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A note on the implicit function theorem and differentials

1 The implicit function theorem[1]

In economics we often consider problems of the following kind: if a system of equations is intended to define some endogenous variables as functions of the remaining exogenous variables, what are the partial derivatives of these functions? This raises the mathematical questions: do these functions exist, if so, are they differentiable, and if so, can we provide useful formulas for the partial derivatives?

1.1 The implicit function theorem for two variables

Consider the equation

$$f(x, y) = 0. \tag{1}$$

It could represent an isoquant (level curve) $F(K, L) = \bar{Y}$ in which case $f(x, y) \equiv F(x, y) - \bar{Y} = 0$. If f is specified, we might be able to solve the equation for y expressed as some explicit function of x, interpreted as the exogenous variable (the "known"). Even if f is not specified, we might be able to "solve" the equation for the endogenous variable (the "unknown") y expressed as some "implicit" function of x.

Theorem 1 (implicit function theorem, special version). Suppose that f is C^1 in an open set S in $\mathbb{R}^2$.[2] Let $(x_0, y_0) \in S$ satisfy (1) and assume that $f_2'(x_0, y_0) \neq 0$.[3] Then there exists an interval $I_1 = (x_0 - \delta, x_0 + \delta)$ and an interval $I_2 = (y_0 - \varepsilon, y_0 + \varepsilon)$ (with $\delta > 0$ and $\varepsilon > 0$) such that $I_1 \times I_2 \subseteq S$ and:

(i) for every x in I_1 the equation (1) has a unique solution in I_2 which defines y as a (generally only "implicit") function $y = \varphi(x)$ in I_1;

[1] This relies primarily on K. Sydsæter, P. Hammond, A. Seierstad, and A. Strøm, *Further Mathematics for Economic Analysis*, Second edition, Prentice Hall, Essex 2008.

[2] A function f is *of class* C^k, or simply C^k, in the set $S \subset \mathbb{R}^n$ if all partial derivatives of f of order $\leq k$ are continuous in S.

[3] Note that when we write *pure math*, we use apostrophes to indicate partial derivatives. But when we set up and analyze *economic models*, we usually skip the apostrophes unless they are necessary to avoid confusion.

(ii) φ is C^1 in I_1 with derivative

$$\varphi'(x) = -\frac{f_1'(x, \varphi(x))}{f_2'(x, \varphi(x))}.$$

Theorem 1 is also applicable if we start from an equation with one endogenous and several exogenous variables (cf. the saving problem of the young in Section 3.3). Consider the equation $f(x, y, z) = 0$, where f is C^1 in an open set S in $\mathbb{R}^3$, and x and y are exogenous variables while z is the endogenous variable. If $(x_0, y_0, z_0) \in S$ satisfies the equation and $f_3'(x_0, y_0, z_0) \neq 0$, this equation defines z as an implicit C^1 function $z = \psi(x, y)$ of x and y. Keeping y fixed at some constant level $\bar{y}$, we may apply Theorem 1 to get an expression for $\psi_1(x, \bar{y})$. And keeping x fixed at some constant level $\bar{x}$, we may apply the theorem to get an expression for $\psi_2(\bar{x}, y)$. Finally, $\bar{y}$ and $\bar{x}$ are varied within the relevant ranges.

1.2 The implicit function theorem (general version)

Consider a system of m equations with n exogenous variables and m endogenous variables

$$f^1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f^m(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = 0,$$

or, in vector notation,

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}. \tag{2}$$

Theorem 2 (implicit function theorem, general version). Suppose $f^1, \ldots, f^n$ are C^1 functions of $(\mathbf{x}, \mathbf{y})$ in an open set S in $\mathbb{R}^n \times \mathbb{R}^m$. Let $(\mathbf{x}^0, \mathbf{y}^0) \in S$ satisfy (2) and assume that the Jacobian determinant of $\mathbf{f}$ with respect to $\mathbf{y}$ is different from 0 at $(\mathbf{x}^0, \mathbf{y}^0)$, i.e.,

$$|J_y(\mathbf{x}, \mathbf{y})| = \begin{vmatrix} \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_1} & \cdots\cdots & \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_1} & \cdots\cdots & \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_m} \end{vmatrix} \neq 0 \quad \text{at} \quad (\mathbf{x}, \mathbf{y}) = (\mathbf{x}^0, \mathbf{y}^0).$$

Then there exists neighborhoods B_1 in $\mathbb{R}^n$ around $\mathbf{x}^0$ and B_2 in $\mathbb{R}^m$ around $\mathbf{y}^0$, respectively, with $B_1 \times B_2 \subseteq S$ such that $|J(\mathbf{x}, \mathbf{y})| \neq 0$ for all $(\mathbf{x}, \mathbf{y}) \in B_1 \times B_2$, and such that:

