2:

while q remains constant at the level q^*. As time proceeds, K converges to K^*, where net investment is zero.

g) The time profiles of q_t, I_t, and K_t are shown in Fig. 2.2.

Comment. In the steady state we have $q^* > 1$, and the marginal productivity of capital is $F_K(K^*, L^*) = f'(\bar{k}) > r + \delta$. In spite of this inequality, why is there no incentive to increase K further? Because the marginal cost of doing that exceeds the marginal benefit. Indeed, we would have

$$I_t > \mathcal{M}\left(\frac{f'(\bar{k})}{r+\delta}\right) = \delta K^*,$$

which, since $G''(I) > 0$, would imply

$$\begin{aligned} 1 + G'(I_t) \quad &> \quad \frac{f'(\bar{k})}{r+\delta} = q^*, \qquad \text{i.e.} \\ \text{MC} &> \text{MB}. \end{aligned}$$

h) From (2.10) follows

$$K^* = \frac{\mathcal{M}(q^*)}{\delta} = \frac{\mathcal{M}\left(\frac{f'(\bar{k})}{r+\delta}\right)}{\delta} = \frac{\mathcal{M}\left(\frac{f'(k(w))}{r+\delta}\right)}{\delta},$$

where the last equality follows from (2.6). Applying the chain rule, we get

$$\begin{aligned} \frac{\partial K^*}{\partial r} &= \frac{\mathcal{M}\left(\frac{f'(k(w))}{r+\delta}\right)}{\delta}\mathcal{M}'\frac{-f'(\bar{k})}{(r+\delta)^2} < 0, \\ \frac{\partial K^*}{\partial w} &= \frac{\mathcal{M}\left(\frac{f'(k(w))}{r+\delta}\right)}{\delta}\mathcal{M}'\frac{f''(k(w))}{r+\delta}k'(w) = \frac{\mathcal{M}\left(\frac{f'(k(w))}{r+\delta}\right)}{\delta}\mathcal{M}'\frac{-1}{(r+\delta)k(w)} < 0. \end{aligned}$$

These signs also follow graphically from Fig. 2.1. A higher r (higher interest costs) reduces q^* and thereby shifts the $\dot{q} = 0$ locus downward, while the $\dot{K} = 0$ locus is not affected. Hence the steady-state point E shifts South-West, whereby K^* is reduced. (One might question whether it is correct to let $\bar{k}$ be unaffected vis-a-vis an increase in r. The answer is that it *is* correct. The reason is the presence of convex installation costs. These disconnect r from the determination of the optimal k, which is, under CRS, determined alone by w.)

A higher w (higher labor costs) implies higher capital-labor ratio, $k(w)$, hence lower $f'(k(w))$ and thereby a lower q^*, while the $\dot{K} = 0$ locus is not affected. So the steady-state point E again shifts South-West, which implies a reduced K^*. The positive substitution effect on K/L of a higher w is thus (under perfect competition) more than neutralized by a negative level effect of the higher cost of production implied by the higher labor cost. Consequently the "desired capital stock" K^* decreases. The explanation is that the present value of expected future marginal gross operating profits is reduced by the higher cost of production.

i) In view of (2.10), optimal net investment is

$$I_t^n \equiv I_t - \delta K_t = \mathcal{M}(q^*) - \delta K_t = \delta K^* - \delta K_t = \delta(K^* - K_t).$$

This result has traditionally been called the *capital adjustment principle* because it describes the gradual adjustment of actual to desired capital. The principle says that optimal net investment is proportional to the difference between desired and actual capital. Net investment is thus positive (negative) as long as the actual capital stock is below (above) the desired capital stock. The adjustment takes time due to the strictly convex installation costs. (In the absence of these costs, the "desired capital" would be reached immediately by purchasing capital in a bulk. Mathematically this would amount to an upward *jump* in K.)

