Christian Groth

A suggested solution to the problem set at the exam in Advanced Macroeconomics December 19, 2016

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates (a) accurate and thorough understanding of the concepts, methods, and models in the course, (b) knowledge of the major empirical regularities for aggregate economic variables, and (c) ability to use these theoretical tools and this empirical knowledge to answer macroeconomic questions.

1. Solution to Problem 1

Y_t grows at a given constant rate $g_Y \geq 0$, and the real interest rate in the economy is a constant $r > g_Y$. Further notation is:

$$\begin{aligned} G_t &= \text{real government spending on goods and services in period } t, \\ GBD_t &= \text{real government budget deficit in period } t, \\ B_t &= \text{real public debt (all short-term) at the start of period } t. \end{aligned}$$

There is no money-financing of budget deficits, and tax revenue is

$$T_t = \tau(Y_t + rB_t), \qquad 0 < \tau < 1, \qquad t = 0, 1, 2, \ldots. \tag{*}$$

a) We have

$$GBD_t = rB_t + G_t - T_t = rB_t + G_t - \tau(Y_t + rB_t) = (1-\tau)rB_t + G_t - \tau Y_t, \tag{GBD}$$

and

$$B_{t+1} = (1+r)B_t + G_t - T_t = (1 + (1-\tau)r)B_t + G_t - \tau Y_t. \tag{DGBC}$$

[1]The solution below contains more details and more precision than can be expected at a three hours exam.

As an implication,

$$B_{t+1} = B_t + GBD_t. \tag{1.1}$$

We are told that $G_t = \gamma Y_t$, $t = 0, 1, 2, \ldots$, where γ is a constant, $0 < \gamma < 1$. Moreover, $B_0 > 0$, and b_t denotes the debt-income ratio, i.e., $b_t \equiv B_t/Y_t$.

b) A given fiscal policy is called *sustainable* if by applying its spending and tax rules forever, the government stays solvent.

A simple operational criterion for fiscal sustainability is that the fiscal policy is compatible with upward boundedness of the public debt-to-income ratio. An additional requirement may be mentioned, namely that the actual upper bound is not "too" worrisome in the sense of being able to generate a danger of self-fulfilling expectations of default.

c) We have

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{(1 + (1-\tau)r)B_t + \gamma Y_t - \tau Y_t}{(1+g_Y)Y_t} = \frac{1 + (1-\tau)r}{1+g_Y} b_t - \frac{\tau - \gamma}{1+g_Y}.$$

So the law of motion is

$$b_{t+1} = \frac{1 + (1-\tau)r}{1+g_Y} b_t - \frac{\tau - \gamma}{1+g_Y}. \tag{1.2}$$

d) By the hint, given $(1-\tau)r > g_Y$, (1.2) has the solution

$$b_t = (b_0 - b^*) \left(\frac{1 + (1-\tau)r}{1+g_Y} \right)^t + b^*, \tag{1.3}$$

where $b_0 \equiv B_0/Y_0 > 0$ is historically given, and b^* is consistent with (1.2) if $b_{t+1} = b_t = b^*$. So

$$b^* = -\frac{\tau - \gamma}{1+g_Y} \left(1 - \frac{1 + (1-\tau)r}{1+g_Y} \right)^{-1} = \frac{\tau - \gamma}{(1-\tau)r - g_Y}. \tag{1.4}$$

We are told that the minimum value, $\hat{\tau}$, of τ required for fiscal sustainability is such that $(1-\hat{\tau})r > g_Y$. Hence, with $\tau = \hat{\tau}$, the growth factor $[1 + (1-\tau)r/(1+g_Y)]$ in (1.3) exceeds 1, implying that fiscal sustainability requires $b^* \geq b_0$. So, $\hat{\tau}$ should be such that $b^* = b_0$, i.e., such that

$$\frac{\hat{\tau} - \gamma}{(1-\hat{\tau})r - g_Y} = b_0,$$

which implies

$$\begin{aligned} \hat{\tau} - \gamma &= rb_0 - \hat{\tau}rb_0 - g_Y b_0 \Rightarrow (1 + rb_0)\hat{\tau} = \gamma + (r - g_Y)b_0 \Rightarrow \\ \hat{\tau} &= \frac{\gamma + (r - g_Y)b_0}{1 + rb_0}. \end{aligned}$$

