Chapter 6

Long-run aspects of fiscal policy and public debt

We consider an economy with a government providing public goods and services. It finances its spending by taxation and borrowing. The term *fiscal policy* refers to the government's decisions about spending and the financing of this spending, be it by taxes or debt issue. The government's choice concerning the level and composition of its spending and how to finance it, may aim at:

1 affecting resource allocation (provide public goods that would otherwise not be supplied in a sufficient amount, correct externalities and other markets failures, prevent monopoly inefficiencies, provide social insurance);

2 affecting income distribution, be it a) within generations or b) between generations;

3 contribute to macroeconomic stabilization (dampening of business cycle fluctuations through aggregate demand policies).

The design of fiscal policy with regard to the aims 1 and 2 at a disaggregate level is a major theme within the field of public economics. Macroeconomics studies ways of dealing with aim 3 as well as big-picture aspects of 1 and 2, like overall policies to maintain and promote sustainable prosperity.

In this chapter we address fiscal sustainability and long-run implications of debt finance. This relates to one of the conditions that constrain public financing instruments. To see the issue of fiscal sustainability in a broader context, Section 6.1 provides an overview of conditions and factors that constrain public financing instruments. Section 6.2 introduces the basics of government budgeting and Section 6.3 defines the concepts of *government solvency* and *fiscal sustainability.* In Section 6.4 the analytics of debt dynamics is presented. As an example, the

Stability and Growth Pact of the EMU (the Economic and Monetary Union of the European Union) is discussed. Section 6.5 looks more closely at the link between government solvency and the government's *No-Ponzi-Game condition* and *intertemporal budget constraint.* In Section 6.6 we widen public sector accounting by introducing separate operating and capital budgets so as to allow for proper accounting of public investment. A theoretical claim, known as the *Ricardian equivalence* proposition, is studied in Section 6.7. The question whether Ricardian equivalence is likely to be a good approximation to reality, is addressed, applying the Diamond OLG framework extended with a public sector.

6.1 An overview of government spending and financing issues

Before entering the more specialized sections, it is useful to have a general idea about circumstances that condition public spending and financing. These circumstances include:

(i) financing by debt issue is constrained by the need to remain solvent and avoid catastrophic debt dynamics;

(ii) financing by taxes is limited by problems arising from:

- (a) distortionary supply-side effects of many kinds of taxes;
- (b) tax evasion (cf. the rise of the shadow economy, tax havens used by multinationals, etc.).

(iii) time lags in spending as well as taxing may interfere with attempts to stabilize the economy (recognition lag, decision lag, implementation lag, and effect lag);

(iv) credibility problems due to time-inconsistency;

(v) conditions imposed by political processes, bureaucratic self-interest, lobbying, and rent seeking.

Point (i) is the main focus of sections 6.2-6.6. Point (ii) is briefly considered in Section 6.4.1 in connection with the *Laffer curve.* In Section 6.6 point (iii) is briefly commented on. The remaining points, (iv) - (v), are not addressed specifically in this chapter. They should always be kept in mind, however, when discussing fiscal policy in practice. Hence some remarks at the end of the chapter.

Now to the specifics of government budget accounting and debt financing.

6.2 The government budget

We generally perceive the *public sector* (or the *nation state*) as consisting of the *national government* and a *central bank.* In economics the term "government" does not generally refer to the particular administration in office at a point in time. The term is rather used in a broad sense, encompassing both legislation and central and local administration. The aspects of legislation and administration in focus in macroeconomics are the rules and decisions concerning spending on public consumption, public investment, transfers, and subsidies on the expenditure side and on levying taxes and incurring debts on the financing side. Within certain limits the national government has usually delegated the management of the nation's currency to the central bank, a separate governmental institution, often called the monetary authority. Yet, from an overall macroeconomic point of view it is useful to treat "government budgeting" as covering the public sector as a whole: the consolidated government and central bank. Government bonds held by the central bank are thus excluded from what we call "government debt".

The basics of government budget accounting cannot be described without including money, nominal prices, and inflation. Elementary aspects of money and inflation will therefore be included in this section. We shall not, however, consider money and inflation in any systematic way until later chapters. Whether the economy considered is a closed or open economy will generally not be important in this chapter. We use the terms *government debt* and *public debt* synonymously.

Table 6.1 lists key variables of government budgeting.

Table 6.1. List of main variable symbols

Symbol	*Meaning*
Y_t	real GDP (= real GNP if the economy is closed)
C_t^g	public consumption
I_t^g	public fixed capital investment
G_t	$\equiv C_t^g + I_t^g$ real public purchases (spending on goods and services)
X_t	real transfer payments
$\tilde{T}_t$	real gross tax revenue
T_t	$\equiv \tilde{T}_t - X_t$ real net tax revenue
M_t	the monetary base (currency and bank reserves in the central bank)
P_t	price level (in money) for goods and services (the GDP deflator)
D_t	nominal net public debt (including possible debt of local government)
B_t	$\equiv \frac{D_t}{P_{t-1}}$ real net public debt
b_t	$\equiv \frac{B_t}{Y_t}$ government debt-to-income ratio
i_t	nominal short-term interest rate
Δx_t	$\equiv x_t - x_{t-1}$ (where x is some arbitrary variable)
π_t	$\equiv \frac{\Delta P_t}{P_{t-1}} \equiv \frac{P_t - P_{t-1}}{P_{t-1}}$ inflation rate
$1 + r_t$	$\equiv \frac{P_{t-1}(1+i_t)}{P_t} \equiv \frac{1+i_t}{1+\pi_t}$ real short-term interest rate

Note that Y_t, G_t, and T_t are quantities defined *per period*, or more generally, *per time unit,* and are thus flow variables. On the other hand, M_t, D_t, and B_t are stock variables, that is, quantities defined at a given point in time, here at the *beginning* of period t. We measure D_t and B_t *net* of financial claims held by the government. Almost all countries have positive government net debt, but in principle $D_t < 0$ is possible.[1] The monetary base, M_t, is currency plus fully liquid deposits in the central bank held by the private sector at the beginning of period t; M_t is by definition nonnegative.

Until further notice, we shall in this chapter ignore uncertainty and default risk. We shall also ignore the fact that government bonds are usually more liquid (easier to quickly convert into cash) than other financial assets. Under these circumstances the market interest rate on government bonds must be the same as that on other interest-bearing assets. There is thus only one interest rate, i_t, in the economy. For ease of exposition we imagine that all government bonds are *one-period bonds.* That is, each government bond promises a payout equal to one unit of account at the end of the period and then the bond expires. Given the interest rate, i_t, the market value of a bond at the start of period t is $v_t = 1/(1 + i_t)$. If the number of outstanding bonds (the quantity of bonds) in

[1] If $D_t < 0$, the government has positive net financial claims on the private sector and earns interest on these claims − which is then an additional source of government revenue besides taxation.

period t is q_t, the government debt has face value (value at maturity) equal to q_t. The market value at the start of period t of this quantity of bonds will be $D_t \equiv q_t v_t = q_t/(1+i_t)$. The nominal expenditure to be made at the end of the period to redeem the outstanding debt can then be written

$$q_t = D_t(1+i_t). \tag{6.1}$$

This is the usual way of writing the expenditure to be made, namely as *if* the government debt were like a given bank loan of size D_t with a variable rate of interest. We should not forget, however, that given the quantity, q_t, of the bonds, the value, D_t, of the government debt at the issue date depends negatively on i_t.

Anyway, the total nominal government expenditure in period t can be written

$$P_t(G_t + X_t) + D_t(1+i_t).$$

It is common to refer to this expression as expenditure "in period t". Yet, in a discrete time model (with a period length of a year or a quarter corresponding to typical macroeconomic data) one has to imagine that the *payment* for goods and services delivered in the period occurs either at the beginning or the end of the period. We follow the latter interpretation and so the nominal price level P_t for period-t goods and services refers to payment occurring at the *end* of period t. As an implication, the real value, B_t, of government debt at the beginning of period t (= end of period $t-1$) is D_t/P_{t-1}. This may look a little awkward but is nevertheless meaningful. Indeed, D_t is a *stock* of liabilities at the beginning of period t while P_{t-1} is a price referring to a flow *paid for* at the *end* of period $t-1$ which is essentially the same point in time as the beginning of period t. Anyway, whatever timing convention is chosen, some kind of awkwardness will always arise in discrete time analysis. This is because the discrete time approach artificially treats the continuous flow of time as a sequence of discrete points in time.[2]

The government's expenditure is financed, effectively, by a combination of taxes, bonds issue, and increase in the monetary base:

$$P_t\tilde{T}_t + D_{t+1} + \Delta M_{t+1} = P_t(G_t + X_t) + D_t(1+i_t). \tag{6.2}$$

By rearranging we have

$$\Delta D_{t+1} + \Delta M_{t+1} = P_t(G_t + X_t - \tilde{T}_t) + i_t D_t. \tag{6.3}$$

Although in many developed countries the central bank is prohibited from buying government bonds directly from the government, it may buy them from private

[2] In a theoretical model this kind of problems is avoided when government budgeting is formulated in continuous time, cf. Chapter 13.

entities shortly after these have bought them from the government. Over the year the newly issued government debt may thus be more or less "monetarized".

In customary government budget accounting the nominal *government budget deficit, GBD*, is defined as the excess of total government spending over government revenue, $P\tilde{T}$. That is, according to this definition the right-hand side of (6.3) is the nominal budget deficit in period t, GBD_t. The first term on the right-hand side, $P_t(G_t + X_t - \tilde{T}_t)$, is named the nominal *primary budget deficit* (non-interest spending less taxes). The second term, i_tD_t, is called the nominal *debt service*. Similarly, $P_t(\tilde{T}_t - X_t - G_t)$ is called the nominal *primary budget surplus*. A negative value of a "deficit" thus amounts to a positive value of a corresponding "surplus", and a negative value of a "surplus" amounts to a positive value of a corresponding "deficit".

We immediately see that this accounting deviates from "normal" principles. Business companies typically have sharply separated capital and operating budgets. In contrast, the budget deficit defined above treats that part of G which represents government *net investment* as parallel to government consumption. Government net investment is attributed as an *expense* in a single year's account. According to "normal" principles it is only the *depreciation* on the public capital that should figure as an expense. Likewise, the above accounting does not consider that a part of D, or perhaps more than D, may be backed by the value of public physical capital. And if the government sells a physical asset to the private sector, the sale will appear as a reduction of the government budget deficit while in reality it is merely a conversion of an asset from a physical form to a financial form. The expense and asset aspects of government net investment are thus not properly dealt with in the standard public accounting.[3]

With the exception of Section 6.6 we will nevertheless stick to the traditional vocabulary. Where this might create logical difficulties, it helps to imagine that:

(a) all of G *is* public consumption, i.e., $G_t = C_t^g$ for all t;

(b) there is *no* public physical capital.

Now, from (6.2) and the definition $T_t \equiv \tilde{T}_t - X_t$ (net tax revenue) follows that

[3]Another anomaly is related to the fact that some countries, for instance Denmark, have large implicit government assets due to deferred taxes on the part of personal income invested in pension funds. If the government then decides to reverse the deferred taxation (as the Danish government did 2012 and 2014 to comply better with the 3%-deficit rule of the Stability and Growth Pact of the EMU), the official budget deficit is reduced. But essentially, all that has happened is that one government asset has been replaced by another.

real government debt at the beginning of period $t+1$ is:

$$\begin{aligned} B_{t+1} &\equiv \frac{D_{t+1}}{P_t} = G_t + X_t - \tilde{T}_t + (1+i_t)\frac{D_t}{P_t} - \frac{\Delta M_{t+1}}{P_t} \\ &= G_t - T_t + (1+i_t)\frac{D_t/P_{t-1}}{P_t/P_{t-1}} - \frac{\Delta M_{t+1}}{P_t} = G_t - T_t + \frac{1+i_t}{1+\pi_t}B_t - \frac{\Delta M_{t+1}}{P_t} \\ &\equiv (1+r_t)B_t + G_t - T_t - \frac{\Delta M_{t+1}}{P_t}. \end{aligned} \tag{6.4}$$

This is the law of motion of real government debt.

The last term, $\Delta M_{t+1}/P_t$, in (6.4) is *seigniorage,* i.e., public sector revenue obtained by issuing base money (ignoring the diminutive cost of printing money). To get a sense of this variable, suppose real output grows at the constant rate g_Y so that $Y_{t+1} = (1+g_Y)Y_t$. Then the public debt-to-income ratio can be written

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r_t}{1+g_Y}b_t + \frac{G_t - T_t}{(1+g_Y)Y_t} - \frac{\Delta M_{t+1}}{P_t(1+g_Y)Y_t}. \tag{6.5}$$

Apart from the growth-correcting factor, $(1+g_Y)^{-1}$, the last term is the seigniorage-income ratio,

$$\frac{\Delta M_{t+1}}{P_t Y_t} = \frac{\Delta M_{t+1}}{M_t}\frac{M_t}{P_t Y_t}.$$

If in the long run the base money growth rate, $\Delta M_{t+1}/M_t$, as well as the nominal interest rate (i.e., the opportunity cost of holding money) are constant, then the velocity of money and its inverse, the money-nominal income ratio, $M_t/(P_tY_t)$, are also likely to be roughly constant. So is, therefore, the seigniorage-income ratio.[4] For the more developed countries this ratio tends to be a fairly small number although not immaterial. For emerging economies with poor institutions for collecting taxes seigniorage matters more.[5]

The U.S. has a single monetary authority, the central bank, and a single fiscal authority, the treasury. The seigniorage created is immediately transferred from the first to the latter. The Eurozone has a single monetary authority but

[4] A reasonable money demand function is $M_t^d = P_tY_te^{-\alpha i}$, $\alpha > 0$, where i is the nominal interest rate. With clearing in the money market, we thus have $M_t/(P_tY_t) = e^{-\alpha i}$. In view of $1+i \equiv (1+r)(1+\pi)$, when r and π are constant, so is i and, thereby, $M_t/(P_tY_t)$.

[5] In the U.S. over the period 1909-1950s seigniorage fluctuated a lot and peaked 4 % of GDP in the 1930s and 3 % of GDP at the end of WW II. But over the period from the late 1960s to 1986 seigniorage fluctuated less around an average close to 0.5 %.of GDP (Walsh, 2003, p. 177). In Denmark seigniorage was around 0.2 % of GDP during the 1990s (*Kvartalsoversigt 4. kvartal 2000*, Danmarks Nationalbank). In Bolivia, up to the event of hyperinflation 1984-85, seigniorage reached 5 % of GDP and more than 50 % of government revenue (Sachs and Larrain, 1993).

multiple fiscal authorities, namely the treasuries of the member countries. The seigniorage created by the ECB is every year shared by the national central banks of the Eurozone countries in proportion to their equity share in the ECB. And the national central banks then transfer their share to the national treasuries. This makes up a ΔM_{t+1} term for the consolidated public sector of the individual Eurozone countries.

In monetary unions and countries with their own currency, government budget deficits are thus, from a macroeconomic point of view, generally financed both by debt creation and money creation, as envisioned by the above equations. Nonetheless, from now on, for simplicity, in this chapter we will predominantly ignore the seigniorage term in (6.5) and only occasionally refer to the modifications implied by taking it into account.

We thus proceed with the simple government accounting equation:

$$B_{t+1} - B_t = r_t B_t + G_t - T_t, \tag{DGBC}$$

where the right-hand side is the *real budget deficit.* This equation is often called the *dynamic government budget constraint* (or DGBC for short). It is in fact just an accounting identity conditional on $\Delta M = 0$. It says that if the real budget deficit is positive and there is essentially no financing by money creation, then the real public debt grows. We come closer to a *constraint* when combining (DGBC) with the requirement that the government stays *solvent.*

A terminological remark before proceeding: One is tempted to call the right-hand side of (DGBC) the *real* budget deficit. And there is nothing wrong with that as long as one keeps in mind that right-hand side of (DGBC) is *not* the same as the nominal budget deficit deflated by P_t. Indeed,

$$r_t B_t + G_t - T_t = \left(\frac{1+i_t}{1+\pi_t} - 1\right)\frac{D_t}{P_{t-1}} + G_t - T_t = \frac{i_t - \pi_t}{1+\pi_t}\frac{D_t}{P_{t-1}} + G_t - T_t = \frac{GBD_t - \pi_t D_t}{P_t},$$

by definition of the nominal budget deficit GBD_t. The reason that the term $\pi_t D_t$ is subtracted is that *inflation* curtails the increase in *real* debt, given the nominal interest rate

6.3 Government solvency and fiscal sustainability

To be *solvent* means being able to meet the financial commitments as they fall due. In practice this concept is closely related to the government's No-Ponzi-Game condition and intertemporal budget constraint (to which we return in Section 6.5), but at the theoretical level it is more fundamental.

We may view the public sector as an infinitely-lived agent in the sense that there is no last date where all public debt has to be repaid. Nevertheless, as we shall see, there tends to be stringent constraints on government debt creation in the long run.

6.3.1 The critical role of the growth-corrected interest factor

Very much depends on whether the real interest rate in the long-run is higher than the growth rate of GDP or not.

To see this, suppose the country considered has positive government debt at time 0 and that the government levies taxes equal to its non-interest spending:

$$\tilde{T}_t = G_t + X_t \quad \text{or} \quad T_t \equiv \tilde{T}_t - X_t = G_t \text{ for all } t \geq 0. \tag{6.6}$$

So taxes cover only the primary expenses while interest payments (and debt repayments when necessary) are financed by issuing new debt. That is, the government attempts a permanent *roll-over* of the debt including the interest due for payment. In view of (DGBC), this implies that $B_{t+1} = (1 + r_t)B_t$, saying that the debt grows at the rate r_t. Assuming, for simplicity, that $r_t = r$ (a given constant), the law of motion for the public debt-to-income ratio is

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y}\frac{B_t}{Y_t} \equiv \frac{1+r}{1+g_Y} b_t, \qquad b_0 > 0,$$

where we have maintained the assumption of a constant output growth rate, g_Y. The solution to this linear difference equation then becomes

$$b_t = b_0 (\frac{1+r}{1+g_Y})^t,$$

where we consider both r and g_Y as exogenous. We see that the growth-corrected interest rate, $\frac{1+r}{1+g_Y} - 1 \approx r - g_Y$ (for g_Y and r "small") plays a key role. There are contrasting cases to discuss.

Case 1: $r > g_Y$. In this case, $b_t \to \infty$ for $t \to \infty$. Owing to compound interest, the debt grows so large in the long run that the government will be unable to find buyers for the newly issued debt. Permanent debt roll-over is thus not feasible. Imagine for example an economy described by the Diamond OLG model. Here the buyers of the debt are the young who place part of their saving in government bonds. But if the stock of these bonds grows at a higher rate than income, the saving of the young cannot in the long run keep track with the fast-growing government debt. In this situation the private sector will understand that bankruptcy is threatening and nobody will buy government bonds except at a low price, which means a high interest rate. The high interest rate only aggravates the problem. That is, the fiscal policy (6.6) breaks down. Either the government defaults on the debt or T must be increased or G decreased (or both) until the growth rate of the debt is no longer higher than g_Y.

If the debt is denominated in the country's own currency, an alternative way out is of course a shift to money financing of the budget deficit, that is, seigniorage. When capacity utilization is high, this leads to rising inflation and thus the real value of the debt is eroded. Bond holders will then demand a higher nominal interest rate, thus aggravating the fiscal difficulties. The economic and social chaos of *hyperinflation* threatens.[6] The hyperinflation in Germany 1922-23 peaked in Nov. 1923 at 29,525% per month; it eroded the real value of the huge government debt of Germany after WW I by 95 percent.

Case 2: $r = g_Y$. If $r = g_Y$, we get $b_t = b_0$ for all $t \geq 0$. Since the debt, increasing at the rate r, does not increase faster than national income, the government has no problem finding buyers of its newly issued bonds − the government stays solvent. Thereby the government is able to finance its interest payments simply by issuing new debt. The growing debt is passed on to ever new generations with higher income and saving and the debt roll-over implied by (6.6) can continue forever.

Case 3: $r < g_Y$. Here we get $b_t \to 0$ for $t \to \infty$, and the same conclusion holds *a fortiori*.

In Case 2 as well as Case 3, where the interest rate is not higher than the growth rate of the economy, the government can thus pursue a permanent debt roll-over policy as implied by (6.6) and still remain solvent. But in Case 1, permanent debt roll-over is impossible and sooner or later the interest payments must be tax financed.

Which of the cases is relevant in real life? Fig. 6.1 shows for Denmark (upper panel) and the US (lower panel) the time paths of the real short-term interest rate and the GDP growth rate, both on an annual basis. Overall, the levels of the two are more or less the same, although on average the interest rate is in Denmark slightly higher but in the US somewhat lower than the growth rate. (Note that the interest rates referred to are not the average rate of return in the economy but a proxy for the lower interest rate on government bonds.)

Nevertheless, many macroeconomists believe there is good reason for paying attention to the case $r > g_Y$, also for a country like the US. This is because we live

[6] In economists' standard terminology "hyperinflation" is present when the inflation rate exceeds 50 percent *per month*. As we shall see in Chapter 18, the monetary financing route comes to a dead end if the needed seigniorage reaches the backward-bending part of the "seigniorage Laffer curve".

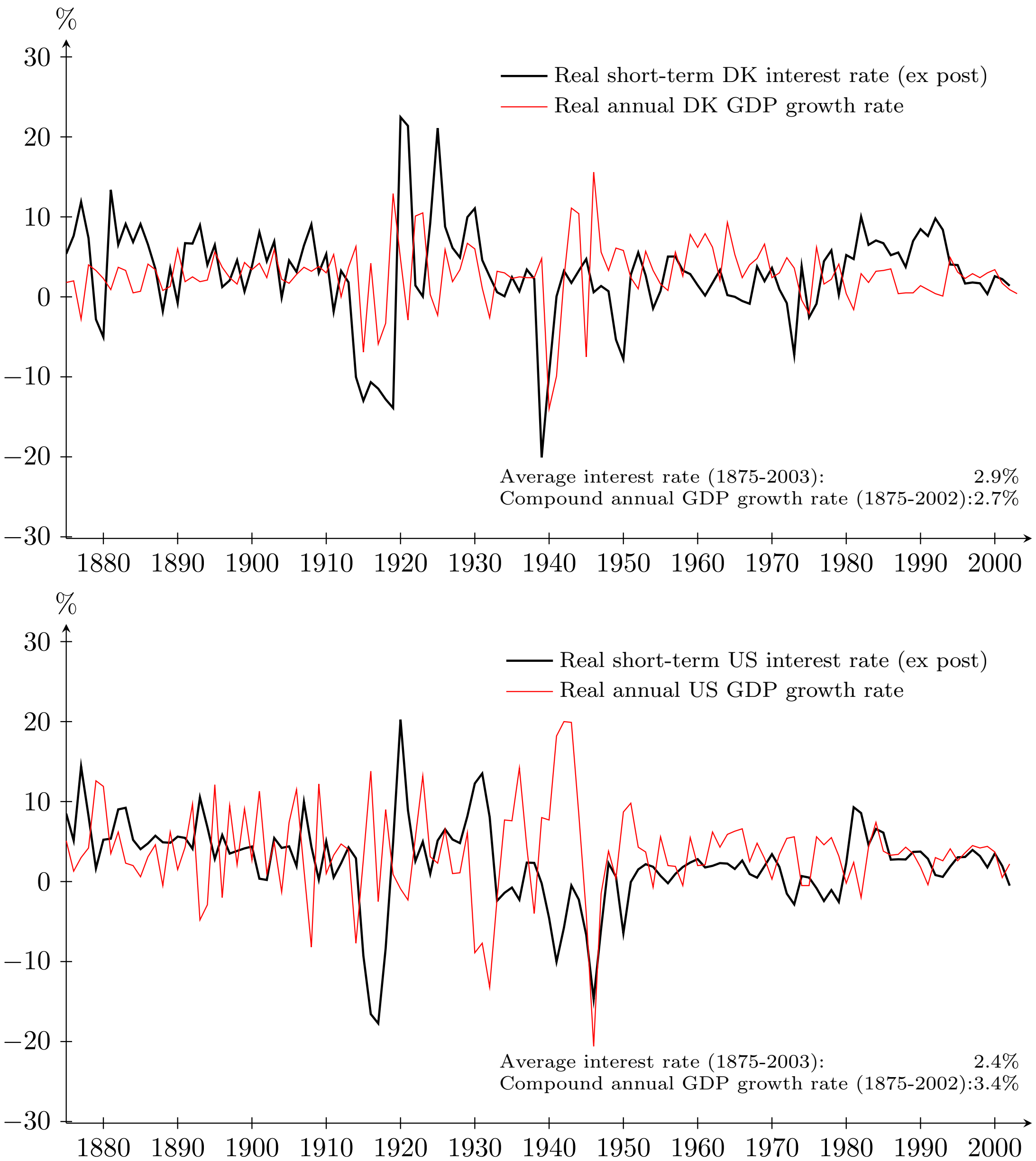

Figure 6.1: Real short-term interest rate and annual growth rate of real GDP in Denmark and the US since 1875. The real short-term interest rate is calculated as the money market rate minus the contemporaneous rate of consumer price inflation. Source: Abildgren (2005) and Maddison (2003).

in a world of *uncertainty,* with many different interest rates, and imperfect credit markets, aspects the above line of reasoning has not incorporated. The prudent debt policy needed whenever, under certainty, $r > g_Y$ can be shown to apply to a larger range of circumstances when uncertainty is present (see Literature notes). To give a flavor we may say that a prudent debt policy is needed when the average interest rate on the public debt exceeds $g_Y - \varepsilon$ for some "small" but positive ε.[7] On the other hand there is a different feature which draws the matter in the opposite direction. This is the possibility that a tax, $\tau \in (0, 1)$, on interest income is in force so that the net interest rate on the government debt is $(1 - \tau)r$ rather than r.

6.3.2 Sustainable fiscal policy

The concept of sustainable fiscal policy is closely related to the concept of government solvency. As already noted, to be *solvent* means being able to meet the financial commitments as they fall due. A given fiscal policy is called *sustainable* if by applying its spending and tax rules forever, the government stays solvent. "Sustainable" conveys the intuitive meaning. The issue is: can the current tax and spending rules continue forever?

To be more specific, suppose G_t and T_t are determined by fiscal policy rules represented by the functions

$$G_t = \mathcal{G}(x_{1t}, ..., x_{nt}, t), \quad \text{and} \quad T_t = \mathcal{T}(x_{1t}, ..., x_{nt}, t),$$

where $t = 0, 1, 2, \ldots,$ and x_{1t},..., x_{nt} are key macroeconomic and demographic variables (like national income, old-age dependency ratio, rate of unemployment, extraction of natural resources, say oil from the North Sea, etc.). In this way a given fiscal policy is characterized by the rules $\mathcal{G}(\cdot)$ and $\mathcal{T}(\cdot)$. Suppose further that we have an economic model, $\mathcal{M}$, of how the economy functions.

DEFINITION Let the current period be period 0 and let the public debt at the beginning of period 0 be given. Then, given a forecast of the evolution of the demographic and foreign economic environment in the future and given the economic model $\mathcal{M}$, the fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is said to be *sustainable* relative to this model if the forecast calculated on the basis of $\mathcal{M}$ is that the government stays solvent under this policy. The fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is called *unsustainable*, if it is not sustainable.

This definition of fiscal sustainability is silent about the presence of uncertainty. Without going into detail about this difficult issue, suppose the model $\mathcal{M}$ is stochastic and let ε be a "small" positive number. Then we may say that

[7] This is only a "rough" characterization, see, e.g., Blanchard and Weil (2001).

the fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ with 100-ε percent probability is *sustainable* relative to the model $\mathcal{M}$ if the forecast calculated on the basis of $\mathcal{M}$ is that with 100-ε percent probability the government stays solvent under this policy.

Governments, rating agencies, and other institutions evaluate sustainability of fiscal policy on the basis of simulations of giant macroeconometric models. Essentially, the operational criterion for sustainability is whether the fiscal policy can be deemed compatible with upward boundedness of the public debt-to-income ratio. Normally, the income measure applied here is GDP. Other measures are conceivable such as GNP, taxable income, or after-tax income. Moreover, even if a debt spiral is not (yet) underway in a given country, a high *level* of the debt-income ratio may in itself be worrisome. This is because a high level of debt under certain conditions may trigger a spiral of self-fulfilling expectations of default. We come back to this in the section to follow.

Owing to the increasing pressure on public finances caused by factors such as reduced birth rates, increased life expectancy, and a fast-growing demand for medical care, many industrialized countries have for a long time been assessed to be in a situation where their fiscal policy is not sustainable (Elmendorf and Mankiw 1999). The implication is that sooner or later one or more expenditure rules and/or tax rules (in a broad sense) will probably have to be changed.

Two major kinds of strategies have been suggested. One kind of strategy is the *pre-funding strategy.* The idea is to prevent sharp future tax increases by ensuring a fiscal consolidation prior to the expected future demographic changes. Another strategy (alternative or complementary to the former) is to attempt a gradual increase in the labor force by letting the age limits for retirement and pension increase along with expected lifetime − this is the *indexed retirement strategy.* The first strategy implies that current generations bear a large part of the adjustment cost. In the second strategy the costs are shared by current and future generations in a way more similar to the way the benefits in the form of increasing life expectancy are shared. We shall not go into detail about these matters here, but refer the reader to a large literature about securing fiscal sustainability in the ageing society, see Literature notes.

6.4 Debt arithmetic

A key tool for evaluating fiscal sustainability is *debt arithmetic*, i.e., the analytics of debt dynamics. The previous section described the important role of the growth-corrected interest rate. The next subsection considers the minimum primary budget surplus required for fiscal sustainability in different situations.

6.4.1 The required primary budget surplus

Ignoring the seigniorage term $\Delta M_{t+1}/P_t$ in the dynamic government budget identity (6.4) and assuming a constant interest rate r, we have:

$$B_{t+1} = (1+r)B_t - (T_t - G_t), \qquad \text{(DGBC)}$$

where $T_t - G_t \equiv \tilde{T}_t - X_t - G_t$ is the primary budget surplus in real terms. Suppose aggregate income, Y_t, grows at a given constant rate g_Y. Let the spending-to-income ratio, G_t/Y_t, and the (net) tax revenue-to-income ratio, T_t/Y_t, be constants, γ and τ, respectively. We assume that interest income on government bonds is not taxed. It follows that the public debt-to-income ratio $b_t \equiv B_t/Y_t$ (from now just denoted debt-income ratio) changes over time according to

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y} b_t - \frac{\tau - \gamma}{1+g_Y}, \qquad (6.7)$$

where we have assumed a constant interest rate, r. There are (again) three cases to consider.

Case 1: $r > g_Y$. As emphasized above this case is generally considered the one of most practical relevance. And it is in this case that *latent debt instability* is present and the government has to pay attention to the danger of runaway debt dynamics. To see this, note that the solution of the linear difference equation (6.7) is

$$b_t = (b_0 - b^*)\left(\frac{1+r}{1+g_Y}\right)^t + b^*, \quad \text{where} \qquad (6.8)$$

$$b^* = -\frac{\tau-\gamma}{1+g_Y}\left(1 - \frac{1+r}{1+g_Y}\right)^{-1} = \frac{\tau-\gamma}{r-g_Y} \equiv \frac{s}{r-g_Y}, \qquad (6.9)$$

where s is the *primary surplus as a share of GDP*. Here b_0 is historically given. But the steady-state debt-income ratio, b^*, depends on fiscal policy. The important feature is that the growth-corrected interest factor is in this case higher than 1 and has the exponent t. Therefore, if fiscal policy is such that $b^* < b_0$, the debt-income ratio exhibits geometric growth. The solid curve in the topmost panel in Fig. 6.2 shows a case where fiscal policy is such that $\tau - \gamma < (r - g_Y)b_0$ whereby we get $b^* < b_0$ when $r > g_Y$, so that the debt-income ratio, b_t, grows without bound. This reflects that with $r > g_Y$, compound interest is stronger than compound growth. The sequence of discrete points implied by our discrete-time model is in the figure smoothed out as a continuous curve.

The American economist and Nobel Prize laureate George Akerlof (2004, p. 6) came up with this analogy:

"It takes some time after running off the cliff before you begin to fall. But the law of gravity works, and that fall is a certainty".

Somewhat surprisingly, perhaps, when $r > g_Y$, there can be debt explosion in the long run even if $\tau > \gamma$, namely if $0 < \tau - \gamma < (r - g_Y)b_0$. Debt explosion can also arise if $b_0 < 0$, namely if $\tau - \gamma < (r - g_Y)b_0 < 0$.

The only way to avoid the snowball effects of compound interest when the growth-corrected interest rate is positive is to ensure a primary budget surplus as a share of GDP, $\tau - \gamma$, high enough such that $b^* \geq b_0$. So the *minimum* primary surplus as a share of GDP, $\hat{s}$, required for fiscal sustainability is the one implying $b^* = b_0$, i.e., by (6.9),

$$\hat{s} = (r - g_Y)b_0. \tag{6.10}$$

If by adjusting τ and/or γ, the government obtains $\tau - \gamma = \hat{s}$, then $b^* = b_0$ whereby $b_t = b_0$ for all $t \geq 0$ according to (6.8), cf. the second from the top panel in Fig. 6.2. The difference between $\hat{s}$ and the actual primary surplus as a share of GDP is named the *primary surplus gap* or the *sustainability gap*.

Note that $\hat{s}$ will be larger:

- the higher is the initial level of debt, b_0; and,
- when $b_0 > 0$, the higher is the growth-corrected interest rate, $r - g_Y$.

Delaying the adjustment increases the size of the needed policy action, since the debt-income ratio, and thereby $\hat{s}$, will become higher in the meantime.

For fixed spending-income ratio γ, the minimum tax-to-income ratio needed for fiscal sustainability is

$$\hat{\tau} = \gamma + (r - g_Y)b_0. \tag{6.11}$$

Given b_0 and γ, this tax-to-income ratio is sometimes called the *sustainable tax rate*. The difference between this rate and the actual tax rate, τ, indicates the size of the needed tax adjustment, were it to take place at time 0, assuming a given γ.

Suppose that the debt build-up can be − and is − prevented already at time 0 by ensuring that the primary surplus as a share of income, $\tau - \gamma$, at least equals $\hat{s}$ so that $b^* \geq b_0$. The solid curve in the midmost panel in Fig. 6.2 illustrates the resulting evolution of the debt-income ratio if b^* is at the level corresponding to the hatched horizontal line while b_0 is unchanged compared with the top panel. Presumably, the government would in such a state of affairs relax its fiscal policy after a while in order not to accumulate large government financial net wealth. Yet, the pre-funding strategy vis-a-vis the fiscal challenge of population ageing (referred to above) is in fact based on accumulating some positive public financial net wealth as a buffer before the substantial effects of population ageing set in. In

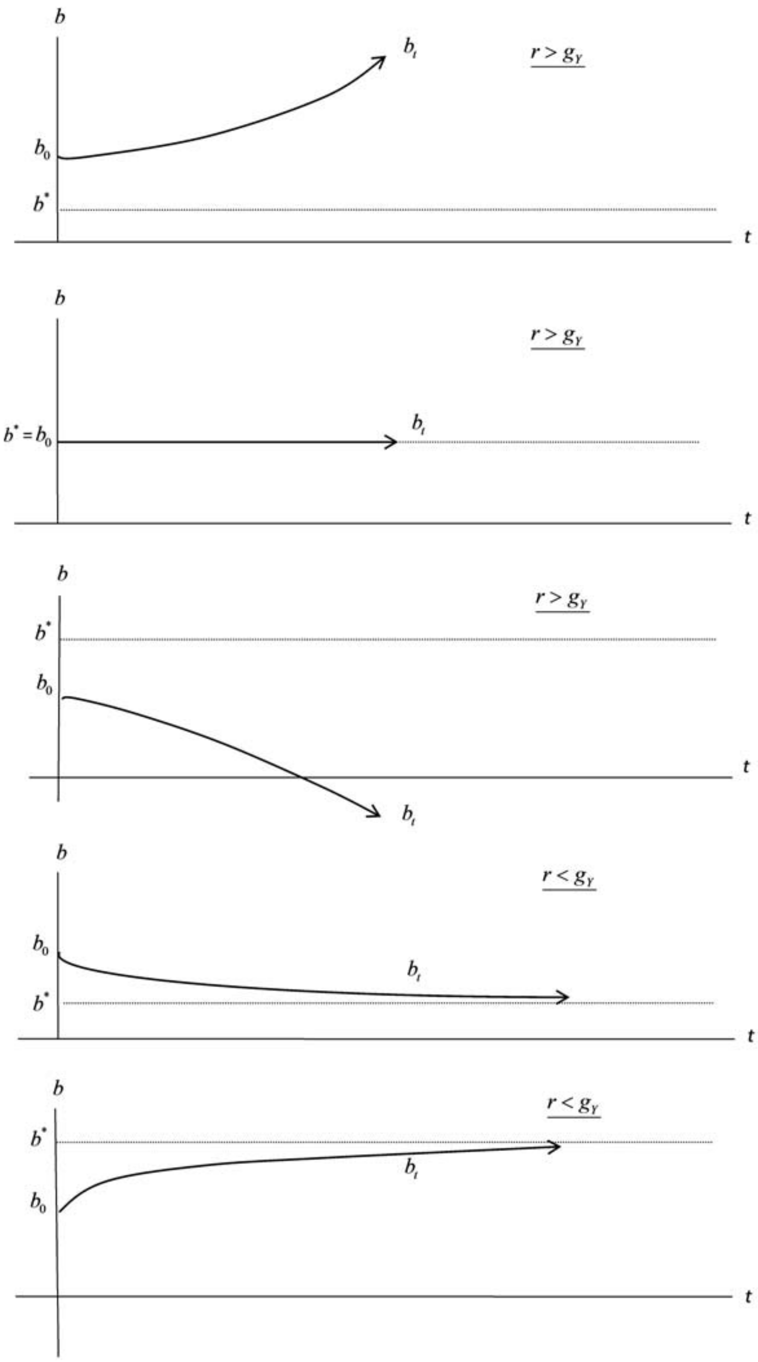

Figure 6.2: Evolution of the debt-income ratio, depending on the sign of $b_0 - b^*$, in the cases $r > g_Y$ (the three upper panels) and $r < g_Y$ (the two lower panels), respectively.

this context, the higher the growth-corrected interest rate, the shorter the time needed to reach a given positive net wealth position.

Case 2: $r = g_Y$. In this knife-edge case there is still a danger of runaway dynamics, but of a less explosive form. The formula (6.8) is no longer valid. Instead the solution of (6.7) is $b_t = b_0 + [(\gamma - \tau)/(1 + g_Y)]\, t = b_0 - [(\tau - \gamma)/(1 + g_Y)]\, t$. Here, a non-negative primary surplus is both necessary and sufficient to avoid $b_t \to \infty$ for $t \to \infty$.

Case 3: $r < g_Y$. This is the case of stable debt dynamics. The formula (6.8) is again valid, but now implying that the debt-income ratio is non-explosive. Indeed, $b_t \to b^*$ for $t \to \infty$, whatever the level of the initial debt-income ratio and whatever the sign of the budget surplus. Moreover, when $r < g_Y$,

$$b^* = \frac{\tau - \gamma}{r - g_Y} \lesseqgtr 0 \text{ for } \tau - \gamma \gtreqless 0. \qquad (*)$$

So, if there is a forever positive primary surplus, the result is a negative long-run debt, i.e., a positive government financial net wealth in the long run. And if there is a forever negative primary surplus, the result is not debt explosion but just convergence toward some positive long-run debt-income ratio. The second from bottom panel in Fig. 6.2 illustrates this case for a situation where $b_0 > b^*$ and $b^* > 0$, i.e., $\tau - \gamma < 0$, by (*). When the GDP growth rate continues to exceed the interest rate on government debt, a large debt-income ratio can be brought down quite fast, as witnessed by the evolution of both UK and US government debt in the first three decades after the second world war. Indeed, if the growth-corrected interest rate remains negative, permanent debt roll-over can handle the financing, and taxes need never be levied.[8]

Finally, the bottom panel in Fig. 6.2 shows the case where, with a *large* primary deficit ($\tau - \gamma < 0$ but large in absolute value), excess of output growth over the interest rate still implies convergence towards a constant debt-income ratio, albeit a high one.

In this discussion we have treated r as exogenous. But r may to some extent be dependent on prolonged budget deficits. Indeed, in Chapter 13 we shall see that with prolonged budget deficits, r tends to become higher than otherwise. Everything else equal, this reduces the likelihood of Case 2 and Case 3.

Laffer curve*

We return to Case 1 because we have ignored supply-side effects of taxation, and such effects could be important in Case 1.

[8]On the other hand, we should not forget that this analysis presupposes absence of uncertainty. As touched on in Section 6.3.1, in the presence of uncertainty and therefore existence of many interest rates, the issue becomes more complicated.

A *Laffer curve* (so named after the American economist Arthur Laffer, 1940-) refers to a hump-shaped relationship between the income tax rate and the tax revenue. For simplicity, suppose the (gross) tax revenue equals taxable income times a given average tax rate. A 0% tax rate and likely also a 100% tax rate generate no tax revenue. As the tax rate increases from a low initial level, a rising tax revenue is obtained. But after a certain point some people may begin to work less (in the legal economy), stop reporting all their income, and stop investing. So it is reasonable to think of a tax rate above which the tax revenue begins to decline.

While Laffer was wrong about where USA was "on the curve" (see, e.g., Fullerton 2008), and while, strictly speaking, there is no such thing as *the* Laffer curve and *the* tax rate,[9] Laffer's intuition is hardly controversial. Ignoring, for simplicity, transfers, we therefore now assume that for a *given* tax *system* there is a gross tax-income ratio, τ_L, above which the tax revenue declines. Then, if the presumed sustainable tax-income ratio, $\hat{\tau}$, in (6.11) exceeds τ_L, the tax revenue aimed at can not be realized.

To see what the value of τ_L could be, suppose aggregate taxable income before tax is a function, φ, of the net-of-tax share $1-\tau$. Then tax revenue is

$$\tilde{T} = \tau \cdot \varphi(1-\tau) \equiv R(\tau) \ ,$$

which we assume is a hump-shaped function of τ in the interval $[0,1]$. Taking logs and differentiating w.r.t. τ gives the first-order condition $R'(\tau)/R(\tau) = 1/\tau - \varphi'(1-\tau)/\varphi(1-\tau) = 0$, which holds for $\tau = \tau_L$, the tax-income ratio that maximizes R. It follows that $1/\tau_L = \varphi'(1-\tau_L)/\varphi(1-\tau_L)$, hence

$$\frac{1-\tau_L}{\tau_L} = \frac{1-\tau_L}{\varphi(1-\tau_L)}\varphi'(1-\tau_L) \equiv \mathrm{E}\ell_{1-\tau}\varphi(1-\tau_L).$$

Rearranging gives

$$\tau_L = \frac{1}{1+\mathrm{E}\ell_{1-\tau}\varphi(1-\tau_L)}.$$

If the elasticity of income w.r.t. $1-\tau$ is given as 0.4,[10] we get $\tau_L \simeq 0.7$. Thus, if the required tax-income ratio, $\hat{\tau}$, calculated on the basis of (6.11) (under the simplifying assumption of no transfers), exceeds 0.7, fiscal sustainability can not be obtained by just raising taxation.

[9] A lot of contingencies are involved: income taxes are typically progressive (i.e., average tax rates rise with income); it matters whether a part of tax revenue is spent to reduce tax evasion, etc.

[10] As suggested for the U.S. by Gruber and Saez (2002).

The level of the debt-income ratio and self-fulfilling expectations of default

We again consider Case 1: $r > g_Y$. As incumbent chief economist at the IMF, Olivier Blanchard remarked in the midst of the 2010-2012 debt crisis in the Eurozone:

> "The higher the level of debt, the smaller is the distance between solvency and default".[11]

The background for this remark is the following. There is likely to be an upper bound for the tax-income ratio deemed politically or economically feasible by the government as well as the market participants. Similarly, a lower bound for the spending-income ratio is likely to exist, be it for economic or political reasons. In the present framework we therefore let the government face the constraints $\tau \leq \bar{\tau}$ and $\gamma \geq \bar{\gamma}$, where $\bar{\tau}$ is the least upper bound for the tax-income ratio and $\bar{\gamma}$ is the greatest lower bound for the spending-income ratio. We assume that $\bar{\tau} > \bar{\gamma}$. Then the actual primary surplus, s, can at most equal $\bar{s} \equiv \bar{\tau} - \bar{\gamma}$.

Suppose that at first the situation in the considered country is as in the second from the top panel in Fig. 6.2. That is, initially, $b_0 > 0$ and

$$s = \tau - \gamma = \hat{s} = (r - g_Y)b_0 \leq \bar{s} \equiv \bar{\tau} - \bar{\gamma}, \tag{6.12}$$

with $b_0 > 0$. Define $\bar{r}$ to be the value of r satisfying

$$(\bar{r} - g_Y)b_0 = \bar{s}, \text{ i.e., } \bar{r} = \frac{\bar{s}}{b_0} + g_Y. \tag{6.13}$$

Thereby $\bar{r}$ is the maximum level of the interest rate consistent with absence of an explosive debt-income ratio.

According to (6.12), fundamentals (tax- and spending-income ratios, growth-corrected interest rate, and initial debt) are consistent with absence of an explosive debt-income ratio as long as r is unchanged. Nevertheless, financial investors may be worried about default if b_0 is high. Investors are aware that a rise in the actual interest rate, r, can always happen and that if it does, a situation with $r > \bar{r}$ is looming, in particular if the country has high debt. The larger is b_0, the lower is the critical interest rate, $\bar{r}$, as witnessed by (6.13).

The worrying scenario is that the fear of default triggers a risk premium, and if the resulting level of the interest rate on the debt, say r', exceeds $\bar{r}$, unpleasant debt dynamics like that in the top panel of Fig. 6.2 set in. To r' corresponds a new value of the primary surplus, say $\hat{s}'$, defined by $\hat{s}' = (r' - g_Y)b_0$. So $\hat{s}'$ is the

[11] Blanchard (2011).

minimum primary surplus (as a share of GDP) required for a non-accelerating debt-income ratio in the new situation. With $b_0 > 0$ and $r' > \bar{r}$, we get

$$\hat{s}' = (r' - g_Y)b_0 > (\bar{r} - g_Y)b_0 = \bar{s},$$

where $\bar{s}$ is given in (6.12). The government could possibly increase its primary surplus, s, but at most up to $\bar{s}$, and this will not be enough since the required primary surplus, $\hat{s}'$, exceeds $\bar{s}$. The situation would be as illustrated in the top panel of Fig. 6.2 with b^* given as $\bar{s}/(r' - g_Y) < b_0$.

That is, *if* the actual interest rate should rise above the critical interest rate, $\bar{r}$, runaway debt dynamics would take off and debt default follow. A fear that it *may* happen may be enough to trigger a fall in the market price of government bonds which means a rise in the actual interest rate, r. So financial investors' fear can be a self-fulfilling prophesy. Moreover, as we saw in connection with (6.13), the risk that r becomes greater than $\bar{r}$ is larger the larger is b_0.

It is not so that across countries there is a common threshold value for a "too large" public debt-to-income ratio. This is because variables like $\bar{\tau}$, $\bar{\gamma}$, r, and g_Y, as well as the net foreign debt position and the current account deficit (not in focus in this chapter), *differ* across countries. Late 2010 Greece had (gross) government debt of 148 percent of GDP and the interest rate on 10-year government bonds skyrocketed. Conversely Japan had (gross) government debt of more than 200 percent of GDP while the interest rate on 10-year government bonds remained very low.

Finer shades

1. As we have just seen, even when in a longer-run perspective a solvency problem is unlikely, self-fulfilling expectations can here and now lead to default. Such a situation is known as a *liquidity crisis* rather than a true *solvency crisis.* In a liquidity crisis there is an acute problem of insufficient cash to pay the next bill on time ("cash-flow insolvency") because borrowing is difficult due to actual and potential creditors' *fear* of default. A liquidity crisis can be braked by the central bank stepping in and acting as a "lender of last resort" by printing money. In a country with its own currency, the central bank can do so and thereby prevent a bad self-fulfilling expectations equilibrium to unfold.[12]

[12]In a monetary union which is not also a fiscal union (think of the eurozone), the situation is more complicated. A single member country with large government debt (or large debt in commercial banks for that matter) may find itself in an acute liquidity crisis without its own means to solve it. Indeed, the elevation of interest rates on government bonds in the Southern part of the eurozone in 2010-2012 can be seen as a manifestation of investors' fear of the governments running into difficulties of paying their way. The elevation was not reversed until the European Central Bank in September 2012 declared its willingness to effectively act as a "lender of last resort" (on a conditional basis), see Box 6.2 in Section 6.4.2.

2. In the above analysis we simplified by assuming that several variables, including γ, τ, and r, are constants. The upward trend in the old-age dependency ratio, due to a decreased birth rate and rising life expectancy, together with a rising request for medical care is likely to generate upward pressure on γ. Thereby a high initial debt-income ratio becomes *more* challenging.

3. On the other hand, rB_t is income to the private sector and can be taxed at the same average tax rate τ as factor income, Y_t. Then the benign inequality is no longer $r \leq g_Y$ but $(1-\tau)r \leq g_Y$, which is more likely to hold. Taxing interest income is thus supportive of fiscal sustainability (cf. Exercise B.28).

4. Having ignored seigniorage, there is an upward bias in our measure (6.10) of the *minimum* primary surplus as a share of GDP, $\hat{s}$, required for fiscal sustainability when $r > g_Y$. Imposing stationarity of the debt-income ratio at the level $\bar{b}$ into the general debt-accumulation formula (6.5), multiplying through by $1 + g_Y$, and cancelling out, we find

$$\hat{s} = (r - g_Y)\bar{b} - \frac{\Delta M_{t+1}}{P_t Y_t} = (r - g_Y)\bar{b} - \frac{\Delta M_{t+1}}{M_t} \cdot \frac{M_t}{P_t Y_t}.$$

With $r = 0.04$, $g_Y = 0.03$, and $\bar{b} = 0.60$, we get $(r - g_Y)\bar{b} = 0.006$. With a seigniorage-income ratio even as small as 0.003, the "true" required primary surplus is 0.003 rather than 0.006. As long as the seigniorage-income ratio is approximately constant, our original formula, given in (6.10), for the required primary surplus as a share of GDP is in fact valid if we interpret τ as the (tax+seigniorage)-income ratio.

5. Having assumed a constant g_Y, we have ignored business cycle fluctuations. Allowing for booms and recessions, the *timing* of fiscal consolidation in a country with a structural primary surplus gap ($\hat{s} - s > 0$) becomes a crucial issue. The case study in the next section will be an opportunity to touch upon this issue.

6.4.2 Case study: The Stability and Growth Pact of the EMU

The European Union (EU) is approaching its aim of establishing a "single market" (unrestricted movement of goods and services, workers, and financial capital) across the territory of its member countries, 28 sovereign nations. Nineteen of these have joined the common currency, the euro. They constitute what is known as the Eurozone with the European Central Bank (ECB) as supranational institution responsible for conducting monetary policy in the Eurozone. The Eurozone countries as well as the nine EU countries outside the Eurozone (including UK, Denmark, Sweden, and Poland) are, with minor exceptions, required to abide with a set of *fiscal rules*, first formulated already in the Treaty of Maastrict from

1992. In that year a group of European countries decided a road map leading to the establishment of the euro in 1999 and a set of criteria for countries to join. These fiscal rules included a deficit rule as well as a debt rule. The *deficit rule* says that the annual nominal government budget deficit must not be above 3 percent of nominal GDP. The *debt rule* says that the government debt should not be above 60 percent of GDP. The fiscal rules were upheld and in minor respects tightened in the *Stability and Growth Pact* (SGP) which was implemented in 1997 as the key fiscal constituent of the Economic and Monetary Union (EMU). The latter name is a popular umbrella term for the fiscal and monetary legislation of the EU. The EU member countries that have adopted the euro are often referred to as "the full members of the EMU".

Some of the EU member states (Belgium, Italy, and Greece) had debt-income ratios above 100 percent since the early 1990s − and still have. Committing to the requirement of a gradual reduction of their debt-income ratios, they became full members of the EMU essentially from the beginning (that is, 1999 except Greece, 2001). The 60 percent debt rule of the SGP is to be understood as a long-run ceiling that, by the stock nature of debt, can not be accomplished here and now if the country is highly indebted.

The deficit and debt rules (with associated detailed contingencies and arrangements including ultimate pecuniary fines for defiance) are meant as discipline devices aiming at "sound budgetary policy", alternatively called "fiscal prudence". The motivation is protection of the ECB against political demands to loosen monetary policy in situations of fiscal distress. A fiscal crisis in one or more of the Eurozone countries, perhaps "too big to fail", could set in and entail a state of affairs approaching default on government debt and chaos in the banking sector with rising interest rates spreading to neighboring member countries (a negative externality). This could lead to open or concealed political pressure on the ECB to inflate away the real value of the debt, thus challenging the ECB's one and only concern with "price stability".[13] Or a fiscal crisis might at least result in demands on the ECB to curb soaring interest rates by purchasing government bonds from the country in trouble. In fact, such a scenario is close to what we have seen in southern Europe in the wake of the Great Recession triggered by the financial crisis starting 2007. Such "bailing out" could give governments incentives to be relaxed about deficits and debts (a "moral hazard" problem). And the lid on deficit spending imposed by the SGP should help to prevent needs for "bailing out" to arise.

[13] In recent years the ECB has interpreted "price stability" as a consumer price inflation rate "below, but close to, 2 percent per year over the medium term".

The link between the deficit and the debt rule

Whatever the virtues or vices of the design of the deficit and debt rules, one may ask the plain question: what is the arithmetical relationship, if any, between the 3 percent and 60 percent tenets?

First a remark about measurement. The measure of government debt, called the EMU debt, used in the SGP criterion is based on the book value of the financial liabilities rather than the market value. In addition, the EMU debt is more of a *gross* nature than the theoretical net debt measure represented by our D. The EMU debt measure allows fewer of the government financial assets to be subtracted from the government financial liabilities.[14] In our calculation and subsequent discussion we ignore these complications.

Consider a deficit rule saying that the (total) nominal budget deficit must never be above $\alpha \cdot 100$ percent of nominal GDP. By (6.3) with ΔM_{t+1} "small" enough to be ignored, this deficit rule is equivalent to the requirement

$$D_{t+1} - D_t = GBD_t = i_t D_t + P_t(G_t - T_t) \leq \alpha P_t Y_t. \tag{6.14}$$

In the SGP, $\alpha = 0.03$. Here we consider the general case: $\alpha > 0$. To see the implication for the (public) debt-to-income ratio in the long run, let us first imagine a situation where the deficit ceiling, α, is always *binding* for the economy we look at. Then $D_{t+1} = D_t + \alpha P_t Y_t$ and so

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} \equiv \frac{D_{t+1}}{P_t Y_{t+1}} = \frac{D_t}{(1+\pi)P_{t-1}(1+g_Y)Y_t} + \frac{\alpha}{1+g_Y},$$

assuming constant output growth rate, g_Y, and inflation rate π. This reduces to

$$b_{t+1} = \frac{1}{(1+\pi)(1+g_Y)} b_t + \frac{\alpha}{1+g_Y}. \tag{6.15}$$

Assuming that $(1+\pi)(1+g_Y) > 1$ (as is normal over the medium run), this linear difference equation has the stable solution

$$b_t = (b_0 - b^*)\left(\frac{1}{(1+\pi)(1+g_Y)}\right)^t + b^* \to b^* \text{ for } t \to \infty, \tag{6.16}$$

where

$$b^* = \frac{(1+\pi)}{(1+\pi)(1+g_Y) - 1}\alpha. \tag{6.17}$$

[14] For instance for Denmark the difference between the EMU and the net debt is substantial. In 2013 the Danish EMU debt was 44.6% of GDP while the government net debt was 5.5% of GDP (Danish Ministry of Finance, 2014).

Consequently, if the deficit rule (6.14) is always binding, the debt-income ratio tends in the long run to be proportional to the deficit bound α. The factor of proportionality is a decreasing function of the long-run growth rate of real GDP and the inflation rate. This result confirms the general tenet that if there is economic growth, perpetual budget deficits need not lead to fiscal problems.

If on the other hand the deficit rule is *not* always binding, then the budget deficit is on average smaller than above so that the debt-income ratio will in the long run be *smaller* than b^*.

The conclusion is the following. With one year as the time unit, suppose the deficit rule has $\alpha = 0.03$ and that $g_Y = 0.03$ (which by the architects of the Maastrict Treaty was considered the "natural" GDP growth rate) and $\pi = 0.02$ (which is the upper end of the inflation interval aimed at by the ECB). Suppose further the deficit rule is never violated. Then in the long run the debt-income ratio will be *at most* $b^* = 1.02 \times 0.03/(1.02 \times 1.03 - 1) \approx 0.60$. This is in agreement with the debt rule of the SGP according to which the maximum value allowed for the debt-income ratio is 60%.

Although there is nothing sacred about either of the numbers 0.60 or 0.03, they are mutually consistent, given $\pi = 0.02$ and $g_Y = 0.03$.

We observe that the deficit rule (6.14) implies that:

- The upper bound, b^*, on the long-run debt income ratio is lower the higher is inflation. The reason is that the growth factor $\beta \equiv [(1+\pi)\ \ (1+g_Y)]^{-1}$ for b_t in (6.15) depends negatively on the inflation rate, π. So does therefore b^* since, by (6.16), $b^* \equiv \alpha(1+g_Y)^{-1}(1-\beta)^{-1}$.

- For a *given* π, the upper bound on the long-run debt income ratio is *independent* of both the nominal and real interest rate (this follows from the indicated formula for the growth factor for b_t and the fact that $(1+i)(1+r)^{-1} = 1+\pi$).

The debate about the design of the SGP

In addition to the aimed long-run implications, by its design the SGP has short-run implications for the economy. Hence an evaluation of the SGP cannot ignore the way the economy functions in the short run. How changes in government spending and taxation affects the economy depends on the "state of the business cycle": is the economy in a boom with full capacity utilization or in a slump with slack aggregate demand?

Much of the debate about the SGP has centered around the consequences of the deficit rule in an economic recession triggered by a collapse of aggregate demand (for instance due to private deleveraging in the wake of a banking crisis).

Although the Eurozone countries are economically quite different, they are subject to the same one-size-fits-all monetary policy. Facing dissimilar shocks, the single member countries in need of aggregate demand stimulation in a recession have by joining the euro renounced on both interest rate policy and currency depreciation.[15] The only policy tool left for demand stimulation is therefore fiscal policy. Instead of a supranational fiscal authority responsible for handling the problem, it is up to the individual member countries to act − and to do so within the constraints of the SGP.

On this background, the critiques of the deficit rule of the SGP include the following points. (It may here be useful to have at the back of one's mind the simple Keynesian income-expenditure model, where output is below capacity and demand-determined whereas the general price level is sticky.)

Critiques 1. When considering the need for fiscal stimuli in a recession, a ceiling at 0.03 is too low unless the country has almost no government debt in advance. Such a deficit rule gives too little scope for counter-cyclical fiscal policy, including the free working of the *automatic fiscal stabilizers* (i.e., the provisions, through tax and transfer codes, in the government budget that automatically cause tax revenues to fall and spending to rise when GDP falls).[16] As an economy moves towards recession, the deficit rule may, bizarrely, force the government to tighten fiscal policy although the situation calls for stimulation of aggregate demand. The pact has therefore sometimes been called the "Instability and Depression Pact" − it imposes a *wrong timing* of fiscal consolidation.[17]

2. Since what really matters is long-run fiscal sustainability, a deficit rule should be designed in a more flexible way than the 3% rule of the SGP. A meaningful deficit rule would relate the deficit to the *trend* nominal GDP, which we may denote $(PY)^*$. Such a criterion would imply

$$GBD \leq \alpha(PY)^*. \tag{6.18}$$

[15] Denmark is in a similar situation. In spite of not joining the euro after the referendum in 2000, the Danish krone has been linked to the euro through a fixed exchange rate since 1999.

[16] Over the first 13 years of existence of the euro even Germany violated the 3 percent rule five of the years.

[17] The SGP has an exemption clause referring to "exceptional" circumstances. These circumstances were originally defined as "severe economic recession", interpreted as an annual fall in real GDP of at least 1-2%. By the reform of the SGP in March 2005, the interpretation was changed into simply "negative growth". Owing to the international economic crisis that broke out in 2008, the deficit rule was thus suspended in 2009 and 2010 for most of the EMU countries. But the European Commission brought the rule into effect again from 2011, which according to many critics was much too early, given the circumstances.

Then

$$\frac{GBD}{PY} \leq \alpha \frac{(PY)^*}{PY}.$$

In recessions the ratio $(PY)^*/(PY)$ is high, in booms it is low. This has the advantage of allowing more room for budget deficits when they are needed − without interfering with the long-run aim of stabilizing government debt below some specified ceiling.

3. A further step in this direction is a rule directly in terms of the *structural* or *cyclically adjusted* budget deficit rather than the actual year-by-year deficit. The cyclically adjusted budget deficit in a given year is defined as the value the deficit would take in case actual output were equal to trend output in that year. Denoting the cyclically adjusted budget deficit GBD^*, the rule would be

$$\frac{GBD^*}{(PY)^*} \leq \alpha.$$

In fact, in its original version as of 1997 the SGP contained an *additional* rule like that, but in the very strict form of $\alpha \approx 0$. This requirement was implicit in the directive that the cyclically adjusted budget "should be close to balance or in surplus". By this requirement it is imposed that the debt-income ratio should be close to zero in the long run. Many EMU countries certainly had − and have − larger cyclically adjusted deficits. Taking steps to comply with such a low structural deficit ceiling may be hard and endanger national welfare by getting in the way of key tasks of the public sector. The minor reform of the SGP endorsed in March 2005 allowed more contingencies, also concerning this structural bound. By the more recent reform in 2012, the Fiscal Pact, the lid on the cyclically adjusted deficit-income ratio was raised to 0.5% and to 1.0% for members with a debt-income ratio "significantly below 60%". These are still quite small numbers. Abiding by the 0.5% or 1.0% rule implies a long-run debt-income ratio of at most 10% or 20%, respectively, given structural inflation and structural GDP growth at 2% and 3% per year, respectively.[18]

4. Regarding the *composition* of government expenditure, critics have argued that the SGP pact entails a problematic disincentive for public investment. The view is that a fiscal rule should be based on a proper accounting of public investment instead of simply ignoring the composition of government expenditure. We consider this issue in Section 6.6 below.

5. At a more general level critics have contended that policy rules and surveillance procedures imposed on sovereign nations will hardly be able to do their job unless they encompass stronger incentive-compatible elements. Enforcement mechanisms are bound to be week. The SGP's threat of pecuniary fines to a

[18] Again apply (6.17).

country which during a recession has difficulties to reduce its budget deficit lacks credibility and has, at the time of writing (June 2015), not been made use of so far. Moreover, abiding by the fiscal rules of the SGP prior to the Great Recession was certainly no guarantee of not ending up in a fiscal crisis in the wake of a crisis in the banking sector, as witnessed by Ireland and Spain. A seemingly strong fiscal position can vaporize fast, particularly if banks, "too big to fail", need be bailed out.

Counter-arguments Among the counter-arguments raised against the criticisms of the SGP has been that the potential benefits of the proposed alternative rules are more than offset by the costs in terms of reduced simplicity, measurability, and transparency. The lack of flexibility may even be a good thing because it helps "tying the hands of elected policy makers". Tight rules are needed because of a *deficit bias* arising from short-sighted policy makers' temptation to promise spending without ensuring the needed financing, especially before an upcoming election. These points are sometimes linked to the view that market economies are generally self-regulating: Keynesian stabilization policy is not needed and may do more harm than good.

Box 6.1. The 2010-2012 debt crisis in the Eurozone

What began as a banking crisis became a deep economic recession combined with a government debt crisis.

At the end of 2009, in the aftermath of the global economic downturn, it became evident that Greece faced an acute debt crisis driven by three factors: high government debt, low ability to collect taxes, and lack of competitiveness due to cost inflation. Anxiety broke out about the debt crisis spilling over to Spain, Portugal, Italy, and Ireland, thus widening bond yield spreads in these countries vis-a-vis Germany in the midst of a serious economic recession. Moreover, the solvency of big German and French banks that were among the prime creditors of Greece was endangered. The major Eurozone governments and the International Monetary Fund (IMF) reached an agreement to help Greece (and thereby its creditors) with loans and guarantees for loans, conditional on the government of Greece imposing yet another round of harsh fiscal austerity measures. The elevated bond interest rates of Greece, Italy, and Spain were not convincingly curbed, however, until in August-September 2012 the president of the ECB, Mario Draghi, launched the "Outright Monetary Transactions" (OMT) program according to which, under certain conditions, the ECB will buy government bonds in the secondary market with the aim of "safeguarding an appropriate monetary policy transmission and the singleness of the monetary policy" and with "no ex ante quantitative limits". Considerably reduced government bond spreads followed and so the sheer announcement of the program seemed effective in its own right. Doubts raised by the German Constitutional Court about its legality vis-à-vis Treaties of the European Union were finally repudiated by the European Court of Justice mid-June 2015. At the time of writing (late June 2015) the OMT program has not been used in practice. Early 2015, a different massive program for purchases of government bonds, including long-term bonds, in the secondary market as well as private asset-backed bonds was decided and implemented by the ECB. The declared aim was to brake threatening deflation and return to "price stability", by which is meant inflation close to 2 percent per year.