3. Solution to Problem 3

For convenience we repeat the equations of the model:

$$y_t = \mu - \beta r_t^e, \qquad \mu > 0, \beta > 0, \tag{3.1}$$

$$\dot{\pi}_t = \delta(y_t - y^*), \qquad \delta > 0, \; y^* > 0, \qquad \pi_0 \text{ given}, \tag{3.2}$$

$$r_t^e \equiv i_t + \omega(\mu) - \pi_t^e, \qquad \omega(\mu) \geq 0, \qquad \omega'(\mu) < 0, \tag{3.3}$$

and

$$\begin{aligned} i_t &= \max\left[0, \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi})\right], \qquad (3.4) \\ \text{where } \hat{\imath} &\equiv \hat{r} - \omega(\bar{\mu}) + \hat{\pi}, \quad \hat{\pi} > 0, \quad 0 < \omega(\bar{\mu}) < \hat{r} + \hat{\pi}, \quad \alpha_1 \geq 0, \quad \alpha_2 > 1. \end{aligned}$$

The shift parameter μ reflects the "state of confidence". In "normal" times μ takes the value $\bar{\mu}$. Our dynamic model has five endogenous variables: $y_t, \pi_t, \pi_t^e, r_t^e$, and i_t. Remaining symbols are parameters. A subset of these are linked through the definition

$$\hat{r} \equiv \frac{\bar{\mu} - y^*}{\beta}.$$

The model as presented consists of only four equations, (3.1), (3.2), (3.3), and (3.4), while there are five endogenous variables. One may "close" the model by adding adaptive expectations or rational expectations or some other expectations formation hypothesis. We leave the model "open" in this regard.

a) Combining (3.1) and (3.3), equilibrium in the output market can be written

$$y_t = \mu - \beta(i_t + \omega(\mu) - \pi_t^e), \qquad \text{(IS)}$$

or in inverted form:

$$i_t = \frac{\mu}{\beta} - \omega(\mu) - \frac{y_t}{\beta} + \pi_t^e. \qquad \text{(IS')}$$

Assuming the zero lower bound on the interest rate is not binding, the Taylor rule (3.4) gives

$$i_t = \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) \equiv \alpha_0 + \alpha_1 y_t + \alpha_2 \pi_t^e, \qquad \text{(MP)}$$

where MP stands for monetary policy. At any given point in time, π_t^e and π_t are historically given.

For fixed t, the combinations of y_t and i_t that are consistent with equilibrium in the output market are given by the equation (IS). In Fig. 3.1 these combinations are depicted as the downward-sloping IS curve. The upward-sloping MP curve in Fig. 3.1 represents the combinations of y_t and i_t that are consistent with the Taylor rule (MP), presupposing $\alpha_1 > 0$. If $\alpha_1 = 0$, the MP curve is horizontal. The point of intersection between the IS and MP curves represents the short-run equilibrium, (y_t, i_t), at time t.

b) If expected inflation shifts upward, both the IS curve and the MP curve move upwards, the latter more than the former because $\alpha_2 > 1$ so that the new equilibrium has lower output. On the one hand, the higher expected inflation *tends* to reduce the expected real interest rate and thereby stimulate output demand. On the other hand, following the

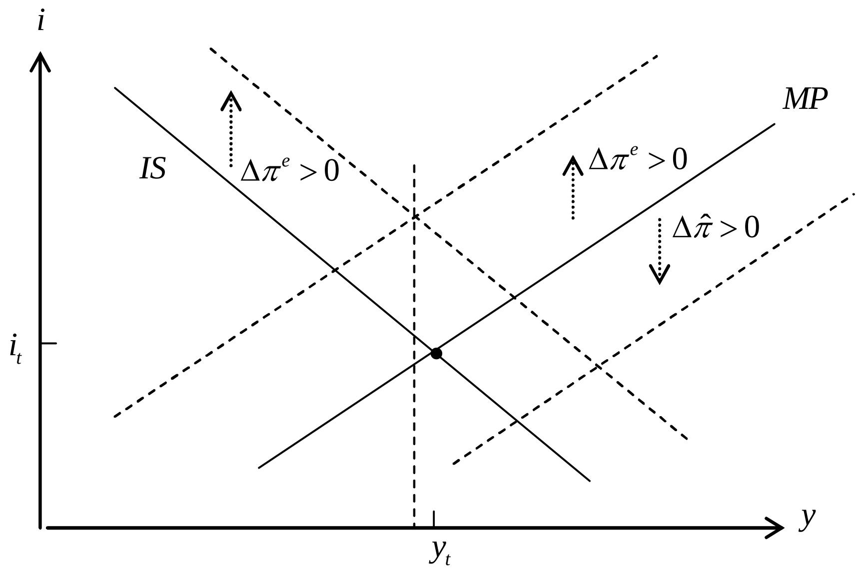

Figure 3.1:

Taylor rule the central bank counteracts this by a rise in the policy rate i_t, indeed a rise *larger* than that of expected inflation. So, in response to the higher expected inflation the central bank effectively *raises* the expected real interest rate. Thereby output demand, hence output, is dampened and the undesired increase in inflation averted.