e) In case of no taxation of interest income from holding government bonds the law of motion of b_t would be

$$b_{t+1} = \frac{1+r}{1+g_Y} b_t - \frac{\tau - \gamma}{1+g_Y}.$$

Given $r > g_Y$, this has the solution

$$b_t = (b_0 - b^*) \left(\frac{1+r}{1+g_Y} \right)^t + b^*, \qquad \text{where } b^* = \frac{\tau - \gamma}{r - g_Y}.$$

Hence, in this case the minimum value, $\hat{\tau}'$, of the tax-income ratio needed for fiscal sustainability is the value resulting in $b^* = b_0$. Consequently,

$$\hat{\tau}' = \gamma + (r - g_Y) b_0 > \hat{\tau} = \frac{\gamma + (r - g_Y) b_0}{1 + r b_0}. \tag{1.5}$$

With no taxation of interest income from holding government bonds, a higher income tax rate is thus required for fiscal sustainability.

f) A simple measure of the *sustainability gap* at time 0 is

$$gap = \hat{\tau} - \tau.$$

We have

$$\hat{\tau} = \frac{\gamma + (r - g_Y) b_0}{1 + r b_0} = \frac{\gamma + r b_0}{1 + r b_0} - \frac{g_Y b_0}{1 + r b_0}.$$

So

$$\frac{\partial gap}{\partial g_Y} = \frac{\partial \hat{\tau}}{\partial g_Y} = -\frac{b_0}{1 + r b_0} < 0. \tag{1.6}$$

g) Yes, given the present model, a higher productivity growth rate is helpful for fiscal sustainability to the extent it raises g_Y. This is what (1.6) shows.

The reason is that higher g_Y also means higher growth of the tax revenue. Hence, for a given interest rate on government debt, it becomes easier for the tax revenue to keep track with a rising debt. So higher productivity growth reduces the sustainability gap and provides scope for a lower tax rate.

Does the model take into account that along with a higher g_Y (hence higher growth in incomes), also growth in government expenses is likely to rise and possibly to the same extent? Yes, this is taken into account by maintaining unchanged γ.

(What the model does not take into account is the special feature of the Danish tax system that taxation of the part of current income invested in pension funds is deferred until retirement. To the extent this postponed portion of the tax revenue depends more on compound interest than on productivity growth, it may not follow suit when productivity growth is raised. The grading is not dependent on this point being mentioned or not. Especially foreign students may not be aware of this special feature of the Danish tax system.)

2. Solution to Problem 2

We consider a simple short-run model for a closed economy. Private aggregate demand, D, is the sum of private consumption, C, and investment, I :

$$\begin{aligned} D(Y_t, R_t, \tau, \alpha) &\equiv C(Y_t - (\tau + T(Y_t)), R_t, \alpha) + I(Y_t, R_t, \alpha), \text{ where} \\ 0 &< D_Y < 1, D_R < 0, -1 < D_\tau < 0, D_\alpha > 0, \qquad (*) \end{aligned}$$

and Y is aggregate income (= aggregate output), $T(Y)$ a tax function, τ a parameter reflecting fiscal tightness, R_t is the real long-term interest rate, and α an indicator of the general "state of confidence".

Time is continuous and we have the following further equations:

$$\dot{Y}_t = \lambda(D(Y_t, R_t, \tau, \alpha) + G - Y_t), \qquad \lambda > 0, \tag{2.1}$$

$$\frac{M_t}{P_t} = L(Y_t, i_t), \qquad L_Y > 0, \ L_i < 0, \tag{2.2}$$

$$q_t = \int_t^\infty 1 \cdot e^{-R_t(s-t)} ds = \frac{1}{R_t}, \tag{2.3}$$

$$\frac{1 + \dot{q}_t^e}{q_t} = r_t^e, \tag{2.4}$$

$$r_t^e \equiv i_t - \pi_t^e, \tag{2.5}$$

$$\pi_t = \pi, \tag{2.6}$$

where the superscript e denotes subjective expectation, q_t = the real price of a consol paying one unit of output per time unit forever, G = government spending on goods and services, M_t = the monetary base (there is no private banking sector), P_t = output price, i_t = nominal short-term interest rate, r_t = real short-term interest rate, and $\pi_t \equiv \dot{P}_t/P_t$ = rate of inflation. The variables λ, τ, G, α, and π are exogenous constants. The initial values Y_0 and P_0 are pre-determined.