So much about the monetary policy response. What about fiscal policy? On the basis of the SGP, the EU Commission imposed "fiscal consolidation" initiatives to be carried out in most EU countries in the period 2011-2013 (some of the countries were required to start already in 2010). With what consequences? By many observers, partly including the research department of the IMF, the initiatives were judged self-defeating. When at the same time comprehensive deleveraging in the private sector is going on, "austerity" policy deteriorates aggregate demand further and raises unemployment. Thereby, instead of budget deficits being decreased, it is the denominator of the debt-income ratio, $D/(PY)$, that is decreased. Fiscal multipliers are judged to be large ("in the 0.9 to 1.7 range since the Great Recession", according to IMF's *World Economic Outlook*, Oct. 2012) in a situation of idle resources where monetary policy aims at low interest rates; and negative spillover effects through trade linkages when "fiscal consolidation" is synchronized across countries. The unemployment rate in the Eurozone countries was elevated from 7.5 percent in 2008 to 12 percent in 2013. The British economists, Holland and Portes (2012), concluded: "It is ironic that, given that the EU was set up in part to avoid coordination failures in economic policy, it should deliver the exact opposite".

The whole crisis has pointed to a basic difficulty faced by the Eurozone. In spite of the member countries being economically very different sovereign nations, they are subordinate to the same one-size-fits-all monetary policy without sharing a federal government ready to use fiscal instruments to mitigate regional consequences of country-specific shocks. Adverse demand shocks may lead to sharply rising budget deficits in some countries, and financial investors may loose confidence and so elevate government bond interest rates. A liquidity crisis may arise, thereby amplifying adverse shocks. Even when a common negative demand shock hits all the member countries in a similar way, and a general relaxation of both monetary and fiscal policy is called for, there is the problem that the individual countries, in fear of boosting their budget deficit and facing the risk of exceeding the deficit or debt limit, may wait for the others to initiate fiscal expansion. The possible consequence of this "free rider" problem is general under-stimulation of the economies.

The dismal experience regarding the ability of the Eurozone to handle the Great Recession has incited proposals along at least two dimensions. One dimension is about allowing the ECB greater scope for acting as a "lender of last resort". The other dimension is about centralizing a larger part of the national budgets into a common union budget (see, e.g., De Grauwe, 2014). (END OF BOX)

6.5 Solvency, the NPG condition, and the intertemporal government budget constraint

Up to now we have considered the issue of government solvency from the perspective of dynamics of the government debt-to-income ratio. It is sometimes useful to view government solvency from another angle – the intertemporal budget constraint (GIBC). Under a certain condition stated below, the intertemporal budget constraint is, essentially, as relevant for a government as for private agents.

A simple condition closely linked to whether the government's intertemporal budget constraint is satisfied or not is what is known as the government's No-Ponzi-Game (NPG) condition. It is convenient to first focus on this condition. We concentrate on government *net* debt, measured in *real* terms, and ignore seigniorage.

6.5.1 When is the NPG condition necessary for solvency?

Consider a situation with a given constant interest rate, r. Suppose taxes are lump sum or at least that there is no tax on interest income from owning government bonds. Then the government's *NPG condition* is that the present discounted value of the public debt in the far future is not positive, i.e.,

$$\lim_{t\to\infty} B_t(1+r)^{-t} \leq 0. \tag{NPG}$$

This condition says that government debt is not allowed to grow in the long run at a rate as high as (or even higher than) the interest rate.[19] That is, a fiscal policy satisfying the NPG condition rules out a permanent debt rollover. Indeed, as we saw in Section 6.3.1, with $B_0 > 0$, a permanent debt rollover policy (financing all interest payments and perhaps even also part of the primary government spending) by debt issue leads to $B_t \geq B_0(1+r)^t$ for $t = 0, 1, 2, \ldots$. Substituting into (NPG) gives $\lim_{t\to\infty} B_t \geq B_0(1+r)^t(1+r)^{-t} = B_0 > 0$, thus violating (NPG).

The designation No-Ponzi-Game condition refers to a guy from Boston, Charles Ponzi, who in the 1920s made a fortune out of an investment scam based on the

[19] If there is effective taxation of interest income at the rate $\tau_r \in (0,1)$, then the after-tax interest rate, $(1-\tau_r)r$, is the relevant discount rate, and the NPG condition would read $\lim_{t\to\infty} B_t\left[1+(1-\tau_r)r\right]^{-t} \leq 0$.

chain-letter principle. The principle is to pay off old investors with money from new investors, keeping the remainder of that money to oneself. Ponzi was sentenced to many years in prison for his transactions; he died poor − and without friends!

To our knowledge, this kind of financing behavior is nowhere forbidden for the government as it generally is for private agents. But under "normal" circumstances a government *has* to plan its expenditures and taxation so as to comply with its NPG condition since otherwise not enough lenders will be forthcoming.

As the state is in principle infinitely-lived, however, there is no final date where all government debt should be over and done with. Indeed, the NPG condition does not even require that the debt has ultimately to be non-increasing. The NPG condition "only" says that the debtor, here the government, can not let the debt grow forever at a rate as high as (or higher than) the interest rate. For instance the U.K. as well as the U.S. governments have had positive debt for centuries − and high debt after both WW I and WW II.

Suppose Y (GDP) grows at the given constant rate g_Y (actually, for most of the following results it is enough that $\lim_{t\to\infty} Y_{t+1}/Y_t = 1 + g_Y$). We have:

PROPOSITION 1 Interpret "solvency" as absence of an for ever accelerating debt-income ratio, $b_t \equiv B_t/Y_t$. Then:

(i) if $r > g_Y$, solvency requires (NPG) satisfied;

(ii) if $r \leq g_Y$, the government can remain solvent without (NPG) being satisfied.

Proof. When $b_t \neq 0$,

$$\lim_{t\to\infty} \frac{b_{t+1}}{b_t} \equiv \lim_{t\to\infty} \frac{B_{t+1}/Y_{t+1}}{B_t/Y_t} = \lim_{t\to\infty} \frac{B_{t+1}/B_t}{Y_{t+1}/Y_t} = \lim_{t\to\infty} \frac{B_{t+1}/B_t}{1+g_Y}. \tag{6.19}$$

Case (i): $r > g_Y$. If $\lim_{t\to\infty} B_t \leq 0$, then (NPG) is trivially satisfied. Assume $\lim_{t\to\infty} B_t > 0$. For this situation we prove the statement by contradiction. Suppose (NPG) is not satisfied. Then, $\lim_{t\to\infty} B_t(1+r)^{-t} > 0$, implying that $\lim_{t\to\infty} B_{t+1}/B_t \geq 1 + r$. In view of (6.19) this implies that $\lim_{t\to\infty} b_{t+1}/b_t \geq (1+r)/(1+g_Y) > 1$. Thus, $b_t \to \infty$, which violates solvency. By contradiction, this proves that solvency implies (NPG) when $r > g_Y$.

Case (ii): $r \leq g_Y$. Consider the permanent debt roll-over policy $T_t = G_t$ for all $t \geq 0$, and assume $B_0 > 0$. By (DGBC) of Section 6.2 this policy yields $B_{t+1}/B_t = 1 + r$; hence, in view of (6.19), $\lim_{t\to\infty} b_{t+1}/b_t = (1 + r)/(1 + g_Y) \leq 1$. The policy consequently implies solvency. On the other hand the solution of the difference equation $B_{t+1} = (1 + r)B_t$ is $B_t = B_0(1 + r)^t$. Thus $B_t(1 + r)^{-t} = B_0 > 0$ for all t, thus violating (NPG). □

Hence imposition of the NPG condition on the government relies on the interest rate being in the long run higher than the growth rate of GDP. If instead $r \leq g_Y$, the government can cut taxes, run a budget deficit, and postpone the tax burden indefinitely. In that case the government can thus run a Ponzi Game and still stay solvent. Nevertheless, as alluded to earlier, if uncertainty is added to the picture, there will be many different interest rates, and matters become more complicated. Then qualifications to Proposition 1 are needed (Blanchard and Weil, 2001). The prevalent view among macroeconomists is that imposition of the NPG condition on the government is generally warranted.

While in the case $r > g_Y$, the NPG condition is *necessary* for solvency, it is *not sufficient*. Indeed, we could have

$$1 + g_Y < \lim_{t\to\infty} B_{t+1}/B_t < 1 + r. \tag{6.20}$$

Here, by the upper inequality, (NPG) is satisfied, yet, by the lower inequality together with (6.19), we have $\lim_{t\to\infty} b_{t+1}/b_t > 1$ so that the debt-income ratio explodes.

EXAMPLE 1 Let GDP $= Y$, a constant, and $r > 0$; so $r > g_Y = 0$. Let the budget deficit in real terms equal $\varepsilon B_t + \alpha$, where $0 \leq \varepsilon < r$ and $\alpha > 0$. Assuming no money-financing of the deficit, government debt evolves according to $B_{t+1} - B_t = \varepsilon B_t + \alpha$ which implies a simple linear difference equation:

$$B_{t+1} = (1+\varepsilon)B_t + \alpha. \tag{*}$$

Case 1: $\varepsilon = 0$. Then the solution of (*) is

$$B_t = B_0 + \alpha t, \tag{**}$$

B_0 being historically given. Then $B_t(1+r)^{-t} = B_0(1+r)^{-t} + \alpha t(1+r)^{-t} \to 0$ for $t \to \infty$. So, (NPG) is satisfied. Yet the debt-GDP ratio, B_t/Y, goes to infinity for $t \to \infty$. That is, in spite of (NPG) being satisfied, solvency is not present. For $\varepsilon = 0$ we thus get the insolvency result even though the lower *strict* inequality in (6.20) is *not* satisfied. Indeed, (**) implies $B_{t+1}/B_t = 1 + \alpha/B_t \to 1$ for $t \to \infty$ and $1 + g_Y = 1$.

Case 2: $0 < \varepsilon < r$. Then the solution of (*) is

$$B_t = (B_0 + \frac{\alpha}{\varepsilon})(1+\varepsilon)^t - \frac{\alpha}{\varepsilon} \to \infty \text{ for } t \to \infty,$$

if $B_0 > -\alpha/\varepsilon$. So $B_t/Y \to \infty$ for $t \to \infty$ and solvency is violated. Nevertheless $B_t(1+r)^{-t} \to 0$ for $t \to \infty$ so that (NPG) holds.

The example of this case fully complies with both strict inequalities in (6.20) because $B_{t+1}/B_t = 1 + \varepsilon + \alpha/B_t \to 1 + \varepsilon$ for $t \to \infty$. □

An approach to fiscal budgeting that *ensures* debt stabilization and thereby solvency is the following. First impose that the cyclically adjusted primary budget surplus as a share of GDP equals a constant, s. Next adjust taxes and/or spending such that $s \geq \hat{s} = (r - g_Y)b_0$, ignoring short-run differences between Y_{t+1}/Y_t and $1 + g_Y$ and between r_t and its long-run value, r. As in (6.10), $\hat{s}$ is the minimum primary surplus as a share of GDP required to obtain $b_{t+1}/b_t \leq 1$ for all $t \geq 0$ (Example 2 below spells this out in detail). This $\hat{s}$ is a measure of the burden that the government debt imposes on tax payers. If the policy steps needed to realize at least $\hat{s}$ are not taken, the debt-income ratio will grow, thus worsening the fiscal position in the future by increasing $\hat{s}$.

6.5.2 Equivalence of NPG and GIBC

The condition under which the NPG condition is necessary for solvency is also the condition under which the government's intertemporal budget constraint is necessary. To show this we let t denote the current period and $t + i$ denote a period in the future. As above, we ignore seigniorage. Debt accumulation is then described by

$$B_{t+1} = (1 + r)B_t + G_t + X_t - \tilde{T}_t, \qquad \text{where } B_t \text{ is given.} \tag{6.21}$$

The *government intertemporal budget constraint* (GIBC), as seen from the beginning of period t, is the requirement

$$\sum_{i=0}^{\infty}(G_{t+i} + X_{t+i})(1 + r)^{-(i+1)} \leq \sum_{i=0}^{\infty}\tilde{T}_{t+i}(1 + r)^{-(i+1)} - B_t. \tag{GIBC}$$

This condition requires that the present value (PV) of current and expected future government spending does not exceed the government's net wealth. The latter equals the PV of current and expected future tax revenue minus existing government debt. By the symbol $\sum_{i=0}^{\infty} x_i$ we mean $\lim_{I\to\infty} \sum_{i=0}^{I} x_i$. Until further notice we assume this limit exists.

What connection is there between the dynamic accounting relationship (6.21) and the intertemporal budget constraint, (GIBC)? To find out, we rearrange

(6.21) and use forward substitution to get

$$\begin{aligned}
B_t &= (1+r)^{-1}(\tilde{T}_t - X_t - G_t) + (1+r)^{-1}B_{t+1} \\
&= \sum_{i=0}^{j}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}) + (1+r)^{-(j+1)}B_{t+j+1} \\
&= \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}) + \lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} \\
&\leq \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}), \qquad (6.22)
\end{aligned}$$

if and only if the government debt ultimately grows at a rate less than r so that

$$\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} \leq 0. \qquad (6.23)$$

This latter condition is exactly the NPG condition above (replace t in (6.23) by 0 and j by $t-1$). And the condition (6.22) is just a rewriting of (GIBC). We conclude:

PROPOSITION 2 Given the book-keeping relation (6.21), then:

(i) (NPG) is satisfied if and only if (GIBC) is satisfied;

(ii) there is strict equality in (NPG) if and only if there is strict equality in (GIBC).

We know from Proposition 1 that in the "normal case" where $r > g_Y$, (NPG) is needed for government solvency. The message of (i) of Proposition 2 is then that also (GIBC) need be satisfied. Given $r > g_Y$, to appear solvent a government has to realistically plan taxation and spending profiles such that the PV of current and expected future primary budget surpluses matches the current debt, cf. (6.22). Otherwise debt default is looming and forward-looking investors will refuse to buy government bonds or only buy them at a reduced price, thereby aggravating the fiscal conditions.[20]

In view of the remarks around the inequalities in (6.20), however, satisfying the condition (6.22) is only a necessary condition (if $r > g_Y$), not in itself a sufficient condition for solvency. A simple condition under which satisfying the

[20] Government debt defaults have their own economic as well as political costs, including loss of credibility. Yet, they occur now and then. Recent examples include Russia in 1998 and Argentina in 2001-2002. During 2010-12, Greece was on the brink of debt default. At the time of writing (June 2015) such a situation has turned up again for Greece.

condition (6.22) *is* sufficient for solvency is that both G_t and T_t are proportional to Y_t, cf. Example 2.

EXAMPLE 2 Consider a small open economy facing an exogenous constant real interest rate r. Suppose that at time t government debt is $B_t > 0$, GDP is growing at the constant rate g_Y, and $r > g_Y$. Assume $G_t = \gamma Y_t$ and $T_t \equiv \tilde{T}_t - X_t = \tau Y_t$, where γ and τ are positive constants. What is the minimum size of the primary budget surplus as a share of GDP required for satisfying the government's intertemporal budget constraint as seen from time t? Inserting into the formula (6.22), with strict equality, yields $\sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tau-\gamma)Y_{t+i} = B_t$. This gives $\frac{\tau-\gamma}{1+g_Y}Y_t\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} = \frac{\tau-\gamma}{r-g_Y}Y_t = B_t$, where we have used the rule for the sum of an infinite geometric series. Rearranging, we conclude that the required primary surplus as a share of GDP is

$$\tau - \gamma = (r - g_Y)\frac{B_t}{Y_t}.$$

This is the same result as in (6.10) above if we substitute $\hat{s} = \tau - \gamma$ and $t = 0$. Thus, maintaining G_t/Y_t and T_t/Y_t constant while satisfying the government's intertemporal budget constraint ensures a constant debt-income ratio and thereby government solvency. □

On the other hand, if $r \leq g_Y$, it follows from propositions 1 and 2 together that the government can remain solvent *without* satisfying its intertemporal budget constraint (at least as long as we ignore uncertainty).[21] The background for this fact may become more apparent when we recognize how the condition $r \leq g_Y$ affects the constraint (GIBC). Indeed, to the extent that the tax revenue tends to grow at the same rate as national income, we have $\tilde{T}_{t+i} = \tilde{T}_t(1+g_Y)^i$. Then

$$\sum_{i=0}^{\infty}\tilde{T}_{t+i}(1+r)^{-(i+1)} = \frac{\tilde{T}_t}{1+g_Y}\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)},$$

which is clearly infinite if $r \leq g_Y$. The PV of expected future tax revenues is thus unbounded in this case. Suppose that also government spending, $G_{t+i} + X_{t+i}$, grows at the rate g_Y. Then the evolution of the primary surplus is described by $\tilde{T}_{t+i} - X_{t+i} - G_{t+i} = (\tilde{T}_t - (G_t + X_t))(1+g_Y)^i$, $i = 1, 2, \ldots$. Although in this case also the PV of future government spending is infinite, (6.22) shows that any

[21] Of course, this statement is a contradiction in terms if one thinks of "solvency" in the standard *financial* sense where solvency requires that the debt does not exceed the present value of future surplusses, i.e., that (6.22) holds. As noted in Section 6.3 we use the term *solvency* in the broader meaning of "being able to meet the financial commitments as they fall due".

positive initial primary budget surplus, $\tilde{T}_t - (G_t + X_t)$, ever so small can repay any level of initial debt in finite time.

In (GIBC) and (6.23) we allow strict inequalities to obtain. What is the interpretation of a strict inequality here? The answer is:

COROLLARY OF PROPOSITION 2 Given the book-keeping relation (6.21), then strict inequality in (GIBC) is equivalent to the government in the long run accumulating positive net financial wealth.

Proof. Strict inequality in (GIBC) is equivalent to strict inequality in (6.22), which in turn, by (ii) of Proposition 2, is equivalent to strict inequality in (6.23), which is equivalent to $\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} < 0$. This latter inequality is equivalent to $\lim_{j\to\infty} B_{t+j+1} < 0$, that is, positive net financial wealth in the long run. Indeed, by definition, $r > -1$, hence $\lim_{j\to\infty}(1+r)^{-(j+1)} \geq 0$. $\square$

It is common to consider as the *regular case* the case where the government does not attempt to accumulate positive net financial wealth in the long run and thereby become a net creditor vis-à-vis the private sector. Returning to the assumption $r > g_Y$, in the regular case fiscal solvency thus amounts to the requirement

$$\sum_{i=0}^{\infty} \tilde{T}_{t+i}(1+r)^{-(i+1)} = \sum_{i=0}^{\infty}(G_{t+i} + X_{t+i})(1+r)^{-(i+1)} + B_t, \qquad \text{(GIBC')}$$

which is obtained by rearranging (GIBC) and replacing weak inequality with strict equality. It is certainly *not* required that the budget is balanced all the time. The point is "only" that for a given planned expenditure path, a government should plan realistically a stream of future tax revenues the PV of which matches the PV of planned expenditure *plus* the current debt.

We may rewrite (GIBC') as

$$\sum_{i=0}^{\infty} \left(\tilde{T}_{t+i} - (G_{t+i} + X_{t+i})\right)(1+r)^{-(i+1)} = B_t. \qquad \text{(GIBC")}$$

This expresses the basic principle that when $r > g_Y$, solvency requires that *the present value of planned future primary surpluses equals the initial debt.* We have thus shown:

PROPOSITION 3 Consider the regular case. Assume $r > g_Y$. Then:

(i) if debt is positive today, the government has to run a positive primary budget surplus for a sufficiently long time in the future;

(ii) if an unplanned deficit arises so as to create an unexpected rise in public debt, then higher taxes than otherwise must be levied in the future.

Finer shades

1. If the real interest rate varies over time, all the above formulas remain valid if $(1+r)^{-(i+1)}$ is replaced by $\Pi_{j=0}^{i}(1+r_{t+j})^{-1}$.

2. We have essentially ignored seigniorage. Under "normal" circumstances seigniorage is present and this relaxes (GIBC") somewhat. Indeed, as noted in Section 6.2, the money-nominal income ratio, M/PY, tend to be roughly constant over time, reflecting that money and nominal income tend to grow at the same rate. So a rough indicator of g_M is the sum $\pi + g_Y$. Seigniorage is $S \equiv \Delta M/P = g_M M/P = sY$, where s is the seigniorage-income ratio. Taking seigniorage into account amounts to subtracting the present value of expected future seigniorage, PV(S), from the right-hand side of (GIBC"). With s constant and Y growing at the constant rate $g_Y < r$, PV(S) can be written

$$\begin{aligned}
\text{PV(S)} &= \sum_{i=0}^{\infty} S_{t+i}(1+r)^{-(i+1)} = s\sum_{i=0}^{\infty} Y_{t+i}(1+r)^{-(i+1)} = \frac{sY_t}{1+g_Y}\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} \\
&= \frac{sY_t}{1+g_Y}\frac{1+g_Y}{1+r}\frac{1}{1-\frac{1+g_Y}{1+r}} = \frac{sY_t}{r-g_Y},
\end{aligned}$$

where the second to last equality comes from the rule for the sum of an infinite geometric series. So the right-hand side of (GIBC") becomes $B_t - sY_t/(r-g_Y) \equiv [b_t - s/(r-g_Y)]\, Y_t$.[22]

3. Should a public deficit rule not make a distinction between public consumption and public investment? This issue is taken up in the next section.

6.6 A proper accounting of public investment*

Public investment as a share of GDP has been falling in the EMU countries since the middle of the 1970s, in particular since the run-up to the euro 1993-97. This later development is seen as in part induced by the deficit rule of the Maastrict Treaty (1992) and the Stability and Growth Pact (1997) which, like the customary government budget accounting we have considered up to now, attributes government gross investment as an expense in a single year's operating account instead of just the depreciation of the public capital. Already Musgrave (1939) recommended applying separate capital and operating budgets. Thereby government net investment will be excluded from the definition of the public "budget deficit". And more meaningful deficit rules can be devised.

[22] In a recession where the economy is in a liquidity trap, the non-conventional monetary policy called Quantitative Easing may partly take the form of seigniorage. This is taken up in Chapter 24.

To see the gist of this, we partition G into public consumption, C^g, and public investment, I^g, that is, $G = C^g + I^g$. Public investment produces public capital (infrastructure etc.). Denoting the public capital K^g we may write

$$\Delta K^g = I^g - \delta K^g, \tag{6.24}$$

where δ is a (constant) capital depreciation rate. Let the annual (direct) financial return per unit of public capital be r_g. This is the sum of user fees and the like. Net government revenue, T', now consists of net tax revenue, T, plus the direct financial return $r_g K^g$.[23] In that now only interest payments and the capital depreciation, δK^g, along with C^g, enter the operating account as "true" expenses, the "true" budget deficit is $rB + C^g + \delta K^g - T'$, where $T' = T + r_g K^g$.

We impose a rule requiring balancing the "true structural budget" in the sense that on average over the business cycle

$$T' = rB + C^g + \delta K^g \tag{6.25}$$

should hold. The spending on public investment of course enters the debt accumulation equation which now takes the form

$$\Delta B = rB + C^g + I^g - T'.$$

Substituting (6.25) into this, we get

$$\Delta B = I^g - \delta K^g = \Delta K^g, \tag{6.26}$$

by (6.24). So the balanced "true structural budget" implies that public net investment is financed by an increase in public debt. Other public spending is tax financed.

Suppose public capital keeps pace with trend GDP, Y_t^*, that is, $\Delta K^g/K^g = g_Y > 0$. So the ratio K^g/Y^* remains constant at some level $h > 0$. Then (6.26) implies

$$B_{t+1} - B_t = K_{t+1}^g - K_t^g = g_Y K_t^g = g_Y h Y_t^*. \tag{6.27}$$

What is the implication for the evolution of the debt-to-trend-income ratio, $\hat{b}_t \equiv B_t/Y_t^*$, over time? By (6.27) together with $Y_{t+1}^* = (1 + g_Y)Y_t^*$ follows

$$\hat{b}_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}^*} = \frac{B_t}{(1+g_Y)Y_t^*} + \frac{g_Y h}{1+g_Y} \equiv \frac{1}{1+g_Y}\hat{b}_t + \frac{g_Y h}{1+g_Y}.$$

[23] There is also an indirect financial return deriving from the fact that better infrastructure may raise efficiency in the supply of public services and increase productivity in the private sector and thereby the tax base. While such expected effects matter for a cost-benefit analysis of a public investment project, from an accounting point of view they will be included in the net tax revenue, T, in the future.

This linear first-order difference equation has the solution

$$\hat{b}_t = (\hat{b}_0 - \hat{b}^*)(1+g_Y)^{-t} + \hat{b}^*, \qquad \text{where } \hat{b}^* = \frac{1}{1+g_Y}\hat{b}^* + \frac{g_Y h}{1+g_Y} = h,$$

assuming $g_Y > 0$. Then $\hat{b}_t \to h$ for $t \to \infty$. Run-away debt dynamics is precluded.[24] Moreover, the ratio B_t/K_t^g, which equals $\hat{b}_t/h$, approaches 1. Eventually the public debt is in relative terms thus backed by the accumulated public capital.

Fiscal sustainability is here ensured *in spite of* a positive "budget deficit" in the *traditional* sense of Section 6.2 and given by ΔB in (**??**). This result holds even when $r_g < r$, which is perhaps the usual case. Still, the public investment may be worthwhile in view of indirect financial returns as well as non-financial returns in the form of the utility contribution of public goods and services.

Additional remarks

1. The deficit rule described says only that the "true structural budget" should be balanced "on average" over the business cycle. This invites deficits in slumps and surpluses in booms. Indeed, in economic slumps government borrowing is usually cheap. As Harvard economist Lawrence Summers put it: "Idle workers + Low interest rates = Time to rebuild infrastructure" (Summers, 2014).

2. When separating government consumption and investment in budget accounting, a practical as well as theoretical issue arises: where to draw the border between the two? A sizeable part of what is investment in an economic sense is in standard public sector accounting categorized as "public consumption": spending on education, research, and health are obvious examples. Distinguishing between such categories and public consumption in a narrower sense (administration, judicial system, police, defence) may be important when economic growth policy is on the agenda. Apart from noting the issue, we shall not pursue the matter here.

3. That *time lags*, cf. point (iii) in Section 6.1, are a constraining factor for fiscal policy is especially important for macroeconomic stabilization policy aiming at dampening business cycle fluctuations. If the lags are ignored, there is a risk that government intervention comes too late and ends up amplifying the fluctuations instead of dampening them. In particular the *monetarists*, lead by Milton Friedman (1912-2006), warned against this risk, pointing out the "long and variable lags". Other economists find awareness of this potential problem relevant but point to ways to circumvent the problem. During a recession there is for instance the option of reimbursing a part of last year's taxes, a policy that can be quickly implemented. In addition, the ministries of Economic affairs

[24] This also holds if $g_Y = 0$. Indeed, in this case, (6.27) implies $B_{t+1} = B_t = B_0$.

can have plans concerning upcoming public investment ready for implementation and carry them out when expansive fiscal policy is called for. More generally, legislation concerning taxation, transfers, and other spending can be designed with the aim of strengthening the automatic fiscal stabilizers.

6.7 Ricardian equivalence?

Having so far concentrated on the issue of fiscal sustainability, we shall now consider how budget policy affects resource allocation and intergenerational distribution. The role of budget policy for economic activity within a time horizon corresponding to the business cycle is not the issue here. The focus is on the longer run: does it matter for aggregate consumption and aggregate saving in an economy with full capacity utilization whether the government finances its current spending by (lump-sum) taxes or borrowing?

There are two opposite answers in the literature to this question. Some macroeconomists tend to answer the question in the negative. This is the *debt neutrality* view, also called the *Ricardian equivalence* view. The influential American economist Robert Barro is in this camp. Other macroeconomists tend to answer the question in the positive. This is the *debt non-neutrality* view or *absence of Ricardian equivalence* view. The influential French-American economist Olivier Blanchard is in this camp.

The two different views rest on two different models of the economic reality. Yet the two models have a *common* point of departure:

1) $r > g_Y$;

2) fiscal policy satisfies the intertemporal budget constraint with strict equality:
$$\sum_{t=0}^{\infty} \tilde{T}_t(1+r)^{-(t+1)} = \sum_{t=0}^{\infty}(G_t + X_t)(1+r)^{-(t+1)} + B_0, \tag{6.28}$$
where the initial debt, B_0, and the planned path of $G_t + X_t$ are given;

3) agents have rational (model consistent) expectations;

4) at least some of the taxes are lump sum and only these are varied in the thought experiment to be considered;

5) no financing by money;

6) credit market imperfections are absent.

For a given planned time path of $G_t + X_t$, equation (6.28) implies that a tax cut in any period has to be met by an increase in future taxes of the same present discounted value as the tax cut.

6.7.1 Two differing views

Ricardian equivalence

The *Ricardian equivalence* view is the conception that government debt is *neutral* in the sense that for a given time path of future government spending, aggregate private consumption is unaffected by a temporary tax cut. The temporary tax cut does not make the households feel richer because they expect that the ensuing rise in government debt will lead to higher taxes in the future. The essential claim is that the timing of (lump-sum) taxes does not matter. The name *Ricardian equivalence* comes from a – seemingly false – association of this view with the early nineteenth-century British economist David Ricardo. It is true that Ricardo articulated the possible logic behind debt neutrality. But he suggested several reasons that debt neutrality would not hold in practice and in fact he warned against high public debt levels (Ricardo, 1969, pp. 161-164). Therefore it is doubtful whether Ricardo was a Ricardian.

Debt neutrality was rejuvenated, however, by Robert Barro in a paper entitled "Are government bonds net wealth [of the private sector]?", a question which Barro answered in the negative (Barro 1974). Barro's debt neutrality view rests on a representative agent model, that is, a model where the household sector is described as consisting of a fixed number of infinitely-lived forward-looking "dynasties". With perfect financial markets, a change in the timing of taxes does not change the PV of the infinite stream of taxes imposed on the individual dynasty. A cut in current taxes is offset by the expected higher future taxes. Though current government saving ($T - G - rB$) goes down, private saving and bequests left to the members of the next generation go up equally much.

More precisely, the logic of the debt neutrality view is as follows. Suppose, for simplicity, that the government waits only 1 period to increase taxes and then does so in one stroke. Then, for each unit of account current taxes are reduced, taxes next period are increased by $(1+r)$ units of account. The PV as seen from the end of the current period of this future tax increase is $(1+r)/(1+r) = 1$. As $1-1 = 0$, the change in the time profile of taxation will make the dynasty feel neither richer nor poorer. Consequently, its current and planned future consumption will be unaffected. That is, its current saving goes up just as much as its current taxation is reduced. In this way the altruistic parents make sure that the next generation is fully compensated for the higher future taxes. Current private consumption in

Ricardian non-equivalence The old saying that "in life only death and tax are certain" fits the Ricardian non-equivalence view well. Many economists dissociate themselves from representative agent models because of their problematic description of the household sector. Instead attention is drawn to overlapping generations models which emphasize finite lifetime and life-cycle behavior of human beings and lead to a refutation of Ricardian equivalence. The essential point is that those individuals who benefit from lower taxes today will only be a fraction of those who bear the higher tax burden in the future. As taxes levied at different times are thereby levied at partly different sets of agents, the timing of taxes generally matters. The current tax cut makes current tax payers feel wealthier and so they increase their consumption and decrease their saving. The present generations benefit and future tax payers (partly future generations) bear the cost in the form of access to less national wealth than otherwise. With another formulation: under full capacity utilization government deficits have a crowding-out effect because they compete with private investment for the allocation of saving.

The next subsection provides an example showing in detail how a change in the timing of taxes affects aggregate private consumption in an overlapping generations life-cycle framework.

6.7.1 A small open OLG economy with a temporary budget deficit

We consider a Diamond-style overlapping generations (OLG) model of a small open economy (henceforth named SOE) with a government sector. The relationship between SOE and international markets is described by the same four assumptions as in Chapter 5.3:

(a) Perfect mobility of goods and financial capital across borders.

(b) No uncertainty and domestic and foreign financial claims are perfect substitutes.

(c) No need for means of payment, hence no need for a foreign exchange market.

(d) No labor mobility across borders.

The assumptions (a) and (b) imply *real interest rate equality.* That is, in equilibrium the real interest rate in SOE must equal the real interest rate, r, in the world financial market. By saying that SOE is "small" we mean it is small enough to not affect the world market interest rate as well as other world market factors. We imagine that all countries trade one and the same homogeneous good. International trade will then be only *intertemporal* trade, i.e., international borrowing and lending of this good.

We assume that r is constant over time and that $r > n \geq 0$. We let L_t denote the size of the young generation and assume $L_t = L_{-1}(1+n)^{t+1}$, $t = 0, 1, 2, \ldots$. Each young supplies one unit of labor inelastically, hence L_t is aggregate labor supply. Assuming full employment and ignoring technical progress, gross domestic product, GDP, is $Y_t = F(K_t, L_t)$.

Firms' behavior and the equilibrium real wage

GDP is produced by an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t, L_t) = L_t F(k_t, 1) \equiv L_t f(k_t),$$

where K_t and L_t are input of capital and labor, respectively, and $k_t \equiv K_t/L_t$. Technological change is ignored. Imposing perfect competition, profit maximization gives $\partial Y_t/\partial K_t = f'(k_t) = r+\delta$, where δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$. When f satisfies the condition $\lim_{k\to 0} f'(k) > r + \delta > \lim_{k\to\infty} f'(k)$, there is always a solution for k_t in this equation and it is unique (since $f'' < 0$) and constant over time (as long as r and δ are constant). Thus,

$$k_t = f'^{-1}(r+\delta) \equiv k, \text{ for all } t \geq 0, \tag{6.29}$$

where k is the desired capital-labor ratio, given r. The endogenous stock of capital, K_t, is determined by the equation $K_t = kL_t$, where, in view of clearing in the labor market, L_t can be interpreted as both employment and labor supply (exogenous).

The desired capital-labor ratio, k, also determines the equilibrium real wage before tax:

$$w_t = \frac{\partial Y_t}{\partial L_t} = f(k_t) - f'(k_t)k_t = f(k) - f'(k)k \equiv w, \tag{6.30}$$

a constant. GDP will evolve over time according to

$$Y_t = f(k)L_t = f(k)L_0(1+n)^t = Y_0(1+n)^t.$$

The growth rate of Y thus equals the growth rate of the labor force, i.e., $g_Y = n$.

Some national accounting for an open economy with a public sector

Since we ignore labor mobility across borders, gross national product (= gross national income) in SOE is

$$GNP_t = GDP_t + r \cdot NFA_t = Y_t + r \cdot NFA_t,$$

where NFA_t is net foreign assets at the beginning of period t. If $NFA_t > 0$, SOE has positive net claims on resources in the rest of the world, it may be in the form of direct ownership of production assets or in the form of net financial claims. If $NFA_t < 0$, the reason may be that part of the capital stock, K_t, in SOE is directly owned by foreigners or these have on net financial claims on the citizens of SOE (in practice usually a combination of the two).

Gross national saving is

$$S_t = Y_t + rNFA_t - C_t - G_t = Y_t + rNFA_t - (c_{1t}L_t + c_{2t}L_{t-1}) - G_t, \quad (6.31)$$

where G_t is government consumption in period t, and c_{1t} and c_{2t} are consumption by a young and an old in period t, respectively. In the open economy, generally, gross investment, I_t, differs from gross saving.

National wealth, V_t, of SOE at the beginning of period t is, by definition, national assets minus national liabilities,

$$V_t \equiv K_t + NFA_t.$$

National wealth is also, by definition, the sum of private financial (net) wealth, A_t, and government financial (net) wealth, $-B_t$. We assume the government has no physical assets and B_t is government (net) debt. Thus,

$$V_t \equiv A_t + (-B_t). \quad (6.32)$$

We may also view *national* wealth from the perspective of national *saving. First*, when the young save, they accumulate *private* financial wealth. The private financial wealth at the start of period $t+1$ must in our Diamond framework equal the (net) saving by the young in the previous period, S_{1t}^N, and the latter must equal *minus* the (net) saving by the old in the next period, S_{2t+1}^N :

$$A_{t+1} = s_t L_t \equiv S_{1t}^N = -S_{2t+1}^N. \quad (6.33)$$

The notation in this section of the chapter follows the standard notation for the Diamond model, and so s_t stands for the saving by the young individual in period t, not the primary budget surplus as in the previous sections.

Second, the increase in *national* wealth equals by definition net *national* saving, S_t^N, which in turn equals the sum of net saving by the private sector, $S_{1t}^N + S_{2t}^N$, and the net saving by the public sector, S_{gt}^N. So

$$\begin{aligned} V_{t+1} - V_t &= S_t - \delta K_t = S_t^N \equiv S_{1t}^N + S_{2t}^N + S_{gt}^N = A_{t+1} + (-A_t) + (-GBD_t) \\ &= A_{t+1} - A_t - (B_{t+1} - B_t), \end{aligned}$$

where the second to last equality comes from (6.33) and the identity $S_{gt}^N \equiv -GBD_t$, while the last equality reflects the maintained assumption that budget deficits are fully financed by debt issue.

Government and household behavior

We assume that the role of the government sector is to deliver public goods and services in the amount G_t in period t. Think of non-rival goods like "rule of law", TV-transmitted theatre, and other public services free of charge. Suppose G_t grows at the same rate as Y_t :

$$G_t = G_0(1+n)^t,$$

where G_0 is given, $0 < G_0 < F(K_0, L_0)$. We may think of G_t as being produced by the same technology as the other components of GDP, thus involving the same unit production costs. We ignore that the public good may affect productivity in the private sector (otherwise G should in principle appear as a third argument in the production function F).

To get explicit solutions, we specify the period utility function to be CRRA: $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, where $\theta > 0$. To keep things simple, the utility of the public good enters individuals' life-time utility additively. Thereby it does not affect marginal utilities of private consumption. There is a tax on the young as well as the old in period t, τ_1 and τ_2, respectively. These taxes are *lump sum* (levied on individuals irrespective of their economic behavior). Until further notice, the taxes are time-independent. Possibly, τ_1 or τ_2 is negative, in which case there is a transfer to either the young or the old.

The consumption-saving decision of the young will be the solution to the following problem:

$$\begin{aligned}
\max U(c_{1t}, c_{2t+1}) &= \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + v(G_t) + (1+\rho)^{-1}\left[\frac{c_{2t+1}^{1-\theta} - 1}{1-\theta} + v(G_{t+1})\right] \text{ s.t.} \\
c_{1t} + s_t &= w - \tau_1, \\
c_{2t+1} &= (1+r)s_t - \tau_2, \\
c_{1t} &\geq 0, c_{1t+1} \geq 0,
\end{aligned}$$

where the function v represents the utility contribution of the public good. The implied Euler equation can be written

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta}.$$

Inserting the two budget constraints and solving for s_t, we get

$$s_t = \frac{w - \tau_1 + \left(\frac{1+\rho}{1+r}\right)^{1/\theta}\tau_2}{1 + (1+\rho)\left(\frac{1+r}{1+\rho}\right)^{(\theta-1)/\theta}} \equiv s_0 = s(w, r, \tau_1, \tau_2), \qquad t = 0, 1, 2, \ldots,$$

This shows how saving by the young depends on the preference parameters θ and ρ and on labor income and the interest rate. Further, saving by the young is constant over time.

Before considering the solution for c_{1t} and c_{2t+1}, it is convenient to introduce the *intertemporal* budget constraint of an individual belonging to generation t and consider the value of the individual's after-tax *human wealth*, h_t, evaluated at the end of period t. This is the present (discounted) value, as seen from the end of period t, of *disposable lifetime income* (the "endowment").obtainable by a member of generation t. In the present case we get

$$c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - \tau_1 - \frac{\tau_2}{1+r} \equiv h, \tag{6.34}$$

where h on the right-hand side is the time independent value of h_t under the given circumstances.[26] To ensure that $h > 0$, we must assume that τ_1 and τ_2 in combination are of "moderate" size.

The solutions for consumption in the first and the second period, respectively, can then be written

$$c_{1t} = w - \tau_1 - s_t = \hat{c}_1(r)h \tag{6.35}$$

and

$$c_{2t+1} = \hat{c}_2(r)h, \tag{6.36}$$

where

$$\hat{c}_1(r) \equiv \frac{1+\rho}{1+\rho+\left(\frac{1+r}{1+\rho}\right)^{(1-\theta)/\theta}} \in (0,1) \text{ and} \tag{6.37}$$

$$\hat{c}_2(r) \equiv \left(\frac{1+r}{1+\rho}\right)^{1/\theta}\hat{c}_1(r) = \frac{1+r}{1+(1+\rho)\left(\frac{1+r}{1+\rho}\right)^{(\theta-1)/\theta}} \tag{6.38}$$

are the marginal (= average) propensities to consume out of wealth.[27]

Given r, both in the first and the second period of life is individual consumption proportional to individual human wealth. This is as expected in view of the homothetic lifetime utility function. If $\rho = r$, then $\hat{c}_1(r) = \hat{c}_2(r) = (1+r)/(2+r)$, that is, there is complete consumption smoothing.

The tax revenue in period t is $T_t = \tau_1 L_t + \tau_2 L_{t-1} = (\tau_1 + \tau_2/(1+n))L_t$. Let $B_0 = 0$ and let the "benchmark path" be a path along which the budget is and remains *balanced* for all t, i.e.,

$$T_t = (\tau_1 + \frac{\tau_2}{1+n})L_0(1+n)^t = G_t = G_0(1+n)^t.$$

[26] With technical progress, the real wage would be rising over time and so would h_t.

[27] By calculating backwards from (6.38) to (6.37) to (**??**), the reader will be able to confirm that the calculated s, c_{1t} and c_{2t+1} are consistent.

In this "benchmark policy regime" the tax code (τ_1, τ_2) thus satisfies $(\tau_1+\tau_2/(1+n))L_0 = G_0$. Given L_0, consistency with $h > 0$ in (6.34) requires a "not too large" G_0.

Along the benchmark path, aggregate private consumption grows at the same constant rate as GDP and public consumption, the rate n. Indeed,

$$C_t = c_{1t}L_t + \frac{c_{2t}}{1+n}L_t = (c_{1t} + \frac{c_{2t}}{1+n})L_0(1+n)^t = C_0(1+n)^t.$$

In view of (6.33) and the absence of government debt, also *national wealth* grows at the rate n :

$$V_t = A_t - B_t = A_t - 0 = s_{t-1}L_{t-1} = s_0 L_{t-1} = s_0 L_{-1}(1+n)^t = V_0(1+n)^t, \qquad t = 0, 1, \ldots. \tag{6.39}$$

Consequently, national wealth per old, V_t/L_{t-1}, is constant over time (recall, we have ignored technical progress).

6.7.2 A one-off tax cut

As an alternative to the benchmark path, consider the case where an unexpected one-off cut in taxation by z units of account takes place in period 0 for every individual, whether young or old. What are the consequences of this? The tax cut amounts to creating a budget deficit in period 0 equal to

$$GBD_0 = rB_0 + G_0 - T_0' = G_0 - T_0' = T_0 - T_0' = (L_0 + L_{-1})z,$$

where the value taken by a variable along this *alternative path* is marked with a prime. At the start of period 1, there is now a government debt $B_1' = (L_0+L_{-1})z$. In the benchmark path we had $B_1 = 0$. Since we assume $r > n = g_Y$, government solvency requires that the present value of future taxes, as seen from the beginning of period 1, rises by $(L_0 + L_{-1})z$, cf. (6.28). Suppose this is accomplished by raising the tax on all individuals from period 1 onward by m. Then

$$\Delta T_t = (L_t + L_{t-1})m = (L_0 + L_{-1})(1+n)^t, \qquad t = 1, 2, \ldots.$$

Suppose the government in period 0 credibly announces that the way it will tackle the arisen debt is by his policy. So also the young in period 0 are aware of the future tax rise.

As solvency requires that the present value of future taxes, as seen from the beginning of period 1, rises by $(L_0 + L_{-1})z$, the required value of m will satisfy

$$\sum_{t=1}^{\infty} \Delta T_t(1+r)^{-t} = \sum_{t=1}^{\infty}(L_0 + L_{-1})(1+n)^t m(1+r)^{-t} = (L_0 + L_{-1})z.$$

This gives

$$m\sum_{t=1}^{\infty}\left(\frac{1+n}{1+r}\right)^{t}=z.$$

As $r>n$, from the rule for the sum of an infinite geometric series follows that

$$m=\frac{r-n}{1+n}z\equiv\bar{m}. \tag{6.40}$$

As an example, let $r=0,02$ and $n=0.005$ per year. Then $\bar{m}\simeq 0.015{\cdot}z$.

The needed rise in future taxes is thus higher the higher is the interest rate r. This is because the interest burden of the debt will be higher. On the other hand, a higher population growth rate, n, reduces the needed rise in future taxes. This is because the interest burden per capita is mitigated by population growth. Finally, a greater tax cut, z, in the first period implies greater tax rises in future periods. (It is assumed throughout that z is of "moderate" size in the sense of not causing $\bar{m}$ to violate the condition $h_t'>0$. The requirement is $0<z<(1+r)(1+n)h/\left[(2+r)(r-n)\right]$.)

Effect on the consumption path

In period 0 the tax cut unambiguously benefits the old. Their increase in consumption equals the saved tax:

$$c_{20}'-c_{20}=z>0. \tag{6.41}$$

The young in period 0 know that per capita taxes next period will be increased by $\bar{m}$. In view of the tax cut in period 0, the young nevertheless experience an increase in after-tax human wealth equal to

$$\begin{aligned}h_0'-h_0 &= \left(w-\tau_1+z-\frac{\tau_2+\bar{m}}{1+r}\right)-\left(w-\tau_1-\frac{\tau_2}{1+r}\right)\\ &= \left(1-\frac{r-n}{(1+r)(1+n)}\right)z \qquad \text{(by (6.40))}\\ &= \frac{1+(2+r)n}{(1+r)(1+n)}z>0. \end{aligned}\tag{6.42}$$

Consequently, through the *wealth effect* this generation enjoys increases in consumption through life equal to

$$c_{10}'-c_{10} = \hat{c}_1(r)(h_0'-h_0)>0, \qquad \text{and} \tag{6.43}$$

$$c_{21}'-c_{21} = \hat{c}_2(r)(h_0'-h_0)>0, \tag{6.44}$$

by (6.35) and (6.36), respectively. The two generations alive in period 0 thus gain from the temporary budget deficit.

All *future* generations are worse off, however. These generations do not benefit from the tax relief in period 0, but they have to bear the future cost of the tax relief by a *reduction* in individual after-tax human wealth. Indeed, for $t = 1, 2, \ldots,$

$$\begin{aligned} h'_t - h_t &= h'_1 - h = w - \tau_1 - \bar{m} - \frac{\tau_2 + \bar{m}}{1+r} - \left(w - \tau_1 - \frac{\tau_2}{1+r}\right) \\ &= -\left(\bar{m} + \frac{\bar{m}}{1+r}\right) = -\frac{2+r}{1+r}\bar{m} < 0. \end{aligned} \tag{6.45}$$

All things considered, since both the young and the old in period 0 increase their consumption, aggregate consumption in period 0 rises. Ricardian equivalence thus *fails*.

Effect on wealth accumulation*

How does aggregate *private* saving in period 0 respond to the temporary tax cut? Consider first the old in period 0. Along both the benchmark path and the alternative path the old entered period 0 with the financial wealth A_0 and they leave the period with zero financial wealth. So their aggregate net saving is $S_{20}^N = -A_0$ in both fiscal regimes. The young in period 0 increase their consumption in response to the temporary tax cut. At the same time they *increase* their period-0 saving. Indeed, from (6.44) and the period budget constraint as old follows

$$\begin{aligned} 0 &< c'_{21} - c_{21} = (1+r)s'_0 - (\tau_2 + \bar{m}) - ((1+r)s_0 - \tau_2) \\ &= (1+r)(s'_0 - s_0) - \bar{m} < (1+r)(s'_0 - s_0), \end{aligned}$$

thus implying $s'_0 - s_0 > 0$. The explanation is that the individuals have a preference for consumption smoothing in that $\theta > 0$. So the young in period 0 want to smooth out the increased consumption possibilities resulting from the increase in their human wealth. To be able to increase consumption as old, their extra saving, with interest, must exceed what is needed to pay the extra tax $\bar{m}$ in period 1. It is the tax cut that makes it possible for the young to increase both consumption and saving in period 0.

The impact on national wealth in period 1 The higher saving by the young in period 0 implies higher aggregate *private* financial wealth per old at the beginning of period 1, since $A'_1/L_0 = s'_0 > s_0 = A_1/L_0$. Nevertheless, gross *national* saving, cf. (6.31), is clearly lower than in the benchmark case. Indeed, $C_0' > C_0$ implies

$$S_0' = F(K_0, L_0) + r \cdot NFA_0 - C_0' - G_0 < F(K_0, L_0) + r \cdot NFA_0 - C_0 - G_0 = S_0.$$

That gross national saving is lower is not inconsistent with the just mentioned rise in *private* saving in period 0 compared to the benchmark path. A counterpart of the increased *private* saving is the *public dissaving*, reflecting that the tax cut in period 0 creates a budget deficit one-to-one. Since the increased disposable income implied by the tax cut is used partly to increase private saving *and* partly to increase private consumption, the rise in private saving is *smaller* than the public dissaving. So *total* or *national* saving in period 0 is reduced.

Consequently, we have:

(i) *National wealth* at the start of period 1 is lower in the debt regime than in the no-debt regime.

By how much? In the benchmark regime the national wealth at the start of period 1 is $V_1 = V_0 + S_0^N = V_0 + S_0 - \delta K_0$. This exceeds national wealth in the debt regime by

$$\begin{aligned} V_1 - V_1' &= S_0 - S_0' = C_0' - C_0 = c_{10}' L_0 + c_{20}' L_{-1} - (c_{10} L_0 + c_{20} L_{-1}) \\ &= (c_{10}' - c_{10}) L_0 + (c_{20}' - c_{20}) L_{-1} \\ &= \hat{c}_1(r)(h_0' - h_0) L_0 + z L_{-1} \quad \text{(by (6.43) and (6.41))} \\ &= \left(\hat{c}_1(r) \frac{1 + (2+r)n}{1+r} + 1 \right) \frac{1}{1+n} L_0 z > 0. \quad \text{(by (6.42))} \end{aligned} \tag{6.46}$$

Later consequences As revealed by (6.45), all future generations (those born in period $1, 2, \dots$) are worse off along the alternative path. This gives rise to two further claims:

(ii) *National wealth per old* along the alternative path, V_t'/L_{t-1}, will remain constant from period 2 onward at a level below that along the path without government debt.

(iii) The constant level along the alternative path from period 2 onward will even be below the level in period 1.

To substantiate these two claims, consider $V_t' \equiv A_t' - B_t'$. In Appendix A it is shown that *government debt* per old will from period 1 onward satisfy

$$\frac{B_t'}{L_{t-1}} = \frac{B_1'}{L_0} = \frac{(L_0 + L_{-1})z}{L_0} = \frac{2+n}{1+n} z, \qquad t = 1, 2, \dots,$$

and thus be constant. So government debt grows at the rate of population growth. In addition, Appendix A shows that *private* financial wealth per old is constant from period 2 onward and satisfies

$$\frac{A_t'}{L_{t-1}} = s_{t-1}' = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z, \qquad t = 2, 3, \ldots.$$

It follows that *national* wealth per old from period 2 onward will be

$$\begin{aligned} \frac{V_t'}{L_{t-1}} &\equiv \frac{A_t'}{L_{t-1}} - \frac{B_t'}{L_{t-1}} = s_{t-1}' - \frac{2+n}{1+n}z = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z - \frac{2+n}{1+n}z \\ &= s_0 - \left(1 - \hat{c}_1(r)\frac{r-n}{1+r}\right)\frac{2+r}{1+n}z = \frac{V_2'}{L_1} < s_0 = \frac{V_2}{L_1} = \frac{V_1}{L_0} \qquad t = 2, 3, \ldots \end{aligned} \tag{6.47}$$

where the last two equalities follow from (6.39). This proves our claim (ii).

National wealth per old in period 1 of the debt path is, by (6.46),

$$\begin{aligned} \frac{V_1'}{L_0} &= \frac{V_1}{L_0} - \left(\hat{c}_1(r)\frac{1+(2+r)n}{1+r} + 1\right)\frac{z}{1+n} \\ &= s_0 - \left(\hat{c}_1(r)\frac{1+(2+r)n}{1+r} + 1\right)\frac{z}{1+n} > \frac{V_2'}{L_1}, \end{aligned}$$

where the inequality follows by comparison with (6.47). This proves our claim (iii).

Period 1 is special compared to the subsequent periods. While there is a per capita tax increase by $\bar{m}$ like in the subsequent periods, period 1's old generation still benefits from the higher disposable income in period 0. Hence, in period 2 national wealth per old is even lower than in period 1 but remains constant henceforth.

A closed economy Also in a closed economy would a temporary lump-sum tax cut make the future generations worse off. Indeed, in view of reduced national saving in period 0, national wealth (which in the closed economy equals K) would from period 1 onward be smaller than along the no-debt path. The precise calculations are more complicated because the rate of interest will no longer be a constant.

6.7.3 Widening the perspective

The fundamental point underlined by OLG models is that there is a difference between the public sector's future tax base, including the resources of individuals yet to be born, and the future tax base emanating from individuals alive today.

This may be called the *composition-of-tax-base argument* for a tendency to non-neutrality of shifting the timing of (lump-sum) taxation.[28]

The conclusion that under full capacity utilization budget deficits imply a burden for future generations may be seen in a somewhat different light if persistent technological progress is included in the model. In that case, everything else equal, future generations will generally be better off than current generations. Then it might seem less unfair if the former carry some public debt forward to the latter. In particular this is so if a part of G_t represents spending on infrastructure, education, research, health, and environmental protection. As future generations directly benefit from such investment, it seems fair that they also contribute to the financing. This is the "benefits received principle" known from public finance theory.

A further concern is whether the economy is in a state of full capacity utilization or serious unemployment and idle capital. The above analysis assumes the first. What if the economy in period 0 is in economic depression with high unemployment due to insufficient aggregate demand? Some economists maintain that also in this situation is a cut in (lump-sum) taxes to stimulate aggregate demand futile because it has no real effect. The argument is again that foreseeing the higher taxes needed in the future, people will save more to prepare themselves (or their descendants through higher bequests) for paying the higher taxes in the future. The opposite view is, first, that the composition-of-tax-base argument speaks against this as usual. Second, there is in a depression an additional and quantitatively important factor. The "first-round" increase in consumption due to the temporary tax cut raises aggregate demand. Thereby production and income is stimulated and a further (but smaller) rise in consumption occurs in the "second round" and so on (the Keynesian multiplier process).

This Keynesian mechanism is important for the debate about effects of budget deficits because there are limits to how *large* deviations from Ricardian equivalence the composition-of-tax-base argument can deliver in the long-run life-cycle perspective of OLG models. Indeed, taking into account the sizeable life expectancy of the average citizen, Poterba and Summers (1987) point out that the composition-of-tax-base argument by itself delivers only modest deviations if the issue is timing of taxes over the business cycle. They find that to comply with the data on private saving responses to supposedly exogenous shifts in taxation should be combined with the hypothesis that households are "myopic" than what standard OLG models assume.

Another concern is that in the real world, taxes tend to be distortionary and

[28]In Exercise 6.?? the reader is asked how the burden of the public debt is distributed across generations if the debt should be completely wiped out through a tax increase in only periods 1 and 2.

not lump sum. On the one hand, this should not be seen as an argument against the possible *theoretical* validity of the Ricardian equivalence proposition. The reason is that Ricardian equivalence (in its strict meaning) claims absence of allocational effects of changes in the timing of *lump-sum* taxes.

On the other hand, in a wider perspective the interesting question is, of course, how changes in the timing of *distortionary* taxes is likely to affect resource allocation. Consider first *income taxes*. When taxes are proportional to income or progressive (average tax rate rising in income), they provide insurance through reducing the volatility of after-tax income. The fall in taxes in a recession thus helps stimulating consumption through *reduced precautionary saving* (the phenomenon that current saving tends to rise in response to increased uncertainty, cf. Chapter ??). In this way, replacing lump-sum taxation by income taxation underpins the positive wealth effect on consumption, arising from the composition-of-tax-base channel, of a debt-financed tax-cut in an economic recession.

What about *consumption taxes*? A debt-financed temporary cut in consumption taxes stimulates consumption through a positive wealth effect, arising from the composition-of-tax-base channel. On top of this comes a positive intertemporal substitution effect on current consumption caused by the changed consumer price time profile.

The question whether Ricardian non-equivalence is important from a quantitative and empirical point of view pops up in many contexts within macroeconomics. We shall therefore return to the issue several times later in this book.

6.8 Concluding remarks

(incomplete)

Point (iv) in Section 6.1 hints at the fact that when outcomes depend on forward-looking expectations in the private sector, governments may face a time-inconsistency problem. In this context *time inconsistency* refers to the possible temptation of the government to deviate from its previously announced course of action once the private sector has acted. An example: With the purpose of stimulating private saving, the government announces that it will not tax financial wealth. Nevertheless, when financial wealth has reached a certain level, it constitutes a tempting base for taxation and so a tax on wealth might be levied. To the extent the private sector anticipates this, the attempt to affect private saving in the first place fails. This raises issues of *commitment* and *credibility*. We return to this kind of problems in later chapters.

Finally, point (v) in Section 6.1 alludes to the fact that political processes, bureaucratic self-interest, rent seeking, and lobbying by powerful interest groups interferes with fiscal policy.[29] This is a theme in the branch of economics called *political economy* and is outside the focus of this chapter.

6.9 Literature notes

(incomplete)

Sargent and Wallace (1981) study consequences of − and limits to − a shift from debt financing to money financing of sustained government budget deficits in response to threatening increases in the government debt-income ratio.

How the condition $r > g_Y$, for prudent debt policy to be necessary, is modified when the assumption of no uncertainty is dropped is dealt with in Abel et al. (1989), Bohn (1995), Ball et al. (1998), and Blanchard and Weil (2001). On self-fulfilling sovereign debt crises, see, e.g., Cole and Kehoe (2000).

Readers wanting to go more into detail with the policy-oriented debate about the design of the EMU and the Stability and Growth Pact is referred to the discussions in for example Buiter (2003), Buiter and Grafe (2004), Fogel and Saxena (2004), Schuknecht (2005), and Wyplosz (2005). As to discussions of the actual functioning of monetary and fiscal policy in the Eurozone in response to the Great Recession, see for instance the opposing views by De Grauwe and Ji (2013) and Buti and Carnot (2013). Blanchard and Giavazzi (2004) discuss how proper accounting of public investment would modify the deficit and debt rules of the EMU. Beetsma and Giuliodori (2010) survey recent research of costs and benefits of the EMU.

On the theory of *optimal currency areas,* see Krugman, Obstfeld, and Melitz (2012).

In addition to the hampering of Keynesian stabilization policy discussed in Section 6.4.2, also demographic staggering (due to baby booms succeeded by baby busts) may make rigid deficit rules problematic. In Denmark for instance demographic staggering is prognosticated to generate considerable budget deficits during several decades after 2030 where younger and smaller generations will succeed older and larger ones in the labor market. This is prognosticated to take place, however, without challenging the long-run sustainability of current fiscal policy as assessed by the Danish Economic Council (see the English Summary in De Økonomiske Råd, 2014). This phenomenon is in Danish known as "hængekøjeproblemet" (the "hammock problem").

Sources for last part of Section 6.7 ....

[29] *Rent seeking* refers to attempts to gain by increasing one's share of existing wealth, instead of trying to *produce* wealth.

6.10 Appendix A

In Section 6.7.2 we asserted that along the alternative path the government debt will grow at the same rate as the population. The proof is as follows.

The law of motion of the debt is, for $t = 1, 2, \ldots,$

$$\begin{aligned} B'_{t+1} &= (1+r)B'_t + G_t - T'_t = (1+r)B'_t + G_t - \left(\tau_1 + \frac{\tau_2}{1+n} + \bar{m} + \frac{\bar{m}}{1+n}\right) L_t \\ &= (1+r)B'_t - \left(\bar{m} + \frac{\bar{m}}{1+n}\right) L_t = (1+r)B'_t - \frac{2+n}{1+n}\bar{m}L_t, \end{aligned}$$

where the second line follows from $G_t - (\tau_1 + \tau_2(1+n))L_t = 0$ in view of the balanced budget along the benchmark path. It is convenient to rewrite the law of motion in terms of $x_t \equiv B'_t/L_{t-1}$, i.e., government debt per old. We get

$$x_{t+1} \equiv \frac{B'_{t+1}}{L_t} = \left(\frac{1+r}{1+n}\right) x_t - \frac{2+n}{1+n}\bar{m}, \qquad t = 1, 2, \ldots,$$

where we have used that $L_t = (1+n)L_{t\ 1}$. The solution of this first-order difference equation with constant coefficients is

$$x_t = (x_1 - x^*)\left(\frac{1+r}{1+n}\right)^{t-1} + x^*,$$

with

$$\begin{aligned} x_1 &= \frac{B'_1}{L_0} = \frac{(L_0 + L_{-1})z}{L_0} = \frac{2+n}{1+n}z, \quad \text{and} \\ x^* &= -\frac{2+n}{1+n}\bar{m}\left(1 - \frac{1+r}{1+n}\right)^{-1} = \frac{2+n}{r-n}\bar{m} = \frac{2+n}{1+n}z, \end{aligned}$$

using the solution (6.40) for the tax rise $\bar{m}$. It follows that x_t is constant over time and equals x^*. Hence, from period 1 onward $B'_t/L_{t-1} = (2+n)z/(1+n)$ where z is the per capita tax cut in period 0. □

In Section 6.7.2 we also asserted that along the alternative path the private financial wealth per old will from period 2 onward be constant. The proof is as follows:

For $t = 2, 3, \ldots,$

$$\begin{aligned} \frac{A'_t}{L_{t-1}} &= s'_{t-1} = w - (\tau_1 + \bar{m}) - c'_{1t-1} = w - (\tau_1 + \bar{m}) - \hat{c}_1(r)\left(w - \tau_1 - \bar{m} - \frac{\tau_2 + \bar{m}}{1+r}\right) \\ &= w - \tau_1 - \hat{c}_1(r)(w - \tau_1 - \frac{\tau_2}{1+r}) - \bar{m} + \hat{c}_1(r)\bar{m} + \hat{c}_1(r)\frac{\bar{m}}{1+r} \\ &= s_0 - \left(1 - \hat{c}_1(r)\left(1 + \frac{1}{1+r}\right)\right)\bar{m} = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z, \end{aligned}$$

where we have used (6.33), the period budget constraint of the young along the alternative path, (6.35), (6.34), the period budget constraint of the young along the benchmark path, the constancy of saving by the young along the benchmark path, and finally the solution for the tax rise $\bar{m}$. We see that private financial wealth per old is constant from period 2 onward. □

6.11 Exercises

6.? Consider the OLG model of Section 6.7. a) Show that if the temporary per capita tax cut, z, is sufficiently small, the debt can be completely wiped out through a per capita tax increase in only periods 1 and 2. b) Investigate how in this case the burden of the debt is distributed across generations. Compare with the alternative debt policy described in the text.

Chapter 7

Bequests and the modified golden rule

This chapter modifies the Diamond OLG model by including a *bequest* motive. Depending on what form and what strength the bequest motive has, distinctive new conclusions may arise. Indeed, under certain conditions the long-run real interest rate, for instance, turns out to be determined by a very simple principle, the *modified golden rule*. The chapter also spells out both the logic and the limitations of the hypothesis of Ricardian equivalence (government debt neutrality).

7.1 Bequests

In Diamond's OLG model individuals care only about their own lifetime utility and never leave bequests. Yet, in reality a sizeable part of existing private wealth is attributable to inheritance rather than own life-cycle saving. Among empiricists there is considerable disagreement as to the exact quantities, though. Kotlikoff and Summers (1981) estimate that 70-80 % of private financial wealth in the US is attributable to intergenerational transfers and only 20-30 % to own life-cycle savings. On the other hand, Modigliani (1988) suggests that these proportions more or less should be reversed.

The possible motives for leaving bequests include:

1. "Altruism". Parents care about the welfare of their descendants and leave bequests accordingly. This is the hypothesis advocated by the American economist Robert Barro (1974). Its implications are the main theme of this chapter.