If actual inflation shifts upward, neither the IS curve nor the MP curve move. The reason is that none of these curves depend on actual inflation when the Taylor rule is formulated as it is here. (A Taylor rule is sometimes specified with actual inflation entering instead of expected inflation in (3.4). In that case, the MP curve would move upwards; but as long as *expected* inflation, π_t^e, is given, the IS curve would not move. The short-run equilibrium would then shift *more* to the left than in Fig. 3.1.)

Since in the present model an upward shift in actual inflation has no effect at all, the examinee might choose to also consider a rise in *targeted* inflation, $\hat{\pi}$. This will shift the MP curve downwards, but not affect the IS curve, as indicated in Fig. 3.1. So the short-run equilibrium will shift to the right, ending up with lower policy rate and higher output. The reason is that a higher inflation target, everything else equal, implies a lower policy rate, hence higher output demand and output.

c) Inserting (MP) into (IS) gives

$$y_t = \mu - \beta \left[\hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) + \omega(\mu) - \pi_t^e \right].$$

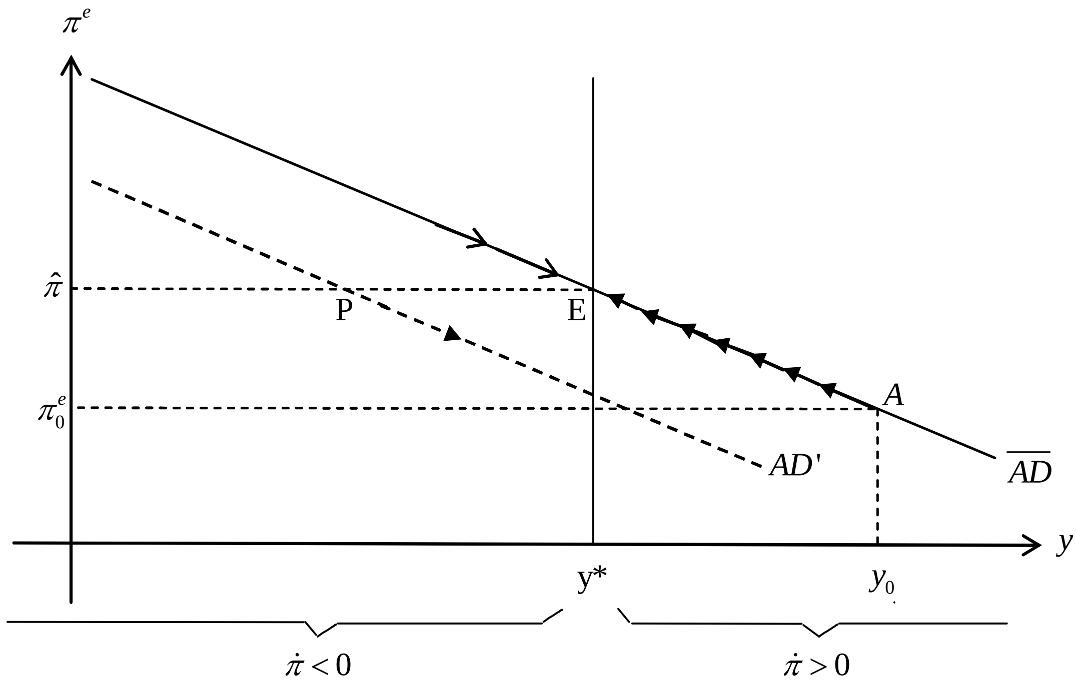

Figure 3.2:

Isolating y_t we thus have

$$y_t = \theta_0(\mu) - \theta_1 \pi_t^e, \tag{AD}$$

where

$$\begin{aligned} \theta_0(\mu) &= \frac{\mu - \beta\left(\hat{\imath} - \alpha_1 y^* + \omega(\mu) - \alpha_2 \hat{\pi}\right)}{1 + \beta\alpha_1}, \\ \theta_1 &= \frac{\beta(\alpha_2 - 1)}{1 + \beta\alpha_1} > 0. \end{aligned} \tag{3.5}$$