The central bank uses the nominal short-term interest rate as policy instrument, maintaining it at a desired level, given current circumstances. Until further notice, let the desired level equal the constant $i^* > 0$.

a) The model is Blanchard's dynamic IS-LM (or IS-MP) model, which describes the adjustment in the "very short run" towards a "short-run equilibrium" with respect to output and interest rates. The adjustment of output to demand takes time and during the adjustment process also demand changes (since the output level and asset prices are among its determinants). There are three financial assets: money, a long-term inflation-indexed bond, and a short-term bond.

Eq. (2.1) tells how output adjusts to demand; the parameter λ is the speed of adjustment. Output demand depends positively on current income (to reflect, e.g., that a fraction of the consumers are credit constrained) and negatively on the long-term real interest rate. In particular investment is likely to depend negatively on this rate. Also consumption tends to be negatively dependent on the long-term rate due to the substitution and wealth effects.

Eq. (2.2) expresses equilibrium in the money market. Real money demand depends positively on Y, a proxy for the number of transactions per time unit, and negatively on the short-term nominal interest rate, the opportunity cost of holding money. The money market and other asset markets are assumed to clear instantaneously.

Eq. (2.3) tells us that the "long-term real interest rate", R_t, is identified with the internal rate of return on the consol.

Eq. (2.4) is a no-arbitrage condition saying that the expected real rate of return on holding the consol one time unit equals the expected real short-term interest rate, r_t^e. So there is no risk premium. Eq. (2.5) defines the expected real interest rate, r_t^e. Finally, eq. (2.6) states the simplifying assumption that the actual rate of inflation is a constant π. So both the initial price level and the inflation rate are "sticky" (two key Keynesian features of the model).

The way the CB is able to maintain i_t at the desired level i^* is through open market operations. The CB has direct control over the monetary base and adjusts it until the actual short-term interest rate equals the one desired. If $i > i^*$, the CB buys short-term bonds from the market, and if $i < i^*$, the CB sells short-term bonds. Given (2.2), the required evolution of the monetary base is $M_t = P_t L(Y_t, i^*) = P_0 e^{\pi t} L(Y_t, i^*)$.

We now assume there is no uncertainty and that expectations are rational (model consistent). This amounts to perfect foresight as there are no stochastic elements in the model. In addition we also assume that speculative bubbles never arise. The absence of bubbles implies that the market price of the consol equals its fundamental value so that we can write

$$R_t = \frac{1}{q_t} = \frac{1}{\int_t^\infty 1 \cdot e^{-\int_t^s r_u du} ds}. \tag{2.7}$$

As shown in lecture notes, this indicates that the long-term rate is a weighted average of the expected future short-term rates, which will prove helpful for intuition below.

b) In view of perfect foresight, $\pi_t^e = E_t\pi_t = \pi_t = \pi$ for all t, implying that $r_t^e = i_t - \pi \equiv r_t = i^* - \pi \equiv \bar{r}$. Similarly, $\dot{q}_t^e = E_t\dot{q}_t = \dot{q}_t$. Substituting into (2.4), we get

$$\frac{1}{q_t} + \frac{\dot{q}_t}{q_t} = R_t - \frac{\dot{R}_t}{R_t} = i_t - \pi = i^* - \pi.$$

Reordering, we get

$$\dot{R}_t = [R_t - (i^* - \pi)]\, R_t. \tag{2.8}$$

We now have two differential equations, (2.1) and (2.8), in the endogenous variables Y_t and R_t. To draw the corresponding phase diagram, note that (2.1) implies

$$\dot{Y} \gtreqless 0 \quad \text{for} \quad D(Y, R, \tau, \alpha) + G \gtreqless Y, \quad \text{respectively.} \tag{2.9}$$

Hence, $\frac{\partial R}{\partial Y}|_{\dot{Y}=0} = (1 - D_Y)/D_R < 0$. The $\dot{Y} = 0$ locus is thus a downward-sloping curve, named the "IS curve" in Fig. 2.1.