2. "Joy of giving". Parents' utility depends not on descendants' utility, as with motive 1, but directly on the size of the bequest. That is, parents

simply have taste for generosity. They enjoy giving presents to their children (Andreoni, 1989, Bossman et al., 2007, Benhabib et al., 2011). Or a more sinister motive may be involved, such as the desire to manipulate your children's behavior (Bernheim et al., 1985).

3. "Joy of wealth". Dissaving may be undesirable even at old age if wealth, or the power and prestige which is associated with wealth, is an independent argument in the utility function (Zou, 1995). Then the profile of individual financial wealth through life may have positive slope at all ages. At death the wealth is simply passed on to the heirs.

In practice an important factor causing bequests is uncertainty about time of death combined with the absence of complete annuity markets. In this situation unintentional bequests arise. Gale and Scholz (1994) find that only about half of net wealth accumulation in the US represents intended transfers and bequests.

How transfers and bequests affect the economy depends on the reasons why they are made. We shall concentrate on a model where bequests reflect the concern of parents for the welfare of their offspring (motive 1 above).

7.2 Barro's dynasty model

We consider a model of overlapping generations linked through altruistic bequests, suggested by Barro (1974). Among the interesting results of the model are that if the extent of altruism is sufficiently high so that the bequest motive is operative in every period, then:

- The differences in age in the population becomes inconsequential; the household sector appears as consisting of a finite number of infinitely-lived dynasties, all alike. In brief, the model becomes a *representative agent model.*
- A simple formula determining the long-run real interest rate arises: the *modified golden rule.*
- *Ricardian equivalence* arises.
- Resource allocation in a competitive market economy coincides with that accomplished by a social planner who has the same intergenerational discount rate as the representative household dynasty.

This chapter considers the three first bullets in detail, while the last bullet is postponed to the next chapter.

7.2.1 A forward-looking altruistic parent

Technology, demography, and market conditions are as in Diamond's OLG model. There is no utility from leisure. Perfect foresight is assumed. Until further notice technological progress is ignored.

The preferences of a member of generation t are given by the utility function

$$U_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) + (1+R)^{-1}(1+n)U_{t+1}. \tag{7.1}$$

Here R is the pure *inter*generational utility discount rate and $1+n$ is the number of offspring per parent in society. So R measures the extent of "own generation preference" (strength of "self preference") and n is the population growth rate. As usual we in (7.1) ignore indivisibility problems and take an average view. The term $u(c_{1t})$ $+(1+\rho)^{-1}u(c_{2t+1})$ is the "own lifetime utility", reflecting the utility contribution from one's own consumption as young, c_{1t}, and as old, c_{2t+1}.The time preference parameter ρ appears as the *intra*generational utility discount rate. Although $\rho > 0$ is plausible, the results to be derived do not depend on this inequality, so we just impose the formal restriction that $\rho > -1$.

The term $(1+R)^{-1}(1+n)U_{t+1}$ in (7.1) is the contribution derived from the utility of the offspring. The intergenerational utility discount factor $(1+R)^{-1}$ is also known as the *altruism* factor.

The *effective* intergenerational utility discount rate is the number $\bar{R}$ satisfying

$$(1+\bar{R})^{-1} = (1+R)^{-1}(1+n). \tag{7.2}$$

We assume $R > n \geq 0$. So $\bar{R}$ is positive and the utility of the next generation is weighed through an *effective* discount factor $(1+\bar{R})^{-1} < 1$. (Mathematically, the model works well with the weaker assumption, $n > -1$; yet it helps intuition to imagine that there always is at least one child per parent.)

We write (7.1) on recursive form,

$$U_t = V_t + (1+\bar{R})^{-1}U_{t+1},$$

where V_t is the "direct utility" $u(c_{1t})+(1+\rho)^{-1}u(c_{2t+1})$, whereas $(1+\bar{R})^{-1}U_{t+1}$ is the "indirect utility" through the offspring's well-being. By forward substitution j periods ahead in (7.1) we get

$$U_t = \sum_{i=0}^{j}(1+\bar{R})^{-i}V_{t+i} \quad + \quad (1+\bar{R})^{-(j+1)}U_{t+j+1}.$$

Provided U_{t+j+1} does not grow "too fast", taking the limit for $j \to \infty$ gives

$$U_t = \sum_{i=0}^{\infty}(1+\bar{R})^{-i}V_{t+i} = \sum_{i=0}^{\infty}(1+\bar{R})^{-i}\left[u(c_{1t+i}) + (1+\rho)^{-1}u(c_{2t+i+1})\right]. \tag{7.3}$$

By $\sum_{i=0}^{\infty}(1+\bar{R})^{-i}V_{t+i}$ we mean $\lim_{j\to\infty}\sum_{i=0}^{j}(1+\bar{R})^{-i}V_{t+i}$, assuming this limit exists. Although each generation cares directly only about the next generation, this series of intergenerational links implies that each generation acts as if it cared about all future generations in the dynastic family, although with decreasing weight. In this way the entire dynasty can be regarded as a single agent with infinite horizon. At the same time the coexisting dynasties are completely alike. So, in spite of starting from an OLG structure, we now have a *representative agent model*, i.e., a model where the household sector consists of completely alike decision makers.

Note that *one-parent* families fit Barro's notion of clearly demarcated dynastic families best. Indeed, the model abstracts from the well-established fact that breeding arises through mating between a man and a woman who come from two *different* parent families. In reality this gives rise to a complex network of interconnected families. Until further notice we ignore such complexities and proceed by imagining reproduction is not sexual.

In each period two (adult) generations coexist: the "young", each of whom supplies one unit of labor inelastically, and the "old" who do not supply labor. Each old is a parent to $1+n$ of the young. And each young is a parent to $1+n$ children, born when the young parent enters the economy and the grandparent retires from the labor market. Next period these children become visible in the model as the young generation in the period.

Let b_t be the bequest received at the end of the first period of "economic life" by a member of generation t from the old parent, belonging to generation $t-1$. In turn this member of generation t leaves in the next period a bequest to the next generation in the family and so on. We will assume, realistically, that negative bequests are ruled out by law; the legal system exempts children from responsibility for parental debts. Thus the budget constraints faced by a young member of generation t are:

$$c_{1t}+s_t = w_t+b_t, \tag{7.4}$$

$$c_{2t+1}+(1+n)b_{t+1} = (1+r_{t+1})s_t, \qquad b_{t+1}\geq 0, \tag{7.5}$$

where s_t denotes saving as young (during work life) out of the sum of labor income and the bequest received (payment for the consumption and receipt of w_t+b_t occur at the end of the period). In the next period the person is an old parent and ends life leaving a bequest, b_{t+1}, to each of the $1+n$ children. Fig. 7.1 illustrates.

What complicates the analysis is that even though a bequest motive is present, the market circumstances may be such that parents do not find it worthwhile to transmit positive bequests. We then have a corner solution, $b_{t+1}=0$. An important element in the analysis is to establish *when* this occurs and when it

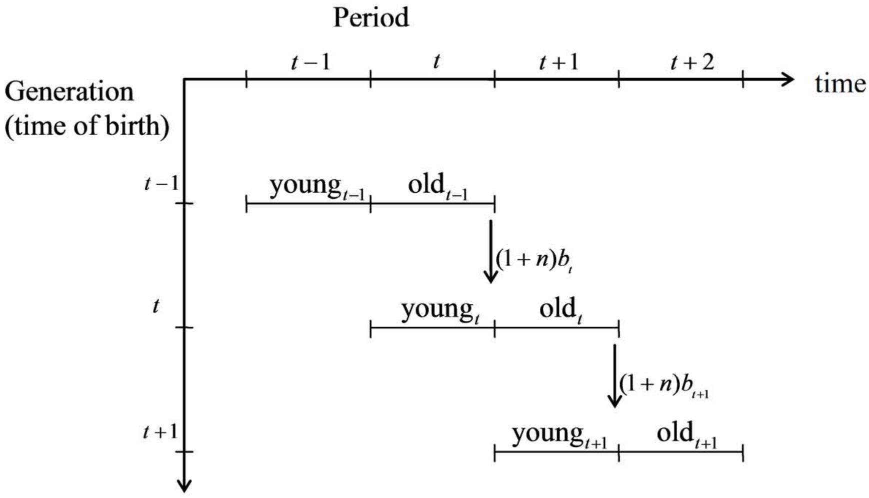

Figure 7.1: The generational structure of the Barro model.

does not. To rule out the theoretical possibility of a corner solution also for s_t, we impose the No Fast Assumption

$$\lim_{c \to 0} u'(c) = \infty. \tag{A1}$$

Given this assumption, the young will always choose $s_t > 0$ as soon as $w_t + b_t > 0$.

Consider a person belonging to generation t. This person has perfect foresight with regard to future wages and interest rates and can compute the optimal choices of the descendants conditional on the bequest they receive. The planning problem is:

$$\begin{aligned} \max U_t &= u(c_{1t}) + (1+\rho)^{-1} u(c_{2t+1}) \\ &\quad + (1+\bar{R})^{-1} \left[u(c_{1t+1}) + (1+\rho)^{-1} u(c_{2t+2}) \right] + \ ... \end{aligned}$$

subject to the budget constraints (7.4) and (7.5) and knowing that the descendants will respond optimally to the received bequest (see below). We insert into U_t the two budget constraints in order to consider the objective of the parent as a function, $\tilde{U}_t$, of the decision variables, s_t and b_{t+1}. We then maximize with respect to s_t and b_{t+1}. First:

$$\begin{aligned} \frac{\partial \tilde{U}_t}{\partial s_t} &= -u'(c_{1t}) + (1+\rho)^{-1} u'(c_{2t+1})(1 + r_{t+1}) = 0, \text{ i.e.,} \\ u'(c_{1t}) &= (1+\rho)^{-1} u'(c_{2t+1})(1 + r_{t+1}). \end{aligned} \tag{7.6}$$

This first-order condition deals with the distribution of own consumption across time. The condition says that in the optimal plan the opportunity cost of saving one more unit as young must equal the discounted utility benefit of having $1+r_{t+1}$ more units for consumption as old.

As to maximizing with respect to the second decision variable, b_{t+1}, we have to distinguish between the case where the bequest motive is operative and the case where it is not. Here we consider the first case and postpone the second for a while.

7.2.2 Case 1: the bequest motive operative ($b_{t+1} > 0$ optimal)

In addition to (7.6) we get the first-order condition

$$\frac{\partial \tilde{U}_t}{\partial b_{t+1}} = (1+\rho)^{-1}u'(c_{2t+1})\left[-(1+n)\right] + (1+\bar{R})^{-1}u'(c_{1t+1}) \cdot 1 = 0, \text{ i.e.,}$$

$$(1+\rho)^{-1}u'(c_{2t+1}) = (1+\bar{R})^{-1}u'(c_{1t+1})(1+n)^{-1}. \tag{7.7}$$

This first-order condition deals with the distribution of consumption across generations in the same period. The condition says that in the optimal plan the parent's utility cost of increasing the bequest by one unit (thereby decreasing the consumption as old by one unit) must equal the discounted utility benefit derived from the next generation having $1/(1+n)$ more units, per member, for consumption in the same period. The factor, $(1+n)^{-1}$, is a "dilution factor" due to total bequests being diluted in view of the $1+n$ children for each parent.

A further necessary condition for an optimal plan is that the bequests are not forever too high, which would imply postponement of consumption possibilities forever. That is, we impose the condition

$$\lim_{i\to\infty}(1+\bar{R})^{-(i-1)}(1+\rho)^{-1}u'(c_{2t+i})(1+n)b_{t+i} = 0. \tag{7.8}$$

Such a terminal condition is called a *transversality condition*; it acts as a necessary first-order condition at the terminal date, here at infinity. Imagine a plan where instead of (7.8) we had $\lim_{i\to\infty}(1+\bar{R})^{-(i-1)}(1+\rho)^{-1}u'(c_{2t+i})(1+n)b_{t+i} > 0$. In this case there would be "over-bequeathing" in the sense that U_t (the sum of the generations' discounted lifetime utilities) could be increased by the ultimate generation consuming more as old and bequeathing less. Decreasing the ultimate bequest to the young, b_{t+i} ($i \to \infty$), by one unit would imply $1+n$ extra units for consumption for the old parent, thereby increasing this parent's utility by $(1+\rho)^{-1}u'(c_{2t+i})(1+n)$. From the perspective of the current generation t this utility contribution should be discounted by the discount factor $(1+\bar{R})^{-(i-1)}$. With a finite time horizon entailing that only $i-1$ future generations (i fixed) were cared about, it would be waste to end up with $b_{t+i} > 0$; optimality would require $b_{t+i} = 0$. The condition (7.8) is an extension of this principle to an infinite horizon.[1]

The optimality conditions (7.6), (7.7), and (7.8) illustrate a general principle of intertemporal optimization. First, no gain should be achievable by a reallocation of resources between two periods or between two generations. This is taken care of by the *Euler equations* (7.6) and (7.7).[2] Second, there should be nothing of value leftover after the "last period", whether the horizon is finite or infinite. This is taken care of by a transversality condition, here (7.8). With a finite horizon, the transversality condition takes the simple form of a condition saying that the intertemporal budget constraint is satisfied with equality. In the two-period models of the preceding chapters, transversality conditions were implicitly satisfied in that the budget constraints were written with $=$ instead of $\leq$.

The reader might be concerned whether in our maximization procedure, in particular regarding the first-order condition (7.7), we have taken the descendants' optimal responses properly into account. The parent should choose s_t and b_{t+1} to maximize U_t, taking into account the descendants' optimal responses to the received bequest, b_{t+1}. In this perspective it might seem inadequate that we have considered only the partial derivative of U_t w.r.t. b_{t+1}, not the total derivative. Fortunately, in view of the *envelope theorem* our procedure is valid. Applied to the present problem, the envelope theorem says that in an interior optimum the total derivative of U_t w.r.t. b_{t+1} *equals* the partial derivative w.r.t. b_{t+1}, evaluated at the optimal choice by the descendants. Indeed, since an objective function "is flat at the top", the descendants' response to a marginal change in the received bequest has a negligible effect on the value of optimized objective function (for details, see Appendix A).

The old generation in period t So far we have treated the bequest, b_t, received by the young in the current period, t, as given. But in a sense also this bequest is a choice, namely a choice made by the old parent in this period, hence endogenous. This old parent enters period t with assets equal to the saving made as young, s_{t-1}, which, if period t is the initial period of the model, is part of the arbitrarily *given* initial conditions of the model. From this perspective the decision problem for this old parent is to choose $b_t \geq 0$ so as to

$$\max \left[(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t\right]$$

[1] Although such a simple extension of a transversality condition from a finite horizon to an infinite horizon is not always valid, it *is* justifiable in the present case. This and related results about transversality conditions are dealt with in detail in the next chapter.

[2] Since $\tilde{U}_t(s_t, b_{t+1})$ is jointly strictly concave in its two arguments, the Euler equations are not only necessary, but also sufficient for a unique interior maximum.

subject to the budget constraint $c_{2t} + (1+n)b_t = (1+r_t)s_{t-1}$ and taking into account that the chosen b_t indirectly affects the maximum lifetime utility to be achieved by the next generation. If the optimal b_t is positive, the choice satisfies the first-order condition (7.7) with t replaced by $t-1$. And if no disturbance of the economy has taken place at the transition from period $t-1$ to period t, the decision by the old in period 0 is simply to do exactly as planned when young in the previous period.

It may seem puzzling that $u(c_{2t})$ is discounted by the factor $(1+\rho)^{-1}$, when standing *in* period t. Truly, when thinking of the old parent as maximizing $(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t$, the utility is discounted back (as usual) to the first period of adult life, in this case period $t-1$. But this is just one way of presenting the decision problem of the old. An alternative way is to let the old maximize the present value of utility as seen from the current period t,

$$\begin{aligned}(1+\rho)\left[(1+\rho)^{-1}u(c_{2t}) + (1+\bar{R})^{-1}U_t\right] &= u(c_{2t}) + (1+\bar{R})^{-1}(1+\rho)U_t \\ &\equiv u(c_{2t}) + (1+\psi)^{-1}U_t,\end{aligned}$$

where the last equality follows by merging the backward and forward discounts, $(1+\bar{R})^{-1}$ and $1+\rho$, respectively, into the coefficient $(1+\psi)^{-1}$. Since both utilities, $u(c_{2t})$ and U_t, arise in the *same* period, the coefficient $(1+\psi)^{-1}$ is no time discount factor but an expression for the degree of unselfishness, see the remark below. As we have just multiplied the objective function by a positive constant, $1+\beta$, the resulting behavior is unaffected.

Remark The effective intergenerational utility discount factor can be decomposed as in (7.2) above, but also as:

$$(1+\bar{R})^{-1} \equiv (1+\rho)^{-1}(1+\psi)^{-1}. \tag{7.9}$$

The sub-discount factor, $(1+\rho)^{-1}$, applies because the prospective utility arrives one period later and is, in this respect, comparable to utility from own consumption when old, c_{2t+1}. The sub-discount factor, $(1+\psi)^{-1}$, can be seen as the *degree of unselfishness* and ψ as reflecting the extent of *selfishness*. Indeed, when ψ is positive, parents are selfish in the sense that, if a parent's consumption when old equals the children's' consumption when young, then the parent prefers to keep an additional unit of consumption for herself instead of handing it over to the next generation (replace $(1+\bar{R})^{-1}$ in (7.7) by (7.9)). □

The equilibrium path

Inserting (7.7) on the right-hand side of (7.6) gives

$$u'(c_{1t}) = (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1+r_{t+1}}{1+n}. \tag{7.10}$$

This is an Euler equation characterizing the optimal distribution of consumption across generations in different periods: in an optimal intertemporal and intergenerational allocation, the utility cost of decreasing consumption of the young in period t (that is, saving and investing one more unit) must equal the discounted utility gain next period for the next generation which, per member, will be able to consume $(1 + r_{t+1})/(1 + n)$ more units.

With perfect competition and neoclassical CRS technology, in equilibrium the real wage and interest rate are

$$w_t = f(k_t) - f'(k_t)k_t \text{ and } r_t = f'(k_t) - \delta, \tag{7.11}$$

respectively, where f is the production function on intensive form, and δ is the capital depreciation rate, $0 \leq \delta \leq 1$. Further, $k_t \equiv K_t/L_t$, where K_t is aggregate capital in period t owned by that period's old, and L_t is aggregate labor supply in period t which is the same as the number of young in that period.

As in the Diamond model, aggregate consumption per unit of labor satisfies the technical feasibility constraint

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} \equiv (c_{1t}L_t + c_{2t}L_{t-1})/L_t = c_{1t} + c_{2t}/(1+n) \\ &= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \end{aligned} \tag{7.12}$$

An *equilibrium path* for $t = 0, 1, 2,\ldots$, is described by the first-order conditions (7.7) ("backwarded" one period) and (7.10), the transversality condition (7.8), the equilibrium factor prices in (7.11), the resource constraint (7.12), and an initial condition in the form of a given $k_0 > 0$. This k_0 may be interpreted as reflecting a given $s_{-1} \geq 0$. Indeed, as in the Diamond model, for every $t = 0, 1,\ldots$, we have

$$k_{t+1} \equiv \frac{K_{t+1}}{L_{t+1}} = \frac{s_t L_t}{L_{t+1}} = \frac{s_t}{1+n}. \tag{7.13}$$

This is simply a matter of accounting. At the beginning of period $t + 1$ the available aggregate capital stock equals the financial wealth of the generation which now is old but was young in the previous period and saved $s_t L_t$ out of that period's labor income plus received transfers in the form of bequests. So $K_{t+1} = s_t L_t$. Indeed, the new young generation of period $t + 1$ own to begin with nothing except their brain and bare hands, although they *expect* to receive a bequest just before they retire from the labor market at the end of period $t+1$. Still another interpretation of (7.13) starts from the general accounting principle for a closed economy that the increase in the capital stock equals aggregate net saving. In turn aggregate net saving is the sum of net saving by the young, S_{1t}, and net saving by the old, S_{2t}:

$$\begin{aligned}
K_{t+1} - K_t &= S_{1t} + S_{2t} = s_t L_t + [r_t s_{t-1} - (c_{2t} + (1+n)b_t)] L_{t-1} \\
&= s_t L_t + [r_t s_{t-1} L_{t-1} - (1+r_t) s_{t-1} L_{t-1}] \qquad \text{(by ((7.5))} \\
&= s_t L_t + (-s_{t-1} L_{t-1}) = s_t L_t - K_t, \qquad (7.14)
\end{aligned}$$

which by eliminating $-K_t$ on both sides gives $K_{t+1} = s_t L_t$.

One of the results formally proved in the next chapter is that, given the technology assumption $\lim_{k\to 0} f'(k) > R + \delta > \lim_{k\to\infty} f'(k)$, an equilibrium path exists and converges to a steady state. In that chapter it is also shown that the conditions listed are exactly those that also describe a certain "command optimum". This refers to the allocation brought about by a social planner having (7.3) as the social welfare function.

Steady state with an operative bequest motive: the modified golden rule

A steady state of the system is a state where, for all t, $k_t = k^*$, $c_t = c^*$, $c_{1t} = c_1^*$, $c_{2t} = c_2^*$, $b_t = b^*$, and $s_t = s^*$. In a steady state with $b^* > 0$, we have the remarkably simple result that the interest rate, r^*, satisfies the *modified golden rule*:

$$1 + r^* = 1 + f'(k^*) - \delta = (1 + \bar{R})(1 + n) \equiv 1 + R. \qquad (7.15)$$

This follows from inserting the steady state conditions ($k_t = k^*$, $c_{1t} = c_1{}^*$, $c_{2t} = c_2{}^*$, for all t) into (7.10) and rearranging. In the *golden rule* of Chapter 3 the interest rate (reflecting the net marginal productivity of capital) equals the output growth rate (here n). The "modification" here comes about because of the strictly positive effective intergenerational discount rate $\bar{R}$, which implies a higher interest rate.

Two things are needed for the economy with overlapping generations linked through bequests to be in a steady state with positive bequests. First, it is necessary that the rate of return on saving matches the rate of return, R, *required* to tolerate a marginal decrease in own current consumption for the benefit of the next generation. This is what (7.10) shows. Second, it is necessary that the rate of return induces an amount of saving such that the consumption of each of the children equals the parent's consumption as young in the previous period. Otherwise the system would not be in a steady state. If in (7.15) "=" is replaced by ">" ("<"), then there would be a temptation to save more (less), thus generating more (less) capital accumulation, thereby pushing the system away from a steady state.

A steady state with an operative bequest motive is unique. Indeed, (7.15) determines a unique k^* (since $f'' < 0$), which gives $s^* = (1 + n)k^*$ by (7.13)

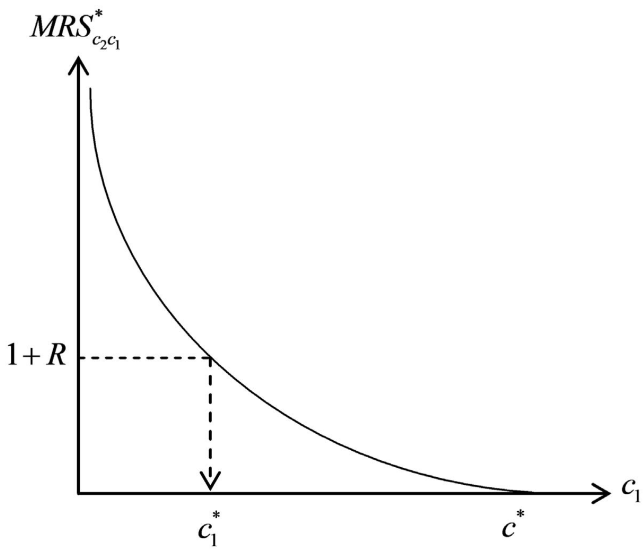

Figure 7.2: Steady state consumption as young.

and determines c^* uniquely from the second line in (7.12). In turn, we can find c_1^* and c_2^* from the Euler equation (7.6) which implies that the marginal rate of substitution of c_2 for c_1 in steady state takes the form

$$MRS^*_{c_2c_1} = \frac{u'(c_1^*)}{(1+\rho)^{-1}u'(c_2^*)} = \frac{u'(c_1^*)}{(1+\rho)^{-1}u'\left((1+n)(c^*-c_1^*)\right)} = 1+r^* = 1+R, \tag{7.16}$$

where the second equality comes from (7.12). Given c^*, $MRS_{c_2c_1}$ is a decreasing function of c_1^* so that a solution for c_1^* in (7.16) is unique. In view of the No Fast Assumption (A1), (7.16) always *has* a solution in c_1^*, cf. Fig. 7.2. Then, from (7.16), $c_2^* = (1+n)(c^* - c_1^*)$. Finally, from the period budget constraint (7.4), $b^* = c_1^* + s^* - w^* = c_1^* + (1+n)k^* - w^*$, where $w^* = f(k^*) - f'(k^*)k^*$, from (7.11).

But what if the market circumstances and preferences in combination are such that the constraint $b_{t+1} \geq 0$ becomes binding? Then the bequest motive is not operative. We get a corner solution such that the equality sign in (7.15) is replaced by $\leq$. The economy behaves like Diamond's OLG model and a steady state of the economy is not necessarily unique. To see these features, we now reconsider the young parent's optimization problem.

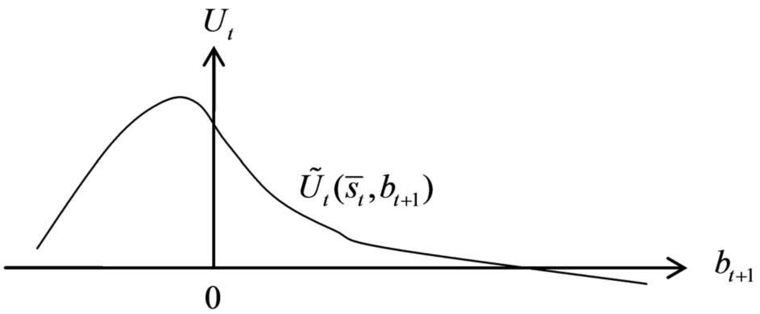

Figure 7.3: Configuration where the constraint $b_{t+1} \geq 0$ is binding (s_t fixed at $\bar{s}_t$).

7.2.3 Case 2: the bequest motive not operative ($b_{t+1} = 0$ optimal)

The first-order condition (7.6) involving s_t is still valid. But the first-order condition involving b_{t+1} becomes an inequality:

$$\frac{\partial \tilde{U}_t}{\partial b_{t+1}} = (1+\rho)^{-1}u'(c_{2t+1})\left[-(1+n)\right] + (1+\bar{R})^{-1}u'(c_{1t+1}) \cdot 1 \leq 0, \quad \text{i.e.,}$$

$$(1+\rho)^{-1}u'(c_{2t+1}) \geq (1+\bar{R})^{-1}u'(c_{1t+1})(1+n)^{-1}. \tag{7.17}$$

This condition says that in an optimal plan the parent's utility cost of increasing the bequest by one unit of account as an old parent must be larger than or equal to the discounted benefit derived from the next generation having $(1+n)^{-1}$ more units for consumption in the same period.

Why can we not exclude the possibility that $\partial \tilde{U}_t/\partial b_{t+1} < 0$ in the first line of (7.17) when $b_{t+1} = 0$ is optimal? To provide an answer, observe first that if we had $\partial \tilde{U}_t/\partial b_{t+1} > 0$ for $b_{t+1} = 0$, then at the prevailing market conditions a state with $b_{t+1} = 0$ could not be an optimum for the individual. Instead positive bequests would be induced. If, however, for $b_{t+1} = 0$, we have $\partial \tilde{U}_t/\partial b_{t+1} \leq 0$, then at the prevailing market conditions the state $b_{t+1} = 0$ *is* optimal for the individual. This is so even if "<" holds, as in Fig. 7.3. Although the market circumstances here imply a *temptation* to decrease b_{t+1} from the present zero level, by law that temptation cannot be realized.

Substituting (7.17) and $r_{t+1} = f'(k_{t+1}) - \delta$ into (7.6) implies that (7.10) is replaced by

$$u'(c_{1t}) \geq (1+\bar{R})^{-1}u'(c_{1t+1})\frac{1+f'(k_{t+1})-\delta}{1+n}. \tag{7.18}$$

Inserting into this the steady state conditions ($k_t = k^*$, $c_{1t} = c_1{}^*$, $c_{2t} = c_2{}^*$, for all t) and rearranging give

$$1 + r^* = 1 + f'(k^*) - \delta \leq (1+\bar{R})(1+n) \equiv 1 + R. \tag{7.19}$$

Since optimal b_{t+1} in case 2 is zero, everything is as if the bequest-motivating term, $(1+R)^{-1}(1+n)U_{t+1}$, were eliminated from the right-hand side of (7.1). Thus, as expected the model behaves like a Diamond OLG model.

7.2.4 Necessary and sufficient conditions for the bequest motive to be operative

An important question is: under what conditions does the bequest motive turn out to be operative? To answer this we limit ourselves to an analysis of a neighborhood of the steady state. We consider the thought experiment: if the bequest term, $(1+\bar{R})^{-1}U_{t+1}$, is eliminated from the right-hand side of (7.1), what will the interest rate be in a steady state? The utility function then becomes

$$\bar{U}_t = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}),$$

which is the standard lifetime utility function in a Diamond OLG model. We will call $\bar{U}_t$ the *truncated utility function* associated with the given true utility function, U_t. The model resulting from replacing the true utility function of the economy by the truncated utility function will be called the *associated Diamond economy.*

It is convenient to assume that the associated Diamond economy is well-behaved, in the sense of having a unique non-trivial steady state. Let r_D denote the interest rate in this Diamond steady state (the suffix D for Diamond). Then:

PROPOSITION 1 *(the cut-off value for the own-generation preference, R)* Consider an economy with a bequest motive as in (7.1) and satisfying the No Fast Assumption (A1). Let $R > n,$ i.e., $\bar{R} > 0$. Suppose that the associated Diamond economy is well-behaved and has steady-state interest rate r_D. Then in a steady state of the economy with a bequest motive, bequests are positive if and only if

$$R < r_D. \tag{*}$$

Proof. From (7.7) we have in steady state of the original economy with a bequest motive:

$$\begin{aligned}
\frac{\partial \tilde{U}_t}{\partial b_{t+1}} &= (1+\rho)^{-1}u'(c_2{}^*)(-(1+n)) + (1+\bar{R})^{-1}u'(c_1{}^*) \\
&= -(1+\rho)^{-1}(1+n)u'(c_2{}^*) \\
&\quad +(1+\bar{R})^{-1}(1+\rho)^{-1}(1+r^*)u'(c_2{}^*) \qquad \text{(from (7.6))} \\
&= (1+\rho)^{-1}u'(c_2{}^*)\left[(1+\bar{R})^{-1}(1+r^*) - (1+n)\right] \gtreqless 0 \text{ if and only if} \\
1+r^* &\gtreqless (1+\bar{R})(1+n), \text{ i.e., if and only if } r^* \gtreqless R, \text{ respectively,} \qquad (**)
\end{aligned}$$

since $(1+\bar{R})(1+n) \equiv 1+R$.

The "if" part: Suppose the associated Diamond economy is in steady state with $r_D > R$. Can this Diamond steady state (where by construction $b_{t+1} = 0$) be an equilibrium also for the original economy? The answer is no because, if we assume it were an equilibrium, then the interest rate would be $r^* = r_D > R$ and, by (**), $\partial\tilde{U}_t/\partial b_{t+1} > 0$. Therefore the parents would raise b_{t+1} from its hypothetical zero level to some positive level, contradicting the assumption that $b_{t+1} = 0$ were an equilibrium.

The "only if" part: Suppose instead that $r_D < R$. Can this Diamond steady state (where again $b_{t+1} = 0$, of course) be an equilibrium also for the original economy with a bequest motive? Yes. From (**) we have $\partial\tilde{U}_t/\partial b_{t+1} < 0$. Therefore the parents would be tempted to decrease their b_{t+1} from its present zero level to some negative level, but that is not allowed. Hence, $b_{t+1} = 0$ is still an equilibrium and the bequest motive does not become operative. Similarly, in a case where $R = r_D$, the situation $b_{t+1} = 0$ is still an equilibrium of the economy with a bequest motive, since we get $\partial\tilde{U}_t/\partial b_{t+1} = 0$ when $b_{t+1} = 0$. □

Thus bequests will be positive if and only if the own-generation preference, R, is sufficiently small or, what amounts to the same, the altruism factor, $(1+R)^{-1}$, is sufficiently large − in short, if and only if parents "love their children enough". Fig. 7.4, where k_{MGR} is defined by $f'(k_{MGR}) - \delta = R$, gives an illustration. If the rate R at which the parent discounts the utility of the next generation is relatively high, then k_{MGR} is relatively low and (*) will not be satisfied. This is the situation depicted in the upper panel of Fig. 7.4 (low altruism). In this case the bequest motive will not be operative and the economy ends up in the Diamond steady state with $k^* = k_D > k_{MGR}$. If on the other hand the own-generation preference, R, is relatively low as in the lower panel (high altruism), then (*) is satisfied, the bequest motive will be operative and motivates more saving so that the economy ends up in a steady state satisfying the modified golden rule. That is, (7.15) holds and $k^* = k_{MGR}$.

Both cases portrayed in Fig. 7.4 have $k_D < k_{GR}$, where k_{GR} is the golden rule capital-labor ratio satisfying $f'(k_{GR}) - \delta = n$. But theoretically, we could equally well have $k_D > k_{GR}$ so that the Diamond economy would be dynamically inefficient. The question arises: does the presence of a bequest motive help to eliminate the potential for dynamic inefficiency? The answer is given by point (i) of the following corollary of Proposition 1.

COROLLARY Let $R > n$. The economy with a bequest motive is:
(i) dynamically inefficient if and only if the associated Diamond economy is dynamically inefficient; and
(ii) the economy has positive bequests in steady state only if it is dynamically

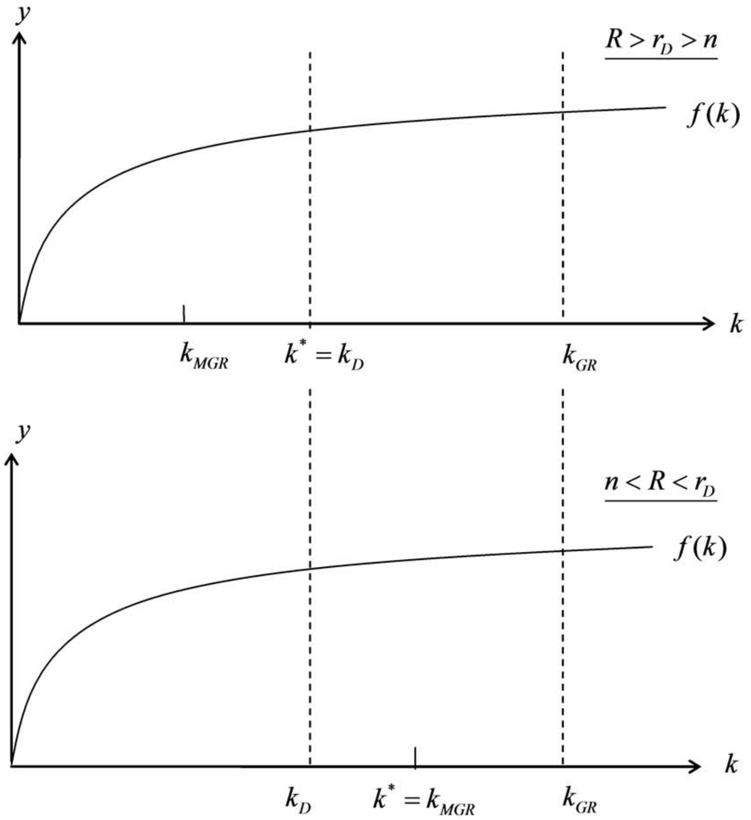

Figure 7.4: How the steady-state capital-labor ratio depends on the size of the own-generation preference R (for a given $k_D < k_{GR}$). Upper panel: high R results in zero bequest. Lower panel: low R induces additional saving and positive bequest so that k_{MGR}, which is larger the lower R is, becomes a steady state instead of k_D (k_{MGR} satisfies $f'(k_{MGR}) - \delta = R > n$ and k_{GR} satisfies $f'(k_{GR} - \delta = n)$.

efficient.

Proof. (i) "if": suppose $r_D < n$. Then, since by assumption $R > n$, $(*)$ is not satisfied. Hence, the bequest motive cannot be operative and the economy behaves like the associated Diamond economy which is dynamically inefficient in view of $r_D < n$.

(i) "only if": suppose $r^* < n$. Then the economy with a bequest motive is dynamically inefficient. Since by assumption $R > n$, it follows that $r^* < R$. So (7.15) is not satisfied, implying by Proposition 1 that bequests cannot be positive. Hence, the allocation is as in the associated Diamond economy which must then also be dynamically inefficient.

(ii) We have just shown that $r^* < n$ implies zero bequests. Hence, if there are positive bequests, we must have $r^* \geq n$, implying that the economy is dynamically efficient. □

The corollary shows that the presence of a bequest motive does not help in eliminating a tendency for dynamic inefficiency. This is not surprising. Dynamic inefficiency arises from perpetual excess saving. A bequest motive cannot be an incentive to save less (unless negative bequests are allowed). On the contrary, when a bequest motive is operative, you are motivated to increase saving as young in order to leave bequests; aggregate saving will be higher. That is why, when $R < r_D$, the economy ends up, through capital accumulation, in a steady state with $r^* = R$, so that r^* is smaller than r_D (though $r^* > n$ still). On the other hand, by the corollary follows also that the bequest motive can only be operative in a dynamically efficient economy. Indeed, we saw that an operative bequest motive implies the modified golden rule (7.15) so that $r^* = R$ where, by assumption, $R > n$ and therefore $k_{MGR} < k_{GR}$ always. (If $R \leq n$, the effective utility discount factor, $(1 + \bar{R})^{-1}$, in (7.3) is no longer less than one, which may result in non-existence of general equilibrium.)

7.3 Bequests and Ricardian equivalence

As we have seen, when the bequest motive is operative, the Barro model becomes essentially a *representative agent* model in spite of starting from an OLG structure. So aggregate household behavior is simply a multiple of the behavior of a single dynasty.

This feature has implications for the issue of Ricardian equivalence.[3] To see this, we add a government sector to the model. We assume that the government

[3] A first discussion of this issue, based on a different model, the Diamond OLG model, appears at the end of Chapter 6.

finances its spending sometimes by lump-sum taxation, sometimes by issuing debt. *Ricardian equivalence*, also called *debt neutrality*, is then present if, for a given path of government spending, a shift between tax and debt financing does not affect resource allocation. That is, a situation with a tax cut and ensuing budget deficit is claimed to be "*equivalent*" to a situation without the tax cut.

Barro (1974) used the above model to substantiate this claim. Faced with a tax cut in period t, the current generations will anticipate higher taxes in the future. Indeed, to cover the government's higher future interest payment, the present value of future taxes will have to rise exactly as much as current taxes are decreased. Assuming the government waits j periods to increase taxes and then does it fully once for all in period $t+j$, for each unit of account current taxes are reduced, taxes j periods ahead are increased by $(1+r)^j$ units of account. The present value as seen from the end of period t of this future tax increase is $(1+r)^j/(1+r)^j = 1$. So the change in the time profile of taxation will neither make the dynasties feel richer or poorer. Consequently, their current and planned future consumption will be unaffected.

The Ricardian Equivalence view is then that to compensate the descendants j generations ahead for the higher taxes, the old generation will save the rise in current after-tax income and leave higher bequests to their descendants (presupposing the bequest motive is operative). And the young generation will increase their saving by as much as *their* after-tax income is raised as a consequence of the tax cut and the higher bequests they expect to receive when retiring. In this way all private agents maintain the consumption level they would have had in the absence of the tax cut. The change in fiscal policy is thus completely nullified by the response of the private sector. The decrease in public saving is offset by an equal increase in aggregate private saving. So national saving as well as consumption remain unaffected.

We now formalize this story, taking population growth and fully specified budget constraints into account. Let

$$\begin{aligned}
G_t &= \text{ real government spending on goods and services in period } t,\\
T_t &= \text{ real tax revenue in period } t,\\
\tau_t &= T_t/L_t = \text{ a lump-sum tax levied on each young in period } t,\\
B_t &= \text{ real government debt as inherited from the end of period } t-1.
\end{aligned}$$

To fix ideas, suppose G_t is primarily eldercare, including health services, and therefore proportional to the number of old, i.e.,

$$G_t = \gamma L_{t-1}, \qquad \gamma > 0. \tag{7.20}$$

We assume the public service enter in a separable way in the lifetime utility function so that marginal utilities of private consumption are not affected by G_t.

The resource constraint of the economy now is

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} \equiv (c_{1t}L_t + c_{2t}L_{t-1})/L_t = c_{1t} + c_{2t}/(1+n) \\ &= f(k_t) - (1+n)^{-1}\gamma + (1-\delta)k_t - (1+n)k_{t+1}. \end{aligned}$$

instead of (7.12) above.

From time to time the government runs a budget deficit (or surplus) and in such cases, the deficit is financed by bond issue (or withdrawal).[4] Along with interest payments on government debt, elder care is the only government expense. That is,

$$B_{t+1} - B_t = rB_t + G_t - T_t, \qquad B_0 \text{ given}, \tag{7.21}$$

where the real interest rate r is for simplicity assumed constant. We further assume $r > n$; this is in accordance with the above result that when a Barro economy is in steady state with positive bequests, then, ignoring technological progress, by (7.15), the interest rate equals the intergenerational utility discount rate R (to ensure existence of general equilibrium, R was in connection with (7.2) assumed larger than n).

In absence of technological progress the steady state growth rate of the economy equals the growth rate of the labor force, that is, $g_Y = n < r$. Since the interest rate is thus higher than the growth rate, to maintain solvency the government must satisfy its intertemporal budget constraint,

$$\sum_{i=0}^{\infty} G_{t+i}(1+r)^{-(i+1)} \leq \sum_{i=0}^{\infty} T_{t+i}(1+r)^{-(i+1)} - B_t. \tag{7.22}$$

This says that the present value of current and expected future government spending is constrained by government wealth (the present value of current and expected future tax revenue minus existing government debt).

Let us concentrate on the "regular" case where the government does not tax more heavily than needed to cover the spending G_t and the debt service, that is, the government does not want to accumulate financial net wealth. Then there is strict equality in (7.22). Applying (7.20) and that $L_{t+i} = L_t(1+n)^i$ and $T_{t+i} = \tau_{t+i}L_{t+i}$, (7.22) with strict equality simplifies to

$$L_t \sum_{i=0}^{\infty} \frac{(1+n)^i}{(1+r)^{i+1}}(\tau_{t+i} - \frac{\gamma}{1+n}) = B_t. \tag{7.23}$$

Suppose that, for some periods, taxes are cut so that $T_{t+i} < G_{t+i} + rB_{t+i}$, that is, a budget deficit is run. Is resource allocation − aggregate consumption

[4] The model ignores money.

and investment – affected? Barro says "no", given that the bequest motive is operative. Each dynasty will then choose the same consumption path $(c_{2t}, c_{1t})_{t=0}^{\infty}$ as it planned before the shift in fiscal policy. The reason is that the change in the time profile of lump-sum taxes will not make the dynasty feel richer or poorer. Aggregate consumption and saving in the economy will therefore remain unchanged.

The proof goes as follows. Suppose there are N dynasties in the economy, all alike. Since L_{t-1} is the total number of old agents in the economy in the current period, period t, each dynasty has L_{t-1}/N old members. Each dynasty must satisfy its intertemporal budget constraint. That is, the present value of its consumption stream cannot exceed the total wealth of the dynasty. In the optimal plan the present value of the consumption stream will equal the total wealth. Thus

$$\frac{L_{t-1}}{N}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=a_t+h_t, \tag{7.24}$$

where a_t is initial financial wealth of the dynasty and h_t is its human wealth (after taxes).[5] Multiplying through in (7.24) by N, we get the intertemporal budget constraint of the *representative dynasty*:

$$L_{t-1}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=Na_t+Nh_t\equiv A_t+H_t,$$

where A_t is aggregate financial wealth in the private sector and H_t is aggregate human wealth (after taxes):

$$H_t\equiv Nh_t=L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i}). \tag{7.25}$$

The financial wealth consists of capital, K_t, and government bonds, B_t. Thus,

$$A_t+H_t=K_t+B_t+H_t. \tag{7.26}$$

Since B_t and H_t are the only terms in (7.26) involving taxes, we consider their sum:

$$\begin{aligned} B_t+H_t &= L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(\tau_{t+i}-\frac{\gamma}{1+n}+w_{t+i}-\tau_{t+i}) \\ &= L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\frac{\gamma}{1+n}), \end{aligned}$$

[5] How to get from the generation budget constraints, $c_{2t}+(1+n)b_t=(1+r)s_{t-1}$ and $(1+n)c_{1t}+(1+n)s_t=(1+n)(w_t-\tau_t+b_t)$, to the intertemporal budget constraint of the dynasty is shown in Appendix B.

where we have used (7.23) and (7.25). We see that the time profile of τ has disappeared and cannot affect $B_t + H_t$. Hence, the total wealth of the dynasty is unaffected by a change in the time profile of taxation.

As a by-product of the analysis we see that higher initial government debt has no effect on the sum, $B_t + H_t$, because H_t becomes equally much lower. This is what Barro (1974) meant by answering "no" to the question: "Are government bonds net [private] wealth?" (the title of his paper).

When the bequest motive is operative, Ricardian equivalence will also manifest itself in relation to a pay-as-you-go pension system. Suppose the mandatory contribution of the young (the workers) is raised in period t, before the old generation has decided the size of the bequest to be left to the young. Then, if the bequest motive is operative, the old generation will use the increase in their pension to leave higher bequests. In this way the young generation is compensated for the higher contribution they have to pay to the pay-as-you-go system. As a consequence all agents' consumption remain unchanged and so does resource allocation. Indeed, within this framework with perfect markets, as long as the bequest motive is operative, a broad class of lump-sum government fiscal actions can be nullified by offsetting changes in private saving and bequests.

Discussion

According to many macroeconomists, the modified golden rule and the Ricardian Equivalence result have elegance, but are hardly good approximations to reality:

1. The picture of the household sector as a set of dynasties, all alike, seems remote from reality. Universally, only a fraction of a country's population leave bequests.[6] In the last section of Chapter 6 we considered the "pure" case assumed in the Diamond OLG model where a bequest motive is entirely absent. In that case, because the new generations are then *new* agents, and the future taxes are levied partly on these new agents, the future taxes are no longer equivalent to current taxes. This is the *composition-of-the-tax-base argument* for Ricardian Non-equivalence. This argument is also relevant for "mixed" cases where only a fraction of the population leave bequests or where the bequests are motivated in other ways than assumed in the Barro model. Moreover, in a world of uncertainty bequests may simply be accidental rather than planned.

Even if there is a bequest motive of the altruistic form assumed by Barro, it will only be operative if it is strong enough, as we saw in Section 7.2.4.

2. When the bequest motive is not operative, Ricardian equivalence breaks down. Consider a situation where the constraint $b_{t+1} \geq 0$ is binding. Then

[6] Wolf (2002) found that in 1998 around 30 per cent of US households of "age" above 55 years (according to the age of the head of the household) reported to have received wealth transfer.

there will be no bequests. Parents would in fact like to pass on *debt* to the next generation. They are hindered by law, however. But the government can do what the private agents would like to do but cannot. Specifically, if for example $r_D < n$, there will be no bequests in the Barro economy and we have $r^* = r_D < n$ in steady state. So there is dynamic inefficiency. Yet the government can avoid this outcome and instead achieve $r^* = n$ within sight by, for example, a fiscal policy continually paying transfers to the old generation, financed by creation of public debt. The private agents cannot nullify this beneficial fiscal policy − and have no incentive to try doing so. However, the logical validity of this point notwithstanding, its practical relevance is limited since empirical evidence of dynamic inefficiency seems absent.[7]

3. Another limitation of Barro's analysis has to do with the dynastic-family story which portrays families as clearly demarcated and harmonious infinitely-lived entities. This view abstracts from the fact that:

(a) families are not formed by inbreeding, but by marriage of two individuals coming from two different parent families;

(b) the preferences of distinct family members may conflict; think for example of the schismatic family feud of the Chicago-based Pritzker family.[8]

Point (a) gives rise to a complex network of interconnected families. In principle, and perhaps surprisingly, this observation need not invalidate Ricardian equivalence. As Bernheim and Bagwell (1988) ironically remark, the problem is that "therein lies the difficulty". Almost all elements of fiscal policy, even on-the-face-of-it distortionary taxes, tend to become neutral. This is because virtually all the population is interconnected through chains involving parent-child linkages. Bernheim and Bagwell (1988, p. 311) conclude that in this setting, "Ricardian equivalence is merely one manifestation of a much more powerful and implausible neutrality theorem".[9]

Point (b) gives rise to broken linkages among the many linkages. This makes it difficult to imagine that Ricardian equivalence comes up.

4. From an econometric point of view, by and large Barro's hypothesis does not seem to do a good job in explaining how families actually behave. If altruistically linked extended family members really did pool their incomes across generations when deciding how much each should consume, then the amount that any particular family member consumes would depend only on the present discounted value of total future income stream of the extended family, not on that

[7] On the other hand, this latter fact *could* be a *consequence* of the described fiscal policy.

[8] The Pritzker family is one of the wealthiest American families, owning among other things the Hyatt hotel chain. In the early 2000s a long series of internal battles and lawsuits across generations resulted in the family fortune being split between 11 family members. For an account of this and other family owned business wars, see for example Gordon (2008).

[9] Barro (1989) answers the criticism of Bernheim and Bagwell.

person's share of that total. To state it differently: if the dynasty hypothesis were a good approximation to reality, then the ratio of the young's consumption to that of the parents should not depend systematically on the ratio of the young's *income* to that of the parents. But the empirical evidence goes in the opposite direction − own resources *do* matter (see Altonji et al. 1992 and 1997).

To conclude: The debt neutrality view is of interest as a theoretical benchmark case. In practice, however, tax cuts and debt financing by the government seem to make the currently alive generations feel wealthier and stimulate their consumption. Bernheim (1987) reviews the theoretical controversy and the empirical evidence of Ricardian equivalence. He concludes with the empirical estimate, for the US, that private saving offset only roughly half the decline in government saving that results from a shift from taxes to deficit finance.

We shall come across the issue of Ricardian equivalence or non-equivalence in other contexts later in this book.

7.4 The modified golden rule when there is technological progress*

Heretofore we have abstracted from technological progress. What does the modified golden rule look like when it is recognized that actual economic development is generally accompanied by technological progress?

To find out we extend the Barro model with Harrod-neutral technological progress. As we want consistency with Kaldor's stylized facts, we assume that technological progress is Harrod-neutral:

$$Y_t = F(K_t, \mathcal{T}_t L_t),$$

where F is a neoclassical aggregate production function with CRS and $\mathcal{T}_t$ (not to be confused with tax revenue T_t) is the technology level, which is assumed to grow exogenously at the constant rate $g > 0$, that is, $\mathcal{T}_t = \mathcal{T}_0(1+g)^t$, $\mathcal{T}_0 > 0$. Apart from this (and the specification of $u(c)$ below), everything is as in the simple Barro model analyzed above. Owing to equilibrium in the factor markets, K_t and L_t can be interpreted as predetermined variables, given from the supply side.

We have

$$\tilde{y}_t \equiv \frac{Y_t}{\mathcal{T}_t L_t} \equiv \frac{y_t}{\mathcal{T}_t} = F(\frac{K_t}{\mathcal{T}_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0,$$

where $\tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t) \equiv k_t/\mathcal{T}_t$. The dynamic aggregate resource constraint,

$K_{t+1} = (1-\delta)K_t + Y_t - C_t$, can now be written

$$\frac{K_{t+1}}{\mathcal{T}_t L_t} = (1+g)(1+n)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t, \tag{7.27}$$

where $\tilde{c}_t \equiv C_t/(\mathcal{T}_t L_t) \equiv c_t/\mathcal{T}_t$, the per capita "technology-corrected" consumption level. With perfect competition we have the standard equilibrium relations

$$r_t = \frac{\partial Y_t}{\partial K_t} - \delta = \frac{\partial \left[\mathcal{T}_t L_t f(\tilde{k}_t)\right]}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{7.28}$$

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{\partial \left[\mathcal{T}_t L_t f(\tilde{k}_t)\right]}{\partial L_t} = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right]\mathcal{T}_t \equiv \tilde{w}(\tilde{k}_t)\mathcal{T}_t. \tag{7.29}$$

We want the model to comply with Kaldor's stylized facts. Thus, among other things the model should be consistent with a constant rate of return in the long run. Such a state requires $\tilde{k}_t$ to be constant, say equal to $\tilde{k}^*$; then $r_t = f'(\tilde{k}^*) - \delta \equiv r^*$. When $\tilde{k}_t$ is constant, then also $\tilde{c}_t$ and $\tilde{w}_t$ are constant, by (7.27) and (7.29), respectively. In effect, the capital-labor ratio k_t, output-labor ratio y_t, consumption-labor ratio c_t, and real wage w_t, all grow at the same rate as technology, the constant rate g. So a constant $\tilde{k}_t$ implies a balanced growth path with constant rate of return.

To be capable of maintaining a balanced growth path (and thereby be consistent with Kaldor' stylized facts) when the bequest motive is operative, the Barro model needs that the period utility function, $u(c)$, is a CRRA function. Indeed, when the bequest motive is operative, the Barro model becomes essentially a representative agent model in spite of its OLG structure. And it can be shown (see Appendix C) that for existence of a balanced growth path in a representative agent model with Harrod-neutral technological progress, we have to assume that the period utility function, $u(c)$, is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0, \tag{7.30}$$

where θ is the constant (absolute) elasticity of marginal utility (if $\theta = 1$, (7.30) should be interpreted as $u(c) = \ln c$).

From now on we shall often write the CRRA utility function this way, i.e., without adding the "normalizing" constant $-1/(1-\theta)$. This is immaterial for the resulting consumption/saving behavior, but we save notation and avoid the inconvenience that an infinite sum of utilities, as in (7.3), may not be bounded for the sole reason that an economically trivial constant has been added to the crucial part of the period utility function.

When the bequest motive is operative, the young parent's optimality condition (7.10) is also valid in the new situation with technological progress. Assuming (7.30), we can write (7.10) as $c_{1t}^{-\theta} = c_{1t+1}^{-\theta}(1 + r_{t+1})/(1 + R)$, where we have used that $(1 + \bar{R})(1 + n) \equiv 1 + R$; this implies

$$(\frac{c_{1t+1}}{c_{1t}})^{\theta} = \frac{1 + r_{t+1}}{1 + R}, \tag{7.31}$$

where c_{1t+1}/c_{1t} is 1+ the growth rate (as between generations) of consumption as young. To the extent that the right-hand side of (7.31) is above one, it expresses the excess of the rate of return over and above the intergenerational discount rate. The interpretation of (7.31) is that this excess is needed for $c_{1t+1}/c_{1t} > 1$. For the young to be willing to save and in the next period leave positive bequests, the return on saving must be large enough to compensate the parent for the absence of consumption smoothing (across time as well as across generations). The larger is θ (the desire for consumption smoothing), for a given $c_{1t+1}/c_{1t} > 1$, the larger must r_{t+1} be in order to leave the parent satisfied with not consuming more herself. In the same way, the larger is c_{1t+1}/c_{1t} (and therefore the inequality across generations), for a given θ, the larger must r_{t+1} be in order to leave the parent satisfied with not consuming more.

As observed in connection with (7.14), the old at the beginning of period $t+1$ own all capital in the economy. So the aggregate capital stock equals their saving in the previous period: $K_{t+1} = s_t L_t$. Defining $\tilde{s}_t \equiv s_t/\mathcal{T}_t$, we get

$$\tilde{k}_{t+1} = \frac{\tilde{s}_t}{(1 + g)(1 + n)}. \tag{7.32}$$

Steady state

By (7.32), in steady state $\tilde{s}_t = (1+g)(1+n)\tilde{k}^* \equiv \tilde{s}^*$. The consumption per young and consumption per old in period t add to total consumption in period t, that is, $C_t = L_t c_{1t} + L_t(1 + n)^{-1}c_{2t}$. Dividing through by effective labor gives

$$\tilde{c}_t \equiv \frac{C_t}{\mathcal{T}_t L_t} = \tilde{c}_{1t} + (1 + n)^{-1}\tilde{c}_{2t},$$

where $\tilde{c}_{1t} \equiv c_{1t}/\mathcal{T}_t$ and $\tilde{c}_{2t} \equiv c_{2t}/\mathcal{T}_t$. In this setting we define a steady state as a path along which not only $\tilde{k}_t$ and $\tilde{c}_t$, but also $\tilde{c}_{1t}$ and $\tilde{c}_{2t}$ separately, are constant, say equal to $\tilde{c}_1^*$ and $\tilde{c}_2^*$, respectively. Dividing through by $\mathcal{T}_t$ in the two period budget constraints, (7.4) and (7.5), we get $\tilde{b}_t \equiv b_t/\mathcal{T}_t = \tilde{c}_1^* + \tilde{s}^* - \tilde{w}(\tilde{k}^*) \equiv \tilde{b}^*$. Consequently, in a steady state with an operative bequest motive, bequest per young, b_t, consumption per young, c_{1t}, saving per young, s_t, and consumption per old, c_{2t}, all grow at the same rate as technology, the constant rate g.

But how are $\tilde{k}^*$ and r^* determined? Inserting the steady state conditions $c_{1t+1} = c_{1t}(1+g)$ and $r_{t+1} = r^*$ into (7.31) gives

$$(1+g)^\theta = \frac{1+r^*}{1+R}$$

or

$$1 + r^* = (1+R)(1+g)^\theta \equiv (1+\bar{R})(1+n)(1+g)^\theta. \tag{7.33}$$

This is the *modified golden rule* when there is Harrod-neutral technological progress at the rate g and a constant (absolute) elasticity of marginal utility of consumption θ. The modified golden rule says that for the economy to be in a steady state with positive bequests, it is necessary that the gross interest rate matches the "subjective" gross discount rate, taking account of both (a) the own-generation preference rate R, and (b) the fact that there is aversion (measured by θ) to the lack of consumption smoothing arising from per capita consumption growth at rate g. If in (7.33) "=" were replaced by ">" ("<"), then *more* (less) saving and capital accumulation would take place, tending to push the system away from a steady state. When $g = 0$ (no technological progress), (7.33) reduces to the simple modified golden rule, (7.15).

The effective capital-labor ratio in steady state, $\tilde{k}^*$, satisfies $f'(\tilde{k}^*) - \delta = r^*$, where r^* is given from the modified golden rule, (7.33), when the bequest motive is operative. Assuming f satisfies the Inada conditions, this equation has a (unique) solution $\tilde{k}^* = f'^{-1}(r^* + \delta) = f'^{-1}((1+R)(1+g)^\theta - 1 + \delta) \equiv \tilde{k}_{MGR}$. Since $f'' < 0$, it follows that the higher are R and g, the lower is the modified-golden-rule capital intensity, $\tilde{k}_{MGR}$.

To ensure that the infinite sum of discounted lifetime utilities is bounded from above along the steady state path (so that maximization is possible) we need an *effective* intergenerational discount rate, $\bar{R} \equiv (1+R)/(1+n) - 1$, that is not only positive, but sufficiently large. In the next chapter it is shown that $1 + \bar{R} > (1+g)^{1-\theta}$ is required and that this inequality also ensures that the transversality condition (7.8) holds in a steady state with positive bequests. The required inequality is equivalent to

$$1 + R > (1+n)(1+g)^{1-\theta}, \tag{7.34}$$

which we assume satisfied, in addition to $R > n$.[10] Combining this with (7.33), we thus have

$$1 + r^* = (1+R)(1+g)^\theta > (1+n)(1+g). \tag{7.35}$$

[10] Fortunately, (7.34) is more strict than the restriction $R > n$, used up to now, only if $\theta < 1$, which is not the empirically plausible case.

Let $\tilde{k}_{GR}$ denote the *golden rule* capital intensity, defined by $1 + f'(\tilde{k}_{GR}) - \delta = (1 + n)(1 + g)$.[11] Since $f'' < 0$, we conclude that $\tilde{k}_{MGR} < \tilde{k}_{GR}$. In the "unmodified" golden rule with technological progress, the interest rate (reflecting the net marginal productivity of capital) equals the output growth rate, which with technological progress is $(1+n)(1+g)-1$. The "modification" displayed by (7.33) comes about because both the strictly positive effective intergenerational discount rate $\bar{R}$ and the elasticity of marginal utility enter the determination of r^*. In view of the parameter inequality (7.34), the intergenerational discounting results in a lower effective capital-labor ratio and higher rate of return than in the golden rule. Indeed, in general equilibrium with positive bequests it is impossible for the economy to reach the golden rule.

The condition for positive bequests

Using that $u'(c_{2t}) = c_{2t}^{-\theta} = (\tilde{c}_2^* \mathcal{T}_t)^{-\theta}$ in steady state, the proof of Proposition 1 can be extended to show that bequests are positive in steady state if and only if

$$1 + R = (1 + r^*)(1 + g)^{-\theta} < (1 + r_D)(1 + g)^{-\theta}, \tag{7.36}$$

where the equality follows from (7.35), and r_D is the steady-state interest rate in the associated "well-behaved" Diamond economy. It can moreover be shown that if both (7.36) and (7.34) (as well as $R > n$) hold, and the initial $\tilde{k}$ is in a neighborhood of the modified-golden-rule value, then the bequest motive is operative in every period and the economy converges over time to the modified-golden-rule steady state. This stability result is shown in the next chapter.

Intuition might make us think that a higher rate of technological progress, g, would make the old more reluctant to leave bequests since they know that the future generations will benefit from better future technology. For fixed r_D, this intuition is confirmed by (7.36). The inequality shows that for fixed r_D, a higher rate of technological progress implies a lower cut-off value for R. But r_D is not fixed but an increasing function of g. That is, whether or not it holds that a higher g implies a lower cut-off value for R, depends on which effect is the stronger one, the direct effect in (7.36) of the higher g or the indirect effect through the rise in r_D. See Exercise 7.??

Calibration We shall give a rough informal estimate of the intergenerational discount rate, R, by the method of calibration. Generally, *calibration* means to choose parameter values such that the model matches a list of data characteristics.[12] Given the formula (7.33), we want our estimate of R to be consistent with

[11] See Chapter 4.

[12] A next step (not pursued here) is to consider other data and check whether the model also fits them.

a long-run annual real rate of return, $\tilde{r}^*$, of about 0.05, when reasonable values for the annual rate of technological progress, $\tilde{g}$, and the elasticity of marginal utility of consumption, θ, are chosen. For advanced economies after the Second World War, values with some empirical support are $\tilde{g} = 0.02$ and $\theta \in (1, 5)$. Choosing $\theta = 2$ and taking into account that our model has an implicit period length of about 30 years, we get:

$$\begin{aligned}
1 + g &= (1 + \tilde{g})^{30} = 1.02^{30} = 1.8114 \\
1 + r^* &= (1 + \tilde{r}^*)^{30} = 1.05^{30} = 4.3219 \\
1 + R &= \frac{1 + r^*}{(1 + g)^{\theta}} = \frac{4.3219}{1.8114^2} = 1.3172
\end{aligned}$$

Thus $R = 0.3172$. On a yearly basis the corresponding intergenerational discount rate is then $\tilde{R} = (1 + R)^{1/30} - 1 \simeq 0.0092$. As to $\bar{R}$, with $\tilde{n} = 0.008$, we get $1 + n = (1 + \tilde{n})^{30} = 1.008^{30} = 1.2700$, so that $1 + \bar{R} \equiv (1 + R)/(1 + n) = 1.0371$. This gives an *effective* intergenerational discount rate on an *annual* basis equal to 0.0012.[13]

Do these numbers indicate that we are in a situation where the bequest motive is operative? The answer would be affirmative if and only if $r^* < r_D$, cf. (7.36). Whether the latter inequality holds, depends on the time preference rate, ρ, and the aggregate production function, f. Our empirical knowledge about both is limited. Exercise 7.? considers the Cobb-Douglas case for alternative values of $\theta = 1$.

7.5 Concluding remarks

We have studied Barro's model of overlapping generations linked through altruistic bequests. Barro's insight is that intergenerational altruism may extend households' planning horizon. If the extent of altruism is sufficiently high so that the bequest motive (in the Barro form) is operative in every period, then the model

- becomes a representative agent model and implies Ricardian equivalence;
- results in a simple formula for the long-run real interest rate (the modified golden rule).

What is the implication for intergenerational distribution of welfare? Even ignoring technological progress, a strictly positive intergenerational discount rate,

[13] Control of the calculation: $\frac{1+\tilde{r}^*}{(1+\tilde{n})(1+\tilde{g})^2} = \frac{1.05}{1.008 \cdot 1.02^2} = 1.0012$, hence OK.