(i) for each $\mathbf{x}$ in B_1 there is a unique $\mathbf{y}$ in B_2 satisfying (2); in this way $\mathbf{y}$ is defined "implicitly" on B_1 as a function $\varphi(\mathbf{x}) = (\varphi^1(\mathbf{x}), \ldots, \varphi^m(\mathbf{x}))$ of $\mathbf{x}$;

(ii) φ is C^1 in B_1 with partial derivatives given by

$$\frac{\partial \varphi^i(\mathbf{x})}{\partial x_j} = \frac{\begin{vmatrix} \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_1} \cdots \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_2} \cdots\cdots & \overset{\downarrow\, i\text{'th column}}{-\frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial x_j}} & \cdots \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_{i+1}} \cdots\cdots \frac{\partial f^1(\mathbf{x},\mathbf{y})}{\partial y_n} \\ \vdots \qquad \vdots & \vdots & \vdots \qquad \vdots \\ \vdots \qquad \vdots & \vdots & \vdots \qquad \vdots \\ \vdots \qquad \vdots & \vdots & \vdots \qquad \vdots \\ \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_1} \cdots \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_2} \cdots\cdots & -\frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial x_j} & \cdots \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_{i+1}} \cdots\cdots \frac{\partial f^m(\mathbf{x},\mathbf{y})}{\partial y_n} \end{vmatrix}}{\mid J(x,y) \mid} \tag{3}$$

for $i = 1, \ldots, m, \quad j = 1, \ldots, n$. (The matrix the determinant of which appears in the numerator is identical to the Jacobian $J_y(\mathbf{x}, \mathbf{y})$ but for the replacement of the i'th column of the latter with the m-vector $(-\partial f^1(\mathbf{x}, \mathbf{y})/\partial x_j, -\partial f^2(\mathbf{x}, \mathbf{y})/\partial x_j, \ldots, -\partial f^m(\mathbf{x}, \mathbf{y})/\partial x_j)$.)

2 Differentials[4]

Differentials and Cramer's rule are convenient devices for analytical calculation of partial derivatives of functions implicitly given by a system of equations. At p. 38 in Chapter 2 of the Lecture Notes we see an example. But what is a differential?

2.1 A function of one variable

Let $y = f(x)$ be a differentiable function. Let the symbol dx denote an arbitrary real number to be interpreted as a change in the value of the variable x. Then the expression $f'(x)dx$ at x is called the *differential* of $y = f(x)$ and is denoted dy. That is,

$$dy = f'(x)dx.$$

With Δy denoting the actual change in the value of y when x is changed by dx, we have

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx, \tag{4}$$

where "$\approx$" means "approximately equal to" for dx "small" in absolute value. To clarify the meaning of "approximately equal to" and "small", let ε be defined through the equation

$$f(x + dx) - f(x) = f'(x)dx + \varepsilon dx.$$

[4]The definition of a differential given here follows K. Sydsæter and P. Hammond, *Essential Mathematics for Economic Analysis*. Third edition, Prentice-Hall 2008.

Since f is differentiable, we have $\varepsilon \to 0$ for $dx \to 0$. If dx is very small, then ε is very small, and εdx is "very, very small".

If f is an affine function,"$\approx$" in (4) can be replaced by "=". Otherwise, the differential dy in (4) only provides an approximation to the actual change Δy.

2.2 A function of two variables

Let $z = f(x, y)$ be a C^1 function of two variables. Let the symbols dx and dy denote arbitrary real numbers to be interpreted as changes in the value of x and y, respectively. Then the (total) *differential* of $z = f(x, y)$ at (x, y), denoted dz or df, is defined by

$$dz \ (= df) \ = f_1'(x,y)dx + f_2'(x,y)dy. \tag{5}$$

The actual change in the value of z when x and y are changed by dx and dy, respectively, is

$$\Delta z = f(x + dx, y + dy) - f(x, y) \approx dz = f_1'(x,y)dx + f_2'(x,y)dy,$$

where the approximation "works well" when dx and dy are "small" in absolute value.[5]

3 Solving two equations in terms of differentials

Let also $u = g(x, y)$ be a C^1 function of two variables. In analogy with (5) we can then write

$$du \ (= dg) \ = g_1'(x,y)dx + g_2'(x,y)dy. \tag{6}$$

The immediate interpretation of the equations (5) and (6) is that they *determine* the (value of the) differentials dz and du (i.e., the approximate changes in z and u) from knowledge of the pairs (x, y) and (dx, dy). That is, the "actual" changes dx and dy are seen as "causing" changes in z and u approximately equal to dz and du.