Similarly, (2.8) implies

$$\dot{R} \gtreqless 0 \quad \text{for} \quad R \gtreqless i^* - \pi, \quad \text{respectively.} \tag{2.10}$$

Since $i^* - \pi$ is a constant, the $\dot{R} = 0$ locus is horizontal, cf. Fig. 2.1.

The figure also shows the direction of movement in the different regions according to (2.9) and (2.10). The arrows in the diagram indicate that the steady-state point, E, is a saddle point.[2] This implies that two and only two solution paths – one from each side – converges towards E. These two saddle paths coincide with the $\dot{R} = 0$ locus. We thus have:

i) the steady state is a saddle point;

ii) there is in the dynamic system one predetermined variable, Y, and one jump variable, R;

iii) the saddle path is not parallel to the jump-variable axis, the R axis;

iv) the diverging paths cannot be solutions because there are no speculative bubbles.

It follows that the steady state is saddle-point stable.

Since Y_0 is predetermined, at time $t = 0$ the economy must be somewhere on the vertical line $Y = Y_0$. In addition, from iv) follows that the economy must initially be on

[2] More formally, the determinant of the Jacobian matrix for the right hand sides of the two differential equations, evaluated at the steady-state point ($\bar{Y}$, $\bar{R}$), is $\bar{R}\lambda(D_Y - 1) < 0$.

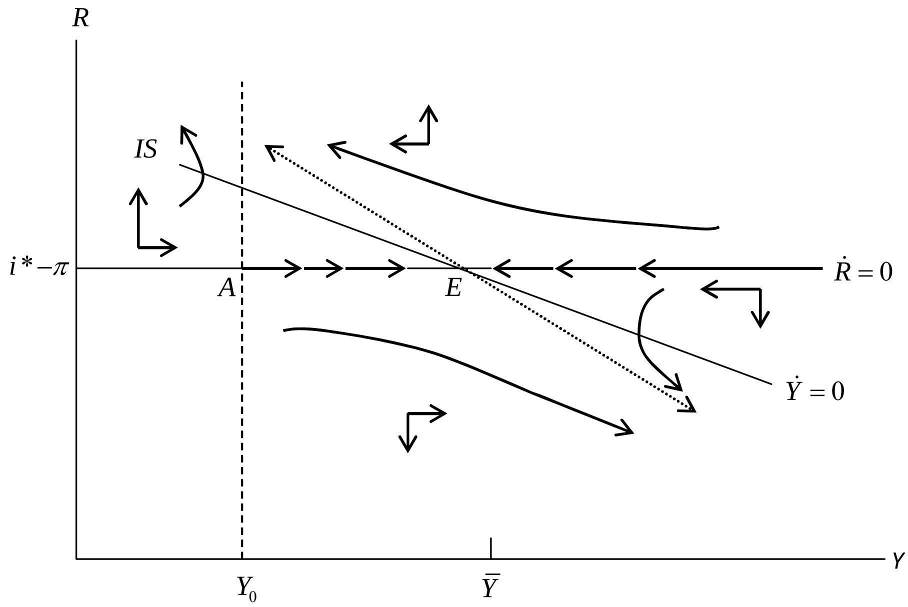

Figure 2.1:

the saddle path. Hence, we are left with the saddle path, the path AE in Fig. 2.1, as the unique solution to the model.

As the point A is below the $\dot{Y} = 0$ locus, for $t = 0$ current output is below current aggregate demand. This leads to increasing output. During this adjustment of output, the long-term rate, R_t, remains unchanged at the level $\bar{R} = i^* - \pi \equiv \bar{r}$. The reason is the *weighted-average principle* displayed by (2.7): R is a weighted average of expected future real short-term rates, and these remain constant at the level $\bar{r} \equiv i^* - \pi$ since i^* is kept constant by monetary policy.