R, does not imply that future generations must end up worse off than current generations. This is because positivity of R does not hinder existence of a stable steady state. The role of R is to determine *what* steady state the economy is heading to, that is, what effective capital-labor ratio and level of per capita consumption is sustainable. Within the constraint displayed by (7.34), a higher R results in a lower steady-state capital intensity, a lower level of per capita consumption (in an economy without technological progress), and a lower position of the upward-sloping time path of per capita consumption in an economy with Harrod-neutral technological progress.

The simplifying assumption behind the modified-golden-rule formula for the long-run interest rate, the representative agent assumption, has been seriously questioned. The modified-golden-rule formula itself finds no empirical support. From the formula (7.35) we should expect to find positive comovements over the medium run between the rate of interest (rate of return) and the productivity growth rate. However, the investigation by Hamilton et al. (2016), covering more than a century and many countries, finds no relationship of that kind.

In the Barro model resource allocation and the coordination of economic behavior across generations is brought about through the market mechanism and an operative bequest motive due to parental altruism. In the next chapter we shall study a situation where the coordination across generations is brought about by a fictional social planner maximizing a social welfare function.

7.6 Literature notes

(incomplete)

The criticism by Bernheim and Bagwell is answered in Barro (1989). A survey of the debt neutrality issues is provided by Dalen (1992), emphasizing "demographic realism".

We have concentrated on the Barro model where bequests reflect the concern of parents for the welfare of their offspring. The Barro model was further developed and analyzed by Buiter (1980), Abel (1987), and Weil (1987) and our treatment above draw on these contributions. Our analysis ruled out circumstances where children help support their parents. This is dealt with in Abel (1987) and Kimball (1987); see also the survey in Bernheim (1987). For a more general treatment of bequests in the economy, see for example Laitner (1997).

Reviews of how to model the distinction between "life-cycle wealth" and "inherited wealth" and of diverging views on the empirical importance of inherited wealth are contained in Kessler and Masson (1988) and Malinvaud (1998a). How much of net wealth accumulation in Scandinavia represents intended transfers and bequests is studied by Laitner and Ohlsson (2001) and Danish Economic

Council (2004).

7.7 Appendix

A. The envelope theorem for an unconstrained maximum

In the solution of the parent's decision problem in the Barro model of Section 7.2 we appealed to the envelope theorem which, in its simplest form, is the principle that in an interior maximum the total derivative of a maximized function w.r.t. a parameter equals the partial derivative w.r.t. that parameter. More precisely:

ENVELOPE THEOREM Let $y = f(a, x)$ be a continuously differentiable function of two variables. The first variable, a, is conceived as a parameter and the other variable, x, as a control variable. Let $g(a)$ be a value of x at which $\frac{\partial f}{\partial x}(a, x) = 0$, i.e., $\frac{\partial f}{\partial x}(a, g(a)) = 0$. Let $F(a) \equiv f(a, g(a))$. Provided $F(a)$ is differentiable, we have

$$F'(a) = \frac{\partial f}{\partial a}(a, g(a)),$$

where $\partial f / \partial a$ denotes the partial derivative of $f(\cdot)$ w.r.t. the first argument.

Proof $F'(a) = \frac{\partial f}{\partial a}(a, g(a)) + \frac{\partial f}{\partial x}(a, g(a))g'(a) = \frac{\partial f}{\partial a}(a, g(a))$, since $\frac{\partial f}{\partial x}(a, g(a)) = 0$ by definition of $g(a)$. $\square$

That is, when calculating the total derivative of a function w.r.t. a parameter and evaluating this derivative at an interior maximum w.r.t. a control variable, the envelope theorem allows us to ignore the second term arising from the chain rule. This is also the case if we calculate the total derivative at an interior minimum. Extension to a function of n control variables is straightforward.[14]

The envelope theorem in action For convenience we repeat the first-order conditions (7.6) and (7.7):

$$\begin{aligned} u'(c_{1t}) &= (1+\rho)^{-1} u'(c_{2t+1})(1 + r_{t+1}), && (*) \\ (1+\rho)^{-1} u'(c_{2t+1}) &= (1+\bar{R})^{-1} u'(c_{1t+1})(1+n)^{-1}. && (**) \end{aligned}$$

We described in Section 7.2 how the parent chooses s_t and b_{t+1} so as to maximize the objective function $\tilde{U}_t$, taking into account the descendants' optimal responses to the received bequest b_{t+1}. We claimed without proof that in view of the envelope theorem, these two at first sight incomplete first-order conditions

[14] For extensions and more rigorous framing of the envelope theorem, see for example Sydsaeter et al. (2006).

are correct, both as they read and for t replaced by $t+i$, $i=1,2,...$, and do indeed characterize an optimal plan.

To clarify the issue we substitute the two period budget constraints of a young into the objective function to get

$$\begin{aligned}\tilde{U}_t(s_t, b_{t+1}) &= u(w_t + b_t - s_t) + (1+\rho)^{-1}u((1+r)s_t \\ &\quad -(1+n)b_{t+1}) + (1+\bar{R})^{-1}\tilde{U}_{t+1}(\hat{s}_{t+1}, \hat{b}_{t+2}),\end{aligned} \tag{7.37}$$

where $\hat{s}_{t+1}$ and $\hat{b}_{t+2}$ are the optimal responses of the next generation and where $\tilde{U}_{t+1}(\cdot)$ can be written in the analogue recursive way, and so on for all future generations. The responses of generation $t+1$ are functions of the received b_{t+1} so that we can write

$$\hat{s}_{t+1} = \hat{s}(b_{t+1}, t+2), \qquad \hat{b}_{t+2} = \hat{b}(b_{t+1}, t+2),$$

where the second argument, $t+2$, represents the influence of w_{t+1} and r_{t+2}.

Our at first sight questionable approach rests on the idea that the smooth function $\tilde{U}_{t+1}(\cdot)$ is flat at an interior maximum so that any small change in the descendants' optimal responses induced by a small change in b_{t+1} has a negligible effect on the value of the function, hence also on the value of $\tilde{U}_t(\cdot)$. A detailed argument goes as follows.

For the first-order conditions (*) and (**), both as they read and for t replaced by $t+i$, $i=1,2,...$, to make up a correct characterization of optimal behavior by a parent who takes the optimal responses by the descendants into account, the first-order conditions must imply that the *total* derivative of the parent's objective function w.r.t. b_{t+1} vanishes. To see whether our "half-way" optimization procedure has ensured this, we first forward the period budget constraint (7.5) one period to get:

$$c_{2t+2} + (1+n)\hat{b}_{t+2} = (1+r_{t+2})\hat{s}_{t+1}. \tag{7.38}$$

Using this expression we substitute for c_{2t+2} in (7.37) and let the function $\hat{U}_t(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})$ represent the right-hand side of (7.37).

Although the parent chooses both s_t and b_{t+1}, only the choice of b_{t+1} affects the next generation. Calculating the total derivative of $\hat{U}_t(\cdot)$ w.r.t. b_{t+1}, we get

$$
\begin{aligned}
& d\hat{U}(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})/db_{t+1} = \\
& \quad (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1})(1 - \frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}}) \\
& \quad +(1+\bar{R})^{-1}\left\{(1+\rho)^{-1}u'(c_{2t+2})\left((1+r_{t+2})\frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}} - (1+n)\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}}\right)\right. \\
& \quad \left. +(1+\bar{R})^{-1}u'(c_{1t+2})\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}}\right\} + ... \\
= & \quad (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1}) \\
& \quad -(1+\bar{R})^{-1}\left[u'(c_{1t+1}) - (1+\rho)^{-1}u'(c_{2t+2})(1+r_{t+2})\right]\frac{\partial \hat{s}_{t+1}}{\partial b_{t+1}} \\
& \quad +(1+\bar{R})^{-1}\left[(1+\rho)^{-1}u'(c_{2t+2})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+2})\right]\frac{\partial \hat{b}_{t+2}}{\partial b_{t+1}} + ... \\
= & \quad (1+\rho)^{-1}u'(c_{2t+1})\left(-(1+n)\right) + (1+\bar{R})^{-1}u'(c_{1t+1}) = 0. \qquad (7.39)
\end{aligned}
$$

The second last equality sign is due to the first-order conditions (*) and (**), first with t replaced by $t+1$, implying that the two terms in square brackets vanish, second with t replaced by $t+i$, $i = 2, 3, \ldots$, implying that also all the remaining terms, represented by "...", vanish (since the latter terms can be written in the same way as the former). The last equality sign is due to (**) as it reads. Thus, also the *total* derivative is vanishing as it should at an interior optimum.

Note that the expression in the last line of the derivation is the *partial* derivative of $\hat{U}_t(\cdot)$, namely $\partial\hat{U}(s_t, b_{t+1}, \hat{s}_{t+1}, \hat{b}_{t+2})/\partial b_{t+1}$. The whole derivation is thus a manifestation of the envelope theorem for an unconstrained maximum: in an interior optimum the total derivative of a maximized function w.r.t. a parameter, here b_{t+1}, equals the partial derivative w.r.t. that parameter.

B. The intertemporal budget constraint of a dynasty

We here show how to derive a dynasty's intertemporal budget constraint as presented in (7.24) of Section 7.3. With lump-sum taxation and constant interest rate, r, the period budget constraints of a member of generation t are

$$
\begin{aligned}
c_{1t} + s_t &= w_t - \tau_t + b_t, \qquad \text{and} \qquad (7.40) \\
c_{2t+1} + (1+n)b_{t+1} &= (1+r)s_t. \qquad (7.41)
\end{aligned}
$$

We isolate s_t in (7.41), substitute into (7.40), and reorder to get

$$
b_t = c_{1t} + \frac{c_{2t+1}}{1+r} - (w_t - \tau_t) + \frac{1+n}{1+r}b_{t+1}.
$$

Then, by forward substitution,

$$\begin{aligned} b_t &= \sum_{i=0}^{j}(\frac{1+n}{1+r})^i\left[c_{1t+i}+\frac{c_{2t+i+1}}{1+r}-(w_{t+i}-\tau_{t+i})\right]+(\frac{1+n}{1+r})^{j+1}b_{t+j+1} \\ &= \sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i\left[c_{1t+i}+\frac{c_{2t+i+1}}{1+r}-(w_{t+i}-\tau_{t+i})\right], \qquad (7.42) \end{aligned}$$

assuming $\lim_{j\to\infty}(\frac{1+n}{1+r})^{j+1}b_{t+j+1}=0$, in view of $r>n$. For every old in any given period there are $1+n$ young. We therefore multiply through in (7.42) by $1+n$ and reorder:

$$(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i}+\frac{c_{2t+i+1}}{1+r})=(1+n)b_t+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i}-\tau_{t+i}).$$

To this we add the period budget constraint of an old member of the dynasty,

$$c_{2t}+(1+n)b_t=(1+r)s_{t-1},$$

and get the consolidated intertemporal budget constraint of the dynasty in period t:

$$c_{2t}+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(c_{1t+i}+\frac{c_{2t+i+1}}{1+r})=(1+r)s_{t-1}+(1+n)\sum_{i=0}^{\infty}(\frac{1+n}{1+r})^i(w_{t+i}-\tau_{t+i}),$$

where $(1+n)b_t$ has been cancelled out on both sides. Dividing through by $1+r$ and reordering gives

$$\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right]=s_{t-1}+(1+n)\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i}).$$

This is the intertemporal budget constraint, as seen from the beginning of period t, of a dynasty with one old member in period t. With L_{t-1} old members, this becomes

$$\begin{aligned} & L_{t-1}\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}\left[c_{2t+i}+(1+n)c_{1t+i}\right] \\ =\ & s_{t-1}L_{t-1}+L_t\sum_{i=0}^{\infty}\frac{(1+n)^i}{(1+r)^{i+1}}(w_{t+i}-\tau_{t+i})=A_t+H_t, \end{aligned}$$

where $A_t=s_{t-1}L_{t-1}$ is the financial wealth in the beginning of period t and H_t is the human wealth, as defined in (7.25). Dividing through by N gives (7.24).

C. Proof that a representative agent model allows balanced growth only if the period utility function is CRRA

This appendix refers to Section 7.4. When the bequest motive is operative, the Barro model becomes a representative agent model where the intergenerational Euler equation (7.10) holds for all dynasties and therefore also at the aggregate level. For convenience we repeat the Euler equation in question here:

$$u'(c_{1t}) = \frac{1 + r_{t+1}}{1 + R} u'(c_{1t+1}), \tag{7.43}$$

in view of $(1 + \bar{R})^{-1} \equiv (1 + R)^{-1}(1 + n)$. In balanced growth, $c_{1t+1} = (1 + g)c_{1t}$, where $g > 0$ and $r_{t+1} = r^*$, so that (7.43) takes the form

$$u'(c_{1t}) = \frac{1 + r^*}{1 + R} u'((1 + g)c_{1t}) \equiv \omega(c_{1t}), \tag{7.44}$$

which must hold for all $c_{1t} > 0$ to be generally consistent with balanced growth. Thus, the derivatives on both sides should also be equal for all $c_{1t} > 0$:

$$u''(c_{1t}) = \omega'(c_{1t}) = \frac{1 + r^*}{1 + R} u''((1 + g)c_{1t})(1 + g). \tag{7.45}$$

Dividing through by $u'(c_{1t})$ in accordance with (7.44) and multiplying by c_{1t} yields

$$\frac{c_{1t}u''(c_{1t})}{u'(c_{1t})} = \frac{(1 + g)c_{1t}u''((1 + g)c_{1t})}{u'((1 + g)c_{1t})},$$

showing that for all $c_{1t} > 0$, the (absolute) elasticity of marginal utility should be the same at the consumption level c_{1t} as at the consumption level $(1 + g)c_{1t}$. It follows that $u(\cdot)$ must be such that the (absolute) elasticity of marginal utility, $\theta(c) \equiv cu''(c)/u'(c)$, is independent of c, i.e., $\theta(c) = \theta > 0$. We know from Chapter 3 that this requires that $u(\cdot)$, up to a positive linear transformation, has the CRRA form $c^{1-\theta}/(1 - \theta)$.

7.8 Exercises

Chapter 9

The intertemporal consumption-saving problem in discrete and continuous time

In the next two chapters we shall discuss – and apply – the continuous-time version of the basic representative agent model, the Ramsey model. As a preparation for this, the present chapter gives an account of the transition from discrete time to continuous time analysis and of the application of optimal control theory to set up and solve the household's consumption/saving problem in continuous time.

There are many fields in economics where a setup in continuous time is preferable to one in discrete time. One reason is that continuous time formulations expose the important distinction in dynamic theory between stock and flows in a much clearer way. A second reason is that continuous time opens up for application of the mathematical apparatus of differential equations; this apparatus is more powerful than the corresponding apparatus of difference equations. Similarly, optimal control theory is more developed and potent in its continuous time version than in its discrete time version, considered in Chapter 8. In addition, many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A).

As a vehicle for comparing continuous time modeling with discrete time modeling we consider a standard household consumption/saving problem. How does the household assess the choice between consumption today and consumption in the future? In contrast to the preceding chapters we allow for an arbitrary number of periods within the time horizon of the household. The period length may thus be much shorter than in the previous models. This opens up for capturing additional aspects of economic behavior and for undertaking the transition to continuous time in a smooth way.

We first specify the market environment in which the optimizing household operates.

9.1 Market conditions

In the Diamond OLG model no loan market was active and wealth effects on consumption or saving through changes in the interest rate were absent. It is different in a setup where agents live for many periods and realistically have a hump-shaped income profile through life. This motivates a look at the financial market and more refined notions related to intertemporal choice.

A perfect loan market Consider a given household or, more generally, a given *contractor*. Suppose the contractor at a given date t wants to take a loan or provide loans to others at the going interest rate, i_t, measured in money terms. So two contractors are involved, a *borrower* and a *lender*. Let the market conditions satisfy the following four criteria:

(a) the contractors face the same interest rate whether borrowing or lending (that is, monitoring, administration, and other transaction costs are absent);

(b) there are many contractors on each side and none of them believe to be able to influence the interest rate (the contractors are price takers in the loan market);

(c) there are no borrowing restrictions other than the requirement on the part of the borrower to comply with her financial commitments;

(d) the lender faces no default risk (contracts can always be enforced, i.e., the borrower can somehow cost-less be forced to repay the debt with interest on the conditions specified in the contract).

A loan market satisfying these idealized conditions is called a *perfect loan market*. In such a market,

1. various payment streams can be subject to comparison in a simple way; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another one with the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment in currency at the *end* of period t. Period t runs from time t to time $t+1$ for $t = 0, 1, ..., T-1$. We *ignore uncertainty* and so i_t is the interest rate on a risk-less loan from time t to time $t+1$. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+i_0} + \frac{x_1}{(1+i_0)(1+i_1)} + \cdots + \frac{x_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \qquad (9.1)$$

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$ and at time 0 wishes to buy a durable consumption good of value PV_0, she can borrow this amount and use a part of the income stream $\{x_t\}_{t=0}^{T-1}$ to repay the debt with interest over the periods $t = 0, 1, 2, ..., T-1$. In general, when Jones wishes to have a time profile on the payment stream different from the income stream, she can attain this through appropriate transactions in the loan market, leaving her with any stream of payments of the same present value as the given income stream.

Real versus nominal rate of return In this chapter we maintain the assumption of perfect competition in all markets, i.e., households take all prices as given from the markets. In the *absence of uncertainty,* the various assets (real capital, stocks, loans etc.) in which households invest give the same rate of return in equilibrium. The good which is traded in the loan market can be interpreted as a (risk-less) *bond*. The borrower issues bonds and the lender buys them. In this chapter all bonds are assumed to be short-term, i.e., one-period bonds. For every unit of account borrowed at the end of period $t-1$, the borrower pays back with certainty (1 + short-term interest rate) units of account at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued at the end of the current period and the obtained loans are spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to offering a variable-rate demand deposit like in a bank.[2]

Our analysis will be in real terms, that is, inflation-corrected terms. In principle the unit of account is a fixed bundle of consumption goods. In the simple macroeconomic models to be studied in this and most subsequent chapters, such

[1] We use "present value" as synonymous with "present discounted value". As usual our timing convention is such that PV_0 denotes the time-0 value of the payment stream, including the discounted value of the payment (or dividend) indexed by 0.

[2] Unless otherwise specified, this chapter uses terms like "loan market", "credit market", and "bond market" interchangeably. As uncertainty is ignored, it does not matter whether we have personalized loan contracts or tradable securities in mind.

a bundle is reduced to *one* consumption good. The models simply assume there *is* only one consumption good in the economy. In fact, there will only be *one produced good*, "the" output good, which can be used for both consumption and capital investment. Whether our unit of account is seen as the consumption good or the output good is thus immaterial.

The *real* (net) rate of return on an investment is the rate of return in units of the output good. More precisely, the *real rate of return* in period t, r_t, is the (proportionate) rate at which the *real* value of an investment, made at the end of period $t-1$, has grown after one period.

The link between this rate of return and the more commonplace concept of a nominal rate of return is the following. Imagine that at the end of period $t-1$ you make a bank deposit of value V_t euro. The *real value* of the deposit when you invest is then V_t/P_{t-1}, where P_{t-1} is the price in euro of the output good at the end of period $t-1$. If the nominal short-term interest rate is i_t, the deposit is worth $V_{t+1} = V_t(1+i_t)$ euro at the end of period t. By definition of r_t, the factor by which the deposit in real terms has expanded is

$$1 + r_t = \frac{V_{t+1}/P_t}{V_t/P_{t-1}} = \frac{V_{t+1}/V_t}{P_t/P_{t-1}} = \frac{1+i_t}{1+\pi_t}, \tag{9.2}$$

where $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ is the inflation rate in period t. So the real (net) *rate* of return on the investment is $r_t = (i_t - \pi_t)/(1+\pi_t) \approx i_t - \pi_t$ for i_t and π_t "small". The number $1 + r_t$ is called the real interest *factor* and measures the rate at which current units of output can be traded for units of output one period later.

In the remainder of this chapter we will think in terms of *real* values and completely ignore monetary aspects of the economy.

9.2 Maximizing discounted utility in discrete time

As mentioned, the consumption/saving problem faced by the household is assumed to involve only one consumption good. The composition of consumption in each period is not part of the problem. What remains is the question how to distribute consumption over time.

The intertemporal utility function

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. Period 0 ("the initial period") need not refer to the "birth" of the household but is just an arbitrary period within the lifetime of the household.

We assume the preferences of the household can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure. The latter assumption implies that the labor supply of the household in each period is inelastic. The time-separability itself just means that the intertemporal utility function is additive, i.e., $U(c_0, c_1, \ldots, c_{T-1}) = u^{(0)}(c_0) + u^{(1)}(c_1) + \ldots + u^{(T-1)}(c_{T-1})$, where $u^{(t)}(c_t)$ is the utility contribution from period-t consumption, $t = 0, 1, \ldots, T-1$. In addition we assume *geometric utility discounting,* meaning that utility obtained t periods ahead is converted into a present equivalent by multiplying by the *discount factor* $(1+\rho)^{-t}$, where ρ is a constant utility *discount rate.* So $u^{(t)}(c_t) = u(c_t)(1+\rho)^{-t}$, where $u(c)$ is a time-independent period utility function. Together, these two assumptions amount to

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \ \ldots \ + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \quad (9.3)$$

The period utility function is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$. As explained in Box 9.1, only *linear* positive transformations of the period utility function are admissible.

As (9.3) indicates, the number $1+\rho$ tells how many units of utility in the next period the household insists on "in return" for a decrease of one unit of utility in the current period. So, a $\rho > 0$ will reflect that if the chosen level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher the earlier it arrives. This explains why ρ is named the *rate of time preference* or, even more to the point, the *rate of impatience.* The utility discount factor, $1/(1+\rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t.[3] Admittedly, this assumption that the utility discount rate is a *constant* is questionable. There is a growing body of evidence suggesting that the utility discount rate is usually *not* a constant, but declining with the time distance from the current period to the future periods within the horizon. This phenomenon is referred to as "present bias" or "hyperbolic discounting". Justified or not, macroeconomics often, as a first approach, ignores the problem and assumes a constant ρ to keep things simple. Most of the time we will follow this practice.

It is generally believed that human beings are impatient and that realism

[3] Multiplying through in (9.3) by $(1+\rho)^{-1}$ would make the objective function appear in a way similar to (9.1) in the sense that also the first term in the sum becomes discounted. At the same time the ranking of all possible alternative consumption paths would remain unaffected. For ease of notation, however, we use the form (9.3) which is more standard. Economically, there is no difference.

therefore speaks for assuming ρ positive.[4] The results to be derived in this chapter do not require a positive ρ, however. So we just impose the definitional constraint in discrete time: $\rho > -1$.

Box 9.1. Admissible transformations of the period utility function

When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function U in (9.3) is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, U will express the same ranking between alternative consumption paths as any increasing transformation of U. The period utility function, u, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility of period utility units" is legitimate. That is, the *size* (not just the sign) of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95\left[u(c_1') - u(c_1)\right] \Leftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality. (The notation $x \succ y$ means that x is strictly preferred to y.)

Only a *linear* positive transformation of the utility function u, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged. This is because a linear positive transformation does not affect the *ratios* of marginal utilities (the marginal rates of substitution across time).

The saving problem in discrete time

Suppose the household considered has income from two sources: work and financial wealth. Let a_t denote the real value of (net) financial wealth held by the household at the beginning of period t (a for "assets"). We treat a_t as predetermined at time t and in this respect similar to a variable-interest deposit with a bank. The initial financial wealth, a_0, is thus *given*, independently of what in

[4] If uncertainty were included in the model, $(1 + \rho)^{-1}$ might be interpreted as (roughly) reflecting the probability of surviving to the next period. In this perspective, $\rho > 0$ is definitely a plausible assumption.

interest rate is formed in the loan market. And a_0 can be positive as well as negative (in the latter case the household is initially in debt).

The labor income of the household in period t is denoted $w_t \geq 0$ and may follow a typical life-cycle pattern, first rising, then more or less stationary, and finally vanishing due to retirement. Thus, in contrast to previous chapters where w_t denoted the real wage per unit of labor, here a broader interpretation of w_t is allowed. Whatever the time profile of the amount of labor delivered by the household through life, in this chapter, where the focus is on individual saving, we regard this time profile, as well as the hourly wage as exogenous. The present interpretation of w_t will coincide with the one in the other chapters if we imagine that the household in each period delivers one unit of labor.

To avoid corner solutions, we impose the No Fast Assumption $\lim_{c\to 0} u'(c) = \infty$. Since uncertainty is by assumption ruled out, the problem is to choose a plan $(c_0, c_1, \ldots, c_{T-1})$ so as to maximize

$$U = \sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t} \qquad \text{s.t.} \tag{9.4}$$

$$c_t \geq 0, \tag{9.5}$$

$$a_{t+1} = (1+r_t)a_t + w_t - c_t, \qquad a_0 \text{ given,} \tag{9.6}$$

$$a_T \geq 0, \tag{9.7}$$

where r_t is the interest rate. The control region (9.5) reflects the definitional non-negativity of the control variable, consumption. The dynamic equation (9.6) is an accounting relation telling how financial wealth moves over time. Indeed, income in period t is $r_t a_t + w_t$ and saving is then $r_t a_t + w_t - c_t$. Since saving is by definition the same as the increase in financial wealth, $a_{t+1} - a_t$, we obtain (9.6). Finally, the terminal condition (9.7) is a solvency requirement that no financial debt be left over at the terminal date, T. We may call this decision problem the *standard discounted utility maximization problem* with a *perfect loan market* and *no uncertainty.*

Solving the problem

To solve the problem, let us use the *substitution method.*[5] From (9.6) we have $c_t = (1+r_t)a_t + w_t - a_{t+1}$, for $t = 0, 1, \ldots, T-1$. Substituting this into (9.4), we obtain a function of $a_1, a_2, \ldots, a_T$. Since $u' > 0$, saturation is impossible and so an optimal solution cannot have $a_T > 0$. Hence we can put $a_T = 0$ and the problem is reduced to an essentially unconstrained problem of maximizing a function $\tilde{U}$

[5] Alternative methods include the *Maximum Principle* as described in the previous chapter or *Dynamic Programming* as described in Math Tools.

w.r.t. $a_1, a_2, \ldots, a_{T-1}$. Thereby we indirectly choose $c_0, c_1, \ldots, c_{T-2}$. Given a_{T-1}, consumption in the last period is trivially given as

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1},$$

ensuring

$$a_T = 0, \tag{9.8}$$

the terminal optimality condition, necessary when $u'(c) > 0$ for all $c \geq 0$ (saturation impossible).

To obtain first-order conditions we put the partial derivatives of $\tilde{U}$ w.r.t. a_{t+1}, $t = 0, 1, \ldots, T-2$, equal to 0:

$$\frac{\partial \tilde{U}}{\partial a_{t+1}} = (1 + \rho)^{-t}\left[u'(c_t) \cdot (-1) + (1 + \rho)^{-1}u'(c_{t+1})(1 + r_{t+1})\right] = 0.$$

Reordering gives the Euler equations describing the trade-off between consumption in two succeeding periods,

$$u'(c_t) = (1 + \rho)^{-1}u'(c_{t+1})(1 + r_{t+1}), \qquad t = 0, 1, 2, ..., T-2. \tag{9.9}$$

One of the implications of this condition is that

$$\rho \lesseqqgtr r_{t+1} \text{ causes } u'(c_t) \gtreqqless u'(c_{t+1}), \text{ i.e., } c_t \lesseqqgtr c_{t+1} \tag{9.10}$$

in the optimal plan (due to $u'' < 0$). Absent uncertainty the optimal plan entails either increasing, constant, or decreasing consumption over time depending on whether the rate of time preference is below, equal to, or above the rate of return on saving.

Interpretation The interpretation of (9.9) is as follows. Let the consumption path $(c_0, c_1, \ldots, c_{T-1})$ be our "reference path". Imagine an alternative path which coincides with the reference path except for the periods t and $t+1$. If it is possible to obtain a higher total discounted utility than in the reference path by varying c_t and c_{t+1} within the constraints (9.5), (9.6), and (9.7), at the same time as consumption in the other periods is kept unchanged, then the reference path cannot be optimal. That is, "local optimality" is a necessary condition for "global optimality". So the optimal plan must be such that the current utility loss by decreasing consumption c_t by one unit equals the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit in the current period.

A more concrete interpretation, avoiding the notion of "utility units", is obtained by rewriting (9.9) as

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{9.11}$$

The left-hand side indicates the marginal rate of substitution, MRS, of period-$(t+1)$ consumption for period-t consumption, namely the increase in period-$(t+1)$ consumption needed to compensate for a one-unit marginal decrease in period-t consumption:

$$MRS_{t+1,t} = -\frac{dc_{t+1}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}. \tag{9.12}$$

And the right-hand side of (9.11) indicates the marginal rate of transformation, MRT, which is the rate at which the loan market allows the household to shift consumption from period t to period $t+1$.

So, in an optimal plan MRS must equal MRT. This has implications for the time profile of optimal consumption as indicated by the relationship in (9.10). The Euler equations, (9.9), can also be seen in a comparative perspective. Consider two alternative values of r_{t+1}. The higher interest rate will induce a *negative substitution effect* on current consumption, c_t. There is also an *income effect*, however, and this goes in the *opposite* direction. The higher interest rate makes the present value of a given consumption plan lower. This allows more consumption in all periods for a given total wealth. Moreover, there is generally a third effect of the rise in the interest rate, a *wealth effect*. As indicated by the intertemporal budget constraint in (9.20) below, total wealth includes the present value of expected future after-tax labor earnings and this present value depends *negatively* on the interest rate, cf. (9.15) below.

From the formula (9.12) we see one of the reasons that the assumption of a *constant* utility discount rate is *convenient* (but also restrictive). The marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the same in the two periods.

The formula for MRS between consumption this period and consumption *two* periods ahead is

$$MRS_{t+2,t} = -\frac{dc_{t+2}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-2}u'(c_{t+2})}.$$

This displays one of the reasons that the time-separability of the intertemporal utility function is a *strong* assumption. It implies that the trade-off between consumption this period and consumption two periods ahead is independent of consumption in the interim.

Deriving the consumption function when utility is CRRA The first-order conditions (9.9) tell us about the relative consumption levels over time, not the absolute level. The latter is determined by the condition that initial consumption, c_0, must be highest possible, given that the first-order conditions *and* the constraints (9.6) and (9.7) must be satisfied.

To find an explicit solution we have to specify the period utility function. As an example we choose the CRRA function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$.[6] Moreover we simplify by assuming $r_t = r$, a constant > -1. Then the Euler equations take the form $(c_{t+1}/c_t)^\theta = (1+r)(1+\rho)^{-1}$ so that

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \equiv \gamma, \tag{9.13}$$

and thereby $c_t = \gamma^t c_0$, $t = 0, 1, \ldots, T-1$. Substituting into the accounting equation (9.6), we thus have $a_{t+1} = (1+r)a_t + w_t - \gamma^t c_0$. By backward substitution we find the solution of this difference equation to be

$$a_t = (1+r)^t \left[a_0 + \sum_{i=0}^{t-1}(1+r)^{-(i+1)}(w_i - \gamma^i c_0)\right].$$

Optimality requires that the left-hand side of this equation vanishes for $t = T$. So we can solve for c_0 :

$$c_0 = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}\left[a_0 + \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i\right] = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}(a_0 + h_0), \tag{9.14}$$

where we have inserted the human wealth of the household (present value of expected lifetime labor income) as seen from time zero:

$$h_0 = \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i. \tag{9.15}$$

Thus (9.14) says that initial consumption is proportional to initial *total wealth*, the sum of financial wealth and human wealth at time 0. To allow for positive consumption we need $a_0 + h_0 > 0$.

[6] In later sections of this chapter we let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

In (9.14) γ is not one of the original parameters, but a derived parameter. To express the consumption function only in terms of the original parameters, note that, by (9.14), the propensity to consume out of total wealth depends on:

$$\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i = \begin{cases} \frac{1-\left(\frac{\gamma}{1+r}\right)^T}{1-\frac{\gamma}{1+r}} & \text{when } \gamma \neq 1+r, \\ T & \text{when } \gamma = 1+r, \end{cases} \tag{9.16}$$

where the result for $\gamma \neq 1+r$ follows from the formula for the sum of a finite geometric series. Inserting this together with (9.13) into (9.14), we end up with the expression

$$c_0 = \begin{cases} \frac{(1+r)\left[1-(1+\rho)^{-1/\theta}(1+r)^{(1-\theta)/\theta}\right]}{1-(1+\rho)^{-T/\theta}(1+r)^{(1-\theta)T/\theta}}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \neq 1+r, \\ \frac{1+r}{T}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = 1+r. \end{cases} \tag{9.17}$$

This, together with (9.14), thus says:

Result 1: Consumption is proportional to total wealth, and the factor of proportionality, often called the *marginal propensity to consume out of wealth*, depends on the interest rate r, the time horizon T, and the preference parameters ρ and θ, where ρ is the impatience rate and θ is the strength of the preference for consumption smoothing, respectively.

For the subsequent periods we have from (9.13) that

$$c_t = c_0\left(\left(\frac{1+r}{1+\rho}\right)^{1/\theta}\right)^t, \qquad t = 1, \ldots, T-1. \tag{9.18}$$

EXAMPLE 1 Consider the special case $\theta = 1$ (i.e., $u(c) = \ln c$) together with $\rho > 0$. The upper case in (9.17) is here the relevant one and period-0 consumption will be

$$c_0 = \frac{(1+r)(1-(1+\rho)^{-1})}{1-(1+\rho)^{-T}}(a_0+h_0) \qquad \text{for } \theta = 1.$$

We see that $c_0 \to (1+r)\rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$, assuming the right-hand side of (9.15) converges for $T \to \infty$.

We have assumed that payment for consumption occurs at the end of the period at the price 1 per consumption unit. To compare with the corresponding result in continuous time with continuous compounding (see Section 9.4), we might want to have initial consumption in the same present value terms as a_0 and h_0. That is, we consider $\tilde{c}_0 \equiv c_0(1+r)^{-1} = \rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$. □

So far the expression (9.17) is only a *candidate* consumption function. But in view of strict concavity of the objective function, (9.17) is indeed the unique optimal solution when $a_0 + h_0 > 0$.

The conclusion from (9.17) and (9.18) is that *consumers look beyond current income.* More precisely:

Result 2: Under the idealized conditions assumed, including a perfect loan market and perfect foresight, and given the marginal propensity to consume out of total wealth shown in (9.17), the time profile of consumption is determined by the total wealth and the interest rate (relative to impatience corrected for the preference for consumption smoothing). The time profile of *income* does not matter because consumption can be smoothed over time by drawing on the loan market.

EXAMPLE 2 Consider the special case $\rho = r > 0$ (and still $\theta = 1$). Again the upper case in (9.17) is the relevant one and period-0 consumption will be

$$c_0 = \frac{r}{1-(1+r)^{-T}}(a_0 + h_0).$$

We see that $c_0 \to r(a_0 + h_0)$ for $T \to \infty$, assuming the right-hand side of (9.15) converges for $T \to \infty$. So, with an infinite time horizon current consumption equals the interest on total current wealth. By consuming this the individual or household maintains total wealth intact. This consumption function provides an interpretation of Milton Friedman's *permanent income hypothesis.* Friedman defined "permanent income" as "the amount a consumer unit could consume (or believes it could) while maintaining its wealth intact" (Friedman, 1957). The key point of Friedman's theory was the idea that a random change in current income only affects current consumption to the extent that it affects "permanent income". Replacing Friedman's awkward term "permanent income" by the straightforward "total wealth", this feature is a general aspect of all consumption functions considered in this chapter. In contrast to the theory in this chapter, however, Friedman emphasized credit market imperfections and thought of a "subjective income discount rate" of as much as 33% per year. His interpretation of the empirics was that households adopt a much shorter "horizon" than the remainder of their expected lifetimes (Friedman, 1963, Carroll 2001). □

If the real interest rate varies over time, the discount factor $(1+r)^{-(i+1)}$ for a payment made at the end of period i is replaced by $\Pi_{j=0}^{i}(1+r_j)^{-1}$.

Alternative approach based on the intertemporal budget constraint

There is another approach to the household's saving problem. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint (IBC for short). The present value of the consumption plan $(c_1, ..., c_{T-1})$, as seen from time zero, is

$$PV(c_0, c_1, ..., c_{T-1}) \equiv \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)}. \tag{9.19}$$

This value cannot exceed the household's total initial wealth, $a_0 + h_0$. So the household's *intertemporal budget constraint* is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)} \leq a_0 + h_0. \tag{9.20}$$

In this setting the household's problem is to choose its consumption plan so as to maximize U in (9.4) subject to this budget constraint.

This way of stating the problem is equivalent to the approach above based on the dynamic budget condition (9.6) and the solvency condition (9.7). Indeed, given the accounting equation (9.6), the consumption plan of the household will satisfy the intertemporal budget constraint (9.20) if and only if it satisfies the solvency condition (9.7). And there will be strict equality in the intertemporal budget constraint if and only if there is strict equality in the solvency condition (the proof is similar to that of a similar claim relating to the government sector in Chapter 6.2).

Moreover, since in our specific saving problem saturation is impossible, an optimal solution must imply strict equality in (9.20). So it is straightforward to apply the substitution method also within the IBC approach. Alternatively one can introduce the *Lagrange function* associated with the problem of maximizing $U = \sum_{t=0}^{T-1}(1+\rho)^{-t}u(c_t)$ s.t. (9.20) with strict equality.

Infinite time horizon In the Ramsey model of the next chapter, the idea is used that households may have an *infinite* time horizon. One interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links. The household is then seen as a family dynasty with a time horizon beyond the lifetime of the current members of the family. Barro's bequest model in Chapter 7 is an application of this idea. Given a sufficiently large rate of time preference, it is ensured that the sum of achievable discounted utilities over an infinite horizon is bounded from above.

One could say, of course, that infinity is a long time. The sun will eventually, in some billion years, burn out and life on earth become extinct. Nonetheless, there are several reasons that an infinite time horizon may provide a convenient substitute for finite but remote horizons. First, in many cases the solution to an optimization problem for T "large" is in a major part of the time horizon close to the solution for $T \to \infty$.[7] Second, an infinite time horizon tends to ease aggregation because at any future point in time, remaining time is still infinite. Third, an infinite time horizon may be a convenient notion when in any given period there is a always a positive probability that there will be a next period to be concerned about. This probability may be low, but this can be reflected in a high effective utility discount rate. This idea will be applied in chapters 12 and 13.

We may perform the transition to infinite horizon by letting $T \to \infty$ in the objective function, (9.4) and the intertemporal budget constraint, (9.20). On might think that, in analogy of (9.8) for the case of finite T, the terminal optimality condition for the case of infinite horizon is $\lim_{T\to\infty} a_T = 0$. This is generally not so, however. The reason is that with infinite horizon there is no final date where all debt must be settled. The terminal optimality condition in the present problem with a perfect loan market is simply equivalent to the condition that the intertemporal budget constraint should hold with strict equality.

Like with finite time horizon, the saving problem with infinite time horizon may alternatively be framed in terms of a series of dynamic period-by-period budget identities, in the form (9.6), together with the borrowing limit known as the No-Ponzi-Game condition:

$$\lim_{t\to\infty} a_t \Pi_{i=0}^{t-1}(1+r_i)^{-1} \geq 0.$$

As we saw in Chapter 6.5.2, such a "flow" formulation of the problem is equivalent to the formulation based on the intertemporal budget constraint. We also recall from Chapter 6 that the name Ponzi refers to a guy, Charles Ponzi, who in Boston in the 1920s temporarily became very rich by a loan arrangement based on the chain letter principle. The fact that debts grow without bounds is irrelevant for the lender *if* the borrower can always find new lenders and use their outlay to pay off old lenders with the contracted interest. In the real world, endeavours to establish this sort of financial eternity machine sooner or later break down because the flow of new lenders dries up. Such financial arrangements, in everyday speech known as pyramid companies, are universally illegal.[8] It is exactly such

[7] The turnpike proposition in Chapter 8 exemplifies this.

[8] A related Danish instance, though on a modest scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Northern Jutland is charged with fraud. In an ad she offered to tell the reader, for 200 DKK, how to make easy money. Some hundred people responded and received the reply: do like me".

A more serious present day example is the Wall Street stockbroker, Bernard Madoff, who admitted a Ponzi scheme that is considered to be the largest financial fraud in U.S. history. In 2009 Madoff was sentenced to 150 years in prison. Other examples of large-scale Ponzi games appeared in Albania 1995-97 and Ukraine 2008.

arrangements the No-Ponzi-Game condition precludes.

The terminal optimality condition, known as a *transversality condition*, can be shown[9] to be

$$\lim_{t\to\infty}(1+\rho)^{-(t-1)}u'(c_{t-1})a_t = 0.$$

9.3 Transition to continuous time analysis

In the formulation of a model we have a choice between putting the model in period terms or in continuous time. In the former case, denoted period analysis or discrete time analysis, the run of time is divided into successive periods of equal length, taken as the time-unit. We may index the periods by $i = 0, 1, 2, \ldots.$ Thus, in period analysis financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t + \Delta t)$ refers to financial wealth at time $t + \Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t)-a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$.

Now, *keep the time unit unchanged*, but let the length of the time interval $[t, t + \Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When a is a differentiable function of t, we have

$$\lim_{\Delta t\to 0}\frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{a(t+\Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *time derivative of* a at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{9.21}$$

where $s(t)$ is the saving flow (saving intensity) at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t + 1) - a(t) \approx s(t)$.

[9] The proof is similar to that given in Chapter 8, Appendix C.

As time unit choose one year. Going back to discrete time, if wealth grows at a constant rate g per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \qquad i = 0, 1, 2, \ldots . \tag{9.22}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period, we have

$$a_i = a_0(1+\frac{g}{n})^i. \tag{9.23}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (9.23) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$. Thus $m \to \infty$. In the limit there is continuous compounding and we get

$$a(t) = a_0 e^{gt}, \tag{9.24}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.7182818285....$

The formula (9.24) is the continuous-time analogue to the discrete time formula (9.22) with annual compounding. A geometric growth factor, $(1+g)^i$, is replaced by an exponential growth factor, e^{gt}, and this growth factor is valid for any t in the time interval $(-\tau_1, \tau_2)$ for which the growth rate of a equals the constant g (τ_1 and τ_2 being some positive real numbers).

We can also view the formulas (9.22) and (9.24) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.22) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.24) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$, a continuous function of t. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.25}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.25) is called the *accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[10]

[10] Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.26}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.25) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time, we use compound discounting. We reverse the accumulation formula and go from the compounded or terminal value to the present value, $a(t_0)$. Similarly, given a consumption plan $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt}dt, \tag{9.27}$$

presupposing a constant interest rate, r. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.27) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau}dt.$$

In (9.27) $c(t)$ is discounted by $e^{-rt} \approx (1 + r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (9.19) where it is c_{t-1} that is discounted by $(1 + r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t - 1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

The range for particular parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{9.28}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in period analysis, now $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.28) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$, and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations.* What is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these elementary units we can form *composite measurement units.* Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are thus illegitimate. But one can write $K(t + \Delta t) \approx K(t) + (I(t) - \delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* (or saving *density*), at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

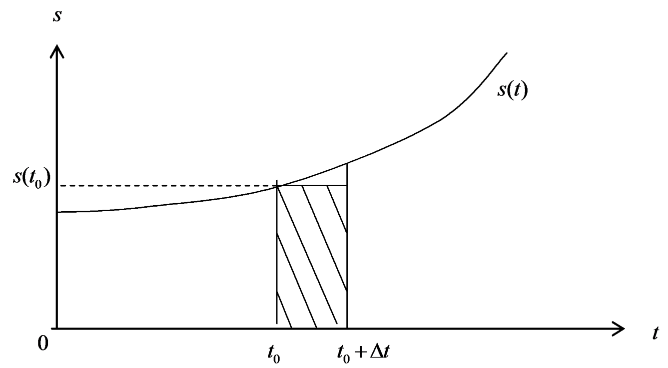

Figure 9.1: With Δt *small* the integral of $s(t)$ from t_0 to $t_0 + \Delta t \approx$ the hatched area.

The choice between discrete and continuous time formulation

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for period analysis rather than continuous time modelling. And the fact that economic actors often think, decide, and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. Moreover, as for instance Allen (1967) argued, it can hardly be said that the *mass* of economic actors think and decide with the same time distance between successive decisions and actions. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. In addition, centralized asset markets equilibrate very fast and respond almost immediately to new information. For such markets a formulation in continuous time seems a good approximation.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.29}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1+\alpha)x_t$, where $1+\alpha < 0$. The solution is $x_t = (1+\alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1+\alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise (together with divergence if $\alpha < -2$) in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

This potential problem can always be avoided, however, by choosing a sufficiently *short* period length in the discrete time model. The solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.29), the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.29). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. Yet, everything else equal, the clearer distinction between stocks and flows in continuous time than in discrete time speaks for the former. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.[11]

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

In continuous time the analogue to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.30}$$

In this context it is common to name the utility flow, u, the *instantaneous utility function*. We still assume that $u' > 0$ and $u'' < 0$. The analogue in continuous time to the intertemporal budget constraint (9.20) is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{9.31}$$

[11] In the latter case, Nobel laureate Robert C. Merton argues in favor of a continuous time formulation (Merton, 1975).

where, as before, a_0 is the historically given initial financial wealth, while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.32}$$

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.31).

Infinite time horizon Transition to infinite horizon is performed by letting $T \to \infty$ in (9.30), (9.31), and (9.32). In the limit the household's, or dynasty's, problem becomes one of choosing a plan, $(c(t))_{t=0}^{\infty}$, which maximizes

$$U_0 = \int_0^{\infty} u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.33}$$

$$\int_0^{\infty} c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.32) approach ∞. With an infinite horizon there may exist technically feasible paths along which the integrals in (9.30), (9.31), and (9.32) go to ∞ for $T \to \infty$. In that case maximization is not well-defined. However, the assumptions we are going to make when working with infinite horizon will guarantee that the integrals converge as $T \to \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high.

Generally we define a person as *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in an economy with a perfect loan market remain "small", they will in general equilibrium remain solvent if and only if their net debt does not exceed the present value of future primary saving.[12] Denoting by d_0 net debt at time 0, i.e., $d_0 \equiv -a_0$, the solvency requirement as seen from time 0 is

$$d_0 \leq \int_0^{\infty} (w(t) - c(t))e^{-\int_0^t r(\tau)d\tau}dt,$$

where the right-hand side is the present value of future primary saving. By the definition in (9.32), we see that this requirement is identical to the intertemporal budget constraint (IBC) which consequently expresses solvency.

[12] By *primary* saving is meant the difference between current non-interest income and current consumption, where non-interest income means labor income and transfers after tax.

The budget constraint in flow terms

The method which is particularly apt for solving intertemporal decision problems in continuous time is based on the mathematical discipline *optimal control theory*. To apply the method, we have to convert the household's budget constraint from the present-value formulation considered above into flow terms.

By mere accounting, in every short time interval $(t,\ t+\Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t)+\dot{a}(t))\Delta t = (r(t)a(t)+w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is the increase per time unit in financial wealth, and thereby the saving intensity, at time t (assuming no robbery). If we divide through by Δt and rearrange, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{9.34}$$

This equation in itself is just a dynamic budget identity. It tells how much and in which direction the financial wealth is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only "restriction", one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until we add the requirement of solvency that we get a *constraint*. When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt is left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (9.31).

When $T \to \infty$, the relevant solvency requirement is the No-Ponzi-Game condition

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

This condition says that the present value of net debt, measured as $-a(t)$, infinitely far out in the future, is not permitted to be positive. We have the following equivalency:

PROPOSITION 1 *(equivalence of NPG condition and intertemporal budget constraint)* Let the time horizon be infinite and assume that the integral (9.32) remains finite for $T \to \infty$. Then, given the accounting relation (9.34), we have:
(i) the requirement (NPG) is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household, or family dynasty, can remain in debt. This would also be an unnatural requirement as the dynasty is infinitely-lived. The condition does imply, however, that there is an upper bound for the speed whereby debt can increase in the long term. The NPG condition says that in the long term, debts are not allowed to grow at a rate as high as (or higher than) the interest rate.

To understand the implication, consider the case with a constant interest rate $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[13] Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run cover their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run. In brief, the borrowers are not allowed to run a Ponzi Game.

9.4.2 Solving the saving problem

The household's consumption/saving problem is one of choosing a path for the *control variable* $c(t)$ so as to maximize a *criterion function,* in the form of an integral, subject to constraints that include a first-order differential equation where the control variable enters, namely (9.34). Choosing a time path for the control variable, this differential equation determines the evolution of the *state variable*, $a(t)$. Optimal control theory, which in Chapter 8 was applied to a related discrete time problem, offers a well-suited apparatus for solving this kind of optimization problems. We will make use of a special case of Pontryagin's *Maximum Principle* (the basic tool of optimal control theory) in its continuous time version. We shall consider both the finite and the infinite horizon case. The only regularity condition required is that the exogenous variables, here $r(t)$ and $w(t)$, are piecewise continuous and that the control variable, here $c(t)$, is piecewise continuous and take values within some given set $\mathbb{C} \subset \mathbb{R}$, called the *control region.*

For a fixed $T < \infty$ the problem is: choose a plan $(c(t))_{t=0}^{T}$ that maximizes

$$\begin{aligned} U_0 &= \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} && (9.35) \\ c(t) &\geq 0, \qquad \text{(control region)} && (9.36) \\ \dot{a}(t) &= r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given}, && (9.37) \\ a(T) &\geq 0. \qquad \text{(solvency requirement)} && (9.38) \end{aligned}$$

With an infinite time horizon, the upper limit of integration, T, in (9.35) is

[13] Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have $d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

interpreted as going to infinity, and the solvency condition (9.38) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Let I denote the time interval $[0,T]$ if $T < \infty$ and the time interval $[0,\infty)$ if $T \to \infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (9.36) and (9.37) for all $t \in I$ as well as the relevant solvency condition, we call $(a(t), c(t))_{t=0}^T$ an *admissible path.* If a given admissible path $(a(t), c(t))_{t=0}^T$ solves the problem, it is referred to as an *optimal path.*[14] We assume that $w(t) > 0$ for all t. No condition on the impatience parameter ρ is imposed (in this chapter).

Necessary conditions for an optimal plan

The solution procedure for this problem is as follows:[15]

1. We set up the *current-value Hamiltonian function* (often just called the *current-value Hamiltonian* or even just the *Hamiltonian)*:

$$H(a, c, \lambda, t) \equiv u(c) + \lambda(ra + w - c),$$

where λ is the *adjoint variable* (also called the *co-state variable*) associated with the dynamic constraint (9.37).[16] That is, λ is an auxiliary variable which is a function of t and is analogous to the Lagrange multiplier in static optimization.

2. At every point in time, we maximize the Hamiltonian w.r.t. the *control variable.* Focusing on an *interior* optimal path,[17] we calculate

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0.$$

For every $t \in I$ we thus have the condition

$$u'(c(t)) = \lambda(t). \tag{9.39}$$

[14] The term "path", sometimes "trajectory", is common in the natural sciences for a solution to a differential equation because one may think of this solution as the path of a particle moving in two- or three-dimensional space.

[15] The four-step solution procedure below is applicable to a large class of dynamic optimization problems in continuous time, see Math tools.

[16] The explicit dating of the time-dependent variables a, c, and λ is omitted where not needed for clarity.

[17] A path, $(a_t, c_t)_{t=0}^T$, is an *interior* path if for no $t \in [0,T)$, the pair (a_t, c_t) is at a boundary point of the set of admissible such pairs. In the present case where a_t is not constrained, except at $t = T$, $(a_t, c_t)_{t=0}^T$, is an *interior* path if $c_t > 0$ for all $t \in [0,T)$.

3. We calculate the partial derivative of H with respect to the *state variable* and put it equal to minus the time derivative of λ plus the discount rate (as it appears in the integrand of the criterion function) multiplied by λ, that is
$$\frac{\partial H}{\partial a} = \lambda r = -\dot{\lambda} + \rho\lambda.$$
This says that, for all $t \in I$, the adjoint variable λ should fulfil the differential equation
$$\dot{\lambda}(t) = (\rho - r(t))\lambda(t). \tag{9.40}$$

4. We now apply the *Maximum Principle* which in the present context says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda = \lambda(t)$ such that for all $t \in I$, (9.39) and (9.40) hold along the path, and the *transversality condition,*
$$\begin{aligned} a(T)\lambda(T) &= 0, \text{ if } T < \infty, \text{ and} \\ \lim_{t\to\infty} a(t)\lambda(t)e^{-\rho t} &= 0, \text{ if “}T = \infty\text{”}, \end{aligned} \tag{TVC}$$
is satisfied.

The transversality condition stated for the *finite* horizon case is general and simply a part of the Maximum Principle. But for the *infinite* horizon case no such *general* mathematical theorem exists. Fortunately, for the household's intertemporal consumption-saving problem, the condition stated in the second row of (TVC) turns out to *be* the true *necessary* transversality condition, as we shall see at the end of this section.

Let us provide an interpretation of the set of optimality conditions. Overall, the Maximum Principle characterizes an optimal path as a path that for every t implies maximization of the Hamiltonian associated with the problem and at the same time satisfies a certain terminal condition. Intuitively, maximization of the Hamiltonian at every t is needed because the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, which comes from the generated change in the state variable (here financial wealth). In the present context "right" means that the trade-off between consuming or saving the marginal unit of account is described in accordance with the opportunities offered by the rate of return vis-a-vis the time preference rate, ρ. Indeed, the optimality condition (9.39) can be seen as a $MC = MB$ condition in utility terms: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation.

Together with the optimality condition (9.40), this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[18] Reordering the differential equation (9.40) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho. \tag{9.41}$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving. Indeed, $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The right-hand side is the *required* marginal rate of return in utility units, ρ. Along an optimal path the two must coincide. The household is willing to save the marginal unit of income only up to the point where the actual return on saving equals the required return.

We may alternatively write the no-arbitrage condition as

$$r = \rho - \frac{\dot{\lambda}}{\lambda}. \tag{9.42}$$

On the left-hand-side appears the actual *real* rate of return on saving and on the right-hand-side the *required real* rate of return. The intuition behind this condition can be seen in the following way. Suppose Mr. Jones makes a deposit of V utility units in a "bank" that offers a proportionate rate of expansion of the utility value of the deposit equal to i (assuming no withdrawal occurs), i.e.,

$$\frac{\dot{V}}{V} = i.$$

This is the actual *utility* rate of return, a kind of "nominal interest rate". To calculate the corresponding "real interest rate" let the "nominal price" of a consumption good be λ utility units. Dividing the number of invested utility units, V, by λ, we get the *real* value, $m = V/\lambda$, of the deposit at time t. The actual *real* rate of return on the deposit is therefore

$$r = \frac{\dot{m}}{m} = \frac{\dot{V}}{V} - \frac{\dot{\lambda}}{\lambda} = i - \frac{\dot{\lambda}}{\lambda}. \tag{9.43}$$

Mr. Jones is just willing to save the marginal unit of income if this actual real rate of return on saving equals the required real rate, that is, the right-hand side of (9.42). In turn, this necessitates that the "nominal interest rate", i, in (9.43) equals the required nominal rate, ρ. The formula (9.43) is analogue to the discrete-time formula (9.2) except that the unit of account in (9.43) is current utility while in (9.2) it is currency.

[18] Recall, a *shadow price* (measured in some unit of account) of a good is, from the point of view of the buyer, the maximum number of units of account that the optimizing buyer is willing to offer for one extra unit of the good.

The first-order conditions (9.39) and (9.40) thus say in what local direction the pair $(a(t), \lambda(t))_{t=0}^{T}$ must move in the (a, λ) plane to be optimal. This "local optimality" condition, which must hold at every t, is generally satisfied by infinitely many admissible paths. In addition, we need "overall optimality". This requires that the "general level" of the path $(a(t), \lambda(t))_{t=0}^{T}$ is such that a specific *terminal* condition holds, namely the transversality condition (TVC).

Let us see how close to an intuitive understanding of (TVC) as a necessary terminal optimality condition we can come. Consider first the *finite horizon* case $T < \infty$. The solvency requirement is $a(T) \geq 0$. Here (TVC) claims that $a(T)\lambda(T) = 0$ is needed for optimality. This condition is equivalent to

$$a(T)\lambda(T)e^{-\rho T} = 0, \tag{9.44}$$

since multiplying by a positive constant, $e^{-\rho T}$, does not change the condition. The form (9.44) has the advantage of being "parallel" to the transversality condition for the case $T \to \infty$. Transparency is improved if we insert (9.39) to get

$$a(T)u'(c(T))e^{-\rho T} = 0. \tag{9.45}$$

Note that the factor $u'(c(T))e^{-\rho T} > 0$.

Could an optimal plan have ">" in (9.45) instead of "="? That would require $a(T) > 0$. But then, within some positive time interval, consumption could be increased without the ensuing decrease in $a(T)$ violating the solvency requirement $a(T) \geq 0$. Thereby, discounted utility, U_0, would be increased, which contradicts that the considered plan is an optimal plan.

Could an optimal plan have "<" in (9.45) instead of "="? That would require $a(T) < 0$. Thereby the solvency requirement $a(T) \geq 0$ would be violated. So the plan is not admissible and therefore cannot be optimal.

From these two observations we conclude that with finite horizon, (9.45) *is* necessary for optimality. This is then also true for the original $a(T)\lambda(T) = 0$ condition in (TVC). So far so good.

Consider now the *infinite horizon* case $T \to \infty$. Here our intuitive reasoning will be more shaky, hence a precise proposition with a formal proof will finally be presented. Anyway, in (9.45) let $T \to \infty$ so as to give the condition

$$\lim_{T\to\infty} a(T)u'(c(T))e^{-\rho T} = 0. \tag{9.46}$$

By (9.39), this says the same as (TVC) for "$T = \infty$" does. In analogy with the finite horizon case, a plan that violates the condition (9.46), by having ">" instead of "=", reveals scope for improvement. Such a plan would amount to "postponing possible consumption *forever*", which cannot be optimal. The postponed consumption possibilities could be transformed to consumption on earth in real time by reducing $a(T)$ somewhat, without violating (NPG).

Finally, our intuitive belief is again that an optimal plan, violating (9.46) by having "<" instead "=", can be ruled out because it will defy the solvency requirement, that is, the (NPG) condition.

These intuitive considerations do not settle the issue whether (TVC) is really necessary for optimality, and if so, why? Luckily, regarding the household's intertemporal consumption-saving problem a simple formal proof, building on the household's intertemporal budget constraint, exists.

PROPOSITION 2 *(the household's necessary transversality condition with infinite time horizon)* Let $T \to \infty$ in the criterion function (9.35) and assume that the human wealth integral (9.32) converges (and thereby remains bounded) for $T \to \infty$. Assume further that the adjoint variable, $\lambda(t)$, satisfies the first-order conditions (9.39) and (9.40). Then:
(i) In an optimal plan, (NPG) is satisfied with strict equality.
(ii) (TVC) is satisfied if and only if (NPG) is satisfied with strict equality.

Proof. (i) By definition, an admissible plan must satisfy (NPG) and, in view of Proposition 1 in Section 9.4.1, thereby also the household's intertemporal budget constraint, (IBC). To be optimal, an admissible plan must satisfy (IBC) with strict equality; otherwise there would be scope for improving the plan by raising $c(t)$ a little in some time interval without decreasing $c(t)$ in other time intervals. By Proposition 1 strict equality in (IBC) is equivalent to strict equality in (NPG). Hence, in an optimal plan, (NPG) is satisfied with strict equality. (ii) See Appendix D. □

In view of this proposition, we can write the transversality condition for $T \to \infty$ as the NPG condition with strict equality:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} = 0. \tag{TVC'}$$

This offers a nice economically interpretable formulation of the necessary transversality condition. In view of the equivalence of the NPG condition with strict equality and the IBC with strict equality, established in Proposition 1, the transversality condition for $T \to \infty$ can also be written this way:

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0. \tag{IBC'}$$

Above we dealt with the difficult problem of stating necessary transversality conditions when the time horizon is infinite. The approach was to try a simple straightforward extension of the necessary transversality condition for the finite horizon case to the infinite horizon case. Such an extension is often valid (as it was here), but not always. There is no completely general all-embracing mathematical theorem about the necessary transversality condition for infinite-horizon optimal control problems. So, with infinite horizon problems care must be taken.

The current-value Hamiltonian versus the present-value Hamiltonian The prefix "current-value" is used to distinguish the current-value Hamiltonian from what is known as the *present-value Hamiltonian.* The latter is defined as $\hat{H} \equiv He^{-\rho t}$ with $\lambda e^{-\rho t}$ substituted by μ, which is the associated (discounted) adjoint variable. The solution procedure is similar except that step 3 is replaced by $\partial \hat{H}/\partial a = -\dot{\mu}$ and $\lambda(t)e^{-\rho t}$ in the transversality condition is replaced by $\mu(t)$. The two methods are equivalent (and if the discount rate is nil, the formulas for the optimality conditions coincide). But for many economic problems the *current-value* Hamiltonian has the advantage that it makes both the calculations and the interpretation slightly simpler. The adjoint variable, $\lambda(t)$, which as mentioned acts as a shadow price of the state variable, becomes a *current* price along with the other prices in the problem, $w(t)$ and $r(t)$. This is in contrast to $\mu(t)$ which is a *discounted* price.

9.4.3 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (9.39) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$. This equation can be written $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by drawing on (9.39) again. Applying (9.40) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{9.47}$$

where $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption,

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0. \tag{9.48}$$

As in discrete time, $\theta(c)$ indicates the strength of the consumer's preference for consumption smoothing. The inverse of $\theta(c)$ measures the *instantaneous intertemporal elasticity of substitution* in consumption, which in turn indicates the willingness to change the time profile of consumption over time when the interest rate changes, see Appendix F.

The result (9.47) says that an optimal consumption plan is characterized in the following way. The household will completely smooth − even out − consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. In this case the household will have to accept a relatively low level of current consumption with the purpose of enjoying higher consumption in the future. The higher the real interest rate relative to the rate of time preference, the more favorable is it to defer consumption − *everything else equal.* The proviso is important. In addition to

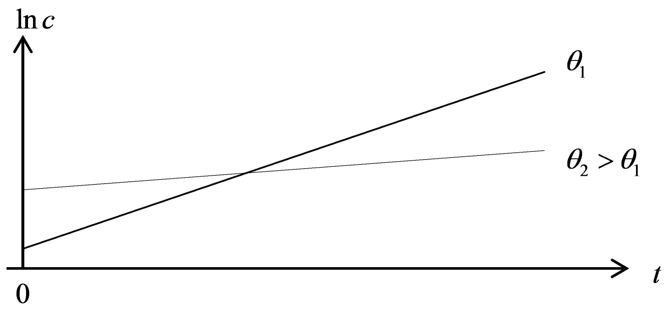

Figure 9.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

the *negative* substitution effect on current consumption of a higher interest rate, there is a *positive* income effect due to the present value of a given intertemporal consumption plan being reduced by a higher interest rate (see (IBC)). On top of this comes a *negative* wealth effect due to a higher interest rate causing a lower present value of expected future labor earnings (again see (IBC)). The special case of a CRRA utility function provides a convenient agenda for sorting these details out, see Example 1 in Section 9.5.

By (9.47) we also see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption for a given value of $r(t) - \rho$. The reason for this is that a strong curvature means that the marginal utility will drop sharply if consumption increases, and will rise sharply if consumption decreases. Fig. 9.2 illustrates this in the CRRA case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high has lower slope, but starts from a higher level, than when θ is low.

The condition (9.47), which holds for all t within the time horizon whether this is finite or infinite, is referred to as the *Keynes-Ramsey rule*. The name springs from the English mathematician Frank Ramsey who derived the rule in 1928, while his mentor, John Maynard Keynes, suggested a simple and intuitive way of presenting it. The rule is the continuous-time counterpart to the consumption Euler equation in discrete time.

The Keynes-Ramsey rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods to the corresponding price ratio. In the present context the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different dates. In Section 9.2 we used the principle to interpret the optimal saving behavior in discrete time. How can the principle be translated into a continuous time setting?

Local optimality in continuous time* Let $(t, t+\Delta t)$ and $(t+\Delta t, t+2\Delta t)$ be two short successive time intervals. The marginal rate of substitution, $MRS_{t+\Delta t,t}$, of consumption in the second time interval for consumption in the first, is[19]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} = \frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))}, \tag{9.49}$$

approximately. On the other hand, by saving $-\Delta c(t)$ more per time unit (where $\Delta c(t) < 0$) in the short time interval $(t, t+\Delta t)$, one can, via the market, transform $-\Delta c(t) \cdot \Delta t$ units of consumption in this time interval into

$$\Delta c(t+\Delta t) \cdot \Delta t \approx -\Delta c(t)\Delta t \; e^{\int_t^{t+\Delta t} r(\tau)d\tau} \tag{9.50}$$

units of consumption in the time interval $(t+\Delta t, t+2\Delta t)$. The marginal rate of transformation is therefore

$$\begin{aligned} MRT_{t+\Delta t,t} &\equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} \approx \\ &= e^{\int_t^{t+\Delta t} r(\tau)d\tau}. \end{aligned}$$

In the optimal plan we must have $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ which gives

$$\frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}, \tag{9.51}$$

approximately. When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (9.11) from discrete time (generally, by a first-order Taylor approximation, we have $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (9.51), dividing through by Δt, inserting (9.50), and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{9.52}$$

With the definition of $\theta(c)$ in (9.48), this is exactly the same as the Keynes-Ramsey rule (9.47) which, therefore, is merely an expression of the general optimality condition $MRS = MRT$. When $\dot{c}(t) > 0$, the household is willing to sacrifice some consumption today for more consumption tomorrow only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines with rising consumption, i.e., the larger is $(-u''/u')\dot{c}$ already. Indeed, a higher c_t in the future than today implies a lower marginal utility of consumption in the future than of consumption today. Saving of the marginal unit of income today is thus only warranted if the rate of return is sufficiently above ρ, and this is what (9.52) indicates.