We can also write the AD equation this way:

$$\pi_t^e = \frac{\theta_0(\mu)}{\theta_1} - \frac{1}{\theta_1} y_t.$$

The AD curve, which represents the relationship (AD) "under normal circumstances", i.e., when $\mu = \bar{\mu}$, is shown in Fig. 3.2, and marked $\overline{\text{AD}}$. Since $-1/\theta_1 < 0$, the AD curve has negative slope.

d) *Case 1: ZLB not binding.*

For a given μ, the AD curve is fixed, and the economy is at some point on the AD curve, depending on the current expected inflation rate. At a given point in time, t, there is a historically given expected inflation rate, π_t^e. Fig. 3.2 shows the case $\pi_t^e = \pi_0^e < \hat{\pi}$. Because of the low expected inflation, monetary policy is slack. Hence, aggregate demand,

and therefore output, is above y^*. As long as μ or π_t^e do not change, the economy stays at the point A. But a change in π_t^e is likely to occur relatively soon, since π_t will be rising in view of the Phillips curve (3.2).

When π_t^e begins to change, so does y. The movement of the economy will be along the AD curve. The latter does not change its position, unless the confidence parameter μ changes its value.

Suppose the economy is in steady state at the point E. Then an adverse demand shock occurs. The background could be a bursting bubble triggering a financial crisis which leads to lower confidence. So now $\mu = \mu' < \bar{\mu}$. The interest spread rises to $\omega' = \omega(\mu') > \omega(\bar{\mu})$, which prompts a reduced $\theta_0(\mu)$. The shock shifts the AD curve leftward to a new position, indicated by AD' in Fig. 3.2. Immediately after the shock the economy shifts its position to the point P in the figure. The implied recession activates the Taylor rule, both via the output gap (if $\alpha_1 > 0$) and, possibly with a delay, via reduced expected and actual inflation generated by the Phillips curve in response to $y < y^*$. The policy rate is reduced and output demand thereby stimulated. The recession is gradually relieved.

Case 2: ZLB binding.

Suppose the adverse demand shock occurring at time t_1 is "large" and implies a reduction in confidence large enough to make the desired interest rate in (MP) negative. Then the ZLB is *binding* and instead of the desired negative interest rate being realized, we get $i_t = 0$. This is what happened in the US and several other countries when the full-blown financial crisis late in 2008 unfolded.

Through the Phillips curve the recession triggers a falling inflation rate. The expected inflation rate is likely to follow a similar downward path. As the nominal interest rate can not be reduced, the expected *real* interest rate *rises,* whereby aggregate demand and output are further reduced. A *vicious spiral* is unfolding if not other types of economic policy is made use of. This could be fiscal policy and/or "unconventional" monetary policy like some form of quantitative easing. An increased inflation target is also likely to help.

4. Solution to Problem 4

a) A believer of the *Schumpeterian story* would expect "total separations", "quits", and "hiring" to rise in the recession because the recession is seen as reflecting high frictional unemployment as workers move from obsolete industries to blossoming industries offering new types of jobs with attractive wages.

A believer of the *Keynesian story* would expect "layoffs and discharges" to rise because firms generally need fewer workers to satisfy the slack demand. A believer of the Keynesian story would expect "hiring" to fall in the recession because firms generally need fewer new workers in a situation with slack demand. Finally, a believer of the Keynesian story would expect "quits" to fall because workers are hesitant to quit as they perceive that vacant alternative jobs are scarce.

b) The data on labor market flows in the U.S. published by the Bureau of Labor Statistics tells the following:

- During the recession in 2001 (associated with the crash of the dot-com boom) as well as during the outbreak of the Great Recession 2008-2009, "total separations", "quits", and "hiring" were systematically *falling.* These three features are exactly the opposite of what the believer of the Schumpeterian story would expect.
- The observed fall in "quits" and "hiring" during the two recessions are in line with what the believer of the Keynesian story would expect. Moreover, the data also shows that "layoffs and discharges" rose considerably during the first years of the Great Recession, again in line with the Keynesian story about the nature of this recession. In the relatively short 2001 recession "layoffs and discharges" showed no clear tendency.

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