The economy gradually approaches the steady state, where $Y = \bar{Y}$ and $R = i^* - \pi \equiv \bar{R} \equiv 1/\bar{q}$. Since q is always a positive number, so must R be, cf. (2.3). Hence, we need $\pi < i^*$.

c) In steady state we have $R = i^* - \pi$ so that the steady-state value, $\bar{Y}$, of output satisfies

$$\bar{Y} = D(\bar{Y}, i^* - \pi, \tau, \alpha) + G. \tag{2.11}$$

Since $0 < D_Y < 1$, a solution for $\bar{Y}$ in this equation is unique. The solution can be illustrated by the "Keynesian cross" in the plane. By curve shifting it then follows from (*) that

$$i^* \uparrow \Rightarrow \bar{Y} \downarrow, \quad G \uparrow \Rightarrow \bar{Y} \uparrow, \quad \text{and } \tau \uparrow \Rightarrow \bar{Y} \downarrow,$$

as expected.

A slightly more precise answer may use total differentiation. The equation (2.11) defines $\bar{Y}$ as an implicit function of i^*, G, and τ. Taking the total differential on both sides of (2.11) w.r.t. these three variables, we get

$$\begin{aligned} 1 \cdot d\bar{Y} &= D_Y d\bar{Y} + D_R di^* + D_\tau d\tau + 1 \cdot dG \Rightarrow \\ (1 - D_{\bar{Y}}) d\bar{Y} &= D_R di^* + D_\tau d\tau + dG. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial \bar{Y}}{\partial i^*} &= \frac{D_R}{1 - D_{\bar{Y}}} < 0, \\ \frac{\partial \bar{Y}}{\partial G} &= \frac{1}{1 - D_{\bar{Y}}} > 0, \\ \frac{\partial \bar{Y}}{\partial \tau} &= \frac{D_\tau}{1 - D_{\bar{Y}}} < 0. \end{aligned}$$

d) It is now assumed that the economy has been in steady state until time $t_1 > 0$. Then an unanticipated large decrease in the state of confidence to $\alpha' < \alpha$ occurs. After this everybody rightly expects the state of confidence to remain unchanged for a long time.

Fig. 2.2 illustrates the evolution of the economy from time t_1 onwards. Immediately after the adverse demand shock aggregate demand is lower than output. The latter is then gradually pulled down toward the new steady state E'.

e) It is now instead assumed that immediately after time t_1, the central bank responds to the confidence shock by lowering the policy rate to the constant $i^{*\prime} < i^*$. The reduction in the policy rate is "moderate" relative to the size of the confidence shock.

Fig. 2.3 illustrates the evolution of the economy from time t_1 onwards. In view of the weighted-average principle, the long-term rate follows the short-term rate, jumps down to $i^{*\prime} - \pi \equiv \bar{r}'$ and remains at this constant level.

The new steady state, E", is to the left of the old, indicating that an economic downturn is not avoided, although reduced. Owing to the reduced interest rate, aggregate demand is lower than output, the latter is gradually pulled down toward the new steady state E".

f) Instead it is now assumed that immediately after time t_1, the central bank responds to the confidence shock by *announcing* that at time $t_2 > t_1$, the policy rate will be reduced to the constant $i^{*\prime} < i^*$. Everybody rightly expects the policy rate to behave as announced.