[19] The underlying analytical steps can be found in Appendix E.

9.4.4 Mangasarian's sufficient conditions

For dynamic optimization problems with one state variable, one control variable, and finite or infinite horizon, the present version of the Maximum Principle delivers a set of first-order conditions and suggests a terminal optimality condition, the transversality condition. The first-order conditions are *necessary* conditions for an interior path to be optimal. With infinite horizon, the necessity of the suggested transversality condition in principle requires a verification in each case. In the present case the verification is implied by Proposition 2. So, up to this point we have only shown that if the consumption/saving problem has an interior solution, then this solution satisfies the Keynes-Ramsey rule and a transversality condition, (TVC').

But are these conditions also *sufficient*? The answer is yes in the present case. This follows from *Mangasarian's sufficiency theorem* (see Math tools) which, applied to the present problem, tells us that if the Hamiltonian is *jointly concave* in (a, c) for every t within the time horizon, then the listed first-order conditions, together with the transversality condition, are also sufficient. Because the instantaneous utility function (the first term in the Hamiltonian) is here strictly concave in c and the second term is linear in (a, c), the Hamiltonian *is* jointly concave in (a, c).

To sum up: if we have found a path satisfying the Keynes-Ramsey rule and (TVC'), we have a *candidate solution.* Applying the Mangasarian theorem, we check whether our candidate *is* an optimal solution. In the present optimization problem it is. In fact the strict concavity of the Hamiltonian with respect to the control variable in this problem ensures that the optimal solution is *unique* (Exercise 9.?).

9.5 The consumption function

We have not yet fully solved the saving problem. The Keynes-Ramsey rule gives only the optimal rate of *change* of consumption over time. It says nothing about the *level* of consumption at any given time. In order to determine, for instance, the level $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many consumption paths satisfying the Keynes-Ramsey rule, the household will choose the "highest" one that also fulfils the solvency requirement (NPG). Thus, the household acts so that strict equality in (NPG) obtains. As we saw in Proposition 2, this is equivalent to the transversality condition being satisfied.

EXAMPLE 1 *(constant elasticity of marginal utility; infinite time horizon).* In the problem in Section 9.4.2 with $T \to \infty$, we consider the case where the elasticity of marginal utility $\theta(c)$, as defined in (9.48), is a constant $\theta > 0$. From Appendix A of Chapter 3 we know that this requirement implies that up to a positive linear transformation the utility function must be of the form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{9.53}$$

This is our familiar CRRA utility function. In this case the Keynes-Ramsey rule implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{9.54}$$

cf. the general accumulation formula, (9.25).

We know from Proposition 2 that the transversality condition is equivalent to the NPG condition being satisfied with strict equality, and from Proposition 1 we know that this condition is equivalent to the intertemporal budget constraint being satisfied with strict equality, i.e.,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0, \tag{IBC'}$$

where h_0 is the human wealth,

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.55}$$

This result can be used to determine $c(0)$.[20] Substituting (9.54) into (IBC') gives

$$c(0)\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt = a_0 + h_0.$$

The consumption function is thus

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt} = \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau}dt} \end{aligned} \tag{9.56}$$

is the marginal propensity to consume out of wealth. We have here assumed that these improper integrals over an infinite horizon are bounded from above for all admissible paths. We see that consumption is proportional to total wealth. The factor of proportionality, often called the *marginal propensity to consume out of wealth*, depends on the expected future interest rates and on the preference

[20] The method also applies if instead of $T = \infty$, we have $T < \infty$.

parameters ρ and θ, that is, the impatience rate and the strength of the preference for consumption smoothing, respectively.

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[21] On the one hand, such an increase makes future consumption cheaper in present value terms. This change in the trade-off between current and future consumption entails a negative *substitution effect* on $c(0)$. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth, cf. (IBC'). This entails a positive *pure income effect* on consumption today as consumption is a normal good. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the pure income effect, and if $\theta > 1$ (large curvature), the reverse will hold. This is because the larger is θ, the stronger is the propensity to smooth consumption over time.

In the intermediate case $\theta = 1$ (the logarithmic case) we get from (9.56) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{9.57}$$

In this special case the marginal propensity to consume is time independent and equal to the rate of time preference. For a given *total* wealth, $a_0 + h_0$, current consumption is thus independent of the expected path of the interest rate. That is, in the logarithmic case the *substitution* and *pure income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents in the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect. Thus, even in the logarithmic case, and *a fortiori* when $\theta < 1$, the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get

$$\begin{aligned} \beta_0 &= [(\theta - 1)r + \rho]/\theta, \\ a_0 + h_0 &= a_0 + w/r. \end{aligned}$$

When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and h_0, hence the total effect on $c(0)$ is even "more

[21] By an increase in the interest rate *level* we mean an upward shift in the time-profile of the interest rate. That is, there is at least one time interval within $[0, \infty)$ where the interest rate is higher than in the original situation and no time interval within $[0, \infty)$ where the interest rate is lower.

negative". When $\theta > 1$, a higher r implies a higher β_0 which more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 3, available empirical studies generally suggest a value of θ somewhat above 1. □

A remark on *fixed-rate loans* and *positive net debt* is appropriate here. Suppose $a_0 < 0$ and assume that this net debt is *not* in the form of a variable-rate loan (as hitherto assumed), but for instance a fixed-rate mortgage loan. Then a rise in the interest rate level implies a lowering of the present value of the debt and thereby *raises* financial wealth and *possibly total wealth.* If so, the rise in the interest rate level implies a *positive* wealth effect on current consumption, thereby "joining" the positive *pure* income effect in *counterbalancing* the negative substitution effect.

EXAMPLE *2* (*constant absolute semi-elasticity of marginal utility; infinite time horizon*). In the problem in Section 9.4.2 with $T \rightarrow \infty$, we consider the case where the sensitivity of marginal utility, measured by the absolute value of the semi-elasticity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α. The utility function must then, up to a positive linear transformation, be of the form,

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{9.58}$$

This is known as the CARA utility function (where the name CARA comes from "Constant Absolute Risk Aversion"). The Keynes-Ramsey rule now becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find, through (IBC') and partial integration, $c(0) = r(a_0 + h_0) - (r - \rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

This hypothesis of a "constant absolute variability aversion" implies that the degree of *relative* variability aversion is $\theta(c) = \alpha c$ and thus greater, the larger is c. The CARA function has been popular in the theory of behavior under uncertainty. One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 3 *(logarithmic utility; finite time horizon; retirement).* We consider a life-cycle saving problem. A worker enters the labor market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known), retires at time $t_1 \in (0, T]$, and does not wish to pass on bequests. For simplicity we assume that $r_t = r > 0$ for all $t \in [0, T]$ and labor income is $w(t) = w > 0$ for $t \in [0, t_1]$, while

$w(t) = 0$ for $t > t_1$. The decision problem is

$$\begin{aligned}
\max_{(c(t))_{t=0}^T} U_0 &= \int_0^T (\ln c(t)) e^{-\rho t} dt \quad \text{s.t.} \\
c(t) &\geq 0, \\
\dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\
a(T) &\geq 0.
\end{aligned}$$

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r - \rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{9.59}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation the solution of which (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt}\left[\frac{w}{r}(1 - e^{-rz}) - \frac{c_0}{\rho}(1 - e^{-\rho t})\right], \tag{9.60}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (9.60), we get

$$c(0) = (\rho w/r)(1 - e^{-rt_1})/(1 - e^{-\rho T}). \tag{9.61}$$

Substituting this into (9.59) gives the optimal consumption plan.[22]

If $r = \rho$, consumption is constant over time at the level given by (9.61). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself. □

The examples illustrate the importance of *forward-looking expectations*, here expectations about future wage income and interest rates. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 1) and through their impact on the present value of expected future labor income (or of expected future dividends on shares or imputed rental income on owner-occupied houses in a more general setup).[23]

To avoid misunderstanding: The examples should *not* be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the

[22] For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (9.56) gives when $a(0) = 0$ and $\theta = 1$.

[23] There exist cases where, due to new information, a shift in expectations occurs so that a discontinuity in a responding endogenous variable results. How to deal with such cases is treated in Chapter 11.

household's maximization problem with infinite horizon. There is generally no guarantee that integrals converge and thus have an upper bound for $T \to \infty$. The evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary, however. It is determined by the requirement of equilibrium. In turn, of course *existence* of an equilibrium *imposes restrictions* on the utility discount rate relative to the potential growth in instantaneous utility. We shall return to these aspects in the next chapter.

9.6 Concluding remarks

(incomplete)

...

The examples above – and the consumption theory in this chapter in general – should only be seen as a first, simple approximation to actual consumption/saving behavior. One thing is that we have not made a distinction between durable and non-durable consumption goods. More importantly, real world factors such as uncertainty and narrow credit constraints (absence of perfect loan and insurance markets) have been ignored. Debt limits imposed on borrowers are usually much sharper than the NPG condition. When such circumstances are included, current income and expected income in the *near* future tend to become important co-determinants of current consumption, at least for a large fraction of the population with little financial wealth.

Short- and medium-run macro models typically attempt to take into account different kinds of such "frictions". There is also a growing interest in incorporating psychological factors such as short-sightedness (in one or another form, including present-bias) into macroeconomic theory.

9.7 Literature notes

(incomplete)

In Chapter 6, where the borrower was a "large" agent with fiscal and monetary policy mandates, namely the public sector, satisfying the intertemporal budget constraint was a necessary condition for solvency (when the interest rate exceeds the growth rate of income), but not a sufficient condition. When the modelled borrowers are "small" private agents as in this chapter, the situation is different. Neoclassical models with perfect markets then usually contain equilibrium mechanisms such that the agents' compliance with their intertemporal budget constraint is sufficient for lenders' willingness and ability to supply the demanded finance. See ...

Abel 1990, p. 726-53.

Epstein-Zin preferences. See Poul Schou.

Present-bias and time-inconsistency. Strots (1956). Laibson, QJE 1997: $1, \alpha\beta, \alpha\beta^2, ...$

Loewenstein and Thaler (1989) survey the evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. This is known as *hyperbolic discounting.*

The assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period utilities discounted at a constant rate are dealt with by Koopmans (1960), Fishburn and Rubinstein (1982), and $-$ in summary form $-$ by Heal (1998).

Borovika, WP 2013, Recursive preferences, separation of risk aversion and IES.

Deaton, A., *Understanding Consumption*, OUP 1992.

On continuous-time finance, see for instance Merton (1990).

Goldberg (1958).

Allen (1967).

To Math Tools: Rigorous and more general presentations of the Maximum Principle in continuous time applied in economic analysis are available in, e.g., Seierstad and Sydsæter (1987), Sydsæter et al. (2008) and Seierstad and Sydsæter (Optimization Letters, 2009, 3, 507-12).

9.8 Appendix

A. Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \dot{z}(t)/z(t) = \dot{x}(t)/x(t) + \dot{y}(t)/y(t)$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. $\square$

The procedure applied in this proof is called "logarithmic differentiation" w.r.t. t : take the log and then the time derivative.

QUOTIENT RULE $z(t) = x(t)/y(t) \Rightarrow \dot{z}(t)/z(t) = \dot{x}(t)/x(t) - \dot{y}(t)/y(t)$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \dot{z}(t)/z(t) = \alpha\dot{x}(t)/x(t)$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small, cf. Appendix A to Chapter 4.

B. Average growth and interest rates

Sometimes we may want to express the accumulation formula in continuous time,

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.62}$$

in terms of then arithmetic average of the growth rates in the time interval $[0, t]$. This is defined as $\bar{g}_{0,t} = (1/t)\int_0^t g(\tau)d\tau$. So we can rewrite (9.62) as

$$a(t) = a(0)e^{\bar{g}_{0,t}t}, \tag{9.63}$$

which has form similar to (9.24). Similarly, let $\bar{r}_{0,t}$ denote the arithmetic average of the (short-term) interest rates from time 0 to time t, i.e., $\bar{r}_{0,t} = (1/t)\int_0^t r(\tau)d\tau$. Then we can write the present value of the consumption stream $(c(t))_{t=0}^T$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$.

The arithmetic average growth rate, $\bar{g}_{0,t}$, coincides with the average *compound* growth rate from time 0 to time t, that is, the number g satisfying

$$a(t) = a(0)e^{gt}, \tag{9.64}$$

for the same $a(0)$ and $a(t)$ as in (9.63).

There is no similar concordance within discrete time modeling.

Discrete versus continuous compounding Suppose the period length is one year so that the given observations, $a_0, a_1 \dots, a_n$, are annual data. There are two alternative ways of calculating an average compound growth rate (often just called the "average growth rate") for the data. We may apply the geometric growth formula,

$$a_n = a_0(1 + G)^n, \tag{9.65}$$

which is natural if the compounding behind the data *is* discrete and occurs for instance quarterly. Or we may apply the exponential growth formula,

$$a_n = a_0e^{gn}, \tag{9.66}$$

corresponding to continuous compounding. Unless $a_n = a_0$ (in which case $g = 0$), the resulting g will be smaller than the average compound growth rate, G, calculated from (9.65) for the same data. Indeed,

$$g = \frac{\ln \frac{a_n}{a_0}}{n} = \ln(1+G) \lessapprox G$$

for G "small", where "$\lessapprox$" means "close to" (by a first-order Taylor approximation about $G = 0$) but "less than" except if $G = 0$. The intuitive reason for "less than" is that a given growth force is more powerful when compounding is continuous. To put it differently: rewriting $(1+G)^n$ into exponential form gives $(1+G)^n = (e^{\ln(1+G)})^n = e^{gn} < e^{Gn}$, as $\ln(1+G) < G$ for all $G \neq 0$.

On the other hand, the difference between G and g is usually unimportant. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$, the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference between G and g will be substantial. During hyperinflation the monthly inflation rate may be, say, $G = 100\%$, but the corresponding g will be only 69%.[24]

C. Proof of Proposition 1 (about equivalence between the No-Ponzi-Game condition and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{9.67}$$

where $a(0) = a_0$ (given), and the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Technical remark. The expression in (NPG) should be understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths in both the (NPG) and (TVC) and therefore replace "$\lim_{t\to\infty}$" by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0,\infty)$ is defined as $\lim_{t\to\infty} \inf \{f(s) | \ s \geq t\}$.[25] As noted in Appendix E of the previous chapter, however, undamped infinitely fluctuating paths do not turn up in "normal" economic

[24] Apart from the discrete compounding instead of continuous compounding, a *geometric* growth factor is equivalent to a "corresponding" *exponential* growth factor. Indeed, we can rewrite the growth factor $(1+g)^t$, $t = 0, 1, 2, \ldots$, into exponential form since $(1+g)^t = (e^{\ln(1+g)})^t = e^{[\ln(1+g)]t}$. Moreover, if g is "small", we have $e^{[\ln(1+g)]t} \approx e^{gt}$.

[25] By "inf" is meant *infimum* of the set, that is, the largest number less than or equal to all numbers in the set.

optimization problems, whether in discrete or continuous time. Hence, we apply the simpler concept "lim" rather than "lim inf". □

On the background of (9.67), Proposition 1 in the text claimed that (NPG) is equivalent to the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

being satisfied, where h_0 is defined as in (9.55) and is assumed to be a finite number. In addition, Proposition 1 in Section 9.4 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG). A plain proof goes as follows.

Proof. Isolate $c(t)$ in (9.67) and multiply through by $e^{-\int_0^t r(\tau)d\tau}$ to obtain

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

Integrate from 0 to $T > 0$ to get $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$\begin{aligned}
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right) \\
&+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),
\end{aligned}$$

where the second equality follows from integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogenous first-order differential equation and then let $T \to \infty$. Since this is a more generally applicable approach, we will show how it works and use it for Claim 1 below (an extended version of Proposition 1) and for the proof of Proposition 2 in the text. Claim 1 will for example prove useful in Exercise 9.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{9.68}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t} g(s)ds} \geq 0 \tag{9.69}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau \leq x_{t_0}. \tag{9.70}$$

Moreover, if and only if (9.69) is satisfied with strict equality, then (9.70) is satisfied with strict equality.

Proof. The linear differential equation (9.68) has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds}d\tau. \tag{9.71}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (9.69) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (9.70). We also see that if and only if (9.69) holds with strict equality, then (9.70) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds}d\tau, \tag{9.72}$$

which we assume is a finite number. Then, given the flow budget identity

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{9.73}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds}d\tau \leq a_{t_0} + h_{t_0}, \tag{9.74}$$

where a strict equality on the left-hand side of "⇔" implies a strict equality on the right-hand side, and vice versa.

Proof. In (9.68), (9.69), and (9.70), let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$. Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the household's necessary transversality condition with an infinite time horizon)

In the differential equation (9.68) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$, and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (9.40) in Section 9.4. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 9.4 yields

$$\lambda(t_0)\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{9.75}$$

From the first-order condition (9.39) in Section 9.4 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (9.75) can be ignored. Thus, (TVC) in Section 9.4 is equivalent to the condition that (NPG) in that section is satisfied with strict equality (let $t_0 = 0 = n$). This proves (ii) of Proposition 2 in the text. □

E. Intertemporal consumption smoothing

We claimed in Section 9.4 that equation (9.49) gives approximately the marginal rate of substitution of consumption in the time interval $(t + \Delta t,\ t + 2\Delta t)$ for consumption in $(t, t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t, etc., even though they are continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}
\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right) \\
&= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right) \\
&= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].
\end{aligned}$$

Requiring unchanged utility integral $U_0 = \bar{U}_0$ is thus approximately the same as requiring $\Delta[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (9.49).

The instantaneous local optimality condition, equation (9.52), can be interpreted on the basis of (9.51). Take logs on both sides of (9.51) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (9.50), and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{9.76}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (9.76) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

F. Elasticity of intertemporal substitution in continuous time

The relationship between the elasticity of marginal utility and the concept of *instantaneous elasticity of intertemporal substitution* in consumption can be exposed in the following way: consider an indifference curve for consumption in the non-overlapping time intervals $(t, t+\Delta t)$ and $(s, s+\Delta t)$. The indifference curve is depicted in Fig. 9.3. The consumption path outside the two time intervals is kept unchanged. At a given point $(c_t\Delta t, c_s\Delta t)$ on the indifference curve, the marginal rate of substitution of s-consumption for t-consumption, MRS_{st}, is given by the absolute slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, MRS_{st} is rising along the curve when c_t decreases (and thereby c_s increases).

Conversely, we can consider the ratio c_s/c_t as a function of MRS_{st} along the given indifference curve. The elasticity of this consumption ratio w.r.t. MRS_{st} as we move along the given indifference curve then indicates the *elasticity of substitution* between consumption in the time interval $(t, t+\Delta t)$ and consumption in the time interval $(s, s+\Delta t)$. Denoting this elasticity by $\sigma(c_t, c_s)$, we thus have:

$$\sigma(c_t, c_s) = \frac{MRS_{st}}{c_s/c_t} \frac{d(c_s/c_t)}{dMRS_{st}} \approx \frac{\frac{\Delta(c_s/c_t)}{c_s/c_t}}{\frac{\Delta MRS_{st}}{MRS_{st}}}.$$

At an optimum point, MRS_{st} equals the ratio of the discounted prices of good t and good s. Thus, the elasticity of substitution can be interpreted as approximately equal to the percentage increase in the ratio of the chosen goods, c_s/c_t, generated by a one percentage increase in the inverse price ratio, holding the utility level and the amount of other goods unchanged. If $s = t + \Delta t$ and the

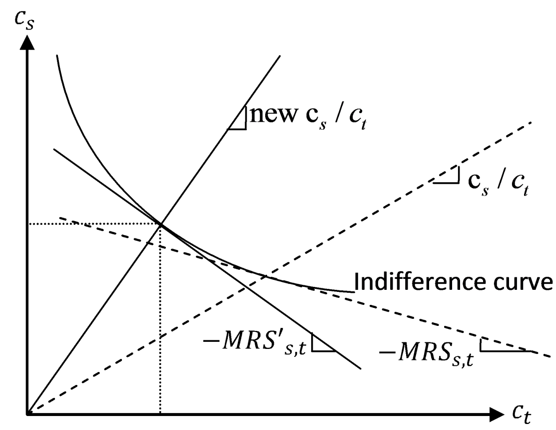

Figure 9.3: Substitution of s-consumption for t-consumption as MRS_{st} increases to MRS'_{st}.

interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (9.51). Inserting MRS_{st} from (9.49) with $t + \Delta t$ replaced by s, we get

$$\begin{aligned}\sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t} \frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}} \\ &= \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))}, \end{aligned} \tag{9.77}$$

since the factor $e^{-\rho(t-s)}$ cancels out.

We now interpret the d's in (9.77) as differentials (recall, the differential of a differentiable function $y = f(x)$ is denoted dy and defined as $dy = f'(x)dx$ where dx is some arbitrary real number). Calculating the differentials we get

$$\sigma(c_t, c_s) \approx \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$ and

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption. It reflects the opposite of the preference for consumption smoothing. Indeed, we see that $\tilde{\sigma}(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility at the consumption level $c(t)$.

9.9 Exercises

9.1 We look at a household (or dynasty) with infinite time horizon. The household's labor supply is inelastic and grows at the constant rate $n > 0$. The household has a constant rate of time preference $\rho > n$ and the individual instantaneous utility function is $u(c) = c^{1-\theta}/(1-\theta)$, where θ is a positive constant. There is no uncertainty. The household maximizes the integral of per capita utility discounted at the rate $\rho - n$. Set up the household's optimization problem. Show that the optimal consumption plan satisfies

$$\begin{aligned}
c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\
\beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta}+n)d\tau} dt}, \qquad \text{and} \\
h_0 &= \int_0^\infty w(t) e^{-\int_0^t (r(\tau)-n)d\tau} dt,
\end{aligned}$$

where $w(t)$ is the real wage per unit of labor and otherwise the same notation as in this chapter is used. *Hint:* apply the corollary to Claim 1 in Appendix C and the method of Example 1 in Section 9.5. As to h_0, start by considering

$$H_0 \equiv h_0 L_0 = \int_0^\infty w(t) L_t e^{-\int_0^t (r(\tau)-n)d\tau} dt$$

and apply that $L(t) = L_0 e^{nt}$.

Chapter 12

Overlapping generations in continuous time

12.1 Introduction

In this chapter we return to issues where life-cycle aspects are important. A representative agent framework is therefore not suitable. We shall see how an overlapping generations (OLG) structure can be made compatible with continuous time analysis.

The reason for the transition to continuous time is the following. The two-period OLG models considered in chapters 3-5 have a coarse notion of time. The implicit length of the period is something in the order of 25-30 years. This implies very rough dynamics. And changes within a shorter time horizon can not be studied. Under special conditions three-period OLG models are analytically obedient, but complex. For OLG models with more than three coexisting generations analytical aggregation is close to unmanageable. Empirical OLG models, for specific economies, with a period length of one or a few years, and thereby many coexisting generations, have been developed. Examples include for the U.S. economy the Auerbach-Kotlikoff (1987) model and for the Danish economy the DREAM model (Danish Rational Economic Agents Model). The dynamics and predictions from this kind of models are studied by numerical simulation on a computer. Governments, large organizations and the financial companies use this type of models to assess how changes in economic policy or in external circumstances are likely to affect the economy.

For basic understanding of economic mechanisms, analytical tractability is important, however. With this in mind, a tractable OLG model with a refined notion of time was developed by the French-American economist, Olivier Blanchard, from Massachusetts Institute of Technology. In a paper from 1985 Blanchard simply suggested an OLG model in *continuous* time, in which people have finite, but uncertain lifetime. The model builds on earlier ideas by Yaari (1965) about life-insurance and is sometimes called the Blanchard-Yaari OLG model. For convenience, we stick to the shorter name *Blanchard OLG model.*

The usefulness of the model derives from its close connection to important facts:

- economic interaction takes place between agents belonging to *many* different age groups;
- agents' working life lasts *many* periods; the present discounted value of expected future labor income is thus a key variable in the system; hereby the wealth effect of a change in the interest rate becomes important;
- owing to uncertainty about remaining lifetime and to retirement from the labor market at old age, a large part of saving is channelled to pension arrangements and various kinds of life-insurance;
- taking finite lifetime into account, the model offers a more realistic approach to the study of long-run effects of government budget deficits and government debt than the Ramsey model;
- by including life expectancy among its parameters, the model opens up for studying effects of demographic changes in the industrialized countries such as increased life expectancy due to improved health conditions.

In the next sections we present and discuss Blanchard's OLG model. A simplifying assumption in the model is that expected remaining lifetime for any individual is independent of age. The simplest version of the model assumes in addition that people stay on the labor market until death. This version is known as the *model of perpetual youth* and is presented in Section 12.2. Later in the chapter we extend the model by including retirement at old age, thereby providing a more distinct life-cycle perspective. Among other things this leads to a succinct theory of the interest rate in the long run. In Section 12.5 we apply the Blanchard framework for a study of national wealth and foreign debt in a small open economy. Key variables are listed in Table 12.1.

The model is in continuous time. Chapter 9 gave an introduction to continuous time analysis. In particular we emphasized that flow variables in continuous time should be interpreted as intensities.

Table 12.1. Key variable symbols in the Blanchard OLG model.

Symbol	*Meaning*
$N(t)$	Size of population at time t
b	Birth rate
m	Death rate (mortality rate)
$n \equiv b - m$	Population growth rate
ρ	Pure rate of time preference
$c(v, t)$	Consumption at time t by an individual born at time v
$C(t)$	Aggregate consumption at time t
$a(v, t)$	Financial wealth at time t by an individual born at time v
$A(t)$	Aggregate financial wealth at time t
$w(t)$	Real wage at time t
$r(t)$	Risk-free real interest rate at time t
$L(t)$	Labor force at time t
$h(v, t)$	PV of expected future labor income by an individual
$H(t)$	Aggregate PV of expected future labor income of people alive at time t
δ	Capital depreciation rate
g	Rate of technological progress
λ	Retirement rate

12.2 The model of perpetual youth

We first portray the household sector. We describe its demographic characteristics, preferences, market environment (including a market for life annuities), the resulting behavior by individuals, and the aggregation across the different age groups. The production sector is as in the previous chapters. But in addition to production firms there are now life insurance companies. Finally, general equilibrium and the dynamic evolution at the aggregate level are studied.

The economy is closed. Perfect competition and rational (model consistent) expectations are assumed throughout. Apart from the uncertain lifetime there is no uncertainty.

12.2.1 Households

Demography

We describe a household as consisting of a single adult whose lifetime is uncertain. Let X denote the remaining lifetime (a stochastic variable) of this person. Then

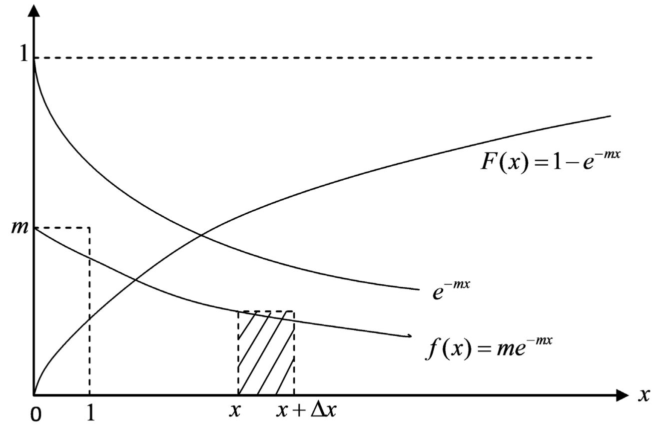

Figure 12.1: The survival probability, e^{-mx}, the exponential cummulative distribution function, $F(x)$, and the associated density function, $f(x)$.

the probability of experiencing X larger than x (an arbitrary positive number) is

$$P(X > x) = e^{-mx}, \tag{12.1}$$

where $m > 0$ is a parameter, reflecting the instantaneous *death rate,* also called *mortality rate.* So (12.1) indicates the *probability of surviving* x more years. The special feature here is that the parameter m is assumed independent of age and the same for all individuals. The reason for introducing this coarse assumption, at least as a first approach, is that it simplifies the analysis a lot by making aggregation easy.

Let us choose one year as our time unit. It then follows from (12.1) that the probability of dying within one year "from now" is approximately equal to m. To see this, note that $P(X \leq x) = 1 - e^{-mx} \equiv F(x)$ is the exponential cumulative distribution function. It follows that the probability density function is $f(x) = F'(x) = me^{-mx}$. We have $P(x < X \leq x + \Delta x) \approx f(x)\Delta x$ for Δx "small". With $x = 0$, this gives $P(0 < X \leq \Delta x) \approx f(0)\Delta x = m\Delta x$. So for a "small" time interval "from now", the probability of dying is approximately proportional to the length of the time interval. And for $\Delta x = 1$, we get $P(0 < X \leq 1) \approx m$, as was to be explained. Fig. 12.1 illustrates.

The expected remaining lifetime is $E(X) = \int_0^\infty xf(x)dx = 1/m$ and is thus the same whatever the current age. This reflects that the exponential distribution is "memory-less". A related unwelcome implication of the assumption (12.1) is that there is no upper bound on *possible* lifetime. Although according to the exponential distribution, the probability of becoming for instance 200 years old is extremely small for values of m consistent with a realistic life expectancy, it is certainly larger than in reality.

Let $N(t)$ be the size of the adult population at time t. We ignore integer problems and consider $N(t)$ as a smooth function of time, t. We assume the events of death are independent across individuals and that the population is "large". Then, by the law of large numbers the actual number of deaths per year at time t is indistinguishable from the expected number, $N(t)m$.[1] Let $b > 0$ denote the *birth rate,* referring to the inflow into the adult population. Like m, b is assumed *constant* over time. Again, appealing to stochastic independence and the law of large numbers, the actual number of births per year at time t is indistinguishable from the expected number, $N(t)b$. So at the aggregate level frequencies and probabilities coincide. By implication, $N(t)$ is growing according to $N(t) = N(0)e^{nt}$, where $n \equiv b - m$ is the population growth rate, a constant. Thus m and b correspond to what demographers call the crude mortality rate and the crude birth rate, respectively.

Let $N(v, t)$ denote the number of people from the birth cohort of the time interval $(v, v + 1)$ still alive at time t (they belong to "vintage" v). Thereby $N(v, t)$ is also the number of people of age $t - v$ at time t, which we perceive as "current time". We have

$$N(v, t) \approx N(v)bP(X > t - v) = N(0)e^{nv}be^{-m(t-v)}. \tag{12.2}$$

Provided parameters have been constant for a long time back in history, from this formula the age composition of the population at time t can be calculated. The number of newborn (age below 1 year) around time t is $N(t, t) \approx N(t)b = N(0)e^{nt}b$. The number of people of age j at time t is approximately

$$N(t - j, t) \approx N(0)e^{n(t-j)}be^{-mj} = N(0)e^{nt}be^{-bj} = N(t)be^{-bj}, \tag{12.3}$$

since $b = n + m$.

Fig. 12.2 shows this age distribution and compares with a stylized empirical age distribution (the hatched curve). The general concavity of the empirical curve and in particular its concentrated "curvature" around 70-80 years' age is not well captured by the theoretical model. Yet the model at least reflects that cohorts of increasing age tend to be smaller and smaller.

By summing over all times of birth we get the total population:

$$\begin{aligned}\int_{-\infty}^{t} N(v, t)dv &= \int_{-\infty}^{t} N(0)e^{nv}be^{-m(t-v)}dv \\ &= N(0)be^{-mt}\int_{-\infty}^{t} e^{(n+m)v}dv = N(0)be^{-mt}\left[\frac{e^{(n+m)v}}{n+m}\right]_{-\infty}^{t} \\ &= N(0)be^{-mt}\frac{e^{(n+m)t} - 0}{b} = N(0)e^{nt} = N(t). \qquad (12.4)\end{aligned}$$

[1]If $\hat{m}$ denotes the frequency of deaths (relative to population), the law of large numbers in this context says that for every $\varepsilon > 0$, $P(|\hat{m} - m| \leq \varepsilon \,|N(t)) \to 1$ as $N(t) \to \infty$.

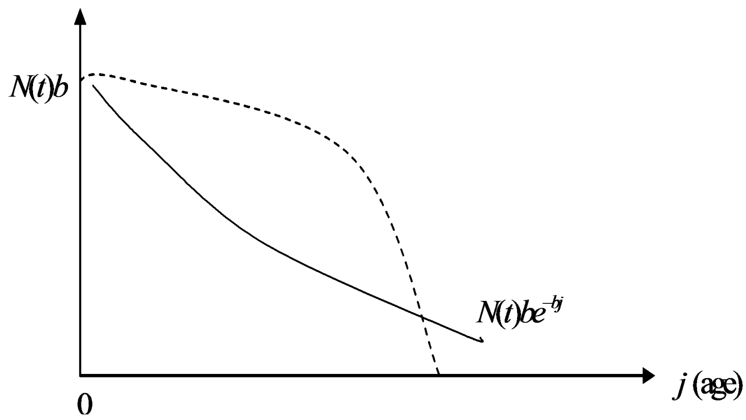

Figure 12.2: Age distribution of the population at time t (the hatched curve depicts a stylized empirical curve).

Preferences

We consider an individual born at time $v \leq t$ and still alive at time t. The consumption flow at time t of the individual is denoted $c(v,t)$. For $s > t$, we interpret $c(v,s)$ as the planned consumption flow at time s in the future. The individual maximizes expected lifetime utility, where the instantaneous utility function is $u(c)$, $u' > 0, u'' < 0$, and the pure rate of time preference (impatience) is a constant $\rho \geq 0$. There is no bequest motive. Expected lifetime utility, as seen from time t, is

$$U_t = E_t \left(\int_t^{t+X} u\left(c(v,s)\right) e^{-\rho(s-t)} ds \right), \tag{12.5}$$

where E_t is the expectation operator conditional on information available at time t. This formula for expected discounted utility is valid for all alive at time t whatever the cohort $v \leq t$ to which they belong; this is due to their common expected remaining lifetime. Hence we can do with only one time index, t, on the symbol U.

There is a convenient way of rewriting the objective function, U_t. Given $s > t$, let $Z(s)$ denote a stochastic variable with two different possible outcomes:

$$Z(s) = \begin{cases} u\left(c(v,s)\right), & \text{if } X > s - t \text{ (i.e., the person is still alive at time } s) \\ 0, & \text{if } X \leq s - t \text{ (i.e., the person is dead at time } s). \end{cases}$$

Then

$$U_t = E_t \left(\int_t^{\infty} Z(s) e^{-\rho(s-t)} ds \right) = \int_t^{\infty} E_t \left(Z(s) e^{-\rho(s-t)} \right) ds.$$

Note that in this context the integration operator $\int_t^\infty (\cdot) ds$ acts like a discrete-time summation operator $\sum_0^\infty$. Hence,

$$\begin{aligned} U_t &= \int_t^\infty e^{-\rho(s-t)} E_t(Z(s)) ds \\ &= \int_t^\infty e^{-\rho(s-t)} \left(u\left(c(v,s)\right) P(X > s-t) + 0 \cdot P(X \le s-t) \right) ds \\ &= \int_t^\infty e^{-(\rho+m)(s-t)} u\left(c(v,s)\right) ds. \end{aligned} \tag{12.6}$$

We see that the expected discounted utility can be written in a way similar to the intertemporal utility function in the deterministic Ramsey model. The only difference is that the pure rate of time preference, ρ, is replaced by an effective rate of time preference, $\rho + m$. This rate is higher, the higher is the death rate m. This reflects that the likelihood of being alive at time s in the future is a decreasing function of the death rate.

For analytical convenience, we let $u(c) = \ln c$.

The market environment

Since every individual faces an uncertain length of lifetime and there is no bequest motive, there will be a demand for assets that pay a high return as long as the investor is alive, but on the other hand is nullified at death. Assets with this property are called *life annuities.* They will be demanded because they make it possible to ensure a high return until the uncertain time of death and to convert potential wealth after death to higher consumption while still alive.

So we assume there is a market for life annuities (also called "reverse" or "negative life insurance") issued by life insurance or pension companies. Consider a depositor who at some point in time buys a life annuity contract for one unit of account. In return the depositor receives $r + \hat{m}$ units of account per year paid continuously until death. Here r is the risk-free interest rate (for simplicity assumed time-independent) and $\hat{m}$ is an *actuarial compensation* over and above that rate. It is a "compensation" for granting the insurance company ownership of the deposit in the event the depositor dies.

How is the actuarial compensation determined in equilibrium? Well, since the economy is large and deaths are assumed stochastically independent, the insurance companies face no aggregate uncertainty. We further assume the insurance companies have negligible administration costs and that there is free entry and exit. We claim that in this case, $\hat{m}$ must in equilibrium equal the mortality rate m. To see this, let the aggregate deposit in the form of life annuity contracts be A units of account and let the number of depositors be N (N "large"). The aggregate revenue to the insurance company on these contracts is then $rA + NmA/N$

per year. The first term is due to A being invested by the insurance company in manufacturing firms, paying the risk-free interest rate r in return (by assumption there is no risk associated with production). The second term is due to Nm of the depositors dying per year. For each depositor who dies there is a transfer, on average A/N, of wealth to the insurance company sector. This is because the deposits are taken over by the insurance company at death (the company's liabilities to those who die are cancelled).

In the absence of administration costs the total costs faced by the insurance company amount to the payout $(r + \hat{m})A$ per year. So total profit is

$$\Pi = rA + NmA/N - (r + \hat{m})A.$$

Under free entry and exit, equilibrium requires $\Pi = 0$. It follows that $\hat{m} = m$. That is, the actuarial compensation equals the mortality rate.

The *conditional* rate of return, $r + m$, obtained by the depositor as long as alive is called the *actuarial rate of interest*. The actuarial rate of interest is called "conditional" because it is conditional upon survival. In contrast, the *expected unconditional rate* of return on holding a life annuity equals r when $\hat{m} = m$ (see Appendix A). A life annuity is said to be *actuarially fair* if it offers the customer the same expected unconditional rate of return as a safe bond. So in this model the life annuities are actuarially fair.[2]

In return for a high conditional rate of return, $r+m$, the estate of the deceased person looses the deposit at the time of death. In this way individuals dying earlier will support those living longer. The market for life annuities is thus a market for *longevity insurance*.

Given r, the actuarial rate of interest will be higher the higher is the mortality rate, m. The intuition is that a higher m implies lower expected remaining lifetime, $1/m$. The expected duration of the life annuity to be paid is therefore shorter. With an unchanged actuarial compensation, this would make issuing these life annuity contracts more attractive to the life insurance companies and competition among them will drive the compensation $\hat{m}$ up until $\hat{m} = m$ again.

As we shall see, in this model, in equilibrium, all financial wealth will be placed in life annuities and earn the conditional rate of return equal to $r + m$ as long as the customer is alive. This is illustrated in Fig. 12.3 where S^N is aggregate net saving and A is aggregate financial wealth. The flows in the diagram are in real terms with the output good as the unit of account.

Whatever name is in practice used for the real world's pension arrangements, many of them have life annuity ingredients and can in a macroeconomic perspective be considered as belonging to the insurance company box in the diagram. Typical Danish "labor market pension" schemes are an example. The stream of

[2] Appendix A considers the case of an age-dependent mortality rate.

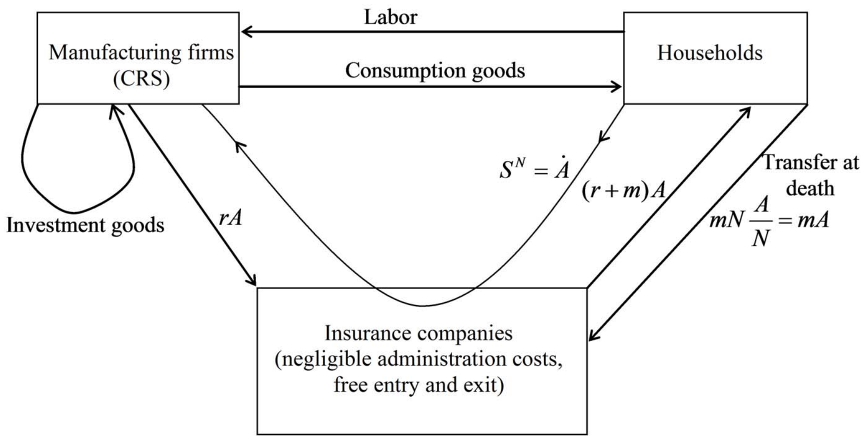

Figure 12.3: Overview of the economy. A is aggregate private financial wealth, S^N is aggregate private net saving.

payouts from such pension arrangements to the customer usually does not start until the customer retires from the labor market, however. This seems in contrast to the model where the flow of dividends to the depositor starts already from the date of purchase of the contract (this is the case of "immediate annuities"). But with perfect credit and annuity markets as simplifying assumed in the model, this difference is immaterial.

What about existence of a market for "ordinary" or "positive" life insurance? In such a market an individual contracts to pay the life insurance company a continuous flow of $\tilde{m}$ units of account per year until death and at death, in return the estate of the deceased person receives one unit of account. Provided the market is active, in equilibrium with free entry and no administration costs, we would have $\tilde{m} = m$ (see Appendix A). In the real world the primary motivation for positive life insurance is care for surviving relatives. But the Blanchard model ignores this motive. Indeed, altruism is absent in the preferences specified in (12.5). Hence there will be no demand for positive life insurance.

The consumption/saving problem

Recall that there is no utility from leisure and, in the present version of the model, no retirement. Hence, labor supply of the individual is inelastic and constant over time. We normalize it to be one unit of labor per year until death.

Let $t = 0$ be the current time and let s denote an arbitrary future point in time. The decision problem for an arbitrary individual born at time $v \leq 0$ is to choose a plan, $(c(v, s))_{s=0}^{\infty}$, so as to maximize expected lifetime utility, U_0, subject to a dynamic budget constraint. The plan is, of course, conditional in the sense of only going to be operative as long as the individual is alive. Letting $u(c) = \ln c$, the decision problem is:

$$
\begin{aligned}
\max U_0 &= \int_0^\infty \ln\left(c\left(v,s\right)\right)e^{-(\rho+m)s}ds \quad \text{s.t.} \\
c\left(v,s\right) &\geq 0, \\
\dot{a}(v,s) &\equiv \frac{\partial a(v,s)}{\partial s} = (r\left(s\right)+m)a\left(v,s\right)+w\left(s\right)-c\left(v,s\right), \qquad (12.7) \\
& \quad \text{where } a(v,0) \text{ is given}, \\
\lim_{s\to\infty} a\left(v,s\right)e^{-\int_0^s (r(\tau)+m)d\tau} &\geq 0. \qquad \text{(NPG)}
\end{aligned}
$$

Labor income per time unit at time s is $w(s)\cdot 1$, where $w(s)$ is the real wage. The variable $a(v,s)$ appearing in the dynamic budget identity (12.7) is real financial wealth at time s and $a\left(v,0\right)$ is the historically given initial financial wealth. Implicit in the way (12.7) and the solvency condition, (NPG), are written is the assumption that the individual *can* procure debt ($a(v,s) < 0$) at the actuarial rate of interest $r(s)+m$. Nobody will offer loans to individuals at the going risk-free interest rate r. There would be a risk that the borrower dies before having paid off the debt including compound interest. But insurance companies will be willing to offer loans at the actuarial rate of interest, $r(s)+m$. As long as the debt is not paid off, the borrower pays the interest rate $r(s)+m$ per time unit. In case the borrower dies before the debt is paid off, the estate is held free of any obligation. In return for this risk the lender receives the actuarial compensation, m, on top of r until the loan is paid off or the borrower dies.

Owing to heterogeneity in an actual population regarding survival probabilities, asymmetric information, and related credit market imperfections, in real world situations this kind of individual loan contracts are rare.[3] This is ignored by the model. But this simplification is not intolerable since, in the context of the model, it turns out that at least in a neighborhood of the steady state, *all* individuals *will save continuously*, that is, *buy* actuarial notes issued by the insurance companies.

All things considered, we end up with a decision problem similar to that in the Ramsey model, namely with an infinite time horizon and a No-Ponzi-Game condition. The only difference is that ρ has been replaced by $\rho+m$ and r by $r+m$. The constraint implied by the NPG condition is that an eventual debt, $-a(v,s)$, is not allowed in the long run to grow at a rate higher than or equal

[3] And to the extent such loans exist, they tend to be associated with an interest cost over and above the sum of the "actuarially fair" rate (the sum of the risk-free rate and the instantaneous mortality rate). Think of what the interest rate on student loans would be in the absence of government support.

to the effective rate of interest $r(s)+m$. This precludes *permanent* financing of interest payments by new loans.

The consumption-saving problem has the same *form* as in the Ramsey model. We can therefore apply the result from Chapter 9 saying that an interior optimal solution must satisfy a set of first-order conditions leading to the Keynes-Ramsey rule. In the present log utility case the latter takes the form

$$\frac{\dot{c}(v,t)}{c(v,t)} \equiv \frac{\partial c(v,t)/\partial t}{c(v,t)} = r(t)+m-(\rho+m) = r(t)-\rho. \tag{12.8}$$

Moreover, the transversality condition,

$$\lim_{t\to\infty} a\left(v,t\right)e^{-\int_0^t(r(\tau)+m)d\tau} = 0, \tag{12.9}$$

must be satisfied by an optimal solution. These conditions are also sufficient for an optimal solution.

The individual consumption function

The Keynes-Ramsey rule itself is only a rule for the rate of change of consumption. We can, however, determine the *level* of consumption in the following way.

We may construct the intertemporal budget constraint that corresponds to the dynamic budget identity (12.7) combined with (NPG). This amounts to a constraint saying that the present value (PV) of the planned consumption stream can not exceed total initial wealth:

$$\int_0^\infty c\left(v,s\right)e^{-\int_0^s(r(\tau)+m)d\tau}ds \leq a(v,0)+h(v,0), \tag{IBC}$$

where $h(v,0)$ is the initial human wealth of the individual. Human wealth is the PV of the expected future labor income and can here, in analogy with (12.6), be written[4]

$$h(v,0) = \int_0^\infty w\left(s\right)e^{-\int_0^s(r(\tau)+m)d\tau}ds = \frac{H(0)}{N(0)} \equiv \bar{h}(0), \tag{12.10}$$

for all $v \leq 0$. Here $H(0)$ is total human wealth at time 0 and $\bar{h}(0)$ is *average* human wealth in the economy. In this version of the Blanchard model there is no retirement and everybody works the same per year until death. In view of the age-independent death probability, expected remaining participation in the labor market is thus the same for all alive. Hence $h(v,0)$ is independent of v and equal to average human wealth.

[4] For details, see Appendix B.

From Proposition 1 of Chapter 9 we know that, given the relevant dynamic budget identity, here (12.7), (NPG) holds if and only if (IBC) holds. Moreover, there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Considering the Keynes-Ramsey rule as a linear differential equation for c as a function of t, the solution formula for consumption is

$$c(v,t) = c(v,0)e^{\int_0^t (r(\tau)-\rho)d\tau}.$$

But so far we do not know $c(v,0)$. Here the transversality condition (12.9) is of help. From Chapter 9 we know that the transversality condition is equivalent to requiring that the NPG condition is not "over-satisfied" which in turn requires strict equality in (IBC). Substituting our formula for $c(v,t)$ into (IBC) with strict equality yields

$$c(v,0)\int_0^\infty e^{\int_0^t (r(\tau)-\rho)d\tau}e^{-\int_0^t (r(\tau)+m)d\tau}dt = a(v,0) + h(v,0),$$

which reduces to $c(v,0) = (\rho + m)\left[a(v,0) + h(v,0)\right]$.

Since initial time is arbitrary and the "effective" time horizon is infinite, we therefore have for any $t \geq 0$ the consumption function

$$c(v,t) = (\rho + m)\left[a(v,t) + h(v,t)\right], \tag{12.11}$$

where $h(v,t)$, in analogy with (12.10), is the PV of the individual's expected future labor income, as seen from time t :

$$h(v,t) = \int_t^\infty w(s)\, e^{-\int_t^s (r(\tau)+m)d\tau}ds = \frac{H(t)}{N(t)} \equiv \bar{h}(t). \tag{12.12}$$

That is, with logarithmic utility the optimal level of consumption is simply proportional to *total* wealth, including human wealth.[5] The factor of proportionality equals the effective rate of time preference, $\rho+m$, and indicates the marginal (and average) propensity to consume out of wealth. The higher is the death rate, m, the shorter is expected remaining lifetime, $1/m$, thus implying a larger marginal propensity to consume (in order to reap the fruits while still alive).

12.2.2 Aggregation

We will now aggregate over the different cohorts, that is, over the different times of birth. Summing consumption over all times of birth, we get aggregate consumption at time t,

$$C(t) = \int_{-\infty}^t c(v,t)N(v,t)dv, \tag{12.13}$$

[5] With a *general* CRRA utility function the marginal propensity to consume out of wealth depends on current and expected future interest rates, as shown in Chapter 9.

where $N(v,t)$ equals $N(v)be^{-m(t-v)}$, cf. (12.2). Similarly, aggregate financial wealth can be written

$$A(t) = \int_{-\infty}^{t} a(v,t)N(v,t)dv, \tag{12.14}$$

and aggregate human wealth is

$$H(t) \equiv N(t)\bar{h}(t) = N(0)e^{nt} \int_{t}^{\infty} w\left(s\right)e^{-\int_t^s (r(\tau)+m)d\tau}ds. \tag{12.15}$$

Since the propensity to consume out of wealth is the same for all individuals, i.e., independent of age, aggregate consumption becomes

$$C(t) = (\rho + m)\left[A(t) + H(t)\right]. \tag{12.16}$$

The dynamics of household aggregates

There are two basic dynamic relations for the household aggregates.[6] The first relation is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \tag{12.17}$$

Note that the rate of return here is $r(t)$ and thereby differs from the conditional rate of return for the individual during lifetime, namely $r(t)+m$. The difference derives from the fact that for the household sector as a whole, $r(t)+m$ is only a *gross* rate of return. The actuarial compensation m is paid by the household sector itself − via the life-insurance companies. There is a transfer of wealth when people die, in that the liabilities of the insurance companies are cancelled. First, $N(t)m$ individuals die per time unit and their average wealth is $A(t)/N(t)$. The implied transfer is in total $N(t)mA(t)/N(t)$ per time unit from those who die. This is what finances the actuarial compensation m to those who are still alive and have placed their savings in life annuity contracts issued by the insurance sector. Hence, the average *net* rate of return on financial wealth for the household sector as a whole is

$$(r(t)+m)A(t) - N(t)m\frac{A(t)}{N(t)} = r(t)A(t),$$

in conformity with (12.17). In short: the reason that (12.17) does not contain the actuarial compensation is that this compensation is only a transfer from those who die to those who are still alive. The unconditional rate of return in the economy is just $r(t)$.

[6] Here we only describe the intuition behind these relations. Their formal derivation is given in Appendix C.

The second important dynamic relation for the household sector as a whole is

$$\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t). \tag{12.18}$$

To interpret this, note that three effects are in play:

1. The dynamics of consumption at the individual level follows the Keynes-Ramsey rule

$$\dot{c}(v,t) \equiv \frac{\partial c(v,t)}{\partial t} = \left(r\left(t\right) - \rho\right) c\left(v,t\right).$$

This explains the term $[r(t) - \rho + n]C(t)$ in (12.18), except for $nC(t)$.

2. The appearance of $nC(t)$ is a trivial population growth effect: defining $C \equiv cN$, we have

$$\dot{C} = (\dot{c}N + c\dot{N}) = (\frac{\dot{c}}{c} + \frac{\dot{N}}{N})cN \equiv (\frac{\dot{c}}{c} + n)C.$$

3. The subtraction of the term $b(\rho + m)A(t)$ in (12.18) is more challenging. This term is due to a *generation replacement effect*. In every short instant some people die and some people are born. The first group has financial wealth, the last group not. The inflow of newborns is $N(t)b$ per time unit and since they have no financial wealth, the replacement of dying people by these young people lowers aggregate consumption. To see by how much, note that the average financial wealth in the population is $A(t)/N(t)$ and the consumption effect of this is $(\rho + m)A(t)/N(t)$, cf. (12.16). This implies, *ceteris paribus,* that the turnover of generations reduces aggregate consumption per time unit by

$$N(t)b(\rho + m)\frac{A(t)}{N(t)} = b(\rho + m)A(t)$$

per time unit. This explains the last term in (12.18). The generation replacement effect makes the growth rate of aggregate consumption smaller than what the Keynes-Ramsey rule suggests.

Whereas the Keynes-Ramsey rule describes individual consumption dynamics, we see that the *aggregate* consumption dynamics do not follow the Keynes-Ramsey rule. The reason is the generation replacement effect. This "compositional effect" is a characteristic feature of overlapping generations models. It distinguishes these models from representative agent models, like the Ramsey model.

12.2.3 The representative firm

As description of the technology, the firms, and the factor markets we apply the simple neoclassical competitive one-sector setup that we have used in previous chapters. The technology of the representative firm in the manufacturing sector is given by

$$Y(t) = F(K(t), T(t)L(t)), \tag{12.19}$$

where F is a neoclassical production function with CRS, and $Y(t), K(t)$, and $L(t)$ are output, capital input, and labor input, respectively, per time unit. The technology level $T(t)$ grows at a constant rate $g \geq 0$, that is, $T(t) = T(0)e^{gt}$, where $T(0) > 0$. Ignoring for the time being the explicit dating of the variables, maximization of profit ($= F_1(K, TL) - (r + \delta)K - wL$) under perfect competition leads to

$$\begin{aligned} \frac{\partial Y}{\partial K} &= F_1(K, TL) = f'(\tilde{k}^d) = r + \delta, & (12.20) \\ \frac{\partial Y}{\partial L} &= F_2(K, TL)T = \left[f(\tilde{k}^d) - \tilde{k}^d f'(\tilde{k}^d)\right] T \equiv \tilde{w}(\tilde{k}^d)T = w, & (12.21) \end{aligned}$$

where $\delta > 0$ is the constant rate of capital depreciation, $\tilde{k}^d \equiv K^d/(TL^d)$ is the desired effective capital-labor ratio, and f is defined by $f(\tilde{k}^d) \equiv F(\tilde{k}^d, 1)$. We have $f' > 0$, $f'' < 0$, and $f(0) = 0$ (the latter condition in view of the upper Inada condition for the marginal productivity of labor, cf. Appendix C to Chapter 2).

Alternatively, we may imagine that the production firms own the capital stock they use and finance their gross investment by issuing bonds, as illustrated in Fig. 12.3. It still holds that total costs per unit of capital is the sum of the interest rate and the capital depreciation rate. The insurance companies use their deposits to buy the bonds issued by the manufacturing firms.

12.2.4 Dynamic general equilibrium (closed economy)

Clearing in the labor market entails $L^d = N$, where N is aggregate labor supply which equals the size of the population. Clearing in the market for capital goods entails $K^d = K$, where K is the aggregate capital stock available in the economy. Hence, in equilibrium $\tilde{k}^d = \tilde{k} \equiv K/(TN)$, which is predetermined at any point in time. The equilibrium factor prices at time t are thus given as

$$\begin{aligned} r(t) &= f'(\tilde{k}(t)) - \delta, \quad \text{and} & (12.22) \\ w(t) &= \tilde{w}(\tilde{k}(t))T(t). & (12.23) \end{aligned}$$

Deriving the dynamic system

We will now derive a dynamic system in terms of $\tilde{k}$ and $\tilde{c} \equiv C/(TN)$. In a closed economy where natural resources (land etc.) are ignored, aggregate financial wealth equals, by definition, the market value of the capital stock, which is $1 \cdot K$.[7] Thus

$$A = K \text{ for all } t.$$

From (12.17) therefore follows:

$$\begin{aligned}
\dot{K} &= \dot{A} = rK + wL - C \\
&= [F_1(K, TL) - \delta] K + F_2(K, TL)TL - C \quad \text{(by (12.20) and (12.21))} \\
&= F_1(K, TL)K + F_2(K, TL)TL - \delta K - C \\
&= F(K, TL) - \delta K - C \quad \text{(by Euler's theorem)} \\
&= Y - \delta K - C. \qquad (12.24)
\end{aligned}$$

So, not surprisingly, we end up with a standard national product accounting relation for a closed economy. In fact we could directly have written down the result (12.24). Its formal derivation here only serves as a check that our product and income accounting is consistent.

To find the law of motion of $\tilde{k} \equiv K/(TN)$, we log-differentiate w.r.t. time (take logs on both sides and differentiate w.r.t. time) so as to get

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = \frac{F(K, TN) - C - \delta K}{K} - (g + n),$$

from (12.24). Multiplying through by $\tilde{k} \equiv K/(TN)$ gives

$$\dot{\tilde{k}} = \frac{F(K, TN) - C}{TN} - (\delta + g + n)\tilde{k} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k},$$

since $\tilde{c} \equiv C/(TN)$. Any path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ satisfying this equation and starting from the historically given initial value, $\tilde{k}_0$, is a technically feasible path. Which of these paths become realized depends on households' effective utility discount rate, $\rho + m$, and the market form, which is here perfect competition.

To find the law of motion of $\tilde{c}$, we first insert (12.22) and $A = K$ into (12.18) to get

$$\dot{C} = \left[f'(\tilde{k}) - \delta - \rho + n\right] C - b(\rho + m)K. \qquad (12.25)$$

Log-differentiating $C/(TN)$ w.r.t. time yields

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{C}}{C} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = f'(\tilde{k}) - \delta - \rho + n - b(\rho + m)\frac{K}{C} - g - n \text{ (from (12.25))} \\
&= f'(\tilde{k}) - \delta - \rho - b(\rho + m)\frac{\tilde{k}}{\tilde{c}} - g,
\end{aligned}$$

[7] There are no capital installation costs and so the value of a unit of installed capital equals the replacement cost per unit before installation. This replacement cost is one.

By rearranging:

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho+m)\tilde{k}.$$

Our two coupled differential equations in $\tilde{k}$ and $\tilde{c}$ constitute the dynamic system of the Blanchard model. Since the parameters n, b, and m are connected through $n \equiv b - m$, one of them should be eliminated to avoid confusion. It is natural to have b and m as the basic parameters and then consider $n \equiv b - m$ as a derived one. Consequently we write the system as

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + b - m)\tilde{k}, \tag{12.26}$$

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho+m)\tilde{k}. \tag{12.27}$$

Observe that initial $\tilde{k}$ equals a predetermined value, $\tilde{k}_0$, while initial $\tilde{c}$ is a forward-looking variable, an endogenous jump variable. Therefore we need more information to pin down the dynamic evolution of the economy. Fortunately, for each individual household we have a transversality condition like that in (12.9). Indeed, for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$, the transversality condition takes the form

$$\lim_{t\to\infty} a(v,t)e^{-\int_{t_0}^{t}(r(\tau)+m)d\tau} = 0. \tag{12.28}$$

In comparison, note that the transversality condition (12.9) was seen from the special perspective of $(v, t_0) = (v, 0)$, which is only of relevance for those alive already at time 0.

Phase diagram

To get an overview of the dynamics, we draw a phase diagram. There are two reference values of $\tilde{k}$, namely the golden rule value, $\tilde{k}_{GR} > 0$, and a certain benchmark value, $\bar{\tilde{k}} > 0$. These are given by

$$f'\left(\tilde{k}_{GR}\right) - \delta = g + b - m = g + n, \qquad \text{and} \qquad f'(\bar{\tilde{k}}) - \delta = \rho + g, \tag{12.29}$$

respectively. Let us for simplicity assume that f satisfies the Inada conditions. Given $b \geq m$, both $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ exist,[8] and they are unique in view of $f'' < 0$. We have $\bar{\tilde{k}} \lesseqgtr \tilde{k}_{GR}$ for $\rho \gtreqless b - m$, respectively. The reason that $\bar{\tilde{k}}$ is an important benchmark value will be apparent in a moment.

[8]Here we use that when $b \geq m$ and $\delta > 0$, $g \geq 0$, and $\rho \geq 0$, then $\delta + g + b - m > 0$ and $\delta + \rho + g > 0$. Then the Inada conditions ensure that the two equations in $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$, respectively, given by (12.29), have a solution.

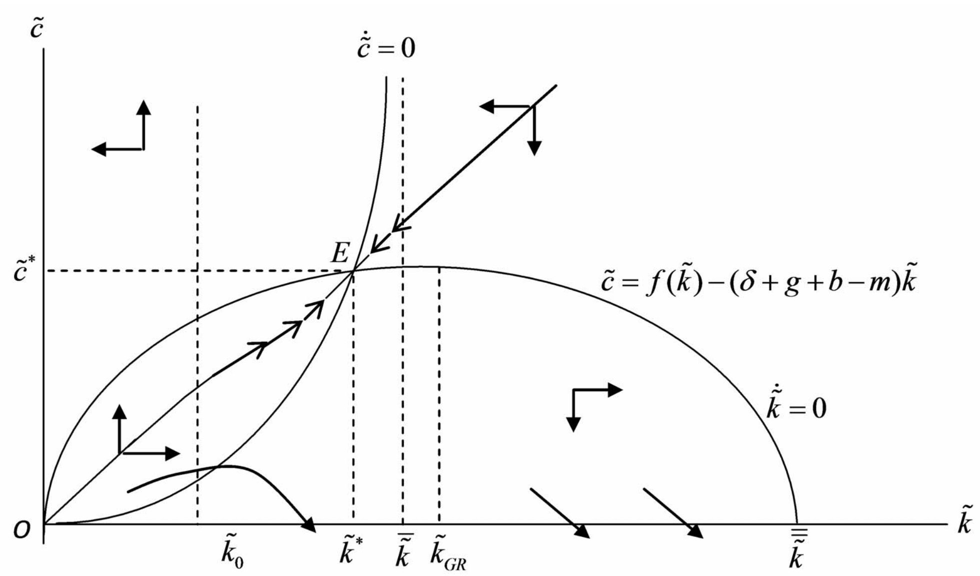

Figure 12.4: Phase diagram of the model of perpetual youth.

Equation (12.26) shows that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + b - m)\tilde{k}. \tag{12.30}$$

The locus $\dot{\tilde{k}} = 0$ is shown in Fig. 12.4. It starts at the origin O, reaches its maximum at the golden rule capital-labor ratio, and crosses the horizontal axis at the capital-labor ratio $\bar{\bar{\tilde{k}}} > \tilde{k}_{GR}$, satisfying $f(\bar{\bar{\tilde{k}}}) = (\delta + g + b - m)\bar{\bar{\tilde{k}}}$. The existence of a $\bar{\bar{\tilde{k}}}$ with this property is guaranteed by the upper Inada condition for the marginal productivity of capital.

The horizontal arrows in the diagram are explained the following way. Imagine that for some fixed value $\tilde{k}_1 < \bar{\bar{\tilde{k}}}$ we draw the vertical line $\tilde{k} = \tilde{k}_1$ in the positive quadrant. We then consider greater and greater values of $\tilde{c}$ along this line. To begin with $\tilde{c}$ is small and therefore, by (12.26), $\dot{\tilde{k}}$ is positive. At the point where the imagined vertical line crosses the $\dot{\tilde{k}} = 0$ locus, we have $\dot{\tilde{k}} = 0$. And above this point we have $\dot{\tilde{k}} < 0$ due to the now large consumption level. In brief, by (12.26) follows

$$\dot{\tilde{k}} \gtreqless 0 \text{ for } \tilde{c} \lesseqgtr f(\tilde{k}) - (\delta + g + b - m)\tilde{k},$$

respectively. Or even briefer: (12.26) implies $\partial\dot{\tilde{k}}/\partial\tilde{c} = -1$. The horizontal arrows thus indicate the *direction* of movement of $\tilde{k}$ in the different regions of the phase diagram as determined by the differential equation (12.26).