Assume, however, that what we know, in addition to the functions f and g, is the pairs (x, y) and (dz, du), while (dx, dy) is the unknown. Then one would normally rearrange the equations so that the known (dz, du) appears alone on the right-hand side:

$$f_1'(x,y)dx + f_2'(x,y)dy \ = \ dz, \tag{7}$$

$$g_1'(x,y)dx + g_2'(x,y)dy \ = \ du. \tag{8}$$

[5]Some elementary texts about mathematics for economists define the differential dz as requiring that dx and dy in (5) are "infinitisimals", or "infinitely small". It is then claimed that Δz becomes equal to dz. These imprecise ideas, originally introduced by Leibniz, have generally been abandoned in mathematics.

Given (x, y) and (dz, du), we have here a system of two *linear* equations in the two unknowns, the variables dx and dy. Suppose that the Jacobian determinant, D, of $f(x, y)$ with respect to x and y differs from 0, i.e.,

$$D = \begin{vmatrix} f'_1(x,y) & f'_2(x,y) \\ g'_1(x,y) & g'_2(x,y) \end{vmatrix} \neq 0. \tag{9}$$

Then, from linear algebra we know that, given the right-hand side of (7)-(8), this system has a unique solution (dx, dy). Indeed, by Cramer's rule, the solution for dx is

$$dx = \frac{\begin{vmatrix} dz & f'_2(x,y) \\ du & g'_2(x,y) \end{vmatrix}}{D}. \tag{10}$$

Similarly, the solution for dy is

$$dy = \frac{\begin{vmatrix} f'_1(x,y) & dz \\ g'_1(x,y) & du \end{vmatrix}}{D}. \tag{11}$$

The solution for dx and dy approximate the changes in x and y "caused" by the changes in z and u, represented by dz and du. The functions f and g might themselves be linear. In that case the solution for dx and dy gives the "true" changes in x and y.

4 Application

Differentials and Cramer's rule are convenient devices for analytical calculation of partial derivatives of functions implicitly given by a system of equations. At p. 38 in Chapter 2 of the Lecture Notes we have an example. The point of departure is the system of equations

$$F_K(K, L) = w_K, \tag{2.29}$$

$$F_L(K, L) = w_L. \tag{2.30}$$

In the text these equations are interpreted as determining the two unknowns, K and L, as implicit functions, say $K = \varphi(w_K, w_L)$ and $L = \psi(w_K, w_L)$, of the exogenous market prices, w_K and w_L.[6] Through a brief procedure the text then finds formulas for the partial derivatives of these two implicit functions. What is the logic behind the procedure?

[6] In the text, p. 38, the two unknowns, interpreted as factor demands, are denoted K^d and L^d, while the two implicit functions are denoted K and L instead of φ and ψ, respectively.

Let us consider the problem in more general terms. We have two equations of the form

$$f(x,y) = z, \tag{12}$$

$$g(x,y) = u, \tag{13}$$

where f and g are functions of (x,y) in an open set S in $\mathbb{R}^2$ with range equal to some intervals, I and J, respectively, in $\mathbb{R}^1$. Moreover, f and g are C^1 in S.

Suppose the point (x_0, y_0, z_0, u_0) in the interior of $S \times I \times J$ satisfies the equations (12) and (13) and that the Jacobian determinant D in (9), evaluated at (x_0, y_0), is not 0. Applied to this setup, the *implicit function theorem* says that there exists a neighborhood B_1 of (z_0, u_0) and a neighborhood B_2 of (x_0, y_0) such that for each (z, u) in B_1 there is a unique (x, y) in B_2 satisfying (12) and (13). In this way x and y are defined implicitly as C^1 functions of z and u which we denote $\varphi(z,u)$ and $\psi(z,u)$. By the implicit function theorem, the partial derivatives of these functions can be directly read off from (10) and (11). We find $\varphi_1'(z,u)$, by inserting $du = 0$ into (10) and rearrange:

$$\varphi_1'(z,u) = \frac{\partial x}{\partial z}(\equiv \lim_{dz\to 0} \frac{dx}{dz}_{|du=0}) = \frac{g_2'(x,y)}{D}.$$

We find $\varphi_2'(z,u)$, by inserting $dz = 0$ into (10) and rearrange:

$$\varphi_2'(z,u) = \frac{\partial x}{\partial u}(\equiv \lim_{du\to 0} \frac{dx}{du}_{|dz=0}) = \frac{-f_2'(x,y)}{D}.$$

In an analogue way we can read off $\psi_1'(z,u)$ and $\psi_2'(z,u)$ from (11):

$$\psi_1'(z,u) = \frac{\partial y}{\partial z}(\equiv \lim_{dz\to 0} \frac{dy}{dz}_{|du=0}) = \frac{-g_1'(x,y)}{D},$$

$$\psi_2'(z,u) = \frac{\partial y}{\partial u}(\equiv \lim_{du\to 0} \frac{dy}{dz}_{|dz=0}) = \frac{f_1'(x,y)}{D}.$$

This illustrates the convenience of using differentials.

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