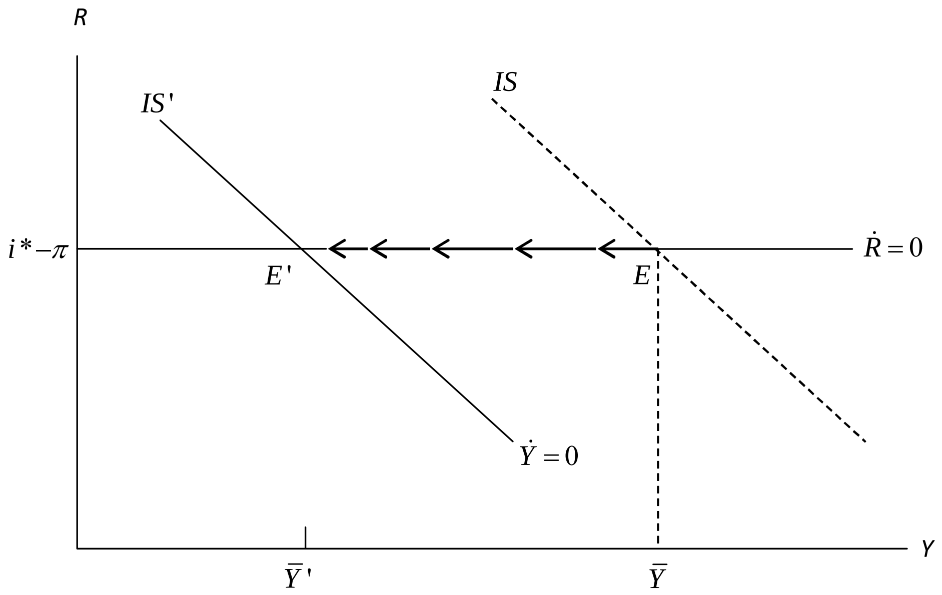

Figure 2.2:

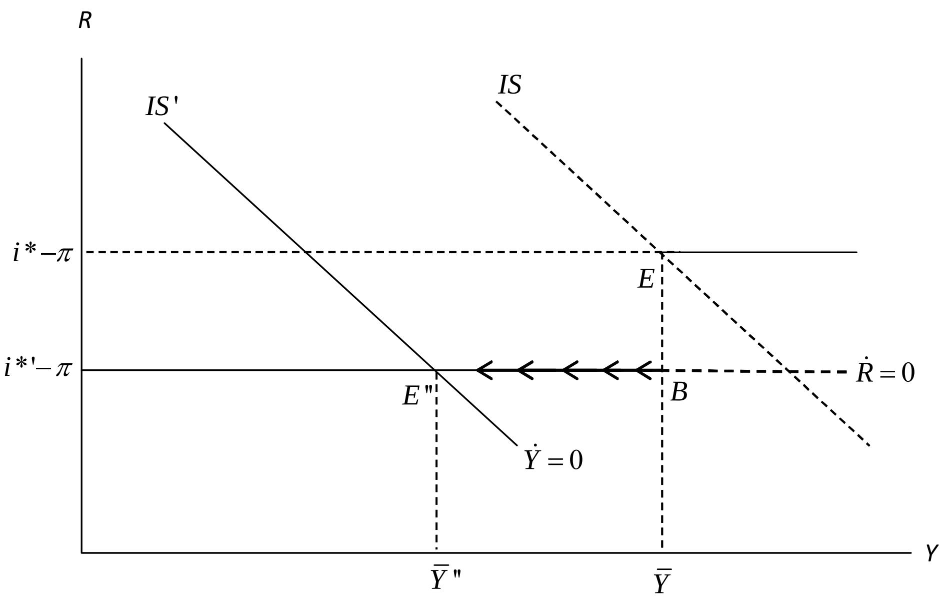

Figure 2.3:

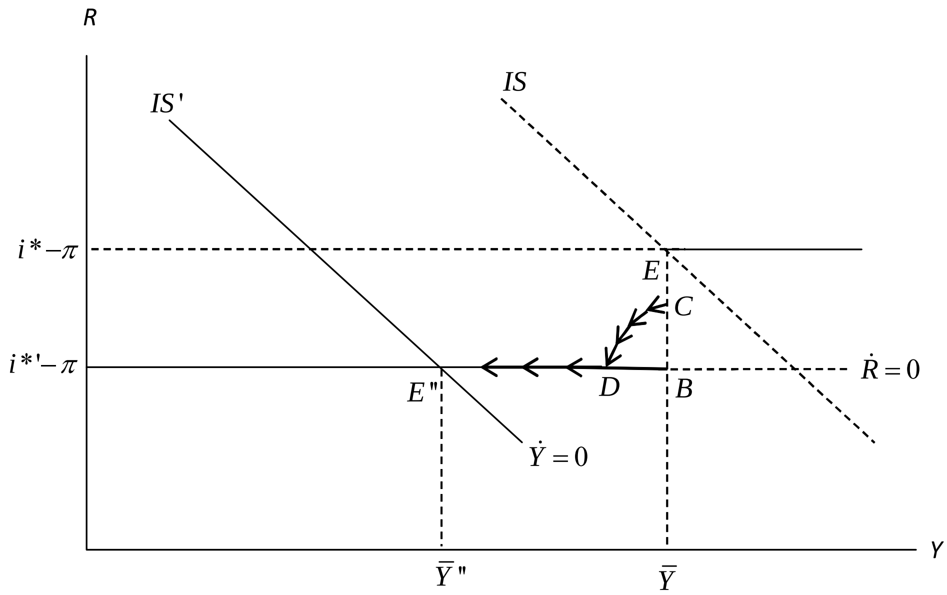

Figure 2.4:

Fig. 2.4 illustrates the evolution of the economy from time t_1 onwards. Fig. 2.5 illustrates the time profiles of of i_t, Y_t, R_t, and r_t for $t \geq 0$.

The credible announcement of a reduction at time t_2 of the policy rate to the new constant $i^{*\prime} < i^*$, implies that the average of expected future real short-term rates, r_t, as seen from time t_1, is somewhere between the old $\bar{r}$ and the ultimate $\bar{r}' \equiv i^{*\prime} - \pi$, which is realized from time t_2 onwards. In view of the weighted-average principle, the long-term rate thus at time t_1 jumps down to a level like indicated by the point C in Figure 2.4. The exact position of C is such that according to the "old dynamics" it takes exactly $t_2 - t_1$ time units for the economy to move from C to some point on the new saddle path in Fig. 2.4. The reason that it must take exactly $t_2 - t_1$ time units for the economy to reach the new saddle path is that otherwise there would be an expected discontinuity in the asset price $q = 1/R$ at time t_2, after which the "new dynamics" will rule. As also noted earlier, from time t_2 onwards, the economy *must* be on the new saddle path since, after time t_2, fiscal and monetary policies are not expected to change, and speculative bubbles are ruled out by assumption. Finally, arbitrage in the market precludes an expected discontinuity in an asset price.

g) As Fig. 2.5 illustrates, in the time interval (t_1, t_2) the "slope of the yield curve" is negative since the long-term rate is smaller than the short-term rate. The intuition is, *first*, that since uncertainty is ignored, the slope of the yield curve depends only on expectations; *second*, since the long-term rate is forward-looking and equals a weighted

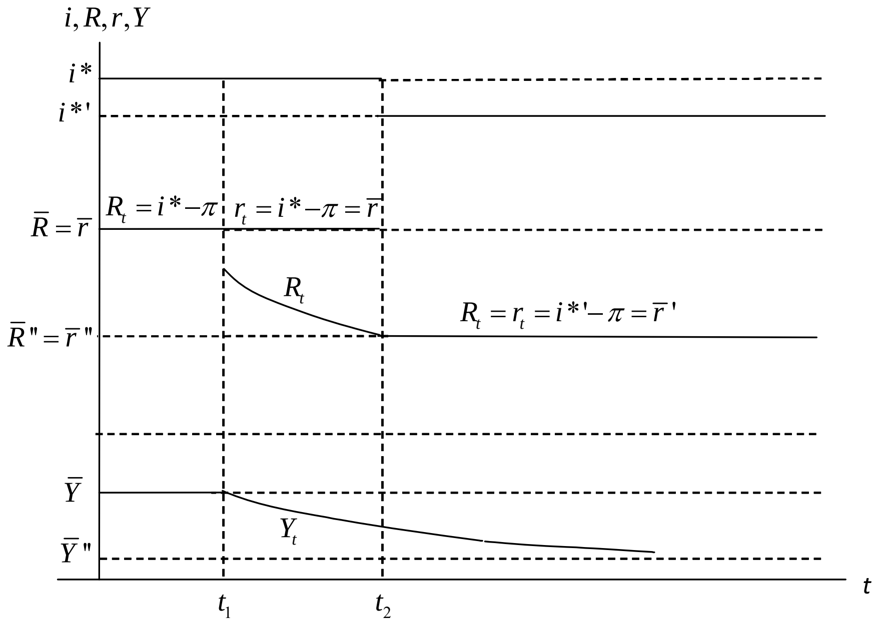

Figure 2.5:

average of the expected future short-term rates, when these are expected to fall in the future, this is incorporated in the long-term rate already today.