Equation (12.27) shows that

$$\dot{\tilde{c}} = 0 \text{ for } \tilde{c} = \frac{b\,(\rho + m)\,\tilde{k}}{f'(\tilde{k}) - \delta - \rho - g}. \tag{12.31}$$

Hence,

$$\text{along the } \dot{\tilde{c}} = 0 \text{ locus, } \lim_{\tilde{k} \to \bar{\tilde{k}}} \tilde{c} = \infty,$$

so that the $\dot{\tilde{c}} = 0$ locus is asymptotic to the vertical line $\tilde{k} = \bar{\tilde{k}}$. Moving along the $\dot{\tilde{c}} = 0$ locus in the other direction, we see from (12.31) that $\lim_{\tilde{k} \to 0} \tilde{c} = 0$, as illustrated in Fig. 12.4.[9]

To explain the vertical arrows in the figure, for some fixed value $\tilde{c}_1$ (not too large) we imagine the corresponding horizontal line $\tilde{c} = \tilde{c}_1$ in the positive quadrant has been drawn. We then consider greater and greater values of $\tilde{k}$ along this line. To begin with, $\tilde{k}$ is small and therefore $f'(\tilde{k})$ is large so that, by (12.27), $\dot{\tilde{c}}$ is positive. At the point where the imagined horizontal line crosses the $\dot{\tilde{c}} = 0$ locus, we have $\dot{\tilde{c}} = 0$. And to the right, we have $\dot{\tilde{c}} < 0$ because $\tilde{k}$ is now large and $f'(\tilde{k})$ therefore small. The vertical arrows thus indicate the *direction* of movement of $\tilde{c}$ in the different regions of the phase diagram as determined by the differential equation (12.27). In brief: by (12.27) follows $\partial\dot{\tilde{c}}/\partial\tilde{k} = f''(\tilde{k})\tilde{c} - b(\rho + m) < 0$.

Steady state Fig. 12.4 also shows the steady-state point E, where the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus. The corresponding effective capital-labor ratio is $\tilde{k}^*$, to which is associated the (technology-corrected) consumption level $\tilde{c}^*$. Given our assumptions, including the Inada conditions, there exists one and only one steady state with positive effective capital-labor ratio. To see this, notice that in steady state the right-hand sides of (12.30) and (12.31) are equal to each other.

[9] The $\dot{\tilde{c}} = 0$ locus is positively sloped everywhere since, by (12.31),

$$\frac{d\tilde{c}}{d\tilde{k}}\Big|_{\dot{\tilde{c}}=0} = b\,(\rho + m)\,\frac{f'(\tilde{k}) - \delta - \rho - g - \tilde{k}f''(\tilde{k})}{\left(f'(\tilde{k}) - \delta - \rho - g\right)^2} > 0, \text{ whenever } f'(\tilde{k}) - \delta > \rho + g.$$

The latter inequality holds whenever $\tilde{k} < \bar{\tilde{k}}$.

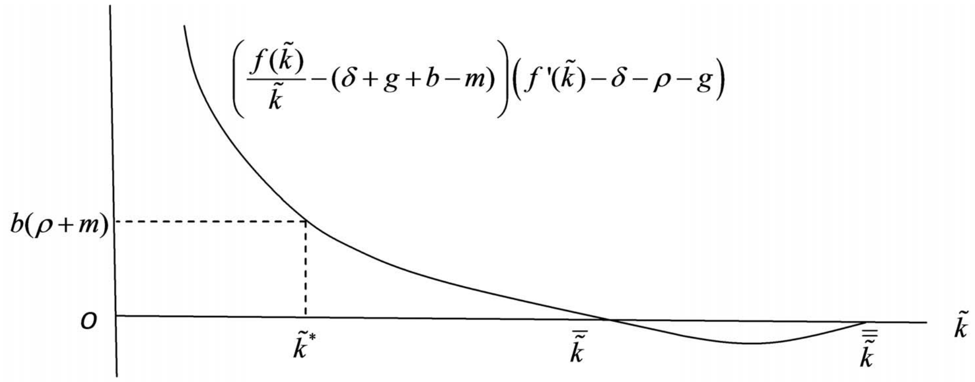

Figure 12.5: Existence of a unique $\tilde{k}^*$.

After ordering this implies

$$\left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + b - m)\right)\left[f'(\tilde{k}) - \delta - \rho - g\right] = b\,(\rho + m)\,. \tag{12.32}$$

The left-hand side of this equation is depicted in Fig. 12.5. Since both the average and marginal productivities of capital are decreasing in $\tilde{k}$, the value of $\tilde{k}$ satisfying the equation is unique, given the requirement $\tilde{k} < \bar{\tilde{k}}$. And such a value exists due to the Inada conditions.[10]

The equilibrium path By equilibrium path is meant the solution (if any) to the model. More precisely, given the model, a path $(\tilde{k}(t),\ \tilde{c}(t))_{t=0}^{\infty}$ is an *equilibrium path* (with perfect foresight), if: (a) the path is *technically feasible,* that is, it satisfies the accounting equation (12.26) and has $\tilde{k}(t) = \tilde{k}_0$, where $\tilde{k}_0$ is the historically given initial effective capital-labor ratio; (b) the path is consistent with *market clearing* for all $t \geq 0$, given firms' profit maximization and households' utility maximization conditional on their expectations and budget constraints; and (c) along the path the evolution of the pair $(w(t),\ r(t))$, where $w(t) = \tilde{w}(\tilde{k}(t))T(t)$ and $r(t) = f'(\tilde{k}(t)) - \delta$, is as expected by the households – *expectations are fulfilled.*

Let us first ask the question: can the steady-state point E, that is, a path with $(\tilde{k}(t), \tilde{c}(t)) = (\tilde{k}^*, \tilde{c}^*)$ for all $t \geq 0$, be an equilibrium path? The answer is "yes", *if* the historically given initial value, $\tilde{k}_0$, happens to equal the steady-state value, $\tilde{k}^*$. Then the first requirement, (a), is fulfilled because the steady state satisfies the accounting equation (12.26) and starts out with $\tilde{k}(0) = \tilde{k}_0$ (which

[10] There is also a *trivial* steady state, namely the origin, which will never be realised as long as initial $\tilde{k}$ is positive.

in this case happens to equal to $\tilde{k}^*$). The second requirement, (b), is fulfilled because, by satisfying both the differential equations, (12.26) and (12.27), *and* the transversality conditions (12.28), the steady state is consistent with market clearing for all $t \geq 0$, given firms' profit maximization and households' utility maximization conditional on their expectations and budget constraints. The third requirement, (c), is fulfilled because expectations are fulfilled in the steady state

Here we claimed, without proof, that the steady state satisfies the transversality condition (12.28) for every $t_0 \geq 0$ and every $v \leq t_0$. A proof is given in Appendix D.

But generally, $\tilde{k}_0$ will differ from $\tilde{k}^*$, for instance we may have $0 < \tilde{k}_0 < \tilde{k}^*$ as in Fig. 12.4. So the equilibrium path cannot coincide with the steady state, at least not to begin with. To find a candidate for the equilibrium path in this case, we consider the dynamics in a neighborhood of the steady state. In Fig. 12.4 the horizontal and vertical arrows in this neighborhood indicate that the steady-state point E is a *saddle point*.[11] Graphically, this means that from the left and the right, respectively, in the phase diagram there is one and only one trajectory − the saddle path − converging toward the steady state, as indicated also in Fig. 12.4. As a construction line we have in the phase diagram drawn the vertical (stippled) line along which $\tilde{k} = \tilde{k}_0 > 0$. Consider the point where this vertical line crosses the saddle path. We claim that the section of the saddle path stretching from this point to E is, by construction, an equilibrium path of the economy. By arguments similar to those we applied for the steady state, we find that this converging path fulfils the requirements (a), (b), and (c) for an equilibrium path. This includes satisfaction of the transversality condition (12.28) for every $t_0 \geq 0$ and every $v \leq t_0$. Truly, in view of continuity, when a transversality condition holds for the steady state, it also holds for any path converging to the steady state.

Thereby we have found a solution to the model, given $0 < \tilde{k}_0 < \tilde{k}^*$. And the ordinate to the point where the vertical line $\tilde{k}_0 = \tilde{k}^*$ crosses the saddle path is the associated equilibrium value of initial consumption, $\tilde{c}(0)$. If instead $\tilde{k}_0 > \tilde{k}^*$, the corresponding section of the upper saddle path makes up the equilibrium path for that case.

Might there exist *other* equilibrium paths? No, the other paths in the phase diagram in Fig. 12.4 violate either the transversality conditions of the households (paths that in the long run point South-East) or their NPG conditions, and therefore also their transversality conditions, (paths that in the long run point North-West).

The conclusion is that there *exists* a solution to the model, and it is *unique.*

[11] To supplement this graphical inspection, a formal proof is given in Appendix E.

Moreover, the solution implies convergence to the steady state. In the language of differential equations this convergence has the form of *conditional* asymptotic stability of the steady state. The conditionality is due to the convergence being dependent on the initial value of the jump variable $\tilde{c}(0)$, cf. the phase diagram. But by assuming perfect foresight into the indefinite future, the model imposes that the market mechanism is able to induce people to choose their initial consumption levels in exact accordance with their transversality conditions, thereby making the technology-corrected average individual consumption level at time 0, $\tilde{c}(0)$, exactly equal to the ordinate of the point where the vertical line $\tilde{k}_0 = \tilde{k}^*$ crosses the saddle path. This is both necessary and sufficient for the convergence.

Saddle-point stability An equivalent way to characterize the dynamics of the model is to say that the unique non-trivial steady state E is *saddle-point stable*.[12] Indeed, the point E satisfies the four definitional conditions for saddle-point stability in a two-dimensional dynamic system with a steady state: (1) the steady state should be a saddle point; (2) one of the two endogenous variables should be predetermined, while the other should be a jump variable; (3) the saddle path should not be parallel to the jump-variable axis; and (4) there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

These four requirements are satisfied by the present model. As noted above, requirement (1) holds. So does (2), since $\tilde{k}$ is a predetermined variable, and $\tilde{c}$ is a jump variable. The saddle path is not parallel to the $\tilde{c}$ axis, and it is the only path that satisfies *all* the conditions of dynamic general equilibrium, thereby ensuring the requirements (3) and (4).

Comments on the solution

Over time the economy moves along the saddle path toward the steady-state point E. If $\tilde{k}_0 < \tilde{k}^* \leq \tilde{k}_{GR}$, as illustrated in Fig. 12.4, then both $\tilde{k}$ and $\tilde{c}$ grow over time until the steady state is "reached". This is just one example, however. We could alternatively have $\tilde{k}_0 > \tilde{k}^*$ and then $\tilde{k}$ would be falling during the adjustment process.

An implication of $\tilde{c}(t) \equiv c(t)/T(t) \to \tilde{c}^*$ for $t \to \infty$ is that per capita consumption grow in the long run at the same rate as technology, the rate g. More precisely, for $t \to \infty$,

$$c(t) \text{ weakly approaches the steady-state path } c^*(t) = \tilde{c}^* T(0) e^{gt}. \qquad (12.33)$$

[12] The formal argument, which is more intricate than for the Ramsey model, is given in Appendix D, where also the arrows indicating paths that cross the $\tilde{k}$-axis are explained.

The qualification "weakly" is added to indicate that we claim no more than that the *ratio* $c(t)/c^*(t)$ $(= \tilde{c}(t)/\tilde{c}^*) \to 1$ for $t \to \infty$. This need not imply that the *difference* between $c(t)$ and $c^*(t)$ converges to nil. The difference can be written $c(t) - c^*(t) \equiv \tilde{c}(t)T(t) - \tilde{c}^*T(t) = (\tilde{c}(t) - \tilde{c}^*)T(t)$, where the first factor in the last expression goes to zero while the second factor goes to $+\infty$ for $t \to \infty$. Which of the factors moves faster depends on circumstances.

Similarly, an implication of $\tilde{k}(t) \to \tilde{k}^*$ for $t \to \infty$ is that

$$\tilde{w}(\tilde{k}(t)) \equiv \frac{w(t)}{T(t)} \to \tilde{w}(\tilde{k}^*) \quad \text{for} \quad t \to \infty, \tag{12.34}$$

where the wage function $\tilde{w}$ is defined in (12.21). This implies that the real wage in the long run grows at the rate g, and, more precisely, that

$w(t)$ weakly approaches the steady-state path $w^*(t) = \tilde{w}(\tilde{k}^*)T(0)e^{gt}$.

These results are similar to those of the Ramsey model. More interesting are the following two observations:

1. *The long-run real interest rate will be higher than in the "corresponding" Ramsey long-run equilibrium when the latter exists.* For $t \to \infty$,

$$r(t) = f'(\tilde{k}(t)) - \delta \to f'(\tilde{k}^*) - \delta = r^* > f'(\bar{\tilde{k}}) - \delta = \rho + g, \tag{12.35}$$

where the inequality follows from $\tilde{k}^* < \bar{\tilde{k}}$. Suppose the Ramsey economy described in Chapter 10 has the same f, ρ and g and has $u(c) = \ln c$, i.e., $\theta = 1$. Then, if and only if $\rho - n > 0$ (cf. the assumption (A1) of Chapter 10), does a long-run equilibrium exist in that Ramsey economy, and its long-run interest rate will be $r_R^* = \rho + g$. Owing to finite lifetime ($m > 0$), the present version of the Blanchard OLG model, with the same ρ and n, unambiguously predicts a higher long-run interest rate than this "corresponding" Ramsey economy has. The positive probability of not being alive at a certain time in the future leads to less saving and therefore less capital accumulation. So the economy ends up with a lower effective capital-labor ratio and thereby a higher real interest rate.[13]

2. *The Perpetual Youth model does not rule out dynamic inefficiency.* From the definition of $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ in (12.29) follows that $\bar{\tilde{k}} \lesseqgtr \tilde{k}_{GR}$ for $\rho \gtreqless n$, respectively, where $n \equiv b - m$. Suppose $0 \leq \rho < n$. Then $\bar{\tilde{k}} > \tilde{k}_{GR}$ and we *can* have $\tilde{k}_{GR} < \tilde{k}^* < \bar{\tilde{k}}$. In contrast, a steady state of a Ramsey economy will always be situated to the left of $\tilde{k}_{GR}$ and dynamic efficiency thereby be ensured.

[13]When retirement at old age is added to the model, this, however, no longer necessarily holds, cf. Section 12.3 below.

How comes this difference? It comes from a parameter restriction imposed in the Ramsey model because it is needed to ensure existence of general equilibrium in that model. This parameter restriction is named (A1) in Chapter 10 and requires that $\rho - n > (1-\theta)g$. The restriction reduces to $\rho - n > 0$ in the case $\theta = 1$, which is the relevant case for comparison between the two models. When $\theta = 1$ in the Ramsey model, its steady-state value of $\tilde{k}$, denoted $\tilde{k}_R^*$, say, satisfies

$$f'(\tilde{k}_R^*) - \delta = \rho + g > n + g = f'(\tilde{k}_{GR}) - \delta,$$

where the inequality comes from the parameter restriction $\rho - n > 0$. *If* we impose $\rho - n > 0$ in the Blanchard OLG model, its steady state will also be to the left, in fact further to the left, of $\tilde{k}_{GR}$, since we get

$$f'(\tilde{k}^*) - \delta > f'(\bar{\tilde{k}}) - \delta = \rho + g > n + g = f'(\tilde{k}_{GR}) - \delta.$$

It is only in the opposite case, namely the case $\rho - n \leq 0$, that the possibility of $\bar{\tilde{k}} \geq \tilde{k}_{GR}$ arises. And in the OLG model, the weak inequality $\rho - n \leq 0$ is *not* in conflict with existence of dynamic general equilibrium. But in the Ramsey model it is. Certainly, in the Ramsey model the weak inequality $\rho - n \leq 0$ leads to $r^* \leq g + n$. The right-hand side of this inequality will equal the growth rate of wL in a hypothetical steady state. So the present value of future labor income of the representative household would be *infinite* and consumption demand thus unbounded. This rules out equilibrium. And it is the reason that the Ramsey model from the beginning only allows $\rho - n > 0$, or more generally $\rho - n > (1-\theta)g$. Hence a Ramsey economy always has steady state to the left of the golden rule and therefore, in contrast to an OLG model, never ends up in dynamic inefficiency.

We can relax the parameter restrictions $\delta > 0$, $\rho \geq 0$, and $b \geq m$ that have hitherto been assumed in the Blanchard OLG model for ease of exposition. To ensure existence of a solution to the household's decision problem, we need that the effective utility discount rate is positive, i.e., $\rho + m > 0$.[14] Further, from the definition of $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ in (12.29) we need $\delta + g + b - m > 0$ and $\delta + \rho + g > 0$ where, by definition, $m > 0$, $\delta \geq 0$, and $g \geq 0$. Hence, as long as $\delta + g > 0$, we can allow $n \equiv b - m$ and/or ρ to be negative (not "too" negative, though) without interfering with the existence of general equilibrium.

Remark on a seeming paradox It might seem like a paradox that the economy can be in steady state and at the same time have $r^* - \rho > g$. By the

[14] Of course we also need that the present discounted value of future labor income is well-defined (i.e., not infinite) and this requires $r^* + m > g$. In view of (12.35), however, this is automatically satisfied when $\rho \geq 0$ and $m > 0$.

Keynes-Ramsey rule, when $r^* - \rho > g$, individual consumption is growing faster than productivity, which grows at the rate g. How can such an evolution be sustained? The answer lies in the fact that we are *not* considering an economy with a representative agent, but an economy with a composition effect in the form of the turnover of the generations. Whereas individual consumption can grow at a relatively high rate, this consumption only exists as long as the individual is alive. Hence per capita consumption, $c \equiv C/N$, behaves differently. From (12.18) we have in steady state

$$\frac{\dot{c}}{c} = r^* - \rho - b(\rho + m)\frac{a}{c} < r^* - \rho,$$

where $a \equiv A/N$ (average financial wealth). The consumption by those who die is replaced by that of the newborn who have less financial wealth, hence lower consumption than the average citizen. To take advantage of $r^* - \rho > g \geq 0$, the young (and in fact everybody) save, thereby becoming gradually richer and improving their standard of living relatively fast. Owing to the generation replacement effect, however, per capita consumption grows at a lower rate. In steady state *this* rate equals g, as indicated by (12.33).

Is this consistent with the aggregate consumption function? The answer is affirmative since by dividing through by $N(t)$ in (12.16) we end up with

$$c(t) = (\rho + m)(k(t) + \bar{h}(t)) \equiv (\rho + m)(\tilde{k}(t) + \tilde{h}(t))T(t) = (\rho + m)(\tilde{k}^* + \tilde{h}^*)T(0)e^{gt}, \tag{12.36}$$

in steady state, where

$$\begin{aligned} \tilde{h}^* &\equiv \left(\frac{\bar{h}(t)}{T(t)}\right)^* = \frac{\int_t^\infty w(s)e^{-(r^*+m)(s-t)}ds}{T(t)} = \frac{\int_t^\infty \tilde{w}(\tilde{k}^*)T(s)e^{-(r^*+m)(s-t)}ds}{T(t)} \\ &= \tilde{w}(\tilde{k}^*)\int_t^\infty e^{g(s-t)}e^{-(r^*+m)(s-t)}ds = \frac{\tilde{w}(\tilde{k}^*)}{r^* + m - g}. \end{aligned} \tag{12.37}$$

The last equality in the first row comes from (12.34); in view of $r^* > \rho + g$, the numerator, $r^* + m - g$, in the second row is a positive constant. Hence, both $\tilde{k}^*$ and $\tilde{h}^*$ are constants. In this way the consumption function in (12.36) confirms the conclusion that per capita consumption in steady state grows at the rate g.

The demographic transition and the long-run interest rate

In the last more than one hundred years the industrialized countries have experienced a gradual decline in the three demographic parameters m, b, and n. Clearly m has gone down, thus increasing life expectancy, $1/m$. But also $n \equiv b - m$ has gone down, hence b has gone down even more than m. What effect on r^* should we expect? A "rough answer" can be based on the Blanchard model.

It is here convenient to consider n and m as the basic parameters and $b \equiv n+m$ as a derived one. So in (12.26) and (12.27) we substitute $b \equiv n+m$. Then there is only one demographic parameter affecting the position of the $\dot{\tilde{k}} = 0$ locus, namely n. Three effects are in play:

a. *Labor-force growth effect.* The lower n results in an upward shift of the $\dot{\tilde{k}} = 0$ locus in Fig. 12.4, hence a tendency to expansion of $\tilde{k}$. This "capital deepening" is due to the fact that slower growth in the labor force implies less capital "dilution".

b. *Life-cycle effect.* Given n, the lower m results in a clockwise turn of the $\dot{\tilde{c}} = 0$ locus in Fig. 12.4. This enforces the tendency to expansion of $\tilde{k}$. The explanation is that the higher life expectancy, $1/m$, increases the incentive to save and thus reduces consumption $C = (\rho+m)(A+H)$. Thereby, capital accumulation is promoted.

c. *Generation replacement effect.* Given m, the lower $b = n+m$ results in lower n, hence a further clockwise turn of the $\dot{\tilde{c}} = 0$ locus in Fig. 12.4. This additional capital deepening is explained by a composition effect. Lower b implies a smaller proportion of young people (with the same human wealth as others, but less financial wealth) in the population, thus leading to smaller H/A, hence smaller $[C/(\rho + m)]/A = (A + H)/A$, by the consumption function. As C/A is thus smaller, $S/A \equiv (Y - C)/A$ will be larger, resulting again in more capital accumulation.

Thus all three effects on the effective capital-labor ratio are positive. Consequently, we should expect a lower marginal productivity of capital and a lower real interest rate in the long run. There are a few empirical long-run studies pointing in this direction (see, e.g., Doménil and Lévy, 1990).

We called our answer to this demographic question a "rough answer". Being based on a *comparative* method, the analysis has its limitations. The comparative method compares the evolution of two distinct economies having the same structure and parameter values except with respect to the parameter the role of which we want to study.

A more appropriate approach would consider dynamic effects of a parameter changing in *historical time* in a given economy. This would be a truly dynamic approach but is much more complex, requiring an extended model with demographic dynamics. In contrast the Blanchard OLG model presupposes a stationary age distribution in the population. That is, the model depicts a situation where m, b, and n have stayed at their current values for a long time and are not changing. A time-dependent n, for example, would require expressions like $N(t) = N(0)e^{\int_0^t n_s ds}$, which gives rise to a much more complicated model.

12.3 Adding retirement

So far the model has assumed that everybody works full-time until death. This is clearly a weakness of a model that is intended to reflect life-cycle aspects of economic behavior. We therefore extend the model by incorporating gradual (but exogenous) retirement from the labor market. Following Blanchard (1985), we assume retirement is *exponential* (thereby still allowing simple analytical aggregation across cohorts).

Gradual retirement and aggregate labor supply

Suppose labor supply, ℓ, per year at time t for an individual born at time v depends only on age, $t - v$, according to

$$\ell(t-v) = e^{-\lambda(t-v)}, \tag{12.38}$$

where $\lambda > 0$ is the *retirement rate*. That is, higher age implies lower labor supply.[15] The graph of (12.38) in $(t-v, \ell)$ space looks like the solid curve in Fig. 12.2 above. Though somewhat coarse, this gives at least a flavour of retirement: old persons don't supply much labor. Consequently an incentive to save for retirement emerges.

Aggregate labor supply now is

$$\begin{aligned}
L(t) &= \int_{-\infty}^{t} \ell(t-v)N(v,t)dv \\
&= \int_{-\infty}^{t} e^{-\lambda(t-v)}N(0)e^{nv}be^{-m(t-v)}dv \qquad \text{(from (12.38) and (12.4))} \\
&= bN(0)e^{-(\lambda+m)t}\int_{-\infty}^{t} e^{(\lambda+b)v}dv = bN(0)e^{-(\lambda+m)t}\frac{e^{(\lambda+b)t}-0}{\lambda+b} \\
&= bN(0)e^{nt}\frac{1}{\lambda+b} = \frac{b}{\lambda+b}N(t). \qquad (12.39)
\end{aligned}$$

For given population size $N(t)$, earlier retirement (larger λ) implies lower aggregate labor supply. Similarly, given $N(t)$, a higher birth rate, b, entails a larger aggregate labor supply. This is because a higher b amounts to a larger fraction of

[15] An alternative interpretation of (12.38) would be that labor *productivity* is a decreasing function of age (as in Barro and Sala-i-Martin, 2004, pp. 185-86).

young people in the population and the young have a larger than average labor supply. Moreover, as long as the birth rate and the retirement rate are constant, aggregate labor supply grows at the same rate as population.

By the specification (12.38) the labor supply per year of a newborn is one unit of labor. In (12.39) we thus measure the labor force in units equivalent to the labor supply per year of one newborn.

The essence of retirement is that the aggregate labor supply depends on the age distribution in the population. The formula (12.39) presupposes that the age distribution has been constant for a long time. Indeed, the derivation of (12.39) assumes that the parameters b, m, and λ took their current values a long time ago so that there has been enough time for the age distribution to reach its steady state.

Human wealth

The present value at time t of expected future labor income for an individual born at time v is

$$\begin{aligned} h(v,t) &= \int_t^\infty w(s)\,\ell(s-v)e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= \int_t^\infty w(s)\,e^{-\lambda(s-v)}e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= e^{-\lambda(t-v)}\int_t^\infty w(s)\,e^{-\lambda(s-t)}e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= e^{-\lambda(t-v)}\int_t^\infty w(s)\,e^{-\int_t^s (r(\tau)+\lambda+m)d\tau}ds = e^{-\lambda(t-v)}h(t,t) \quad (12.40) \end{aligned}$$

where

$$h(t,t) = \int_t^\infty w(s)\,e^{-\int_t^s (r(\tau)+\lambda+m)d\tau}ds, \quad (12.41)$$

which is the human wealth of a newborn at time t (in (12.40) set $v = t$). Hence, aggregate human wealth for those alive at time t is

$$\begin{aligned} H(t) &= \int_{-\infty}^{t} h(v,t)N(v,t)dv = h(t,t)\int_{-\infty}^{t} e^{-\lambda(t-v)}N(v,t)dv \\ &= h(t,t)\int_{-\infty}^{t} e^{-\lambda(t-v)}N(0)e^{nv}be^{-m(t-v)}dv \\ &= h(t,t)bN(0)e^{-(\lambda+m)t}\int_{-\infty}^{t} e^{(\lambda+b)v}dv \\ &= h(t,t)bN(0)e^{-(\lambda+m)t}\frac{e^{(\lambda+b)t}-0}{\lambda+b} \\ &= h(t,t)N(0)e^{nt}\frac{b}{\lambda+b} = h(t,t)N(t)\frac{b}{\lambda+b} = h(t,t)L(t), \end{aligned} \tag{12.42}$$

by substitution of (12.39). That is, aggregate human wealth at time t is the same as the human wealth of a newborn at time t times the size of the labor force at time t. This result is due to the labor force being measured in units equivalent to the labor supply of one newborn.

Combining (12.41) and (12.42) gives

$$H(t) = \frac{b}{\lambda+b}N(t)\int_{t}^{\infty} w\left(s\right)e^{-\int_{t}^{s}(r(\tau)+\lambda+m)d\tau}ds. \tag{12.43}$$

If $\lambda = 0$, this reduces to the formula (12.15) for aggregate human wealth in the simple Blanchard model. We see from (12.43) that the future wage level $w\left(\tau\right)$ is effectively discounted by the sum of the interest rate, the retirement rate, and the death rate. This is not surprising. The sooner you retire and the sooner you are likely to die, the less important to you are the wage levels in the future.

Since the propensity to consume out of wealth is still the same for all individuals, aggregate consumption is, as before,

$$C(t) = (\rho+m)\left[A(t)+H(t)\right]. \tag{12.44}$$

Dynamics of household aggregates

The increase in aggregate financial wealth per time unit is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \tag{12.45}$$

The only difference compared to the simple Blanchard model is that it is now aggregate employment, $L(t)$, rather than population, $N(t)$, that enters the term for aggregate labor income.

The second important dynamic relation for the household sector is the one describing the increase in aggregate consumption per time unit. Instead of $\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t)$ from the simple Blanchard model, we now get

$$\dot{C}(t) = [r(t) - \rho + \lambda + n]C(t) - (\lambda + b)(\rho + m)A(t). \tag{12.46}$$

We see the retirement rate λ enters in two ways. This is because the generation replacement effect now has two sides. On the one hand, as before, the young that replace the old enter the economy with *no financial wealth*. On the other hand now they arrive with *more human wealth* than the average citizen. Through this channel the replacement of generations implies an increase per time unit in human wealth equal to λH, *ceteris paribus*. Indeed, the "rejuvenation effect" on individual labor supply is proportional to labor supply: $\partial\ell(t-v)/\partial v = \lambda\ell(t-v)$, from (12.38). In analogy, with a slight abuse of notation we can express the *ceteris paribus effect* on aggregate consumption as

$$\frac{\partial C}{\partial t} = (\rho + m)\frac{\partial H}{\partial t} = (\rho + m)\lambda H = \lambda(C - (\rho + m)A),$$

where the first and the last equality come from (12.44). This explicates the difference between the new equation (12.46) and the corresponding one from the simple model.[16]

The equilibrium path

With $r = f'(\tilde{k}) - \delta$ and $A = K$, (12.46) can be written

$$\dot{C} = \left[f'(\tilde{k}) - \delta - \rho + \lambda + n\right] C - (\lambda + b)(\rho + m)K. \tag{12.47}$$

Once more, the dynamics of general equilibrium can be summarized in two differential equations in $\tilde{k} \equiv K/(TL) \equiv k/T$ and $\tilde{c} \equiv C/(TN) \equiv c/T$. The differential equation in $\tilde{k}$ can be based on the national product identity for a closed economy: $Y = C + \dot{K} + \delta K$. Isolating $\dot{K}$ and using the definition of $\tilde{k}$, we get

$$\dot{\tilde{k}} = f(\tilde{k}) - \frac{C}{TL} - (\delta + g + n)\tilde{k} = f(\tilde{k}) - \frac{\lambda + b}{b}\tilde{c} - (\delta + g + b - m)\tilde{k}, \tag{12.48}$$

since $C/(TL) \equiv cN/(TL) = (N/L)c/T = (\lambda + b)\tilde{c}/b$ from (12.39).

As to the other differential equation, log-differentiating $\tilde{c}$ w.r.t. time yields

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{C}}{C} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = f'(\tilde{k}) - \delta - \rho + \lambda + n - (\lambda + b)(\rho + m)\frac{K}{C} - g - n,$$

[16] This explanation of (12.46) is only intuitive. A formal derivation can be made by using a method analogous to that applied in Appendix C.

from (12.47). Hence,

$$\begin{aligned}\dot{\tilde{c}} &= \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - (\lambda + b)(\rho + m)\frac{K}{C}\tilde{c} \\ &= \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - (\lambda + b)(\rho + m)\frac{K}{TL}\cdot\frac{L}{N},\end{aligned}$$

implying, in view of (12.39),

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - b(\rho + m)\tilde{k}. \tag{12.49}$$

The transversality conditions of the households are still given by (12.28).

Phase diagram The equation describing the $\dot{\tilde{k}} = 0$ locus is

$$\tilde{c} = \frac{b}{\lambda + b}\left[f(\tilde{k}) - (\delta + g + b - m)\tilde{k}\right]. \tag{12.50}$$

The equation describing the $\dot{\tilde{c}} = 0$ locus is

$$\tilde{c} = \frac{b\,(\rho + m)\,\tilde{k}}{f'(\tilde{k}) - \delta - \rho + \lambda - g}. \tag{12.51}$$

Let the value of $\tilde{k}$ such that the denominator of (12.51) vanishes be denoted $\bar{\tilde{k}}$, that is,

$$f'(\bar{\tilde{k}}) = \delta + \rho - \lambda + g. \tag{12.52}$$

Such a value exists if, in addition to the Inada conditions, the inequality

$$\lambda < \delta + \rho + g$$

holds, saying that the retirement rate is not "too large". We assume this to be true. The $\dot{\tilde{k}} = 0$ and $\dot{\tilde{c}} = 0$ loci are illustrated in Fig. 12.6. The $\dot{\tilde{c}} = 0$ locus is everywhere to the left of the line $\tilde{k} = \bar{\tilde{k}}$ and is asymptotic to this line.

As in the simple Blanchard model, the steady state $(\tilde{k}^*, \tilde{c}^*)$ is saddle-point stable. The economy moves along the saddle path towards the steady state for $t \to \infty$. Hence, for $t \to \infty$,

$$r_t = f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^* > \rho + g - \lambda, \tag{12.53}$$

where the inequality follows from $\tilde{k}^* < \bar{\tilde{k}}$. Again we may compare with the Ramsey model which, with $u(c) = \ln c$, has long-run interest rate equal to $r_R^* = \rho + g$.

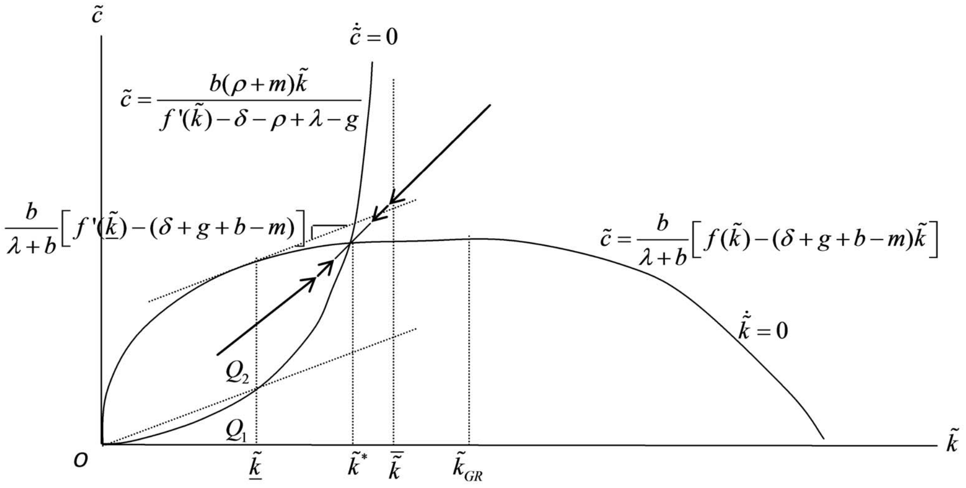

Figure 12.6: Phase diagram of the Blanchard model with retirement.

In the Blanchard OLG model extended with retirement, the long-run interest rate may differ from this value because of two effects of life-cycle behavior, that go in opposite directions. On the one hand, as mentioned earlier, finite lifetime ($m > 0$) leads to a higher effective utility discount rate, hence *less* saving and therefore a tendency for $r^* > \rho + g$. On the other hand, retirement entails an incentive to save *more* (for the late period in life with low labor income). This results in a tendency for $r^* < \rho + g$, everything else equal.

The presence of retirement implies that a new kind of apparent paradox may arise: the growth rate of individual consumption, $c(v,t)$, may be *lower* than that of per capita consumption, $c(t) \equiv \tilde{c}(t)T(t)$. In steady state $c(v,t)$ grows at the rate $r^* - \rho$ as long as the individual is alive, while per capita consumption, $c(t)$, grows at the rate g. In view of (12.53) and $g \geq 0$, the greatest lower bound for the former growth rate is $g - \lambda$ and there is scope for the inequality $g - \lambda < r^* - \rho < g$ to hold. So every individual alive may have a growth rate of consumption below the per capita consumption growth rate, g. The former may even be negative (namely if $g - \lambda < r^* - \rho < 0$) in spite of $g > 0$.[17] How can this be possible? Again the answer lies in the generation replacement effect. On the one hand, $r^* - \rho < 0$ induces every individual to have declining consumption until death (and, outside the model, may be also needs as old *are* smaller). On the other hand, if at the same time $g > 0$, every new generation starts adult life with higher initial consumption than the previous one. This is because technical progress endows new generations with higher initial human wealth than that with which the previous generations entered the economy.

[17] See Exercise 12.??

Another new phenomenon due to retirement is the theoretical possibility of dynamic inefficiency which was absent before. Recall that the golden rule capital-labor ratio, $\tilde{k}_{GR}$, is characterized by

$$f'(\tilde{k}_{GR}) - \delta = g + n,$$

where in the present case $n = b - m$. There are two cases to consider:

Case 1: $\lambda \leq \rho - n$. Then $\rho - \lambda \geq n$, so that $\bar{\tilde{k}} \leq \tilde{k}_{GR}$, implying that $\tilde{k}^*$ is below the golden rule value. In this case the long-run interest rate, r^*, satisfies $r^* = f'(\tilde{k}^*) - \delta > g + n$, that is, the economy is dynamically efficient.

Case 2: $\lambda > \rho - n$. We now have $\rho - \lambda < n$, so that $\bar{\tilde{k}} > \tilde{k}_{GR}$. Hence, it is possible that $\tilde{k}^* > \tilde{k}_{GR}$, implying $r^* = f'(\tilde{k}^*) - \delta < g + n$, so that there is sustained over-saving and the economy is dynamically inefficient. Owing to the retirement, this can arise even when $\rho > n$. A situation with $r^* \leq g + n$ has theoretically interesting implications for solvency and sustainability of fiscal policy, a theme we considered in Chapter 6. On the other hand, as argued in Section 4.2 of Chapter 4, the empirics point to dynamic *efficiency* as the most plausible case.

The reason that a high retirement rate λ (early retirement) *may*, theoretically, lead to over-saving is that early retirement implies a longer span of the period as almost fully retired. Hence there is a need to do more saving for retirement. In general equilibrium, however, there is an effect in the opposite direction. This is that early retirement reduces the labor force relative to population. Thereby the drag on capital accumulation coming from a given level of *per capita* consumption is enlarged, as revealed by the term $((\lambda + b)/b)\tilde{c}$ in (12.48).

12.4 The rate of return in the long run

Blanchard's OLG model provides a succinct and yet multi-facetted theory of the level of the interest rate in the long run. Of course, in the real world there are many different types of uncertainty which simple macro models like the present one ignore. Yet we may interpret the real interest rate of these models as reflecting the general level around which the different interest rates of an economy fluctuate, i.e., a kind of average rate of return.

In this perspective Blanchard's theory of the rate of return differs from the modified golden rule theory from Ramsey's and Barro's models by allowing a role for demographic parameters. The Blanchard model predicts a long-run interest rate in the interval

$$\rho + g - \lambda < r^* < \rho + g + b. \tag{12.54}$$

The left-hand inequality, which reflects the role of retirement, was proved above (see (12.53)). In the right-hand inequality appears the positive birth rate b which allows the interest rate to exceed the level $\rho + g$, i.e., the level of the interest rate in the corresponding Ramsey model. An algebraic proof of this upper bound for r^* is provided in Appendix F. Below is a graphical argument, which is more intuitive.

Proof (sketch) of the upper inequality in (12.54)*

Let $\underline{\tilde{k}} > 0$ be some value of $\tilde{k}$ less than $\tilde{k}^*$. The vertical line $\tilde{k} = \underline{\tilde{k}}$ in Fig. 12.6 crosses the horizontal axis and the $\dot{\tilde{c}} = 0$ locus at the points Q_1 and Q_2, respectively. Adjust the choice of $\underline{\tilde{k}}$ such that the ray OQ_2 is parallel to the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \underline{\tilde{k}}$ (evidently this can always be done). We then have

$$\text{slope of } OQ_2 = \frac{|Q_1Q_2|}{|OQ_1|} = \frac{b}{\lambda + b}\left[f'(\underline{\tilde{k}}) - (\delta + g + n)\right].$$

By construction we also have

$$\text{slope of } OQ_2 = \frac{b(\rho + m)\underline{\tilde{k}}}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} \cdot \frac{1}{\underline{\tilde{k}}},$$

where $\tilde{k}$ cancels out. Equating the two right-hand sides and ordering gives

$$\begin{aligned} \frac{b(\rho + m)}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} &= \frac{b}{\lambda + b}\left[f'(\underline{\tilde{k}}) - (\delta + g + n)\right] \Rightarrow \\ \frac{(\lambda + b)(\rho + m)}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} &= f'(\underline{\tilde{k}}) - (\delta + g + n). \end{aligned} \tag{12.55}$$

This implies a quadratic equation in $f'(\underline{\tilde{k}})$ with the positive solution

$$f'(\underline{\tilde{k}}) = \delta + \rho + g + b. \tag{12.56}$$

Indeed, with (12.56) we have:

$$\begin{aligned} \text{left-hand side of (12.55)} &= \frac{(\lambda + b)(\rho + m)}{\delta + \rho + g + b - \delta - \rho - g + \lambda} \\ &= \frac{(\lambda + b)(\rho + m)}{b + \lambda} = \rho + m, \qquad \text{and} \\ \text{right-hand side of (12.55)} &= \rho + m, \end{aligned}$$

so that (12.55) holds. Now, from $\tilde{k}^* > \underline{\tilde{k}}$ and $f'' < 0$ follows that $f'(\tilde{k}^*) < f'(\underline{\tilde{k}})$. Hence,

$$r^* = f'(\tilde{k}^*) - \delta < f'(\underline{\tilde{k}}) - \delta = \rho + g + b,$$

where the last equality follows from (12.56). This confirms the right-hand inequality in (12.54). □

How r^* depends on parameters

Let us first consider a numerical example.

EXAMPLE 1 Using one year as our time unit, a rough estimate of the rate of technological progress g for the Western countries since World War II is $g = 0.02$. To get an assessment of the birth rate b, we may coarsely estimate $n = b - m$ to be 0.005. An expected lifetime (as adult) around 55 years, equal to $1/m$ in the model, suggests that $m = 1/55 \approx 0.018$. Hence $b = n + m \approx 0.023$. What about the retirement rate λ? An estimate of the labor force participation rate is $L/N = 0.75$, equal to $b/(b + \lambda)$ in the model, so that, from (12.39), $\lambda = b(1 - L/N)/(L/N) \approx 0.008$. Now, guessing $\rho = 0.02$, (12.54) gives $0.032 < r^* < 0.063$. □

As this example illustrates, the interval (12.54) gives only a very rough idea about the level of r^*. And such an interval is of limited help for assessing how r^* depends on the model's parameters.

To dig deeper, given the production function f, let us see if we can determine r^* as an implicit function of the parameters. In steady state, $\tilde{k} = \tilde{k}^*$ and the right-hand sides of (12.50) and (12.51) are equal to each other. After ordering we have

$$\left(\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + b - m)\right)\left[f'(\tilde{k}^*) - \delta - \rho + \lambda - g\right] = (\lambda+b)\,(\rho + m)\,. \quad (12.57)$$

A diagram showing the left-hand side and right-hand side of this equation will look qualitatively like Fig. 12.5 above. The equation defines $\tilde{k}^*$ as an implicit function φ of the parameters g, δ, n, m, ρ, and λ, i.e., $\tilde{k}^* = \varphi(g, b, m, \rho, \lambda, \delta)$. The partial derivatives of φ have the sign structure $\{-, -, ?, -, ?, -\}$ (to see this, use implicit differentiation or simply curve shifting in a graph like Fig. 12.5). Then, from $r^* = f'(\tilde{k}^*) - \delta$ follows $\partial r^*/\partial x = (\partial r^*/\partial \tilde{k}^*)\partial\varphi/\partial x = f''(\tilde{k}^*)\partial\varphi/\partial x$ for $x \in \{g, b, m, \rho, \lambda, \delta\}$. These partial derivatives have the sign structure $\{+, +, ?, +, ?, ?\}$. This tells us how the long-run interest rate qualitatively depends on these parameters.

For example, a higher rate of technological progress results in a higher rate of return, r^*. The higher g is, the greater is the expected future wage income and the associated consumption possibilities even without any current saving. This discourages saving, and thereby capital accumulation, and a lower effective capital-labor ratio in steady state is the result, hence a higher long-run interest rate. In turn this is what is needed to sustain a higher long-run per capita growth rate equal to g. A higher mortality rate has an ambiguous effect on the rate of return in the long run. On the one hand a higher m shifts the $\dot{\tilde{k}} = 0$ curve in Fig. 9.6 upward because of the implied lower labor force growth rate. For given aggregate saving this entails more capital deepening in the economy. On the other hand, a higher m also implies less incentive for saving and therefore a counter-clockwise turn of the $\dot{\tilde{c}} = 0$ curve. The net effect of these two forces on $\tilde{k}^*$, hence on r^*, is ambiguous. But as (12.57) indicates, if b is increased along with m so as to keep n unchanged, $\tilde{k}^*$ falls and r^* rises.

Earlier retirement similarly has an ambiguous effect on the rate of return in the long run. On the one hand a higher λ shifts the $\dot{\tilde{k}} = 0$ curve in Fig. 9.6 downward because the lower labor force participation rate reduces per capita output. On the other hand, a higher λ also implies a clockwise turn of the $\dot{\tilde{c}} = 0$ curve. This is because the need to provide for a longer period as retired implies more saving and capital accumulation in the economy. The net effect of these two forces on $\tilde{k}^*$, hence on r^*, is ambiguous.

Also $\partial r^*/\partial\delta$ can not be signed without further specification, because $\partial r^*/\partial\delta = (\partial r^*/\partial\tilde{k}^*)\partial\varphi/\partial\delta = f''(\tilde{k}^*)\partial\varphi/\partial\delta - \partial\delta/\partial\delta = f''(\tilde{k}^*)\partial\varphi/\partial\delta - 1$, where we cannot apriori tell whether the first term exceeds 1 or not.

Of course, by specifying the production function and assigning numbers to the parameters, numerical solutions can be studied.

Further perspectives

A theoretically important factor for consumption-saving behavior $-$ and thereby r^* $-$ is missing in the version of the Blanchard model considered here. This factor is the desire for consumption smoothing or its inverse, the intertemporal elasticity of substitution in consumption. Since our version of the model is based on the special case $u(c) = \ln c$, the intertemporal elasticity of substitution in consumption is fixed to be 1. Now assume, more generally, that $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$ and $1/\theta$ is the intertemporal elasticity of substitution in consumption. Then the dynamic system becomes three-dimensional and in that way more complicated. Nevertheless it can be shown that a higher θ implies a higher real interest rate in the long run.[18] The intuition is that in an economy with sustained productivity growth, a higher θ means less willingness to offer current consumption for more future consumption and this implies less saving. Thus, $\tilde{k}^*$ becomes lower and r^* higher. Also public debt tends to affect r^* positively in a closed economy, as we will see in the next chapter.

[18] See Blanchard (1985).

We end this section with some general reflections. Economic theory is a set of propositions that are organized in a hierarchic way and have an economic interpretation. A theory of the real interest rate should say something about the factors and mechanisms that determine the level of the interest rate. In fact, in a more realistic setup with uncertainty, it is the level of interest *rates,* including the risk-free rate, that should be determined. We would like the theory to explain both the short-run level of interest rates and the long-run level, that is, the average level over several decades. The Blanchard model can be one part of such a theory as far as long-run interest rates is concerned. Abstracting from monetary factors, nominal price rigidities, and short-run fluctuations in aggregate demand, the model is certainly less reliable as a description of the short run.

Note that the interest rate considered so far is the short-term interest rate. What is important for consumption and, in particular, investment is rather the long-term interest rate (internal rate of return on long-term bonds). With perfect foresight, the long-term rate is just a weighted average of expected future short-term rates.[19] In steady state these rates will be the same and so the present theory applies. In a world with uncertainty, however, the link between the long-term rate and the expected future short-term rates is more difficult to discern, affected as it may be by changing risk and liquidity premia.

12.5 National wealth and foreign debt

We will embed the Blanchard setup in a small open economy (henceforth SOE). The purpose is to study how national wealth and foreign debt in the long run are determined, when technological change is exogenous. Our SOE is characterized by:

(a) Perfect mobility across borders of goods and financial capital.

(b) Domestic and foreign financial claims are perfect substitutes.

(c) The need for means of payment is ignored and so is the need for a foreign exchange market.

(d) No mobility of labor across borders.

(e) Labor supply is inelastic, but age-dependent.

The assumptions (a) and (b) imply interest rate equality (see Section 5.3 in Chapter 5). That is, the interest rate in our SOE must equal the interest rate

[19] See Chapter 22.

in the world market for financial capital. This interest rate is exogenous to our SOE. We denote it r and assume r is positive and constant over time.

Apart from this, households, firms, and market structure are as in the Blanchard model for the closed economy with gradual retirement. We maintain the assumptions of perfect competition, no government sector, and no uncertainty except with respect to individual life lifetime.

Elements of the model

Firms choose effective capital-labor ratio $\tilde{k}(t)$ so that $f'(\tilde{k}(t)) = r+\delta$. The unique solution to this equation is denoted $\tilde{k}^*$. Thus,

$$f'(\tilde{k}^*) = r + \delta. \tag{12.58}$$

How does $\tilde{k}^*$ depend on r? To find out, we interpret (12.58) as implicitly defining $\tilde{k}^*$ as a function of r, $\tilde{k}^* = \varphi(r)$. Taking the total derivative w.r.t. r on both sides of (12.58), then gives $f''(\tilde{k}^*)\varphi'(r) = 1$, from which follows

$$\frac{d\tilde{k}^*}{dr} = \varphi'(r) = \frac{1}{f''(\tilde{k}^*)} < 0. \tag{12.59}$$

With continuous clearing in the labor market, employment equals labor supply, which, as in (12.39), is

$$L(t) = \frac{b}{\lambda + b}N(t), \qquad \text{for all } t,$$

where $N(t)$ is population, $b \equiv n+m$ is the birth rate, and $\lambda \geq 0$ is the retirement rate. We have $\tilde{k}(t) = \tilde{k}^*$, so that

$$K(t) = \tilde{k}^* T(t)L(t) = \tilde{k}^* T(t)\frac{b}{\lambda + b}N(t), \tag{12.60}$$

for all $t \geq 0$. This gives the endogenous stock of physical capital in the SOE at any point in time. If r shifts to a higher level, $\tilde{k}^*$ shifts to a lower level and the capital stock immediately adjusts, as shown by (12.59) and (12.60), respectively. This instantaneous adjustment is a counter-factual prediction of the model; it is a signal that the model ought to be modified so that adjustment of the capital stock takes place more gradually. We come back to this in Chapter 14 in connection with the theory of convex capital adjustment costs. For the time being we ignore adjustment costs and proceed as if (12.60) holds for all $t \geq 0$. This simplification would make short-run results very inaccurate, but is less problematic in long-run analysis.

In equilibrium firms' profit maximization implies the real wage

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = \left[f(\tilde{k}^*) - \tilde{k}^* f'(\tilde{k}^*)\right] T(t) \equiv \tilde{w}(\tilde{k}^*)T(t) = \tilde{w}^* T(t), \tag{12.61}$$

where $\tilde{w}^*$ is the real wage per unit of effective labor. It is constant as long as r and δ are constant. So the real wage, w, per unit of "natural" labor grows over time at the same rate as technology, the rate $g \geq 0$. Notice that $\tilde{w}^*$ depends negatively on r in that

$$\frac{d\tilde{w}^*}{dr} = \frac{d\tilde{w}(\tilde{k}^*)}{d\tilde{k}^*}\frac{d\tilde{k}^*}{dr} = -\tilde{k}^* f''(\tilde{k}^*)\frac{1}{f''(\tilde{k}^*)} = -\tilde{k}^* < 0, \tag{12.62}$$

where we have used (12.59).

From now we suppress the explicit dating of the variables when not needed for clarity. As usual A denotes aggregate private financial wealth. Since the government sector is ignored, A is the same as *national wealth* of the SOE. And since land is ignored, we have

$$A \equiv K - D,$$

where D denotes net foreign debt, that is, financial claims on the SOE from the rest of the world. Then $A_f \equiv -D$ is net holding of foreign assets. Net national income of the SOE is $rA + wL$ and aggregate net saving is $S^N = rA + wL - C$, where C is aggregate consumption. Hence,

$$\dot{A} = S^N = rA + wL - C. \tag{12.63}$$

So far essentially everything is as it would be in a Ramsey (representative agent) model for a small open economy.[20] When we consider the change over time in *aggregate* consumption, however, an important difference emerges. In the Ramsey model the change in aggregate consumption is given simply as an aggregate Keynes-Ramsey rule. But the life-cycle feature arising from the finite horizons leads to something quite different in the Blanchard model. As we saw in Section 12.3 above,

$$\dot{C} = (r - \rho + \lambda + n)C - (\lambda + b)(\rho + m)A, \tag{12.64}$$

where the last term is the generation replacement effect.

[20] The fact that labor supply, L, deviates from population, N, if the retirement rate, λ, is positive, is a minor difference compared with the Ramsey model. As long as λ and b are constant, L is still proportional to N.

The law of motion

All parameters are non-negative and in addition we will throughout, not unrealistically, assume that

$$r > g - m. \tag{A1}$$

This assumption ensures that the model has a solution even for $\lambda = 0$ (see (12.66) below). To obtain a dynamic system capable of being in a steady state, we introduce growth-corrected variables, $\tilde{a} \equiv A/(TN) \equiv a/T$ and $\tilde{c} \equiv C/(TN) \equiv c/T$. Log-differentiating $\tilde{a}$ w.r.t. t gives

$$\begin{aligned} \frac{\dot{\tilde{a}}}{\tilde{a}} &= \frac{\dot{A}}{A} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = \frac{rA + wL - C}{A} - (g+n) \qquad \text{or} \\ \dot{\tilde{a}}(t) &= (r - g - n)\tilde{a}(t) + \tilde{w}^* \frac{b}{\lambda + b} - \tilde{c}(t), \end{aligned} \tag{12.65}$$

where $\tilde{w}^*$ is given in (12.61) and we have used $L(t)/N(t) = b/(\lambda + b)$ from (12.39). We might proceed by using (12.64) to get a differential equation for $\tilde{c}(t)$ in terms of $\tilde{a}(t)$ and $\tilde{c}(t)$ (analogous to what we did for the closed economy). The interest rate is now a constant, however, and then a more direct approach to the determination of $\tilde{c}(t)$ in (12.65) is convenient.

Consider the aggregate consumption function $C = (\rho + m)(A + H)$. Substituting (12.61) into (12.43) gives

$$H(t) = \frac{b}{\lambda + b} N(t)\tilde{w}^* T(t) \int_t^\infty e^{-(r+\lambda+m-g)(\tau - t)} d\tau = \frac{bN(t)\tilde{w}^* T(t)}{\lambda + b} \frac{1}{r + \lambda + m - g}, \tag{12.66}$$

where we have used that (A1) ensures $r + \lambda + m - g > 0$. It follows that

$$\frac{H(t)}{T(t)N(t)} = \frac{b\tilde{w}^*}{(\lambda + b)(r + \lambda + m - g)} \equiv \tilde{h}^*,$$

where $\tilde{h}^* > 0$ by (A1). Growth-corrected consumption can now be written

$$\tilde{c}(t) = (\rho + m)\left(\frac{A(t)}{T(t)N(t)} + \frac{H(t)}{T(t)N(t)}\right) = (\rho + m)(\tilde{a}(t) + \tilde{h}^*). \tag{12.67}$$

Substituting for $\tilde{c}$ into (12.65), inserting $b \equiv n + m$, and ordering gives the law of motion of the economy:

$$\dot{\tilde{a}}(t) = (r - \rho - g - b)\tilde{a}(t) + \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)}. \tag{12.68}$$

The dynamics are thus reduced to *one* differential equation in growth-corrected national wealth; moreover, the equation is linear and even has constant coefficients. If we want it, we can therefore find an explicit solution. Given $\tilde{a}(0) = \tilde{a}_0$ and $r \neq \rho + g + b$, the solution is

$$\tilde{a}(t) = (\tilde{a}_0 - \tilde{a}^*)e^{-(\rho+g+b-r)t} + \tilde{a}^*, \tag{12.69}$$

where

$$\tilde{a}^* = \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)(\rho + g + b - r)}, \tag{12.70}$$

which is the growth-corrected national wealth in steady state. Substitution of (12.69) into (12.67) gives the corresponding time path for growth-corrected consumption, $\tilde{c}(t)$. In steady state growth-corrected consumption is

$$\tilde{c}^* = \frac{(\rho + m)b\tilde{w}^*}{(r + \lambda + m - g)(\rho + g + b - r)}. \tag{12.71}$$

It can be shown that along the paths generated by (12.65), the transversality conditions of the households are satisfied (see Appendix D).

Let us first consider the case of stability. That is, while (A1) is maintained, we assume

$$r < \rho + g + b. \tag{12.72}$$

The phase diagram in $(\tilde{a}, \dot{\tilde{a}})$ space for this case is shown in the upper panel of Fig. 12.7. The lower panel of the figure shows the path followed by the economy in $(\tilde{a}, \tilde{c})$ space, for a given initial $\tilde{a}$ above $\tilde{a}^*$. The equation for the $\dot{\tilde{a}} = 0$ line is

$$\tilde{c} = (r - g - n)\tilde{a} + \tilde{w}^* \frac{b}{\lambda + b},$$

from (12.65). Different scenarios are possible. (Note that all conclusions to follow, and in fact also the above steady-state values, can be derived without reference to the explicit solution (12.69).)

The case of medium impatience In Fig. 12.7, as drawn, it is presupposed that $\tilde{a}^* > 0$, which, given (12.72), requires $r - (g + b) < \rho < r + \lambda - g$. We call this the case of medium impatience. Note that the economy is always at some point on the line $\tilde{c} = (\rho + m)(\tilde{a} + \tilde{h}^*)$, in view of (12.67). If we, as for the closed economy, had based the analysis on *two* differential equations in $\tilde{a}$ and $\tilde{c}$, respectively, then a saddle path would emerge and this would coincide with the $\tilde{c} = (\rho + m)(\tilde{a} + \tilde{h}^*)$ line in Fig. 12.7.[21]

[21] Although the $\dot{\tilde{a}} = 0$ line is drawn with a positive slope, it could alternatively have a negative slope (corresponding to $r < g+n$); stability still holds. Similarly, although growth-corrected per

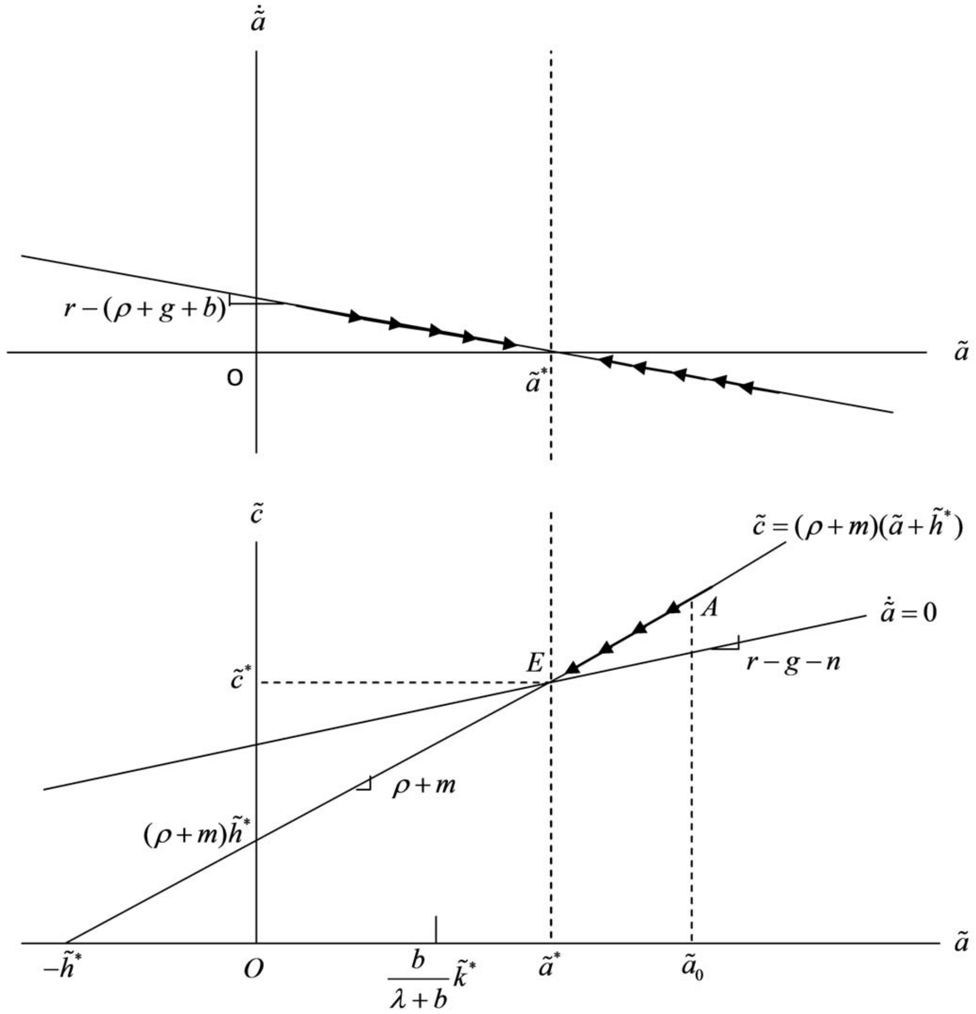

Figure 12.7: Adjustment process for an SOE with *medium impatience*, i.e., $r-(g+b) < \rho < r+\lambda-g$.

The case of high impatience Not surprisingly, a^* in (12.70) is a decreasing function of the impatience parameter ρ. A SOE with $\rho > r + \lambda - g$ (high impatience) has $\tilde{a}^* < 0$. That is, the country ends up with negative national wealth, a scenario which from pure economic logic is definitely possible, if there is a perfect international credit market. One should remember that "national wealth", in its usual definition, also used here, includes only financial wealth. Theoretically it can be negative if at the same time the economy has enough human wealth, H, to make *total* wealth, $A + H$ positive. Since $\tilde{c}^* > 0$, a steady state *must* have $\tilde{h}^* > -\tilde{a}^*$, in view of (12.67).

Negative national wealth of the SOE will reflect that all the physical capital of the SOE and part of its human wealth is in a sense mortgaged. Such an outcome, however, is not likely to be observed in practice. This is so for at least two reasons. First, whereas the analysis assumes a perfect international credit market, in the real world there is limited scope for writing enforceable international financial contracts. Lenders' risk perceptions depend on the level of debt and even within one's own country, access to loans with human wealth as a collateral is limited. Second, long before all physical capital of the impatient SOE is mortgaged or has become directly taken over by foreigners, the government presumably would intervene for *political* reasons.

The case of low impatience Alternatively, if (A1) is strengthened to $r > g+b$, we can have $0 \leq \rho \leq r-(g+b)$. This is the case of low impatience or high patience. Then the stability condition (12.72) is no longer satisfied.

Consider first the generic subcase $0 \leq \rho < r - (g+b)$. In this case the solution formula (12.69) is still valid. The slope of the adjustment path in the upper panel of Fig. 12.7 will now be positive and the $\tilde{c}$ line in the lower panel will be less steep than the $\dot{\tilde{a}} = 0$ line. There is no economic steady state any longer since the $\tilde{c}$ line will no longer cross the $\dot{\tilde{a}} = 0$ line for any positive level of consumption. There is a "fictional" steady-state value, $\tilde{a}^*$, which is negative and unstable. It is only "hypothetical" because it is associated with *negative* consumption, cf. (12.71). With $\tilde{a}_0 > -\tilde{h}^*$, the excess of r over $\rho+g+b$ results in high sustained saving so as to keep $\tilde{a}$ growing *forever*.[22] This means that national wealth, A, grows permanently at a rate higher than $g + n$. The economy grows *large* in the long run. But then, sooner or later, the world market interest rate can no longer be independent of what happens in this economy. The capital deepening resulting from the fast-growing country's capital accumulation will eventually affect the world economy

capita capital, $K/(TN)$ $(L/N)K/(TL) = b\tilde{k}^*/(\lambda + b)$, is in Fig. 12.7 smaller than $\tilde{a}^*$, it could just as well be larger. Both possibilities are consistent with the case of medium impatience.

[22] The reader is invited to draw the phase diagram in $(\tilde{a}, \tilde{c})$ space for this case, cf. Exercise 12.??.

and reduce the gap between r and ρ, so that the incentive to accumulate receives a check − like in a closed economy. Thus, the SOE assumption ceases to fit. Think of China's development since the early 1970s.

The alternative subcase is the knife-edge case $\rho = r - (g + b)$. In this case the solution formula (12.69) is no longer valid. Instead we get

$$\tilde{a}(t) = \tilde{a}(0) + \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)}t \to \infty \quad \text{for} \quad t \to \infty.$$

Foreign assets and debt

Returning to the stability case, where (12.72) holds, let us be more explicit about the evolution of net foreign debt. Or rather, in order to visualize by help of Fig. 12.7, we will consider net foreign assets, $A_f \equiv A - K = -D$. How are growth-corrected net foreign assets determined in the long run? We have

$$\tilde{a}_f \equiv \frac{A_f}{TN} \equiv \frac{A - K}{TN} = \tilde{a} - \frac{L}{N}\frac{K}{TL} = \tilde{a} - \frac{b}{\lambda + b}\tilde{k}^* \text{ (by (12.39))}.$$

Thus, by stability of $\tilde{a}$, for $t \to \infty$

$$\tilde{a}_f \to \tilde{a}_f^* = \tilde{a}^* - \frac{b}{\lambda + b}\tilde{k}^*.$$

The country depicted in Fig. 12.7 happens to have $\tilde{a}_0 > \tilde{a}^* > b\tilde{k}^*/(\lambda + b)$. So growth-corrected net foreign assets decline during the adjustment process. Yet, net foreign assets remain positive also in the long run. The interpretation of the positive $\tilde{a}_f$ is that only a part of national wealth is placed in physical capital in the home country, namely up to the point where the net marginal productivity of capital equals the world market rate of return r.[23] The remaining part of national wealth would result in a rate of return below r if invested within the country and is therefore better placed in the world market for financial capital.

Implicit in the described evolution over time of net foreign assets is a unique evolution of the current account surplus. By definition, the current account surplus, CAS, equals the increase per time unit in net foreign assets, i.e.,

$$CAS \equiv \dot{A}_f = \dot{A} - \dot{K}.$$

[23] The term foreign debt, as used here, need not have the contractual form of debt, but can just as well be equity. Although it may be easiest to imagine that capital in the different countries is always owned by the country's own residents, we do not presuppose this. And as long as we ignore uncertainty, the ownership pattern is in fact irrelevant from an economic point of view.

This says that CAS can also be viewed as the difference between net saving and net investment. Taking the time derivative of $\tilde{a}_f$ gives

$$\dot{\tilde{a}}_f = \frac{TN\dot{A}_f - A_f(T\dot{N} + N\dot{T})}{(TN)^2} = \frac{CAS}{TN} - (g+n)\tilde{a}_f.$$

Consequently, the movement of the growth-corrected current account surplus is given by

$$\begin{aligned} \frac{CAS}{TN} &= \dot{\tilde{a}}_f + (g+n)\tilde{a}_f = \dot{\tilde{a}} + (g+n)\tilde{a} - \frac{(g+n)b}{\lambda+b}\tilde{k}^* \\ &= (r-\rho-m)\tilde{a} + \frac{(r+\lambda-\rho-g)b\tilde{w}^*}{(r+\lambda+m-g)(\lambda+b)} - \frac{(g+n)b}{\lambda+b}\tilde{k}^*, \end{aligned}$$

where the second equality follows from the definition of $\tilde{a}_f$ and the third from (12.68). Yet, in our perfect-markets-equilibrium framework there is no bankruptcy-risk and no borrowing difficulties and so the current account is not of particular interest.

Returning to Fig. 12.7, consider a case where the rate of impatience, ρ, is somewhat higher than in the figure, but still satisfying the inequality $\rho < r+\lambda-g$. Then $\tilde{a}^*$, although smaller than before, is still positive. Since $\tilde{k}^*$ is not affected by a rise in ρ, it is $\tilde{a}_f^*$ that adjusts and might now end up negative, tantamount to net foreign debt, $\tilde{d}^* \equiv -\tilde{a}_f^*$, being positive.

Let us take the US economy as an example. Even if it is not really a small economy, the US economy may be small enough compared to the world economy for the SOE model to have something to say.[24] In the middle of the 1980s the US changed its international asset position from being a net creditor to being a net debtor. Since then, the US net foreign debt as a percentage of GDP has been rising, reaching 22 % in 2004.[25] With an output-capital ratio around 50 %, this amounts to a debt-capital ratio $D/K = (D/Y)Y/K = 11$ %.

A different movement has taken place in the Danish economy (which of course fits the notion of an SOE better). After World War II and until recently, Denmark had positive net foreign debt. In the aftermath of the second oil price shock in 1979-80, the debt rose to a maximum of 42 % of GDP in 1983. After 1991 the debt has been declining, reaching 11 % of GDP in 2004 (a development supported by the oil and natural gas extracted from the North Sea). Since 2011 Denmark has had positive net foreign assets.[26]

[24] The share of the US in world GDP was 29 % in 2003, but if calculated in purchasing power-corrected exchange rates only 21 % (World Economic Outlook, April 2004, IMF). The fast economic growth of, in particular, China and India since the early 1980s has produced a downward trend for the US share.

[25] Source: US Department of Commerce.

[26] Source: Statistics Denmark.

The adjustment speed

By *speed of adjustment* of a variable which converges in a monotonic way is meant the proportionate rate of decline per time unit of its distance to its steady-state value. Defining $\psi \equiv \rho + g + b - r$, from (12.69) we find, for $\tilde{a}(t) \neq \tilde{a}^*$,

$$-\frac{d\left|\tilde{a}(t) - \tilde{a}^*\right|/dt}{\left|\tilde{a}(t) - \tilde{a}^*\right|} = -\frac{d(\tilde{a}(t) - \tilde{a}^*)/dt}{\tilde{a}(t) - \tilde{a}^*} = -\frac{(\tilde{a}_0 - \tilde{a}^*)e^{-\psi t}(-\psi)}{\tilde{a}(t) - \tilde{a}^*} = \psi.$$

Thus, ψ measures the speed of adjustment of growth-corrected national wealth. We get an estimate of ψ in the following way. With one year as the time unit, let $r = 0.04$ and let the other parameters take values equal to those given in the numerical example in Section 12.3. Then $\psi = 0.023$, telling us that 2.3 percent of the gap, $\tilde{a}(t) - \tilde{a}^*$, is eliminated per year.

We may also calculate the *half-life.* By half-life is meant the time it takes for half the initial gap to be eliminated. Thus, we seek the number τ such that

$$\frac{\tilde{a}(\tau) - \tilde{a}^*}{\tilde{a}_0 - \tilde{a}^*} = \frac{1}{2}.$$

From (12.69) follows that $(\tilde{a}(\tau) - \tilde{a}^*)/(\tilde{a}_0 - \tilde{a}^*) = e^{-\psi\tau}$. Hence, $e^{-\psi\tau} = 1/2$, implying that half-life is

$$\tau = \frac{\ln 2}{\psi} \approx \frac{0.69}{0.023} \approx 30 \text{ years.}$$

The conclusion is that adjustment processes involving accumulation of national wealth are slow.

12.6 Concluding remarks

One of the strengths of the Blanchard OLG model compared with the Ramsey model comes to the fore in the analysis of a small open economy. The Ramsey model is a representative agent model so that the Keynes-Ramsey rule holds at both the individual and aggregate level. When applied to a small open economy with exogenous r, the Ramsey model therefore needs the *knife-edge condition* $\rho + \theta g = r$ (where θ is the absolute value of the elasticity of marginal utility of consumption).[27] Indeed, if $\rho + \theta g > r$, the $\tilde{a}$ in a Ramsey economy approaches

[27] A *knife-edge condition* is a condition imposed on a parameter value such that the set of values satisfying this condition has an empty interior in the space of all possible values. For the SOE all four terms entering the Ramsey condition $\rho + \theta g = r$ are parameters. Assuming the condition is satisfied thus amounts to imposing a knife-edge condition, which is unlikely to hold in the real world and which may lead to non-robust results.

a negative number (namely minus the growth-corrected human wealth) and $\tilde{c}$ tends to zero in the long run − an implausible scenario.[28] And if $\rho + \theta g < r$, the economy will tend to grow large relative to the world economy and so, eventually, the SOE framework is no longer appropriate for this economy. It is this lack of robustness which motivates the term "knife-edge" condition. If the parameter values are in a hair's breadth distance from satisfying the condition, qualitatively different behavior of the dynamic system results.

In contrast to the Ramsey model, the Blanchard OLG model deals with life-cycle behavior. Within a "fat" set of parameter values, namely those satisfying the inequalities (A1) and (12.72), it gives robust results for a small open economy.

A further strength of Blanchard's model is that it allows studying effects of alternative age *compositions* of a population. Compared with Diamond's OLG model, the Blanchard model has a less coarse demographic structure and a more refined notion of time. And by taking uncertainty about life-span into account, the model opens up for incorporating markets for life annuities (and similar forms of private pension arrangements). In this way important aspects of reality are included. On the other hand, from an empirical point of view it is a weakness that the propensity to consume out of wealth in the model is the same for a young and an old. In this respect the model lacks a weighty life-cycle feature. This limitation, of cause, comes from the unrealistic premise that the mortality rate is the same for all age groups. Another limitation is that individual asset ownership in the model depends only on age through own accumulated saving. In reality, there is considerable intra-generation differences in asset ownership due to differences in inheritance (Kotlikoff and Summers, 1981; Charles and Hurst, 2003; Danish Economic Council, 2004). Some extensions of the Blanchard OLG model are mentioned in Literature notes.

12.7 Literature notes

Three-period OLG models are under special conditions analytically obedient, see for instance de la Croix and Michel (2002).

Naive econometric studies trying to estimate consumption Euler equations (the discrete time analogue to the Keynes-Ramsey rule) on the basis of aggregate data and a representative agent approach can be seriously misleading. About this, see Attanasio and Weber, RES 1993, 631-469, in particular p. 646.

That Blanchard's OLG model in continuous time becomes three-dimensional if $\theta \neq 1$, is shown in Blanchard (1985). In that article it is also shown that a higher

[28] For a Ramsey-type model with a finite number of infinitely-lived households with different time-preference rates, Uzawa (1968) showed that asymptotically the entire private wealth will be owned by the household with the lowest time-preference rate.

θ implies a higher real interest rate in the long run. That in the perpetual youth model in steady state, individual consumption is growing faster than productivity is due to the young having less financial wealth than the average citizen. We saw that in the Blanchard model with gradual retirement, as in Section 12.3, there is a countervailing effect due to the young having more *human* wealth than the average citizen. Wendner (2010) explores the possibility that the latter effect may dominate and studies how this interferes with the issue of over- or underconsumption created by a "keeping up with the Joneses" externality.

Blanchard (1985) also sketched a more refined life-cycle pattern of the age profile of earned income involving initially rising labor income and then declining labor income with age. This can be captured by assuming that labor supply (or productivity) is the difference between two negative exponentials, $\ell(t-v) = m_1 e^{-\omega_1(t-v)} - m_2 e^{-\omega_2(t-v)}$, where all parameters are positive and $\omega_2/\omega_1 > m_1/m_2 > 1$.

Blanchard's model has been extended in many different directions. Calvo and Obstfeld (1988), Boucekkine et al. (2002), and Heijdra and Romp (2007, 2009) incorporate age-specific mortality. Endogenous education and retirement are included in Boucekkine et al. (2002), Grafenhofer et al. (2005), Sheshinski (2009), and Heijdra and Romp (2009). Matsuyama (1987) includes convex capital adjustment costs. Reinhart (1999) uses the Blanchard framework in a study of endogenous productivity growth. Blanchard (1985), Calvo and Obstfeld (1988), Blanchard and Fischer (1989), and Klundert and Ploeg (1989) apply the framework for studies of fiscal policy and government debt. These last issues will be the topic of the next chapter.

12.8 Appendix

A. Actuarially fair life insurance

Negative life insurance A life annuity contract is defined as *actuarially fair* if it offers the investor the same expected unconditional rate of return as a safe bond. We now check whether the life annuity contracts in equilibrium of the Blanchard model have this property. For simplicity, we assume that the risk-free interest rate is a constant, r.

Buying a life annuity contract at time t means that the depositor invests one unit of account at time t in such a contract. In return the depositor receives a conditional continuous flow of receipts equal to $r+\hat{m}$ per time unit until death. At death the invested unit of account is lost from the point of view of the depositor (or rather the estate of the depositor). The time of death is stochastic, and so the unconditional rate of return, R, is a stochastic variable. Given the constant and age-independent mortality rate m, the expected unconditional return in the short time interval $[t, t+\Delta t)$ is approximately $(r+\hat{m})\Delta t(1-m\Delta t) - 1 \cdot m\Delta t$, where $m\Delta t$ is the approximate probability of dying within the time interval $[t, t+\Delta t)$ and $1-m\Delta t$ is the approximate probability of surviving. The expected unconditional *rate* of return, ER, is the expected return *per time unit* per unit of account invested. Thus,

$$ER \approx \frac{(r+\hat{m})\Delta t(1-m\Delta t) - m\Delta t}{\Delta t} = (r+\hat{m})(1-m\Delta t) - m. \tag{12.73}$$

In the limit for $\Delta t \to 0$, we get $ER = r + \hat{m} - m$. In equilibrium, as shown in Section 12.2.1, $\hat{m} = m$ and so $ER = r$. This shows that the life annuity contracts in equilibrium are actuarially fair.

Positive life insurance To put negative life insurance in perspective, we also considered positive life insurance. We claimed that the charge of $\tilde{m}$ per time unit until death on a positive life insurance contract must in equilibrium equal the death rate, m. This can be shown in the following way. The contract stipulates that the depositor pays the insurance company a premium of $\tilde{m}$ units of account per time unit until death. In return, at death the estate of the deceased person receives one unit of account from the insurance company. The expected revenue obtained by the insurance company on such a contract in the short time interval $[t, t+\Delta t)$ is approximately $\tilde{m}\Delta t(1-m\Delta t) + 0 \cdot m\Delta t$. In the absence of administration costs the expected cost is approximately $0 \cdot (1-m\Delta t) + 1 \cdot m\Delta t$. We find the expected profit *per time unit* to be

$$E\pi \approx \frac{\tilde{m}\Delta t(1-m\Delta t) - m\Delta t}{\Delta t} = \tilde{m} - \tilde{m}m\Delta t - m.$$

In the limit for $\Delta t \to 0$ we get $E\pi = \tilde{m} - m$. Equilibrium with free entry and exit requires $E\pi = 0$, hence $\tilde{m} = m$, as was to be shown.

Like the negative life insurance contract, the positive life insurance contract is said to be *actuarially fair* if it offers the investor (now the insurance company) the same expected unconditional rate of return as a safe bond. In equilibrium it does so. We see this by replacing $\hat{m}$ by $\tilde{m}$ and applying the argument leading to (12.73) once more, this time from the point of view of the insurance company. At time t the insurance company makes a demand deposit of one unit of account in the financial market (or buys a short-term bond) and at the same time contracts to pay one unit of account to a customer at death in return for a flow of contributions, $\tilde{m}$, per time unit from the customer until death. The payout of one unit of account to the estate of the deceased person is financed by cashing the demand deposit (or stopping reinvesting in short-term bonds). Since in equilibrium $\tilde{m} = m$, the conclusion is that $ER = r$.

Age-dependent mortality rates Let X denote the age at death of an individual. Ex ante X is a stochastic variable. Then the *instantaneous mortality rate* for a person of age x, also called the *hazard rate* of death at age x, is defined as

$$m(x) = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x)/\Delta x}{P(X > x)} = \frac{f(x)}{1 - F(x)}, \tag{12.74}$$

where $1 - F(x)$ is the survival function, that is, the unconditional probability of becoming at least x years old. The associated cumulative distribution function is $F(x)$ (the probability of dying before age x). And $f(x) = F'(x)$ is the probability density function. Empirically the *instantaneous mortality rate* is an increasing function of age, x. The classical Gompertz-Makeham formula specifies $m(x)$ as $m(x) = \mu_0 + \mu_1 e^{\mu_2 x}$, where $\mu_0 > 0, \mu_1 > 0$, and $\mu_2 > 0$.

The Blanchard assumption of an age independent mortality rate is the case $\mu_1 = 0$ so that $m(x) = \mu_0 \equiv m$. Indeed, the Blanchard model assumes $1 - F(x) = e^{-mx}$ so that, by (12.74) with $f(x) = me^{-mx}$, we have $m(x) = me^{-mx}/e^{-mx} = m$, a constant.

B. Present value of expected future labor income

Here we show, for the case without retirement ($\lambda = 0$), that the present value, $h(v, t)$, of an individual's expected future labor income can be written as in (12.12) (or as in (12.10) if $t = 0$). For the case with no retirement we have

$$h(v, t) \equiv E_t \int_t^{t+X} w(s) e^{-\int_t^s r(\tau) d\tau} ds, \tag{12.75}$$

where X stands for remaining lifetime, a stochastic variable. The rate of discount for future labor income *conditional on being alive at the moment concerned* is the risk-free interest rate r.

Now, consider labor income of the individual at time $s > t$ as a stochastic variable, $Z(s)$, with two different possible outcomes:

$$Z(s) = \begin{cases} w(s), & \text{if still alive at time } s \\ 0, & \text{if dead at time } s. \end{cases}$$

Then we can rewrite (12.75) as

$$\begin{aligned} h(v, t) &= E_t \int_t^{\infty} Z(s) e^{-\int_t^s r(\tau) d\tau} ds = \int_t^{\infty} E_t(Z(s)) e^{-\int_t^s r(\tau) d\tau} ds \\ &= \int_t^{\infty} w(s) P(X > s - t) + 0 \cdot P(X \leq s - t) e^{-\int_t^s r(\tau) d\tau} ds \\ &= \int_t^{\infty} w(s) e^{-m(s-t)} e^{-\int_t^s r(\tau) d\tau} ds = \int_t^{\infty} w(s) e^{-\int_t^s (r(\tau)+m) d\tau} ds. \end{aligned} \tag{12.76}$$

This confirms (12.12). When we discount the *potential* labor income in all future, the relevant discount rate is the actuarial rate of interest, i.e., the risk-free interest rate *plus* the death rate.

C. Aggregate dynamics

C.1. Aggregate dynamics in the perpetual youth model (Section 12.2)

In Section 12.2.2 we gave an intuitive explanation of why aggregate financial wealth and aggregate consumption follow the rules

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t), \tag{*}$$

and

$$\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t), \tag{**}$$

respectively. Here we will prove these equations, appealing to *Leibniz's formula* for differentiating an integral with respect to a parameter appearing both in the integrand and in the limits of integration.

Leibniz's formula Let $f(v,t)$ and $f_t(v,t)$ be continuous. Suppose too that $g(t)$ and $h(t)$ are differentiable. Then

$$\begin{aligned} F(t) &= \int_{g(t)}^{h(t)} f(v,t)dv \Rightarrow \\ F'(t) &= f(t,h(t))h'(t) - f(t,g(t))g'(t) + \int_{g(t)}^{h(t)} \frac{\partial f(v,t)}{\partial t} dv. \end{aligned}$$

For proof, see, e.g., Sydsæter and Hammond (2008). In case $g(t) = -\infty$, one should replace $g'(t)$ by 0. Similarly, if $h(t) = +\infty$, $h'(t)$ should be replaced by 0.

Proof of (*) The intuitive validity of the accounting rule (*) notwithstanding, we cannot be sure that our concepts and book-keeping are consistent, until we have provided a proof.

Aggregate financial wealth is

$$A(t) = \int_{-\infty}^{t} a(v,t)N(0)e^{nv}be^{-m(t-v)}dv, \tag{12.77}$$

where we have inserted (12.2) into (12.14). Using Leibniz's formula with $g'(t) = 0$ and $h'(t) = 1$, we get

$$\begin{aligned}
\dot{A}(t) &= a(t,t)N(0)e^{nt}b - 0 + \int_{-\infty}^{t} N(0)b\frac{\partial}{\partial t}[a(v,t)e^{nv}e^{-m(t-v)}]dv \\
&= 0 - 0 + N(0)b\int_{-\infty}^{t} e^{nv}[a(v,t)e^{-m(t-v)}(-m) + \frac{\partial a\left(v,t\right)}{\partial t}e^{-m(t-v)}]dv \\
&= -mN(0)b\int_{-\infty}^{t} a(v,t)e^{nv}e^{-m(t-v)}dv + N(0)b\int_{-\infty}^{t} \frac{\partial a\left(v,t\right)}{\partial t}e^{nv}e^{-m(t-v)}dv,
\end{aligned}$$

where the second equality comes from $a(t,t) = 0$, which is due to the absence of bequests. Inserting (12.77), this can be written

$$\dot{A}(t) = -mA(t) + N(0)b\int_{-\infty}^{t} \frac{\partial a\left(v,t\right)}{\partial t}e^{nv}e^{-m(t-v)}dv. \tag{12.78}$$

Thus, the increase per time unit in aggregate financial wealth equals the "intake" (i.e., the increase in financial wealth of those still alive) minus the "discharge" due to death, $mN(t)A(t)/N(t) = mA(t)$.

In (12.78) the term $\partial a\left(v,t\right)/\partial t$ stands for the increase per time unit in financial wealth of an individual born at time v and still alive at time t. By definition this is the same as the saving of the individual, hence the same as income minus consumption. Thus, $\partial a\left(v,t\right)/\partial t = (r(t)+m)a(v,t) + w(t) - c(v,t)$. Substituting this into (12.78) gives

$$\begin{aligned}
\dot{A}(t) &= -mA(t) + N(0)b\left[(r(t)+m)\int_{-\infty}^{t} a(v,t)e^{nv}e^{-m(t-v)}dv\right. \\
&\qquad \left. +w(t)\int_{-\infty}^{t} e^{nv}e^{-m(t-v)}dv - \int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv\right] \\
&= -mA(t) + (r(t)+m)A(t) + w(t)N(t) - C(t), \qquad (12.79)
\end{aligned}$$

by (12.77), (12.4), and (12.13). Reducing the last expression in (12.79) and noting that $N(t) = L(t)$ gives (*). $\square$

We can prove (**) in a similar way:

Proof of (**) The PV, as seen from time t, of future labor income of any individual is as in (12.76), since labor income is independent of age in this simple version of the Blanchard model. Hence, aggregate human wealth is

$$H(t) = N(t)\bar{h}(t) = N(0)e^{nt}\int_{t}^{\infty} w\left(s\right)e^{-\int_t^s (r(\tau)+m)d\tau}ds. \tag{12.80}$$

After substituting (12.3) into (12.13), differentiation w.r.t. t (use again Leibniz's formula with $g'(t) = 0$ and $h'(t) = 1$) gives

$$\begin{aligned}
\dot{C}(t) &= c(t,t)N(0)e^{nt}b - 0 + \int_{-\infty}^{t} N(0)b\frac{\partial}{\partial t}[c(v,t)e^{nv}e^{-m(t-v)}]dv \\
&= c(t,t)N(t)b - 0 + N(0)b\int_{-\infty}^{t} e^{nv}[-c(v,t)e^{-m(t-v)}m + \frac{\partial c\,(v,t)}{\partial t}e^{-m(t-v)}]dv \\
&= (\rho+m)\bar{h}(t)N(t)b - mN(0)b\int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv \\
&\quad +N(0)b\int_{-\infty}^{t} \frac{\partial c\,(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv, \qquad (12.81)
\end{aligned}$$

where the last equality derives from the fact that the consumption function for an individual born at time v is $c(v,t) = (\rho+m)\left[a(v,t) + \bar{h}(t)\right]$, which for $v = t$ takes the form $c(t,t) = (\rho+m)\bar{h}(t)$, since $a(t,t) = 0$. Using (12.80) and (12.13) in (12.81) yields

$$\dot{C}(t) = b(\rho+m)H(t) - mC(t) + N(0)b\int_{-\infty}^{t} \frac{\partial c\,(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv. \qquad (12.82)$$

From the Keynes-Ramsey rule we have $\partial c\,(v,t)\,/\partial t = (r(t)-\rho)c(v,t)$. Substituting this into (12.82) gives

$$\begin{aligned}
\dot{C}(t) &= b(\rho+m)H(t) - mC(t) + N(0)b(r(t)-\rho)\int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv \\
&= b(\rho+m)H(t) - mC(t) + (r(t)-\rho)C(t) \\
&= b(\rho+m)H(t) - bC(t) + nC(t) + (r(t)-\rho)C(t) \\
&= b(\rho+m)H(t) - b(\rho+m)(A(t)+H(t)) + (r(t)-\rho+n)C(t) \\
&= (r(t)-\rho+n)C(t) - b(\rho+m)A(t),
\end{aligned}$$

where the second equality comes from (12.13), the third from $n \equiv b - m$, and the fourth from the aggregate consumption function, $C(t) = (\rho+m)(A(t)+H(t))$. Hereby we have derived (**). □

A more direct proof of (**) An alternative and more direct way of proving (**) may be of interest also in other contexts. The aggregate consumption function immediately gives

$$\dot{C}(t) = (\rho+m)(\dot{A}(t) + \dot{H}(t)). \qquad (12.83)$$

Differentiation of (12.80) w.r.t. t (using Leibniz's Formula with $g'(t) = 1$ and $h'(t) = 0$) gives

$$\begin{aligned}
\dot{H}(t) &= \dot{N}(t)\bar{h}(t) + N(t)\left[-w(t) + \int_t^\infty w\left(s\right) e^{-\int_t^s (r(\tau)+m)d\tau}(r(t)+m)ds\right] \\
&= nH(t) - w(t)N(t) + (r(t)+m)N(t)\int_t^\infty w\left(s\right) e^{-\int_t^s (r(\tau)+m)d\tau}ds \\
&= (r(t)+m+n)H(t) - w(t)N(t),
\end{aligned}$$

where the two last equalities follow from (12.80) and (12.76), respectively. Inserting this together with (*) into (12.83) gives

$$\begin{aligned}
\dot{C}(t) &= (\rho+m)\left[r(t)A(t) + w(t)N(t) - C(t) + (r(t)+m+n)H(t) - w(t)N(t)\right] \\
&= (\rho+m)\left[r(t)A(t) - C(t) + (r(t)+m+n)(\frac{C(t)}{\rho+m} - A(t))\right] \\
&= (\rho+m)r(t)A(t) - (\rho+m)C(t) + (r(t)+m+n)C(t) \\
&\quad -(\rho+m)(r(t)+b)A(t) \\
&= (r(t)-\rho+n)C(t) - b(\rho+m)A(t),
\end{aligned}$$

where the second equality comes from the aggregate consumption function and the third from $b \equiv m+n$. $\square$

C.2. Aggregate dynamics in the model with retirement (Section 12.3-4)

(no text currently available)

D. Transversality conditions and why the diverging paths cannot be equilibrium paths

In Section 12.2.4 we claimed that for every $t_0 \geq 0$ and every $v \leq t_0$, the transversality condition (12.28) is satisfied at the steady-state point E in Fig. 12.4. The following lemma is a key step in proving this.

LEMMA D1. At the steady-state point E, for fixed v, $a(v,t)$ ultimately grows at the rate $r^* - \rho$ if lifetime allows.

Proof. We have

$$
\begin{aligned}
\frac{\partial a(v,t)/\partial t}{a(v,t)} &= \frac{(r(t)+m)a(v,t)+w(t)-c(v,t)}{a(v,t)} \\
&= r(t)+m+\frac{w(t)-(\rho+m)(a(v,t)+\bar{h}(t))}{a(v,t)} \\
&= r(t)-\rho+\frac{w(t)-(\rho+m)\bar{h}(t)}{a(v,t)} \\
&= r(t)-\rho+\frac{\tilde{w}(\tilde{k}(t))-(\rho+m)\tilde{h}(t)}{a(v,t)}T(t). \qquad (12.84)
\end{aligned}
$$

In a small neighborhood of the steady state, where $\tilde{k}(t) \approx \tilde{k}^*$ and $\tilde{h}(t) \equiv \bar{h}(t)/T(t) \approx \tilde{h}^*$, cf. (12.37), the right-hand side of (12.84) can be approximated by

$$
\begin{aligned}
& r^* - \rho + \frac{\left[\tilde{w}(\tilde{k}^*) - (\rho+m)\frac{\tilde{w}(\tilde{k}^*)}{r^*+m-g}\right]T(t)}{a(v,t)} \\
= \; & r^* - \rho + \frac{(r^*-\rho-g)\tilde{w}(\tilde{k}^*)T(t)}{(r^*+m-g)a(v,t)} > r^* - \rho > g, \qquad (12.85)
\end{aligned}
$$

where both inequalities come from (12.35). Thus, at least close to the steady state, $a(v,t)$ grows at a higher rate than technology. It follows that for an imaginary person with infinite lifetime, $T(t)/a(v,t) \to 0$ for $t \to \infty$, so that, by (12.84), $(\partial a(v,t)/\partial t)/a(v,t) \to r^* - \rho$ for $t \to \infty$, as was to be shown. □

At E the discount factor in (12.28) becomes $e^{-(r^*+m)(t-t_0)}$, where $r^* = f'(\tilde{k}^*) - \delta$. In view of $\rho \geq 0$ and $m > 0$, we have $r^* - \rho < r^* + m$. From this, in combination with Lemma D1, follows that all the transversality conditions, (12.28), hold at the steady-state point E and hence also along any path converging to that steady-state point.[29]

Note that, by (12.85), even the limiting value of $\partial a(v,t)/\partial t)/a(v,t)$, $r^* - \rho$, is higher than g. Thus, due to the generation replacement effect, individual financial wealth tends to grow *faster* than average wealth, $A(t)/N(t)$, which for $\lambda = 0$ equals K/L and in the long run grows at the rate g. This explains why transversality conditions can not be checked in the same simple way as in the Ramsey model.

In the text of Section 12.2.4 we also claimed that all the *diverging* paths in the phase diagram of Fig. 12.4 violate the individual transversality conditions (12.28). Let us first explain why in Fig. 12.4 paths which start from below the

[29] The argument can be extended to the case with retirement ($\lambda > 0$) and to the small open economy (even in the case with "low impatience").

stable arm tend to *cross* the $\tilde{k}$-axis. The slope of any path generated by the differential equations for $\tilde{c}$ and $\tilde{k}$, is

$$\frac{d\tilde{c}}{d\tilde{k}} = \frac{d\tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}}{\dot{\tilde{k}}} = \frac{\left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho + m)\tilde{k}}{f(\tilde{k}) - \tilde{c} - (\delta + g + b - m)\tilde{k}}, \tag{12.86}$$

whenever $\dot{\tilde{k}} \neq 0$. Close to the $\tilde{k}$-axis, this slope is positive for $0 < \tilde{k} < \bar{\bar{\tilde{k}}}$ and negative for $\tilde{k} > \bar{\bar{\tilde{k}}}$ (to see this, put $\tilde{c} \approx 0$ in (12.86)). Even *at* the $\tilde{k}$-axis, this holds true. Such paths violate the individual transversality conditions. Indeed, along these paths consumption will in finite time be zero at the same time as both financial wealth and human wealth are far from zero. This indicates that people consume less than what their intertemporal budget constraint allows, which is equivalent to the transversality condition not being satisfied: people over-accumulate. Any individual expecting an evolution of w and r implied by such a path will *deviate* from the consumption level along the path by choosing higher consumption. Hence, the path can not be a perfect foresight equilibrium path.

What about paths starting from above the stable arm in Fig. 12.4? These paths will violate the NPG condition of the individuals (the argument is similar to that used for the Ramsey model in Appendix C of Chapter 10). An individual expecting an evolution of w and r implied by such a path will *deviate* from the consumption level along the path by choosing a *lower* consumption level in order to remain solvent, i.e., comply with the NPG condition.

E. Saddle-point stability in the perpetual youth model

First, we construct the Jacobian matrix of the right-hand sides of the differential equations (12.26) and (12.27), that is, the matrix

$$J(\tilde{k}, \tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta + g + b - m) & -1 \\ f''(\tilde{k})\tilde{c} - b(\rho + m) & f'(\tilde{k}) - \delta - \rho - g \end{bmatrix}$$

The determinant, evaluated at the steady state point $(\tilde{k}^*, \tilde{c}^*)$, is $\det(J(\tilde{k}^*, \tilde{c}^*))$

$$\begin{aligned} &= \left[f'(\tilde{k}^*) - (\delta + g + b - m)\right]\left[f'(\tilde{k}^*) - \delta - \rho - g\right] + \left[f''(\tilde{k}^*)\tilde{c}^* - b(\rho + m)\right] \\ &= \left(\frac{f'(\tilde{k}^*) - (\delta + g + b - m)}{\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + b - m)} - 1\right) b(\rho + m) + f''(\tilde{k}^*)\tilde{c}^* < f''(\tilde{k}^*)\tilde{c}^* < 0, \end{aligned}$$

where the second equality follows from (12.32) whereas the last inequality follows from $f'' < 0$ and the second last from $f'(\tilde{k}^*) < f(\tilde{k}^*)/\tilde{k}^*$.[30] Since the determinant is negative, $J(\tilde{k}^*, \tilde{c}^*)$ has one positive and one negative eigenvalue. Hence the steady state is a saddle point.[31] The remaining needed conditions for (local) saddle-point stability include that the saddle path should not be parallel to the jump-variable axis. This condition holds here, since if the saddle path were parallel to the jump-variable axis, the element in the first row and last column of $J(\tilde{k}^*, \tilde{c}^*)$ would vanish, which it never does here. The remaining conditions for saddle-point stability were confirmed in the text.

An argument analogue to that for the Ramsey model, see Appendix A to Chapter 10, shows that the steady state is in fact *globally* saddle-point stable.

F. The upper bound for r^* in the Blanchard model with retirement

An algebraic proof of the right-hand inequality in (12.54) is given here. Because f satisfies the Inada conditions and $f'' < 0$, the equation

$$f'(\tilde{k}) - \delta = \rho + g + b$$

has a unique solution in $\tilde{k}$. Let this solution be denoted $\underline{\tilde{k}}$.

Since the inequality $r^* < \rho + g + b$ is equivalent with $\underline{\tilde{k}} < \tilde{k}^*$, it is enough to prove the latter inequality. Suppose that on the contrary we have $\underline{\tilde{k}} \geq \tilde{k}^*$. Then, $f'(\tilde{k}^*) - \delta \geq \rho + g + b$ and there exists an $\varepsilon \geq 0$ such that

$$f'(\tilde{k}^*) = \delta + \rho + g + b(1 + \varepsilon). \tag{12.87}$$

This equation is equivalent to

$$\rho + m = f'(\tilde{k}^*) - \delta - g - b(1 + \varepsilon) + m = f'(\tilde{k}^*) - \delta - g - n - b\varepsilon, \tag{12.88}$$

since $n \equiv b - m$. In steady state

$$\tilde{c}^* = \frac{b}{\lambda + b}\left[f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*\right] = \frac{b\,(\rho + m)\,\tilde{k}^*}{f'(\tilde{k}^*) - \delta - \rho - g + \lambda},$$

which implies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + n) = \frac{(\lambda + b)\,(\rho + m)}{f'(\tilde{k}^*) - \delta - \rho - g + \lambda}.$$

[30] To convince yourself of this last inequality, draw a graph of the function $f(\tilde{k})$, reflecting the properties $f' > 0$, $f'' < 0$, and $f(0) \geq 0$.

[31] This also holds in the case with retirement, $\lambda > 0$.

Inserting (12.87) on the right-hand side gives

$$\begin{aligned}
\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + n) &= \frac{(\lambda + b)(\rho + m)}{b(1+\varepsilon) + \lambda} \\
&= \frac{\lambda + b}{b(1+\varepsilon) + \lambda}(f'(\tilde{k}^*) - \delta - g - n - b\varepsilon) \quad \text{(from 12.88))} \\
&\leq f'(\tilde{k}^*) - \delta - g - n - b\varepsilon \qquad \text{(since } \varepsilon \geq 0\text{)}.
\end{aligned}$$

This inequality implies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} \leq f'(\tilde{k}^*) - b\varepsilon.$$

But this last inequality is impossible because of strict concavity of f. Indeed, $f'' < 0$ together with $f(0) = 0$ implies $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$ for all $\tilde{k} > 0$. Thus, from the assumption that $\underline{\tilde{k}} \geq \tilde{k}^*$ we arrive at a contradiction; hence, the assumption must be rejected. It follows that $\underline{\tilde{k}} < \tilde{k}^*$. □

12.9 Exercises

19.10.2016. Christian Groth

A glimpse of theory of the "level of interest rates"

This short note provides a brief sketch of what macroeconomics says about the general level around which rates of return fluctuate. We also give a brief summary of different circumstances that give rise to differences in rates of return on different assets.

In non-monetary models without uncertainty there is in equilibrium only one rate of return, r. If in addition there is: a) perfect competition in all markets, b) the consumption good is physically indistinguishable from the capital good, and c) there are no capital adjustment costs, as in simple neoclassical models (like the Diamond OLG model and the Ramsey model), then the equilibrium real interest rate is at any time equal to the current net marginal productivity of capital evaluated at full employment ($r = \partial Y/\partial K - \delta$ in standard notation). Moreover, under conditions ensuring "well-behavedness" of these models, they predict that in the absence of disturbances, the technology-corrected capital-labor ratio, and thereby the marginal productivity of capital, adjusts over time to some long-run level (on which more below).

Different rates of return In simple neoclassical models with perfect competition and no uncertainty, the equilibrium short-term real interest rate is at any time equal to the net marginal productivity of capital ($r = \partial Y/\partial K - \delta$). In turn the marginal productivity of capital adjusts over time, via changes in the capital intensity, to some long-run level (on this more below). As we saw in Chapter 14, existence of convex *capital installation costs* loosens the link between r and $\partial Y/\partial K$. The convex adjustment costs create a wedge between the price of investment goods and the market value of the marginal unit of installed capital.

When *imperfect competition* in the output markets rules, prices are typically set as a mark-up on marginal cost. This implies a wedge between the net marginal productivity of capital and capital costs. And when *uncertainty* and limited opportunities for risk

	Arithmetic average	Standard deviation	Geometric average
	---------------- Percent ----------------		
Nominal values			
Small Company Stocks	17,3	33,2	12,5
Large Company Stocks	12,7	20,2	10,7
Long-Term Corporate Bonds	6,1	8,6	5,8
Long-Term Government Bonds	5,7	9,4	5,3
Intermediate-Term Government Bonds	5,5	5,7	5,3
U.S. Treasury Bills	3,9	3,2	3,8
Cash	0,0	0,0	0,0
Inflation rate	3,1	4,4	3,1
Real values			
Small Company Stocks	13,8	32,6	9,2
Large Company Stocks	9,4	20,4	7,4
Long-Term Corporate Bonds	3,1	9,9	2,6
Long-Term Government Bonds	2,7	10,6	2,2
Intermediate-Term Government Bonds	2,5	7,0	2,2
U.S. Treasury Bills	0,8	4,1	0,7
Cash	-2,9	4,2	-3,0

Table 1: Average annual rates of return on a range of U.S. asset portfolios, 1926-2001. Source: Stocks, Bonds, Bills, and Inflation: Yearbook 2002, Valuation Edition. Ibbotson Associates, Inc.

diversification are added to the model, a wide spectrum of expected rates of return on different financial assets and expected marginal productivities of capital in different production sectors arise, depending on the risk profiles of the different assets and production sectors. On top of this comes the presence of taxation which may complicate the picture because of different tax rates on different asset returns.

Nominal and real average annual rates of return on a range of U.S. asset portfolios for the period 1926–2001 are reported in Table 1. By a *portfolio* of n assets, $i = 1, 2, \ldots, n$ is meant a "basket", $(v_1, v_2, \ldots, v_n)$, of the n assets in value terms, that is, $v_i = p_i x_i$ is the value of the investment in asset i, the price of which is denoted p_i and the quantity of which is denoted x_i. The total investment in the basket is $V = \sum_{i=1}^n v_i$. If R_i denotes the gross rate of return on asset i, the overall gross rate of return on the portfolio is

$$R = \frac{\sum_i^n v_i R_i}{V} = \sum_{i=1}^n w_i R_i,$$

where $w_i \equiv v_i/V$ is the *weight* or *fraction* of asset i in the portfolio. Defining $R_i \equiv 1 + r_i$, where r_i is the net rate of return on asset i, the net rate of return on the portfolio can be written

$$r = R - 1 = \sum_{i=1}^{n} w_i(1 + r_i) - 1 = \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i r_i - 1 = \sum_{i=1}^{n} w_i r_i.$$

The net rate of return is often just called "the rate of return".

In Table 1 we see that the portfolio consisting of small company stocks throughout the period 1926-2001 had an average annual real rate of return of 13.8 per cent (the arithmetic average) or 9.2 per cent (the geometric average, assuming annual compounding). This is more than the annual rate of return of any of the other considered portfolios. Small company stocks are also seen to be the most volatile. The standard deviation of the annual real rate of return of the portfolio of small company stocks is almost eight times higher than that of the portfolio of U.S. Treasury bills (government zero coupon bonds with 30 days to maturity), with an average annual real return of only 0.8 per cent (arithmetic average) or 0.7 per cent (geometric average) throughout the period. The displayed positive relation between high returns and high volatility is not without exceptions, however. The portfolio of long-term corporate bonds has performed better than the portfolio of long-term government bonds, although they have been slightly less volatile as here measured. The data is historical and expectations are not always met. Moreover, risk depends significantly on the *covariance* of asset returns within the total set of assets and specifically on the correlation of asset returns with the business cycle, a feature that can not be read off from Table 1. Share prices, for instance, are very sensitive to business cycle fluctuations.

The need for means of payment − money − is a further complicating factor. That is, besides dissimilarities in risk and expected return across different assets, also dissimilarities in their degree of liquidity are important, not least in times of financial crisis. The expected real rate of return on cash holding is minus the expected rate of inflation and is therefore negative in an economy with inflation, cf. the last row in Table 1. When agents nevertheless hold cash in their portfolios, it is because the low rate of return is compensated by the *liquidity* services of money. In the Sidrauski model of Chapter 17 this is modeled in a simple way, albeit ad hoc, by including real money holdings directly as an argument in the utility function. Another dimension along which the presence of money interferes with returns is through inflation. Real assets, like physical capital, land, houses, etc. are better protected against fluctuating inflation than are nominally denominated bonds (and money of course).

Without claiming too much we can say that investors facing such a spectrum of rates of return choose a composition of assets so as to balance the need for liquidity, the wish for a high expected return, and the wish for low risk. Finance theory teaches us that adjusted for differences in risk and liquidity, asset returns tend to be the same. This raises the question: at what level? This is where macroeconomics – as an empirically oriented theory about the economy as a whole – comes in.

Macroeconomic theory of the "average rate of return" The point of departure is that market forces by and large may be thought of as anchoring the rate of return of an average portfolio of interest-bearing assets to the net marginal productivity of capital in an aggregate production function, assuming a closed economy. Some popular phrases are:

- the net marginal productivity of capital acts as a centre of gravitation for the spectrum of asset returns; and
- movements of the rates of return are in the long run held in check by the net marginal productivity of capital.

Though such phrases seem to convey the right flavour, in themselves they are not very informative. The net marginal productivity of capital is not a given, but an endogenous variable which, via changes in the capital intensity, adjusts through time to more fundamental factors in the economy.

The different macroeconomic models we have encountered in previous chapters bring to mind different presumptions about what these fundamental factors are.

1. Solow's growth model The Solow growth model leads to the fundamental differential equation (standard notation)

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t,$$

where s is an exogenous and constant aggregate saving-income ratio, $0 < s < 1$. In steady state

$$r^* = f'(\tilde{k}^*) - \delta, \tag{1}$$

where $\tilde{k}^*$ is the unique steady state value of the (effective) capital intensity, $\tilde{k}$, satisfying

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{2}$$

In society there is a debate and a concern that changed demography and less growth in the source of new technical ideas, i.e., the stock of educated human beings, will in the future result in lower n and lower g, respectively, making financing social security more difficult. On the basis of the Solow model we find by implicit differentiation in (2) $\partial\tilde{k}^*/\partial n = \partial\tilde{k}^*/\partial g = -\tilde{k}^*\left[\delta+g+n-sf'(\tilde{k}^*)\right]^{-1}$, which is negative since $sf'(\tilde{k}^*) < sf(\tilde{k}^*)/\tilde{k}^* = \delta+g+n$. Hence, by (1),

$$\frac{\partial r^*}{\partial n} = \frac{\partial r^*}{\partial g} = \frac{\partial r^*}{\partial \tilde{k}^*}\frac{\partial \tilde{k}^*}{\partial n} = f''(\tilde{k}^*)\frac{-\tilde{k}^*}{\delta+g+n-sf'(\tilde{k}^*)} > 0,$$

since $f''(\tilde{k}^*) < 0$. It follows that

$$n\downarrow \text{ or } g\downarrow \Rightarrow r^*\downarrow. \tag{3}$$

A limitation of this theory is of course the exogeneity of the saving-income ratio, which is a key co-determinant of $\tilde{k}^*$, hence of r^*. The next models are examples of different ways of integrating a theory of saving into the story about the long-run rate of return.

2. The Diamond OLG model In the Diamond OLG model, based on a life-cycle theory of saving, we again arrive at the formula $r^* = f'(\tilde{k}^*) - \delta$. Like in the Solow model, the long-run rate of return thus depends on the aggregate production function and on $\tilde{k}^*$. But now there is a logically complete theory about how $\tilde{k}^*$ is determined. In the Diamond model $\tilde{k}^*$ depends in a complicated way on the lifetime utility function and the aggregate production function. The steady state of a well-behaved Diamond model will nevertheless have the same qualitative property as indicated in (3).

3. The Ramsey model Like the Solow and Diamond models, the Ramsey model implies that $r_t = f'(\tilde{k}_t) - \delta$ for all t. But unlike in the Solow and Diamond models, the net marginal productivity of capital now converges in the long run to a specific value given by the *modified golden rule* formula. In a continuous time framework this formula says:

$$r^* = \rho + \theta g, \tag{4}$$

where the new parameter, θ, is the (absolute) elasticity of marginal utility of consumption. Because the Ramsey model is a representative agent model, the Keynes-Ramsey rule holds not only at the individual level, but also at the aggregate level. This is what gives rise to this simple formula for r^*.

Here there is no role for n, only for g. On the other hand, there is an alternative specification of the Ramsey model, namely the "average utilitarianism" specification. In this version of the model, we get $r^* = f'(\tilde{k}^*) - \delta = \rho + n + \theta g$, so that not only a lower g, but also a lower n implies lower r^*.

Also the Sidrauski model, i.e., the monetary Ramsey model of Chapter 17, results in the *modified golden rule* formula (4).

4. Blanchard's OLG model A continuous time OLG model, and thereby a model emphasizing life-cycle aspects of economic behavior, is developed in Blanchard (1985). In that model the net marginal productivity of capital adjusts to a value where, in addition to the production function, technology growth, and preference parameters, also demographic parameters, like birth rate, death rate, and retirement rate, play a role. One of the results is that when $\theta = 1$,

$$\rho + g - \lambda < r^* < \rho + g + b,$$

where λ is the retirement rate (reflecting how early in life the "average" person retire from the labor market) and b is the (crude) birth rate. More precisely, we get

$$r^* = \psi(g, b, m, \rho, \lambda, \delta),$$

where the partial derivatives have the sign structure $(+, +, ?, +, ?, ?)$. The population growth rate is the difference between the birth rate, b, and the (crude) mortality rate, m, so that $n = b - m$. The qualitative property indicated in (3) becomes conditional. It still holds if the fall in n reflects a lower b, but not necessarily if it reflects a higher m.

5. What if technological change is embodied? The models in the list above assume a neoclassical aggregate production function with CRS and *disembodied* Harrod-neutral technological progress, that is,

$$Y_t = F(K_t, T_t L_t) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0. \tag{5}$$

This amounts to assuming that new technical knowledge advances the combined productivity of capital and labor *independently* of whether the workers operate old or new machines.

In contrast, we say that technological change is *embodied* if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Both intuition and empirics suggest that most technological progress is of this form. Indeed, Greenwood et al. (1997) estimate for the U.S. 1950-1990 that embodied technological change explains 60% of the growth in output per man hour.

So a theory of the rate of return should take this into account. Fortunately, this can be done with only minor modifications. We assume that the link between investment and capital accumulation takes the form

$$\dot{K}_t = Q_t I_t - \delta K_t, \tag{6}$$

where I_t is gross investment ($I = Y - C$) and Q_t measures the "quality" (efficiency) of newly produced investment goods. Suppose for instance that

$$Q_t = Q_0 e^{\gamma t}, \qquad \gamma > 0.$$

Then, even if no technological change directly appears in the production function, that is, even if (5) is replaced by

$$Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1,$$

the economy will still experience a rising standard of living.[1] A given level of gross investment will give rise to greater and greater additions to the capital stock K, measured in efficiency units. Since at time t, Q_t capital goods can be produced at the same cost as one consumption good, the price, p_t, of capital goods in terms of the consumption good must in competitive equilibrium equal the inverse of Q_t, that is, $p_t = 1/Q_t$. In this way embodied technological progress results in a steady decline in the relative price of capital equipment.

This prediction is confirmed by the data. Greenwood et al. (1997) find for the U.S. that the relative price of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, a trend that has seemingly been fortified in the wake of the computer revolution.

Along a balanced growth path the constant growth rate of K will now exceed that of Y, and Y/K thus be falling. The output-capital ratio in value terms, $Y/(pK)$, will be constant, however. Embedding these features in a Ramsey-style framework, we find the

[1]We specify F to be Cobb-Douglas, because otherwise a model with embodied technical progress in the form (6) will not be able to generate balanced growth and comply with Kaldor's stylized facts.

long-run rate of return to be[2]

$$r^* = \rho + \theta \frac{\alpha\gamma}{1-\alpha}.$$

This is of the same form as (4) since growth in output per unit of labor in steady state is exactly $g = \alpha\gamma/(1-\alpha)$.

6. Adding uncertainty and risk of bankruptcy Although absent from many simple macroeconomic models, uncertainty and risk of bankruptcy are significant features of reality. Bankruptcy risk may lead to a conflict of interest between share owners and managers. Managers may want less debt and more equity than the share owners because bankruptcy can be very costly to managers who loose a well-paid job and a promising carrier. So managers are unwilling to finance all new capital investment by new debt in spite of the associated lower capital cost (there is generally a lower rate of return on debt than on equity). In this way the excess of the rate of return on equity over that on debt, the equity premium, is sustained.

A rough behavioral theory of the equity premium goes as follows.[3] Firm managers prefer a payout structure with a fraction, s_f, going to equity and the remaining fraction, $1 - s_f$, to debt (corporate bonds). That is, out of each unit of expected operating profit, managers are unwilling to commit more than $1 - s_f$ to bond owners. This is to reduce the risk of a failing payment ability in case of a bad market outcome. And those who finance firms by loans definitely also want debtor firms to have some equity at stake.

We let households' preferred portfolio consist of a fraction s_h in equities and the remainder, $1 - s_h$, in bonds. In view of households' risk aversion and memory of historical stock market crashes, it is plausible to assume that $s_h < s_f$.

As a crude adaptation of for instance the Blanchard OLG model to these features, we interpret the model's r^* as an average rate of return across firms. Let time be discrete and let aggregate financial wealth be $A = pK$, where p is the price of capital equipment in terms of consumption goods. In the frameworks 1 to 4 above we have $p \equiv 1$, but in framework 5 the relative price p equals $1/Q$ and is falling over time. Anyway, given A at time t, the aggregate gross return or payout is $(1 + r^*)A$. Out of this, $(1 + r^*)As_f$ constitutes the gross return to the equity owners and $(1 + r^*)A(1 - s_f)$ the gross return

[2] See Exercise 18.??

[3] The following is inspired by Baker, DeLong, and Krugman (2005). These authors discuss the implied predictions for U.S. rates of return in the future and draw implications of relevance for the debate on social security reform.

to the bond owners. Let r_e denote the rate of return on equity and r_b the rate of return on bonds.

To find r_e and r_b we have

$$\begin{aligned}(1+r_e)As_h &= (1+r^*)As_f,\\ (1+r_b)A(1-s_h) &= (1+r^*)A(1-s_f).\end{aligned}$$

Thus,

$$\begin{aligned}1+r_e &= (1+r^*)\frac{s_f}{s_h} > 1+r^*,\\ 1+r_b &= (1+r^*)\frac{1-s_f}{1-s_h} < 1+r^*.\end{aligned}$$

We may define the *equity premium,* π, by $1+\pi \equiv (1+r_e)/(1+r_b)$. Then

$$\pi = \frac{s_f(1-s_h)}{s_h(1-s_f)} - 1 > 0.$$

Of course these formulas have their limitations. The key variables s_f and s_h will depend on a lot of economic circumstances and should be endogenous in an elaborate model. Yet, the formulas establish a way of organizing one's thoughts about rates of return in a world with asymmetric information and risk of bankruptcy.

There is evidence that in the last decades of the twentieth century the equity premium had become lower than in the long aftermath of the Great Depression in the 1930s.[4] A likely explanation is that s_h had gone up, along with rising confidence. Moreover, the computer and the World Wide Web have made it much easier for individuals to invests in stocks of shares. On the other hand, the recent financial and economic crisis, known as the Great Recession 2008- , and the associated rise in mistrust seems to have halted and possibly reversed this tendency for some time (source ??).

7. Stock market volatility and long swings A stylized fact of stock markets is that stock price indices are quite volatile on a month-to-month, year-to-year, and especially decade-to-decade scale. There are different views about how these swings should be understood. According to the *Efficient Market Hypothesis* the swings just reflect unpredictable changes in the "fundamentals", that is, changes in the present value of rationally expected future dividends. This is for instance the view of Nobel laureate Eugene Fama (1970, 2003) from University of Chicago.

[4] Blanchard (2003, p. 333).

In contrast, Nobel laureate Robert Shiller (1981, 2003, 2005) from Yale University, and others, have pointed to the phenomenon of *excess volatility.* The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Shiller's interpretation of the large stock market swings is that they are due to fads, herding, and shifts in fashions and "animal spirits" (the latter being a notion from Keynes).

In between these two views we have the *rational bubbles* hypothesis of Blanchard (1979, 1982). In Special Note 2 we return to this theme.

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Chapter 13

General equilibrium analysis of public and foreign debt

This chapter reviews long-run dynamics of public and foreign debt in the light of the continuous time OLG model of the previous chapter. Section 13.1 reconsiders the *Ricardian equivalence* issue. In Section 13.2 we extend the enquiry to a *general equilibrium analysis of budget deficits and debt dynamics* in a closed economy. Section 13.3 addresses general equilibrium aspects of public and foreign debt in a small open economy. Issues of *twin deficits* and the current account of a growing economy are considered. In Section 13.4 the assumption of lump-sum taxes is replaced by income taxation in order to examine the relationship between *debt and distortionary taxation.* The theme of *optimal debt* is addressed in Section 13.5, and the concluding Section 13.6 addresses the *time-inconsistency problem* faced by economic policy when outcomes depend on private sector expectations.

13.1 Reconsidering the issue of Ricardian equivalence

Recall that Ricardian equivalence is the claim that, given the (expected) future path of government spending, it does not matter for aggregate private consumption and saving whether the government finances its current spending by lump-sum taxes or borrowing. Whether this claim is an acceptable approximation or not is still a subject of debate among macroeconomists.

As we know from earlier chapters, the representative agent approach and the life-cycle-OLG approach lead to opposite conclusions regarding the issue. In models with a representative household with infinite horizon (the Barro and Ramsey dynasty models) a change in the timing of lump-sum taxes does not change the present value of the infinite stream of taxes imposed on the individual dynasty by a government satisfying its intertemporal budget constraint. A cut in current taxes is offset by the expected higher future taxes. Private saving goes up just as much as current taxes are reduced. This is exactly what is needed for paying the higher taxes in the future and maintain the preferred time path of consumption. Current consumption is thus not affected. And aggregate saving in society as a whole stays the same (the higher government dissaving being matched by higher private saving).

It is otherwise in the life-cycle-OLG models (without an operative Barro-style bequest motive). For instance the Diamond OLG model with a public sector reveals how taxes levied at different times are levied on different sets of agents. In the future some of the currently alive will be gone and there will be newcomers to bear part of the higher future tax burden. A current tax cut thus makes current tax payers feel wealthier and this leads to an increase in their current consumption. So current private consumption in the economy ends up higher. The present generations consequently benefit and future generations bear the cost in the form of smaller national wealth than otherwise.

Because of the more refined notion of time in the Blanchard OLG model from Chapter 12 and its capability of treating wealth effects more aptly, let us see what this model precisely says about the issue. A simple book-keeping exercise will show that the size of the public debt *does* matter. By affecting private wealth, it affects private consumption.

To keep things simple, we ignore retirement ($\lambda = 0$). To avoid notational confusion of the birth rate with the debt-income ratio, the former will in this chapter be denoted β while we still denote the latter by b. As in the previous chapters, B_t will denote net government debt, G_t government spending on goods and services, and T_t net tax revenue, $\tilde{T}_t - X_t$, where $\tilde{T}_t$ is gross tax revenue while X_t is transfers, all in real terms. We assume that the interest rate is in the long run higher than the output growth rate. Hence, to remain solvent the government has to satisfy its intertemporal budget constraint. Ignoring seigniorage and presupposing the government does not plan to procure more tax revenue than needed to satisfy its intertemporal budget constraint, as seen from time 0 (interpreted as "now"), we have the condition

$$\int_0^\infty T_t e^{-\int_0^t r_s ds} dt = \int_0^\infty G_t e^{-\int_0^t r_s ds} dt + B_0, \qquad \text{(GIBC)}$$

where the expected future time paths of G_t and r_t are considered given and B_0 is historically given. In brief, (GIBC) says that the present value of future net tax revenues must equal the sum of the present value of future spending on goods and services and the current level of debt. A temporary cut in taxes in an early time interval after time 0 must be offset in a later time interval by a rise in taxes of the same present value.

Given aggregate private financial wealth, A_0, and aggregate human wealth, H_0, aggregate private consumption is

$$C_0 = (\rho + m)(A_0 + H_0). \tag{13.1}$$

Because of the logarithmic specification of instantaneous utility, the propensity to consume out of wealth is a constant equal to the sum of the pure rate of time preference, ρ, and the mortality rate, m. Human wealth is the present value of expected future net-of-tax labor earnings of those currently alive:

$$H_0 = N_0 \int_0^\infty (w_t - \tau_t) e^{-\int_0^t (r_s + m) ds} dt. \tag{13.2}$$

Here, τ_t is the per capita lump-sum net taxation at time t, i.e., $\tau_t \equiv T_t/N_t \equiv (\tilde{T}_t - X_t)/N_t$, where N_t is the size of the population (here equal to the labor force, which in turn equals employment). The discount rate is the sum of the risk-free interest rate, r_t, and the actuarial compensation which is identical to the mortality rate, m.

To fix ideas, consider a closed economy. In view of the presence of government debt, aggregate private financial wealth in the closed economy is $A_0 = K_0 + B_0$, where K_0 is aggregate (private) physical capital and B_0 is assumed positive. Thus, (13.1) can be written

$$C_0 = (\rho + m)(K_0 + B_0 + H_0), \tag{13.3}$$

where ρ is the pure rate of time preference and m is the mortality rate. We ask whether B_0 is net wealth , for a given K_0, the sum $B_0 + H_0$ depends on the size of B_0, given the expected future path of G_t in (GIBC). We will see that the answer is yes. This is because, contrary to the Ricardian equivalence hypothesis, a higher B_0 is *not* offset by an equally reduced H_0 brought about by the higher future lump-sum taxes. Such a fully offsetting reduction of H_0 will not occur. Therefore C_0 *is* increased. Aggregate consumption depends positively on B_0.

The argument is the following. Rewrite (13.2) as

$$\begin{aligned}
H_0 &= N_0 \int_0^\infty \frac{w_t N_t - T_t}{N_t} e^{-\int_0^t (r_s + m) ds} dt \qquad \text{(from } \tau_t = T_t/N_t\text{)} \\
&= \int_0^\infty (w_t N_t - T_t) e^{-nt} e^{-\int_0^t (r_s + m) ds} dt \qquad \text{(since } N_0 = N_t e^{-nt}\text{)} \\
&= \int_0^\infty (w_t N_t - T_t) e^{-\int_0^t (r_s + n + m) ds} dt = \int_0^\infty (w_t N_t - T_t) e^{-\int_0^t (r_s + \beta) ds} dt,
\end{aligned}$$

using that the population growth rate, n, equals $\beta - m$. Therefore,

$$\begin{aligned} H_0 + B_0 &= \int_0^\infty (w_t N_t - T_t) e^{-\int_0^t (r_s+\beta)ds} dt + B_0 = \int_0^\infty (w_t N_t - G_t) e^{-\int_0^t (r_s+\beta)ds} dt \\ &\quad - \int_0^\infty (T_t - G_t) e^{-\int_0^t (r_s+\beta)ds} dt + B_0. \end{aligned} \tag{13.4}$$

Note that the first integral on the right-hand side of (13.4) is given (independent of a changed time profile of τ_t).

Reordering (GIBC), we have

$$B_0 = \int_0^\infty (T_t - G_t) e^{-\int_0^t r_s ds} dt. \tag{13.5}$$

Hence, the last line of (13.4) can be written

$$\begin{aligned} & - \int_0^\infty (T_t - G_t) e^{-\int_0^t (r_s+\beta)ds} dt + \int_0^\infty (T_t - G_t) e^{-\int_0^t r_s ds} dt \\ = & \int_0^\infty \left((T_t - G_t) e^{-\int_0^t r_s ds} - (T_t - G_t) e^{-\int_0^t r_s ds} e^{-\int_0^t \beta ds} \right) dt \\ = & \int_0^\infty (T_t - G_t) e^{-\int_0^t r_s ds} \left(1 - e^{-\int_0^t \beta ds} \right) dt. \end{aligned} \tag{13.6}$$

From (13.6) then follows

$$H_0 + B_0 = \int_0^\infty (w_t N_t - G_t) e^{-\int_0^t (r_s+\beta)ds} dt + \int_0^\infty (T_t - G_t) e^{-\int_0^t r_s ds} \left(1 - e^{-\int_0^t \beta ds} \right) dt. \tag{13.7}$$

There are two cases regarding the birth rate β to consider: $\beta = 0$ versus $\beta > 0$. The first case turns the Blanchard model into a representative agent model, since the second term on the right-hand side of (13.7) vanishes. Then the remaining term indicates that $H_0 + B_0$ is independent of the time profile of taxes. Only the given time path of G_t matters. A higher B_0 does not affect the $w_t N_t - G_t$ flow, and so the *sum* $H_0 + B_0$ is unaffected. The only effect of a higher B_0 is thus to make H_0 equally much lower so as to leave $H_0 + B_0$ unchanged. The case $\beta = 0$ thus implies Ricardian equivalence.

When $\beta > 0$ (positive birth rate), however, the second term on the right-hand side of (13.7) becomes decisive. In view of (13.5), the primary surplus, $T_t - G_t$, has to be positive for a substantial time interval, when $B_0 > 0$, the more so the larger is B_0. Consequently, the right-hand side of (13.7) is larger the larger is B_0. We conclude:

$$\begin{cases} H_0 + B_0 \text{ is independent of } B_0, \text{ if } \beta = 0, \text{ while} \\ H_0 + B_0 \text{ depends positively on } B_0, \text{ if } \beta > 0. \end{cases} \tag{13.8}$$

The intuition is that when the birth rate is positive, the tax burden in the future falls partly on new generations. Larger holdings of government bonds thus make the current generations feel wealthier in spite of future taxes being raised.

EXAMPLE Let $B_0 > 0$. Suppose T_0 is proportional to G_0 for all $t \geq 0$ with the factor of proportionality $1+\xi$. Then, inserting $T_0 = (1+\xi)G_0$ into (13.7) gives

$$H_0 + B_0 = \int_0^{\infty} (w_t N_t - G_t) e^{-\int_0^t (r_s+\beta)ds} dt + \xi \int_0^{\infty} G_t e^{-\int_0^t r_s ds} \left(1 - e^{-\int_0^t \beta ds}\right) dt,$$

which for $\beta > 0$ is an increasing function of ξ. In turn, ξ is an increasing function of B_0 because inserting $T_0 = (1+\xi)G_0$ into (13.5) and solving for ξ gives $\xi = B_0 / \int_0^{\infty} G_t e^{-\int_0^t r_s ds} dt > 0$. So, for $\beta > 0$, $H_0 + B_0$ depends positively on B_0. □

The result may be seen in the light of the different discount rates involved. The discount rate relevant for the government when discounting future tax receipts and future spending is just the market interest rate, r. But the discount rate relevant for the households currently alive is $r + \beta$. This is because the present generations are, over time, a decreasing fraction of the tax payers, the rate of decrease being larger the larger is the birth rate. In the Barro and Ramsey models the "birth rate" is effectively zero in the sense that no *new* tax payers are born. When the bequest motive (in Barro's form) is operative, those alive today will take the tax burden of their descendents fully into account.

This takes us to the distinction between *new individuals* and *new decision makers*, a distinction related to the fundamental difference between representative agent models and overlapping generations models.

It is neither finite lives nor population growth

It is sometimes claimed that finite lives or the presence of population growth are basic theoretical reasons for the absence of Ricardian equivalence. This is a misunderstanding, however. The distinguishing feature is whether new decision makers continue to enter the economy or not.

To sort this out, let $\bar{\beta}$ be a constant birth rate of *decision makers.* That is, if the population of decision makers is of size N, then $N\bar{\beta}$ is the inflow of new decision makers per time unit.[1] Given the assumption of a perfect credit market, we claim:

$$\text{there is Ricardian equivalence if and only if } \bar{\beta} = 0. \tag{13.9}$$

Indeed, with (13.8) in mind, when $\bar{\beta} = 0$, future taxes have to be paid by those current tax payers who are still alive in the future. In the absence of credit

[1] In view of the law of large numbers, we do not distinguish between expected and actual inflow.

market imperfections the current tax payers will thus respond to deficit finance (deferment of taxation) by increasing current saving out of the currently higher after-tax income. This increase in saving matches the expected extra taxes in the future. So current private consumption is unaffected by the deficit finance.