3. Solution to Problem 3

a) Admittedly, the correct answer depends on how the claim is interpreted.

If "Blanchard's OLG model" is interpreted as the *Perpetual Youth Model*, the answer to the seemingly paradoxical claim, is "True". In that model, by the Keynes-Ramsey rule, individual consumption is in steady state growing at the rate $r^* - \rho$, which in the model necessarily exceeds g, the rate of productivity growth. The latter is also the rate of growth of per capita consumption, $c \equiv C/N$.

That this is possible is due to the absence of a representative agent in the model. Every individual has finite lifetime, and there is a turnover of the generations. After having enjoyed growing consumption through life, due to growing individual wealth, the individual dies and is in a sense "replaced" by a young entering the economy with no financial wealth and therefore comparatively low consumption. The high individual consumption growth only exists as long as the individual is alive. Per capita consumption, c, behaves differently and will in steady state grow at the same rate as productivity.

If, however, "Blanchard's OLG model" is interpreted as the *extended model version* *with retirement*, the answer is "False". This is because the claim about the will then no longer unconditionally hold. There are parameter combinations where it does hold and parameter combinations where it does not.

So there are different ways of answering. Decisive for the grading is the way the answer is framed.

b) False. Counter examples are possible.

Here is an example from the lecture notes, Ch. 31, where *more* price flexibility may be destabilizing in the sense furthering divergence.

Suppose monetary policy is "passive" ("monetarist"). Then, an adverse shock to investors' and firms' general long-term confidence occurs, leading to a downturn of aggregate demand, production, and employment. Through the Phillips curve mechanism, inflation and expected inflation also go down. High price flexibility may turn the incipient recession into a downward wage-price spiral. This is because opposing effects on aggregate demand are in play, giving rise to a *centripetal force* and a *centrifugal force*. On the one hand, the fall in inflation increases real money supply and lowers the nominal rate of interest, thereby stimulating aggregate demand. And in an open economy net exports are stimulated. On the other hand, the fall in *expected* inflation raises the *real* rate of interest,

$$r = i + \omega - \pi^e,$$

for a given short-term nominal rate of interest i (the policy rate) and a given interest differential, $\omega \geq 0$, thereby *reducing* demand. Depending on the circumstances, this effect may be the strongest and lead to a self-sustaining economic contraction. In particular this may happen, when the nominal rate of interest is already low and therefore near the ZLB - or rather - the "slightly below zero" bound.

Remark. Two different interpretations of the term "stabilizing" in the claim are possible. This is because in the macroeconomic literature the term "instability" is used in two somewhat different meanings: either as "divergence" (as in the example above) or as "large statistical variance". Also when one has the latter meaning in mind, do counter examples to the claim exist.

c) In their econometric study Mian & Sufi (2014) explore the question: What explains the large 2007-09 drop in employment in the US? Their conclusion is that deterioration in household balance sheets triggered a sharp fall in aggregate demand, which was the main factor behind the drop in employment:

Lower housing net worth$\Rightarrow$ lower wealth

$\Rightarrow \left\{ \begin{array}{l} \text{consumption} \downarrow \\ \text{value of collateral } \downarrow \Rightarrow \text{credit contraction} \end{array} \right\} \Rightarrow$ consumption $\downarrow\downarrow$ $\Rightarrow$ unemployment $\uparrow$

$\Rightarrow$ consumption $\downarrow\downarrow\downarrow$

Mian & Sufi's empirical study has US counties as the unit of observation. Counties with a larger decline in housing net worth experienced a larger decline in non-tradable employment (outside construction), while the decline in tradable employment showed no correlation with the decline in housing net worth. The authors control for - and reject - alternative explanations such as industry-specific supply side shocks, policy-induced business uncertainty, or credit supply tightening. And they find little evidence of wage adjustment within or emigration out of the hardest hit counties.

—