If $\bar{\beta} > 0$, however, deficit finance means shifting part of the tax burden from current tax payers to new tax payers in the future whom current tax payers do not care about. Even though representative agent models like the Ramsey and Barro models may include population growth in a demographic sense, they have a *fixed* number of dynastic families (decision makers) and whether the *size* of these dynastic families rises (population growth) or not is of no consequence for the question of Ricardian equivalence.

Another implication of (13.9) is that it is not the *finite lifetime* that is decisive for absence of Ricardian equivalence in OLG models. Indeed, even if we imagine the agents in a Blanchard-style model have a zero death rate, there will still be a *positive* birth rate. New decision makers continue to enter the economy through time. When deficit finance occurs, part of the tax burden is shifted to these newcomers.

To be specific, let $\bar{m}$ be a constant and age-independent death rate of existing decision makers. Then $\bar{n} \equiv \bar{\beta} - \bar{m}$ is the growth rate of the number of decision makers. With β, m, and n denoting the birth rate, death rate, and population growth rate, respectively, in the usual *demographic* sense, we have in Blanchard's model $\bar{\beta} = \beta$, $\bar{m} = m$, and $\bar{n} = n$. In the Ramsey model, however, $\bar{\beta} = \bar{m} = \bar{n} = 0 \leq n = \beta - m$. With this interpretation, both the Blanchard and the Ramsey model fit into (13.9). In the Blanchard model every new generation consists of new decision makers, i.e., $\bar{\beta} = \beta > 0$. In that setting, whether or not the population grows, the generations now alive know that the higher taxes in the future implied by deficit finance today will in part fall on the new generations. We therefore have $n \geq 0$, $\bar{\beta} = \bar{n} + \bar{m} \geq \bar{m} > 0$, and in accordance with (13.9) there is not Ricardian equivalence. In the Ramsey model where, in principle, the new generations are not new decision makers since their utility were already taken care of through bequests by their forerunners, there is Ricardian equivalence. This is in accordance with (13.9), since $\bar{\beta} = 0$, whereas $n \geq 0$.

The assumption in the Blanchard model that $\bar{m}$ $(= m)$ is independent of age might be more acceptable if we interpret $\bar{m}$ not as a biological mortality rate but as a *dynasty mortality rate*.[2] Thinking in terms of dynasties allows for *some* intergenerational links through bequests. In this interpretation $\bar{m}$ is the approximate probability that the family dynasty "ends" within the next time interval of unit length (either because members of the family die without children or because the preferences of the current members of the family no longer incorporate a be-

[2] This interpretation was suggested already by Blanchard (1985, p. 225).

quest motive). Then, $\bar{m} = 0$ corresponds to the extreme Barro case where such an event never occurs, i.e., that all existing families are infinitely-lived through intergenerational bequests. Even in this limiting case we can interpret statement (13.9) as telling that if new families still enter the economy ($\bar{\beta} > 0$), then Ricardian equivalence does not hold. How could *new* families enter the economy? One could imagine that immigrants are completely cut off from their relatives in their home country or that a parent only loves the first-born. In that case children who are not first-born, do not, effectively, belong to any preexisting dynasty, but may be linked forward to a chain of their own descendants (or perhaps only their first-born descendants). So *in spite of the infinite horizon of every family alive*, there are *newcomers*; hence, *Ricardian equivalence does not hold.*

Statement (13.9) also implies that if $\bar{\beta} = 0$, then $\bar{m} > 0$ does *not* destroy Ricardian equivalence. It is the difference between the public sector's future tax base (including the resources of individuals yet to be born) and the future tax base emanating from the individuals that are alive today that in the above analysis accounts for non-neutrality of variations over time in the pattern of lump-sum taxation. This reasoning also reminds us that it is immaterial for the validity of (13.9) whether there is productivity growth in the economy or not.

Additional sources of Ricardian non-equivalence

While the above demographic argument against Ricardian equivalence seems logically convincing, it is another question how large *quantitative* deviations from Ricardian equivalence it can deliver. Taking into account the sizeable life expectancy of the average citizen, Poterba and Summers (1987) point out that demography alone delivers only modest deviations if the issue is timing of taxes over the business cycle. Additional sources of deviation that have been put forward in the literature include:

1. *Short-sightedness.* There is evidence that households on average are not as forward-looking as required by the Ricardian equivalence hypothesis. Behavioral economists and experimental economics question that people conform to the assumption of full intertemporal rationality. People seem to have strong "present bias" (Laibson, 1997). With a limited planning horizon (up to five years, say) the effective discount rate becomes high and thereby capable of generating substantial deviation from Ricardian equivalence.

2. *Failure to leave bequests.* Though the bequest motive is certainly of empirical relevance, it is operative for only a minority of the population[3] (primarily

[3] Wolf (2002).

the wealthy families) and it need not have the altruistic form hypothesized by Barro, cf. Chapter 7.

3. *Imperfections in credit markets.* In practice there are imperfections in the credit markets. Many people can not borrow against expected future earnings. When you are credit rationed, you effectively face an interest rate higher than that faced by the government. Then, even if these people expect higher taxes in the future, the present value of the additional taxes is for these people less than the current reduction of taxes. Incurring a debt-financed tax cut the government helps credit-constrained people to tilt their intertemporal consumption by doing what these people would like to do but cannot, namely borrow - and in fact usually the government can do so at a comparatively low interest rate.

4. *Most taxes are not lump sum.* This should *not* be seen as an argument against the possible *theoretical* validity of the Ricardian equivalence hypothesis. Indeed, what the hypothesis claims is that there are no allocational effects of changes in the timing of *lump-sum* taxes. Nevertheless, widening the discussion to distortionary taxes is of course relevant. Towards the end of Chapter 6 we briefly considered both income taxes and consumption taxes.

5. *The Keynesian view.* The Keynesian point is that deviations from Ricardian equivalence tend to be amplified in situations with unemployment and slack aggregate demand. The reason is that otherwise un-utilized resources may be activated by a budget deficit resulting from a tax cut. By stimulating aggregate consumption in the "first round", a temporary tax cut stimulates aggregate demand and thereby production. The higher level of production amounts to higher income and thereby a further rise in consumption in the "second round" - and so on in the Keynesian multiplier process. In a recession also investment may be stimulated in the process due to increased sales. All in all a positive demand spiral arises: $T \downarrow \Rightarrow C \uparrow \Rightarrow Y \uparrow \Rightarrow I \uparrow \Rightarrow Y \uparrow \Rightarrow C \uparrow$ etc.[4]

To sum up, there are good reasons to believe that Ricardian equivalence fails. Of course, this could in some sense be said about nearly all theoretical abstrac-

[4] According to Keynesian theory, a similar multiplier process takes off as a result of a deficit-financed increase in government *spending* on goods and services: $G \uparrow \Rightarrow Y \uparrow \Rightarrow I \uparrow \Rightarrow Y \uparrow \Rightarrow I \uparrow \Rightarrow Y \uparrow$. Here, however, more than just a change in the timing of taxes is involved, namely a change in government spending on goods and services. So, we are outside the domain of the Ricardian Equivalence controversy in the narrow sence. The broader issue of the size of the government spending multiplier in alternative situations is treated later in this book.

tions. But the prevalent view among macroeconomists is that Ricardian equivalence systematically fails in *one* direction: it over-estimates the offsetting reaction of private saving in response to budget deficits. Moreover, relaxing the restrictive assumptions on which the Ricardian equivalence hypothesis rests, tends to *strengthen* the deviation from Ricardian equivalence implied by the simple demographic argument from OLG models.[5]

13.2 Dynamic general equilibrium effects of lasting budget deficits

The above discussion of effects of public debt is *partial equilibrium* analysis. We treated K, r, and w as unaffected by the changes in government debt. But when aggregate saving changes in a closed economy, so does K and generally also r and w. This should be taken into account. We therefore turn to a quantitative assessment of the full dynamic effects of public debt. This requires *general equilibrium* analysis.

We will apply the Blanchard OLG model from Chapter 12. To simplify, we ignore technological progress, population growth, and retirement all together. Therefore $g = n = \lambda = 0$, so that birth rate = mortality rate = m, and employment = population = N (a constant) for all t. Let public spending on goods and services be a constant $\bar{G} > 0$, assumed not to affect marginal utility of private consumption. Suppose all this spending is (and has always been) public *consumption*. There is thus no public capital. Let taxes and transfers be *lump sum* so that we need keep track only of the *net* tax revenue, T, and the consumption-saving trade-off is not affected by taxes.

We consider a closed economy described by

$$\dot{K}_t = F(K_t, N) - \delta K_t - C_t - \bar{G}, \qquad K_0 > 0, \text{ given}, \tag{13.10}$$

$$\dot{C}_t = (F_K(K_t, N) - \delta - \rho)C_t - m(\rho + m)(K_t + B_t), \tag{13.11}$$

$$\dot{B}_t = [F_K(K_t, N) - \delta]\, B_t + \bar{G} - T_t, \qquad B_0 > 0, \text{ given}, \tag{13.12}$$

where we have used the equilibrium relation $r_t = F_K(K_t, N) - \delta$. Here (13.10) is essentially just accounting for a closed economy; (13.11) describes changes in aggregate consumption, taking into account the generation replacement effect; and (13.12) describes how budget deficits give rise to increases in government debt. All government debt is assumed to be short-term and of the same form as a variable-rate loan in a bank. Hence, at any point in time B_t is historically determined and independent of the current and expected future interest rates.

[5] Some empirical evidence was briefly discussed in chapters 6 and 7.

As we shall see, the long-run interest rate will exceed the long-run output growth rate (which is nil). We know from Chapter 6 that in this case, to remain solvent, the government must satisfy its No-Ponzi-Game condition which, as seen from time zero, is

$$\lim_{t\to\infty} B_t e^{-\int_0^t [F_K(K_s,N)-\delta]ds} \leq 0. \tag{13.13}$$

This says that the debt is not in the long run allowed to grow at a rate as high as the long run interest rate. So, a permanent debt-rollover is ruled out.

In addition we assume that households satisfy their transversality conditions. Thereby the aggregate consumption function will be

$$C_t = (\rho + m)(K_t + B_t + H_t), \tag{13.14}$$

with

$$H_t = N \int_t^\infty (w_s - \tau_s) e^{-\int_t^s (r_z+m)dz} ds, \tag{13.15}$$

as in Section 13.1. These formulas will be useful when it comes to interpretation of the dynamics in the economy. For ease of exposition, we let the aggregate production function satisfy the Inada conditions $\lim_{K\to 0} F_K(K,N) = \infty$ and $\lim_{K\to\infty} F_K(K,N) = 0$. We assume $\delta > 0$ and $\rho \geq 0$.

So far the model is incomplete in the sense that there is nothing to pin down the time profile of T_t, except that ultimately the stream of taxes should conform to (13.13). Let us first consider a permanently balanced government budget.

Dynamics under a balanced budget

Suppose that from time 0 the government budget is balanced. Therefore, $\dot{B}_t = 0$ and $B_t = B_0$ for all $t \geq 0$. So (13.12) is reduced to

$$T_t = (F_K(K_t, N) - \delta)B_0 + \bar{G}, \tag{13.16}$$

giving the tax revenue required for the budget to be balanced, when the debt is B_0. This time path of T_t is determined *after* we have determined the time path of K_t and C_t through the two-dimensional system

$$\dot{K}_t = F(K_t, N) - \delta K_t - C_t - \bar{G}, \qquad K_0 > 0, \text{ given}, \tag{13.17}$$

$$\dot{C}_t = [F_K(K_t, N) - \delta - \rho]C_t - m(\rho + m)(K_t + B_0). \tag{13.18}$$

This system is independent of T_t. The implied dynamics can usefully be analyzed by a phase diagram.

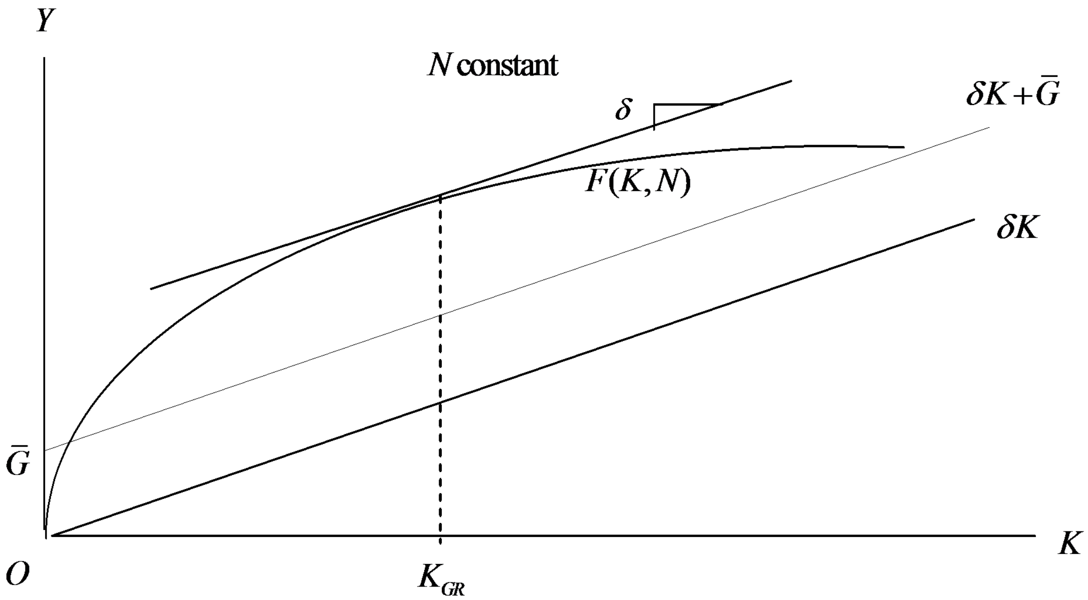

Figure 13.1: Building blocks for a phase diagram.

Phase diagram Equation (13.17) shows that

$$\dot{K} = 0 \quad \text{for} \quad C = F(K, N) - \delta K - \bar{G}. \tag{13.19}$$

The right-hand side of (13.19) is the vertical distance between the $Y = F(K, N)$ curve and the $Y = \delta K + \bar{G}$ line in Fig. 13.1. On the basis of this we can construct the $\dot{K} = 0$ locus in Fig. 13.2. We have indicated two benchmark values of K in the figure, namely the golden rule value K_{GR} and the value $\bar{K}$. These values are defined by

$$F_K\left(K_{GR}, N\right) - \delta = 0, \qquad \text{and} \qquad F_K\left(\bar{K}, N\right) - \delta = \rho,$$

respectively.[6] We have $\bar{K} \leq K_{GR}$, since $\rho \geq 0$ and $F_{KK} < 0$.

From equation (13.18) follows that

$$\dot{C} = 0 \quad \text{for} \quad C = \frac{m\left(\rho + m\right)\left(K + B_0\right)}{F_K(K, N) - \delta - \rho}. \tag{13.20}$$

Hence, for $K \to \bar{K}$ from below we have, along the $\dot{C} = 0$ locus, $C \to \infty$. In addition, for $K \to 0$ from above, we have along the $\dot{C} = 0$ locus that $C \to 0$, in view of the lower Inada condition.

[6] In this setup, where there is neither population growth nor technical progress, the golden rule capital stock is that K which maximizes $C = F(K, N) - \delta K - \dot{K}$ subject to the steady state condition $\dot{K} = 0$.

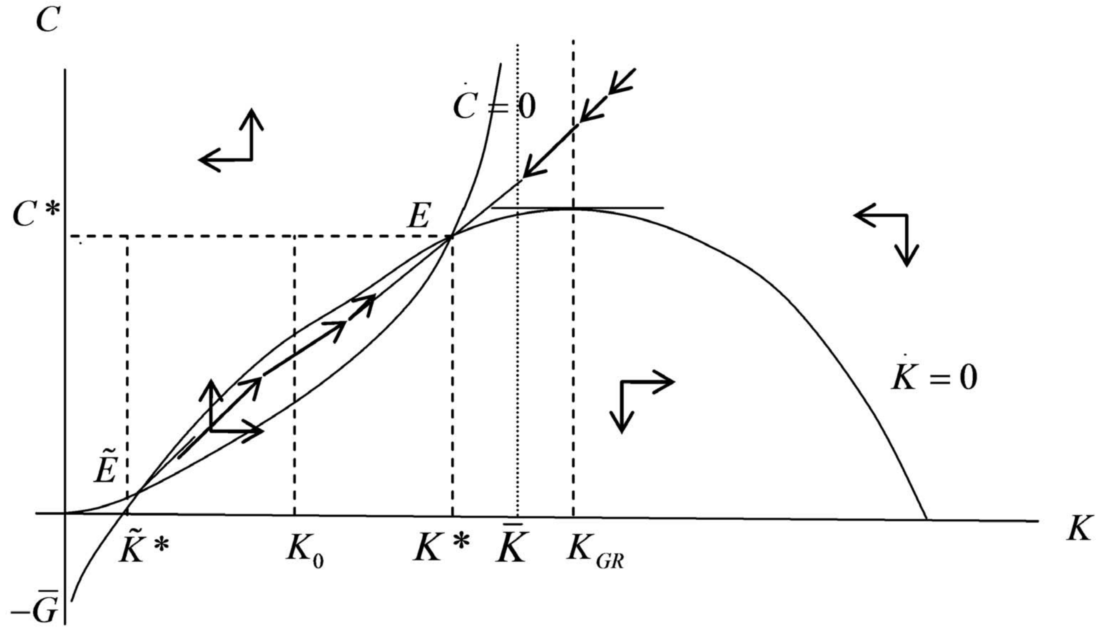

Figure 13.2: Phase diagram under a balanced budget.

Fig. 13.2 also shows the $\dot{C} = 0$ locus. We assume that $\bar{G}$ and B_0 are of "modest" size relative to the production potential of the economy for the given K_0 and N. Then the $\dot{C} = 0$ curve crosses the $\dot{K} = 0$ curve for *two* positive values of K, and the smallest of these is lower than K_0. Fig. 13.2 shows these steady states as the points E and Ẽ with coordinates (K^*, C^*) and $(\tilde{K}^*, \tilde{C}^*)$, respectively, where $\tilde{K}^* < K^* < \bar{K}$.

The direction of movement in the different regions of Fig. 13.2 are indicated by arrows determined by the differential equations (13.17) and (13.18). The steady state E is seen to be a saddle point, whereas Ẽ is a *source*.[7] We assume that $\bar{G}$ and B_0 are "modest" not only relative to the long-run production capacity of the economy but also relative to the given K_0. This means that $\tilde{K}^* < K_0$, as indicated in the figure.[8]

The capital stock is predetermined whereas consumption is a jump variable. Since the slope of the saddle path is not parallel to the C axis, it follows that the

[7] A steady state point with the property that all solution trajectories starting close to it move away from it is called a *source* or a *totally unstable* steady state.

[8] The opposite case, $\tilde{K}^* > K_0$, would reflect that G_0 and B_0 were very large relative to the initial production capacity of the economy, so large, indeed, that aggregate net saving would be chronically negative. Then a forever shrinking capital stock would be in prospect. The economy would in that case *not* converge towards the steady state E. This steady state would only be *locally* saddle-point stable, not globally saddle-point stable.

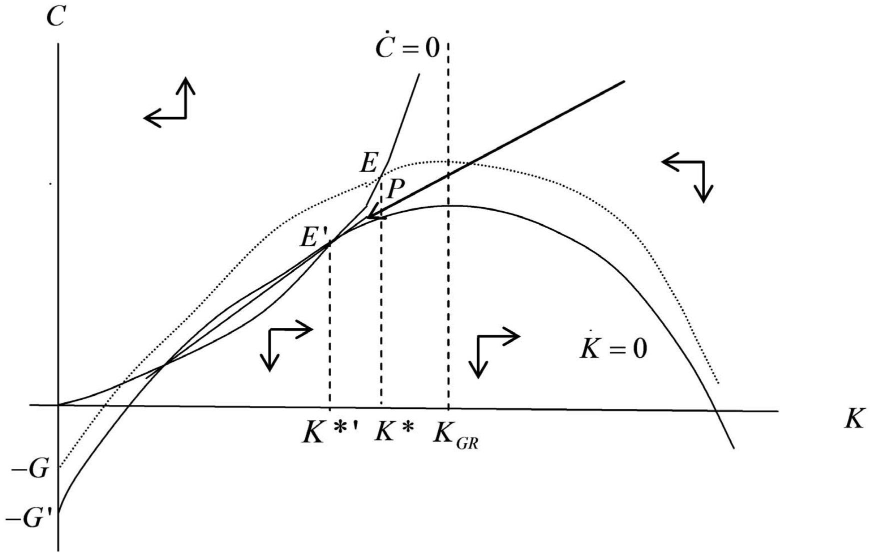

Figure 13.3: Tax-financed shift to higher public consumption.

system is saddle-point stable. The only trajectory consistent with *all* the conditions of general equilibrium (individual utility maximization for given expectations, profit maximization, continuous market clearing, fulfilled expectations) is the saddle path.[9] The other trajectories in the diagram violate the TVCs of the individual households. Hence, initial consumption, C_0, is determined as the ordinate to the point where the vertical line $K = K_0$ crosses the saddle path. Over time the economy moves along the saddle path, approaching the steady state point E with coordinates (K^*, C^*).

Although our main focus will be on effects of budget deficits and changes in the debt, we start with the simpler case of a tax-financed increase in $\bar{G}$.

Unexpected tax-financed shift to a higher level of public consumption Suppose that until time t_1 (> 0) the economy has been in the saddle-point stable steady state E. Hence, for $t < t_1$ we have zero net investment and $r = F_K(K^*, N) - \delta \equiv r^*$. Moreover, as $K^* < \bar{K}$, $r^* > \rho$ (≥ 0).

At time t_1 an unanticipated change in fiscal policy occurs. Public consumption shifts to a new constant level $\bar{G}' > \bar{G}$. Taxes are immediately increased by the same amount so that the budget stays balanced. We assume that everybody rightly expect the new policy to continue forever. The change to a higher G shifts the $\dot{K} = 0$ curve downwards as shown in Fig. 13.3, but leaves the $\dot{C} = 0$ curve

[9] By the same reasoning as in Appendix D of Chapter 12 it can be shown that when $\rho \geq 0$, the transversality conditions of the households will be satisfied in the steady state E, hence along paths converging towards E.

unaffected. At time t_1 when the policy shift occurs, private consumption jumps down to the level corresponding to the point P in Fig. 13.3. The explanation is that the net-of-tax human wealth, H_{t_1}, is immediately reduced as a result of the higher current and expected future taxes.

Owing to the upward-sloping new saddle path, cf. Fig. 13.3, the initial reduction in C is smaller than the increase in G (and T). Therefore net saving becomes negative and K decreases gradually until the new steady state, E*'*, is "reached". To find the long-run effects on K and C we first equalize the right-hand sides of (13.19) and (13.20) and then use implicit differentiation w.r.t. $\bar{G}$ to get

$$\frac{\partial K^*}{\partial \bar{G}} = \frac{r^* - \rho}{C^* F_{KK}^* - (m + r^*)(\rho + m - r^*)} < 0;$$

next, from (13.19), by the chain rule we get

$$\frac{\partial C^*}{\partial \bar{G}} = \frac{\partial C^*}{\partial K^*}\frac{\partial K^*}{\partial \bar{G}} = r^* \frac{\partial K^*}{\partial \bar{G}} - 1 < -1,$$

where $r^* = F_K(K^*, N) - \delta$.[10] In the long run the decrease in C is *larger* than the increase in G because the economy ends up with a smaller capital stock.

Summing up That is, under full capacity utilization a tax-financed shift to higher G crowds out private consumption *and* investment. Private consumption is in the long run crowded out *more* than one to one due to reduced productive capacity. In this way the cost of the higher G falls relatively more on the younger and as yet unborn generations than on the currently elder generations.[11]

Higher public debt

To analyze the effect of a rise in public debt, let us first see how it might come about.

A tax cut Assume again that until time t_1 (> 0) the economy has had a balanced government budget and been in the saddle-point stable steady state E. The level of the public debt in this steady state is $B_0 > 0$ and tax revenue is, by (13.16),

$$T = (F_K(K^*, N) - \delta)B_0 + \bar{G} \equiv T^*,$$

a positive constant in view of $F_K(K^*, L) - \delta = r^* > \rho \geq 0$.

[10] For details, see Appendix B.

[11] This might be different if a part of G were public *investment* (in research and education, say), and this part were also increased.

At time t_1 the government unexpectedly cuts taxes to a lower constant level, $\bar{T}$, holding public consumption unchanged. At least for a while after time t_1 we thus have

$$T_t = \bar{T} < T^*. \tag{13.21}$$

As a result, $\dot{B}_t > 0$. The tax cut make current generations feel wealthier, hence they increase their consumption. They do so in spite of being forward-looking and anticipating that the current fiscal policy sooner or later must come to an end (because it is not sustainable, as we shall see). The prospect of higher taxes in the future dampens the increase in consumption, but does not prevent it, since part of the future taxes will fall on new generations entering the economy.

The rise in C, combined with unchanged G, implies negative net investment so that K begins to fall, implying a rising interest rate, r. For some time, *three* differential equations, determining changes in C, K, and B, are active. Moreover, while (13.10) and (13.12) still hold, (13.11) need not. This is because of the uncertainty about *when* and *how* a fiscal tightening will take place. Anyway, three-dimensional dynamics are complicated and cannot, of course, be illustrated in a two-dimensional phase diagram. Hence, for now we leave the phase diagram.

The fiscal policy $(\bar{G}, \bar{T})$ is not sustainable By definition a fiscal policy (G, T) is *sustainable* if the government stays solvent under this policy. We claim that the fiscal policy $(\bar{G}, \bar{T})$ is *not* sustainable. Relying on principles from Chapter 6, there are at least three different ways to prove this.

Approach 1: Sustained rise in the debt-income ratio. The negative net investment continues. And along with the falling K, we have a falling aggregate income, $Y_t = F(K_t, N)$. So we are in a situation where the interest rate remains larger than the long-run output growth rate which in the absence of growth in technology or labor force is clearly non-positive. The falling K implies falling Y.

The combination of a rising B and falling Y implies a forever rising debt-income ratio, B/Y. The private sector will understand that bankruptcy is threatening and nobody will buy government bonds except at a reduced price, which means a higher interest rate. The high interest rate only aggravates the problem. That is, the fiscal policy $(G, \bar{T})$ breaks down.

An alternative argument is:

Approach 2: The government NPG condition is violated. In view of $K^* < \bar{K} < K_{GR}$, we have $r^* = F_K(K^*, L) - \delta > F_K(\bar{K}, L) - \delta = \rho \geq 0$. After time t_1, K_t is falling, at least for a while. So $K_t < K^*$ and thus $r_t = F_K(K_t, N) - \delta > r^*$ > 0. Thereby the fiscal policy $(\bar{G}, \bar{T})$ implies an interest rate forever larger than the long-run output growth rate which in the absence of growth in technology or labor force is zero. From Chapter 6 we know that in this situation a sustainable fiscal policy must satisfy the NPG condition

$$\lim_{t\to\infty} B_t e^{-\int_{t_1}^{t} r_s ds} \leq 0. \tag{13.22}$$

With a forever positive debt, this requires that there exists an $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \frac{\dot{B}_t}{B_t} < \lim_{t\to\infty} r_t - \varepsilon, \tag{13.23}$$

i.e., the long-run growth rate of the public debt should be less than the long-run interest rate.

The fiscal policy $(\bar{G}, \bar{T})$ violates this condition, however. Indeed, we have, for $t > t_1$,

$$\begin{aligned} \dot{B}_t &= r_t B_t + \bar{G} - \bar{T} \\ &> r^* B_0 + \bar{G} - \bar{T} > r^* B_0 + \bar{G} - T^* = 0, \end{aligned} \tag{13.24}$$

where the first inequality comes from $B_t > B_0 > 0$ and $r_t = F_K(K_t, L) - \delta > r^* = F_K(K^*, L) - \delta$, in view of $K_t < K^*$. This implies $B_t \to \infty$ for $t \to \infty$. Hence, dividing by B_t in (13.24) gives

$$\frac{\dot{B}_t}{B_t} = r_t + \frac{\bar{G} - \bar{T}}{B_t} \to r_t \qquad \text{for} \qquad t \to \infty, \tag{13.25}$$

which violates (13.23). So the fiscal policy $(\bar{G}, \bar{T})$ is not sustainable. The crux of the matter is that in the absence of economic growth, lasting budget deficits indicate an unsustainable fiscal policy.

Approach 3. Yet another way of showing absence of fiscal sustainability is to start out from the intertemporal government budget constraint and check whether the primary budget surplus, $\bar{T} - \bar{G}$, which rules after time t_1, satisfies

$$\int_{t_1}^{\infty} (\bar{T} - \bar{G}) e^{-\int_{t_0}^{t} r_s ds} dt \geq B_{t_1}, \tag{13.26}$$

where $B_{t_1} = B_0 > 0$. Obviously, if $\bar{T} - \bar{G} \leq 0$, (13.26) is not satisfied. Suppose $\bar{T} - \bar{G} > 0$. Then

$$\int_{t_1}^{\infty} (\bar{T} - \bar{G}) e^{-\int_{t_1}^{t} r_s ds} dt < \int_{t_1}^{\infty} (\bar{T} - \bar{G}) e^{-r^*(t-t_1)} dt = \frac{\bar{T} - \bar{G}}{r^*} < B_0 = B_{t_1},$$

where the first inequality comes from $r_t > r^*$, the first equality from carrying out the integration $\int_{t_1}^{\infty} e^{-r^*(t-t_1)} dt$, and, finally, the second inequality from the equality in the second row of (13.24) together with the fact that $\bar{T} < T^*$. So the intertemporal government budget constraint is not satisfied. The current fiscal policy is unsustainable.

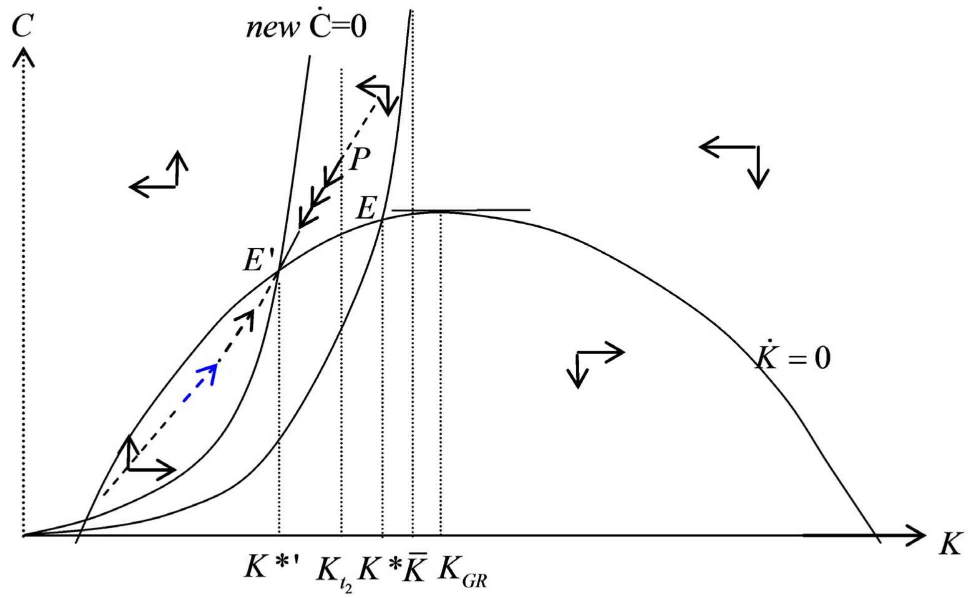

Figure 13.4: The adjustment after fiscal tightening at time t_2, presupposing $K^{*\prime} < K_{t_2}$.

Fiscal tightening and thereafter To avoid default on the debt, sooner or later the fiscal policy must change. This may take the form of lower of public consumption or higher taxes or both.[12] Suppose that the change occurs at time $t_2 > t_1$ in the form of a tax increase so that for $t \geq t_2$ there is again a balanced budget. This new policy is at time t_2 announced to be followed forever and we assume the market participants believe in this and that it holds true, at least "for a long time".

The balanced budget after time t_2 implies

$$T_t = (F_K(K_t, N) - \delta)B_{t_2} + \bar{G}. \tag{13.27}$$

The dynamics are therefore again governed by a two-dimensional system,

$$\dot{K}_t = F(K_t, N) - \delta K_t - C_t - \bar{G}. \tag{13.28}$$

$$\dot{C}_t = [F_K(K_t, N) - \delta - \rho]C_t - m(\rho + m)(K_t + B_{t_2}), \tag{13.29}$$

Consequently phase diagram analysis can again be used.

The phase diagram for $t \geq t_2$ is depicted in Fig. 13.4. The new initial K is K_{t_2}, which is smaller than the previous steady-state value K^* because of the negative net investment in the time interval $[t_1, t_2)$. Relative to Fig. 13.2, the $\dot{K} = 0$ locus is unchanged (since $\bar{G}$ is unchanged). But in view of the new constant debt level

[12] We still ignore financing by seigniorage.

B_{t_2} being higher than B_0, the $\dot{C}=0$ locus has turned counter-clockwise. For any given $K \in (0, \bar{K})$, the value of C required for $\dot{C}=0$ is higher than before, cf. (13.20). The intuition is that for every given K, private financial wealth is higher than before in view of the possession of government bonds being higher. For every given K, therefore, the generation replacement effect on the change in aggregate consumption is greater. Hence, so is the level of aggregate consumption that via the operation of the Keynes-Ramsey rule is required to offset the generation replacement effect and ensure $\dot{C}=0$ (cf. Section 12.2 of the previous chapter).

The new saddle-point stable steady state is denoted E' in Fig. 13.4 and it has capital stock $K^{*\prime} < K^*$ and consumption level $C^{*\prime} < C^*$. As the figure is drawn, K_{t_2} is larger than $K^{*\prime}$. Intuitively, this case may arise if the tax cut at t_1 is "large" so that B_t rises fast in the time interval (t_1, t_2) and causes $K^{*\prime}$ to end up considerably below K^*. The level of consumption immediately after t_2, where the fiscal tightening sets in, is found where the vertical line $K = K_{t_2}$ crosses the new saddle path, i.e., the point P in Fig. 13.4. Immediately before t_2, consumption was at a higher level because the time of arrival of the tax increase was still uncertain. The movement of the economy after t_2 implies gradual lowering of the capital stock and consumption until the new steady state, E', is reached. (More details below in the section on time profiles.)

Alternatively, it can not be ruled out that K_{t_2} is smaller than $K^{*\prime}$ so that the new initial point, P, is to the left of the new steady state, E'. This case is illustrated in Fig. 13.5. Intuitively, this case may arise if the tax cut at t_1 is "small" so that B_t only rises slowly in the time interval (t_1, t_2), thereby causing $K^{*\prime}$ to end up not far below K^*. The low amount of capital at t_2 implies a high interest rate and the fiscal tightening must now be tough. This induces a low consumption level – so low that net investment becomes positive. Then the capital stock and output increase gradually during the adjustment to the steady state E'.[13]

Thus, in both cases the long-run effect of the transitory budget deficit is qualitatively the same, namely that the larger supply of government bonds crowds out physical capital in the private sector. Intuitively, a certain feasible time pro-

[13] A precise determination of conditions under which the case $K^{*\prime} < K_{t_2}$ versus the case $K^{*\prime} > K_{t_2}$ will occur is complicated. Three-dimensional dynamics is complex, and the uncertainty arising from t_2 not being fixed ex ante is a further complication. One might think that the longer the time interval (t_1, t_2) is, the more scope is there for the case $K_{t_2} < K^{*\prime}$ to arise. But the situation is less clear-cut than that because the longer the time interval (t_1, t_2) is, the larger is not only the fall in K but also the rise in B. Still, we might argue that there is a lower bound, $-\delta$, on the proportionate rate of change of the capital stock, whereas there is no comparable upper bound on how fast the government debt can increase. Hence, if the tax cut is substantial and the time interval (t_1, t_2) "small", it may seem likely that the fall in K is "dominated" by the rise in B as in Fig. 13.4. Anyway, numerical simulation and sensitivity analysis should be able to settle the matter but is not pursued here.

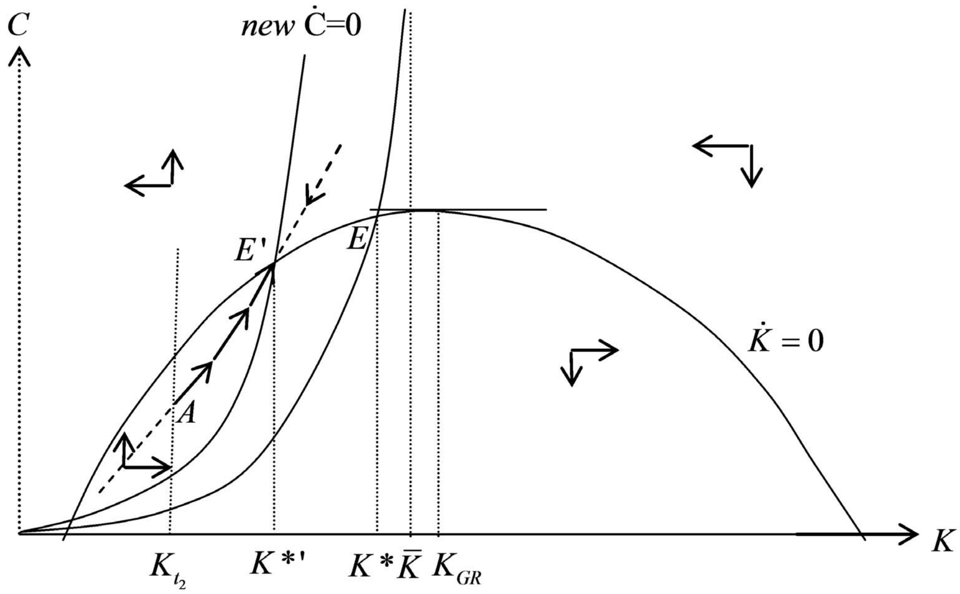

Figure 13.5: The adjustment after fiscal tightening at time t_2, presupposing $K^{*\prime} > K_{t_2}$.

file for financial wealth, $A = K + B$, is desired and the higher is B, the lower is the needed K. To this "stock" interpretation we may add a "flow" interpretation, saying that the budget deficit offers households a saving outlet which is an alternative to capital investment.

To be able to quantify the long-run effects of a change in the debt level on K and C we need the long-run multipliers. By equalizing the right-hand sides of (13.19) and (13.20), with B_0 replaced by $\bar{B}$, and using implicit differentiation w.r.t. $\bar{B}$, we get

$$\frac{\partial K^*}{\partial \bar{B}} = \frac{m\,(\rho + m)}{\mathcal{D}} < 0, \tag{13.30}$$

where $\mathcal{D} \equiv C^* F^*_{KK} - (r^* + m)(\rho + m - r^*) < 0.$[14] Next, by using the chain rule on $C^* = F(K^*, N) - \delta K^* - \bar{G}$ from (13.19), we get

$$\frac{\partial C^*}{\partial \bar{B}} = \frac{\partial C^*}{\partial K^*}\frac{\partial K^*}{\partial \bar{B}} = (F_K(K^*, N) - \delta)\frac{m\,(\rho + m)}{\mathcal{D}} = r^* \frac{m\,(\rho + m)}{\mathcal{D}} < 0.$$

The multiplier $\partial K^*/\partial \bar{B}$ tells us the approximate size of the long-run effect on the capital stock, when a temporary tax cut causes a unit increase in public debt. The resulting change in long-run output is approximately $\partial Y^*/\partial \bar{B} = (\partial Y^*/\partial K^*)(\partial K^*/\partial \bar{B}) = (r^* + \delta) m\,(\rho + m)\,/\mathcal{D} < 0$. The elasticity of long-run out-

[14]For details, see Appendix B.

put with respect to public debt is $(\bar{B}/Y^*)\partial Y^*/\partial\bar{B} = (\bar{B}/Y^*)(r^*+\delta)m\,(\rho+m)\,/\mathcal{D} < 0$.

Under full capacity utilization government deficits have a crowding-out effects because they compete with private investment for the allocation of saving.

These results of course hinge on the assumption of permanent full capacity utilization in the economy (no idle capital, no idle labor). When the economy is far below full resource utilization, allowing a budget deficit to arise helps stimulating aggregate demand, output, and income. The resulting increase in aggregate saving raises the flow of loanable funds and create *downward* pressure on the interest rate.

Time profiles It is also useful to consider the time paths of the variables.

Case 1: $K_{t_2} > K^{*\prime}$. Fig. 13.6 shows stylized aspects of the time profile of T and B, respectively. The upper panel visualizes that the increase in taxation at time t_2 is larger than the decrease at time t_1. As (13.27) shows, this is due to public expenses being larger after t_2 because both the government debt B_t and the interest rate, $F_K(K_t, N_t) - \delta$, are higher. The further gradual rise in T_t towards its new steady-state level is due to the rising interest service along with a rising interest rate, caused by the falling K.

The middle panel of Fig. 13.6 is self-explanatory.

As visualized by the lower panel of Fig. 13.6, the tax cut at time t_1 results in an upward jump in consumption. This implies negative net investment, so that K begins to fall. The size of the upward jump in consumption at time t_1 and the subsequent time path of consumption in the time interval $[t_1, t_2)$ can not be precisely pinned down. We can not even be sure that C will be gradually falling within this time interval. Therefore the downward-sloping time path of C in the lower panel of Fig. 13.6 in this time interval illustrates just one of the possibilities.

The ambiguity arises for the following reason. Though the current generations will immediately feel wealthier and increase their consumption as a result of the tax cut, they have rational expectations and are thereby aware that sooner or later fiscal policy will have to be changed again. As the households may have uncertain and different beliefs about *when* and *how* the fiscal sustainability problem will be remedied, we can not theoretically assign a specific value to the new after-tax human wealth, even less a constant value. What we can tell is that H_{t_1}, and therefore C_{t_1}, will be "somewhat" larger than immediately before time t_1. Also private saving will rise, however. This is because the rise in consumption at time t_1 will be less than the fall in taxes. To see this, imagine first that the households expect a constant level, T, to last for a long time during which also the real interest rate and the real wage remain approximately unchanged. Perceived

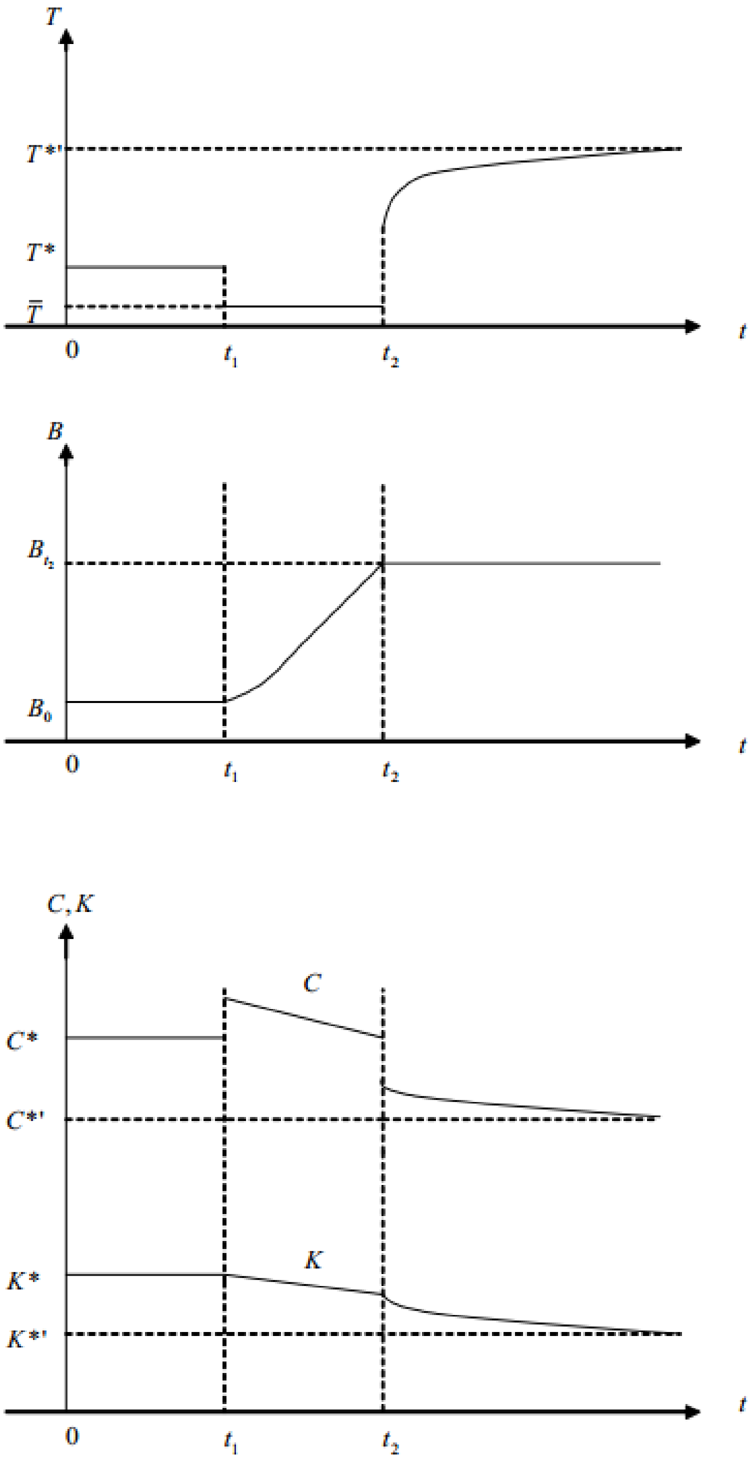

Figure 13.6: Case 1: $K_{t_2} > K^{*\prime}$. Regarding time path of C in the time interval (t_1, t_2) only one possibility shown.

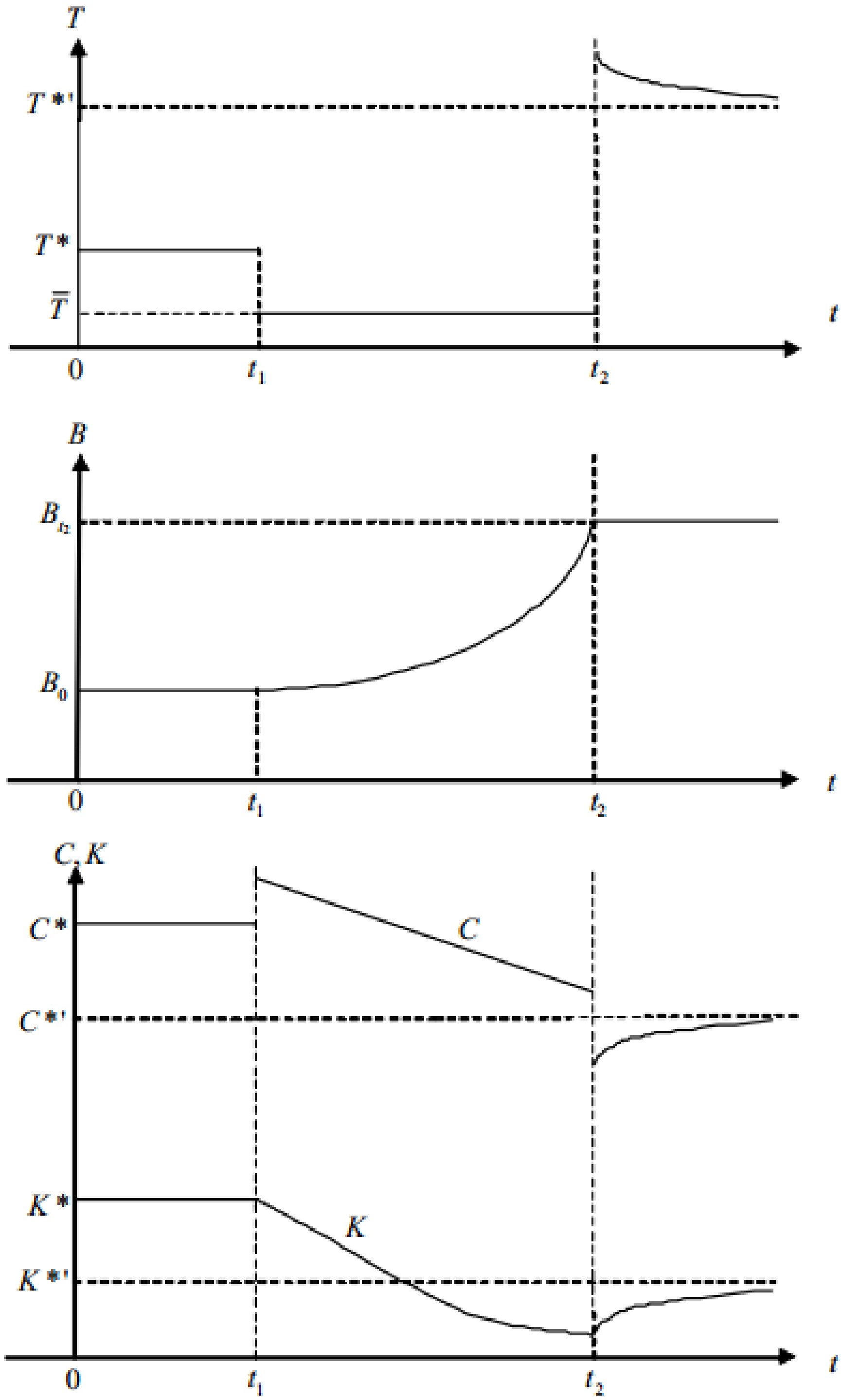

Figure 13.7: Case 2: $K_{t_2} < K^{*\prime}$. Regarding time path of C in the time interval (t_1, t_2) only one possibility shown.

human wealth would then be $H \approx (w^*N - T)/(r^* + m)$, from (13.15). By $C_t = (\rho + m)(A_t + H)$, we would have

$$\Delta C_t \approx dC_t = \frac{\partial C_t}{\partial T} dT = (\rho + m)\frac{\partial H}{\partial T} dT = -\frac{\rho + m}{r^* + m} dT < -dT, \tag{13.31}$$

in view of $dT = \bar{T} - T^* < 0$ and $r^* > \rho$. To the extent that the households expect the new tax level $\bar{T}$ to last a *shorter* time, the boost to H, and therefore also to C, will be *less* than indicated by this equation. The boost to H and C is further dampened by the (correct) anticipation that the ongoing negative net investment will imply a falling K and thereby a falling real wage (due to the falling marginal productivity of labor) and a rising interest rate (due to the rising net marginal productivity of capital). So, at least for a while, there *will* be *positive private saving*, hence rising private financial wealth A. Meanwhile, H *will be falling* after t_1 due to the falling real wage, the rising interest rate, and the fact that the date of likely fiscal tightening is approaching, although uncertain.

So the two components of total wealth, A and H, move in opposite directions. Depending of which of these opposite movements is dominating, consumption will be rising or falling for a while after t_1 (Fig. 13.6 depicts the latter case). Anyway, because the exact time and form of the fiscal tightening is not anticipated, a sharp decrease in the present discounted value of after-tax labor income occurs at time t_2. This induces a downward jump in consumption as indicated in the lower panel of Fig. 13.6. Although the fall in consumption makes room for increased net investment, in Case 1 net investment remains negative so that the fall in K continues after t_2. Therefore, also the real wage continues to fall, implying continued fall in H, cf. (13.14) and (13.15), hence further fall in C, until the new steady-state level is reached.

If the time of the fiscal tightening were anticipated, consumption would not jump at time t_2. But the long-run result would be qualitatively similar.

Case 2: $K_{t_2} < K^{*\prime}$. In this case the tax revenue after t_2 has to exceed what is required in the new steady state. During the subsequent adjustment the taxation level will be gradually falling which reflects the gradual fall in the interest rate generated by the rising K, cf. Fig. 13.5. Private consumption will at time t_2 jump to a level *below* the new (in itself lower) steady state level, $C^{*\prime}$.

The above analysis is in a sense “biased” against budget deficits because it ignores economic growth. Thereby persistent budget deficits necessarily become incompatible with fiscal sustainability. With economic growth, persistent budget deficits are compatible with fiscal sustainability as long as the resulting government debt does not persistently grow faster than GDP. A further limitation of the analysis is its abstraction from the role of Keynesian aggregate demand factors in the process.

13.3 Public and foreign debt: a small open economy

Let the country considered be a small open economy (SOE). Suppose there is perfect substitutability and mobility of goods and financial capital across borders, but no mobility of labor. The main difference compared with the above analysis is then that the interest rate will not be affected by the public debt of the country (as long as its fiscal policy seems sound). Besides making the analysis simpler, this entails a *stronger* crowding out effect of public debt than in the closed economy. The lack of an offsetting increase in the interest rate means absence of the feedback which in a closed economy limits the fall in aggregate saving. In the open economy national wealth equals the stock of physical capital plus net foreign assets. And it is national wealth rather than the capital stock which is crowded out.

The model

The analytical framework is still Blanchard's OLG model with constant population. As above we concentrate on the simple case: $g = \lambda = 0$ and birth rate = mortality rate = $m > 0$. The real interest rate is given from the world financial market and is a constant $r > 0$. Table 13.1 lists key variables for an open economy.

Table 13.1. New variable symbols

A_t^n	$= A_t - B_t = K_t + A_t^f =$ national wealth
$-B_t$	$= -$ government (net) debt = government financial wealth
A_t^f	= net foreign assets (the country's net financial claims on the rest of the world)
D_t	$= -A_t^f =$ net foreign debt
A_t	$= K_t + B_t + A_t^f =$ private financial wealth
$\dot{A}_t$	$= S_t^p =$ private net saving
$-\dot{B}_t$	$= S_t^g = T_t - G - rB_t =$ government net saving = budget surplus
$\dot{A}_t^n$	$= \dot{A}_t - \dot{B}_t = S_t^p + S_t^g = S_t^n =$ aggregate net saving
NX_t	= net exports
$\dot{A}_t^f$	$= \dot{A}_t - \dot{B}_t - \dot{K}_t = NX_t + rA_t^f = CAS_t =$ current account surplus
CAD_t	$= -CAS_t = rD_t - NX_t =$ current account deficit

In view of profit-maximization, the equilibrium capital stock, K^*, satisfies $F_K(K^*, N) = r + \delta$ and is thus a constant. The equilibrium real wage is $w^* = F_L(K^*, N)$. The increase per time unit in real private financial wealth is

$$\dot{A}_t = rA_t + w^*N - T_t - C_t = rA_t + (w^* - \tau_t)N - C_t, \tag{13.32}$$

where $\tau_t \equiv T_t/N$ is a per capita lump-sum tax. The corresponding differential equation for C_t reads $\dot{C}_t = (r-\rho)C_t - m(\rho+m)A_t$. To keep track of consumption in the SOE, however, it is easier to focus directly on the level of consumption:

$$C_t = (\rho + m)(A_t + H_t), \tag{13.33}$$

where H_t is (after-tax) human wealth, given by

$$H_t = N\int_t^\infty (w^* - \tau_s)e^{-(r+m)(s-t)}ds = \frac{Nw^*}{r+m} - N\int_t^\infty \tau_s e^{-(r+m)(s-t)}ds. \tag{13.34}$$

Suppose that from time 0 the government budget is balanced, so that B_t is constant at the level B_0 and $T_t = rB_0 + \bar{G} \equiv T^*$. Consequently,

$$\tau_t = \frac{T^*}{N} = \frac{rB_0 + \bar{G}}{N} \equiv \tau^*. \tag{13.35}$$

Under "normal" circumstances $\tau^* < w^*$, that is, B_0 and $\bar{G}$ are not so large as to leave non-positive after-tax earnings. Then, in view of the constant per capita tax, (13.34) gives

$$H_t = \frac{w^* - \tau^*}{r+m}N \equiv H^* > 0. \tag{13.36}$$

Consequently, (13.32) simplifies to

$$\begin{aligned}\dot{A}_t &= (r-\rho-m)A_t + (w^*-\tau^*)N - (\rho+m)\frac{w^*-\tau^*}{r+m}N \\ &= (r-\rho-m)A_t + \frac{r-\rho}{r+m}(w^*-\tau^*)N. \end{aligned}\tag{13.37}$$

Presupposing $r \neq \rho + m$, this linear differential equation has the solution

$$A_t = (A_0 - A^*)e^{(r-\rho-m)t} + A^*, \tag{13.38}$$

where A^* is the steady-state national wealth,

$$A^* = \frac{(r-\rho)(w^*-\tau^*)N}{(r+m)(\rho+m-r)}. \tag{13.39}$$

(For economic relevance of the solution (13.38) it is required that $A_0 > -H^*$, since otherwise C_0 would be zero or negative in view of (13.33).) Substitution into (13.33) gives steady-state consumption,

$$C^* = \frac{m(\rho+m)(w^*-\tau^*)N}{(r+m)(\rho+m-r)}. \tag{13.40}$$

By an argument similar to that in Appendix D of Chapter 12, it can be shown that the transversality conditions of the individual households are satisfied along the path (13.38).

By (13.37) we see that the steady state, A^*, is asymptotically stable if and only if

$$r < \rho + m. \tag{13.41}$$

Let us consider this case first. The phase diagram describing this case is shown in the upper panel of Fig. 13.8. The lower panel of the figure illustrates the movement of the economy in (A, C) space, given $A_0 < A^*$. The $\dot{A} = 0$ line represents the equation $C = rA + (w^* - \tau^*)N$, which in view of (13.32) must hold when $\dot{A} = 0$. Its slope is lower than that of the line representing the consumption function, $C = (\rho + m)(A + H^*)$. The economy is always at some point on this line.[15] A sub-case of (13.41) is the following case.

Medium impatience: $r - m < \rho < r$

As Fig. 13.8 is drawn, it is presupposed that $A^* > 0$, which, given (13.41), requires $r - m < \rho < r$. This is the case of "medium impatience".

A fiscal easing Imagine that until time $t_1 > 0$ the system has been in the steady state E. At time t_1 an unforeseen tax cut occurs so that at least for some spell of time after t_1 we have $T = \bar{T} < T^*$, hence $\tau = \bar{\tau} \equiv \bar{T}/N < \tau^*$. Since government spending remains unchanged, there is now a budget deficit and public debt begins to rise. We know from the partial equilibrium analysis of Section 13.1 that current generations will feel wealthier and increase their consumption. Like in the similar situation in the closed economy of Section 13.2, we can not assign a specific value to the new after-tax human wealth, even less a constant value. The phase diagram as in Fig. 13.8 is thus no longer applicable and for now we leave phase diagram analysis.

We claim that the rise in consumption at time t_1 will be less than the fall in taxes. This amounts to positive private saving and rising private financial wealth for a while. To see this provisional outcome, imagine first that the agents expect taxation to be at a constant level, T, *forever.* Perceived human wealth would then be $H = (w^*N - T)/(r + m)$, in analogy with (13.36). From $C_t = (\rho + m)(A_t + H)$ we would have

$$dC_t \approx \frac{\partial C_t}{\partial T} dT = (\rho + m)\frac{\partial H}{\partial T} dT = -\frac{\rho + m}{r + m} dT < -dT, \tag{13.42}$$

[15] If we (as for the closed economy) had based the analysis on *two* differential equations in A and C, then a saddle path would arise and this path would coincide with the $C = (\rho + m)(A + H^*)$ line in Fig. 13.8.

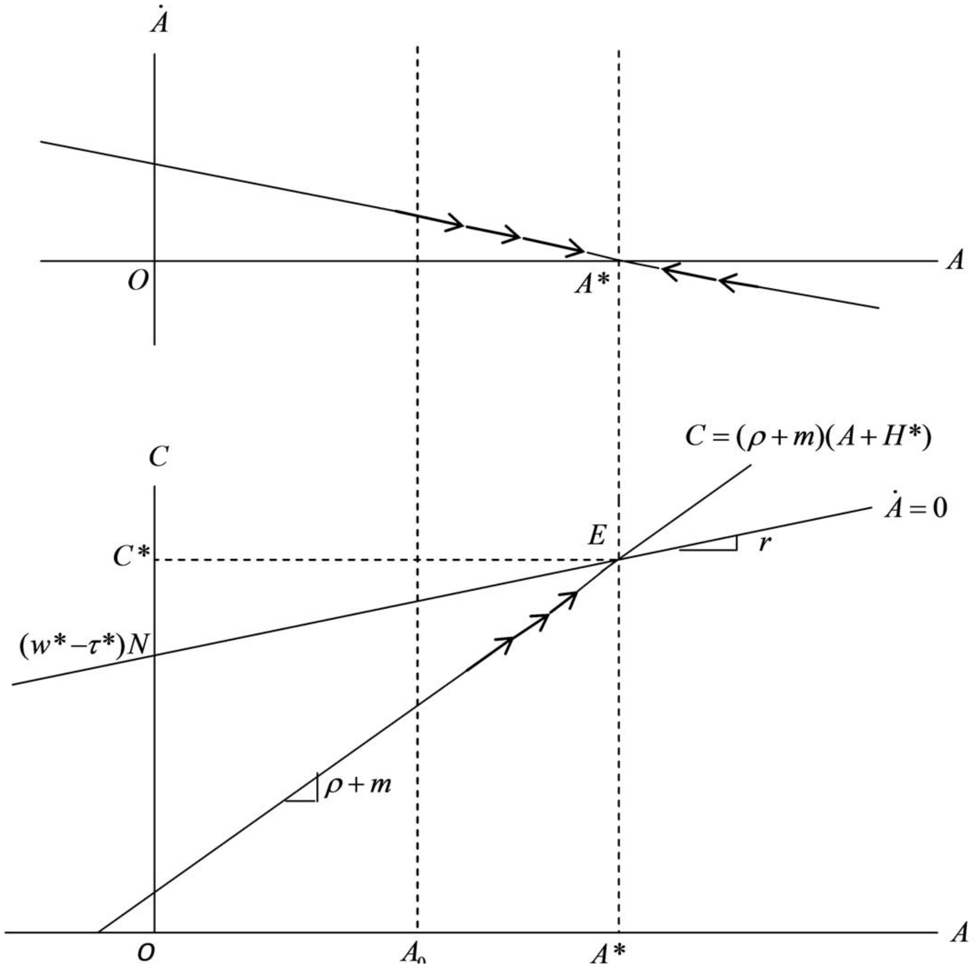

Figure 13.8: Dynamics of an SOE with *medium impatience*, i.e., $r-m < \rho < r$ (balanced budget).

in view of $dT = \bar{T} - T^* < 0$ and $r > \rho$. To the extent that the households expect the new tax level $\bar{T}$ to last a shorter time, the boost to H and C will be *less* than indicated by (13.42). This fortifies the rise in saving and the resulting growth in A.

Fiscal tightening at a higher debt level As hinted at, the fiscal policy $(\bar{G}, \bar{T})$ is not sustainable. It generates a growth rate of government debt which approaches r, whereas income and net exports are clearly bounded in the absence of economic growth.[16] To end the runaway debt spiral a fiscal tightening sooner or later is carried into effect. Suppose this happens at time $t_2 > t_1$. Let the fiscal tightening take the form of a return to a balanced budget with unchanged $\bar{G}$. That is, for $t \geq t_2$ the tax revenue is

$$T = rB_{t_2} + \bar{G} \equiv T^{*\prime} > T^*,$$

where the inequality is due to $B_{t_2} > B_0$. The corresponding per-capita tax is $\tau^{*\prime} \equiv T^{*\prime}/N > \tau^*$.

Since the budget is now balanced, a phase diagram of the same form as in Fig. 13.8 is again valid and is depicted in Fig. 13.9. Compared with Fig. 13.8 the $\dot{A} = 0$ line is shifted downwards because $w^* - \tau^{*\prime}$ is lower than before t_1. For the same reason the new level of human wealth, $H^{*\prime}$, is lower than the old, H^*. So the line representing the consumption function is also shifted down compared to the situation before t_1. Immediately after time t_2 the economy is at some point like P, where the vertical line $A = A_{t_2}$ $(> A^*)$ crosses the new line representing the consumption function. The economy then moves along that line and converges toward the new steady state, E'. At that point we have $A = A^{*\prime} < A^*$ and $C = C^{*\prime} < C^*$.

As a consequence *national wealth* goes down *more* than one to one with the increase in government debt when we are in the medium impatience case. Indeed, for a given level $\bar{B}$ of government debt, long-run national wealth is

$$A^{n*} \equiv A^* - \bar{B}. \tag{13.43}$$

An increase in government debt by $d\bar{B}$ increases national wealth by $\Delta A^{n*} \approx dA^{n*} = (\partial A^*/\partial \bar{B} - 1)d\bar{B} < -d\bar{B}$, since $\partial A^*/\partial \bar{B} < 0$ when $r - m < \rho < r$. The explanation follows from the analysis above. On top of the reduction of government wealth by $d\bar{B}$ there is a reduction of private financial wealth due to the private dissaving during the adjustment process. This dissaving occurs

[16] Indeed, as in the analogue situation for the closed economy, $\dot{B}_t/B_t = r + (\bar{G} - \bar{T})/B_t \to r$ for $t \to \infty$. Because we ignore economic growth, lasting budget deficits indicate an unsustainable fiscal policy.

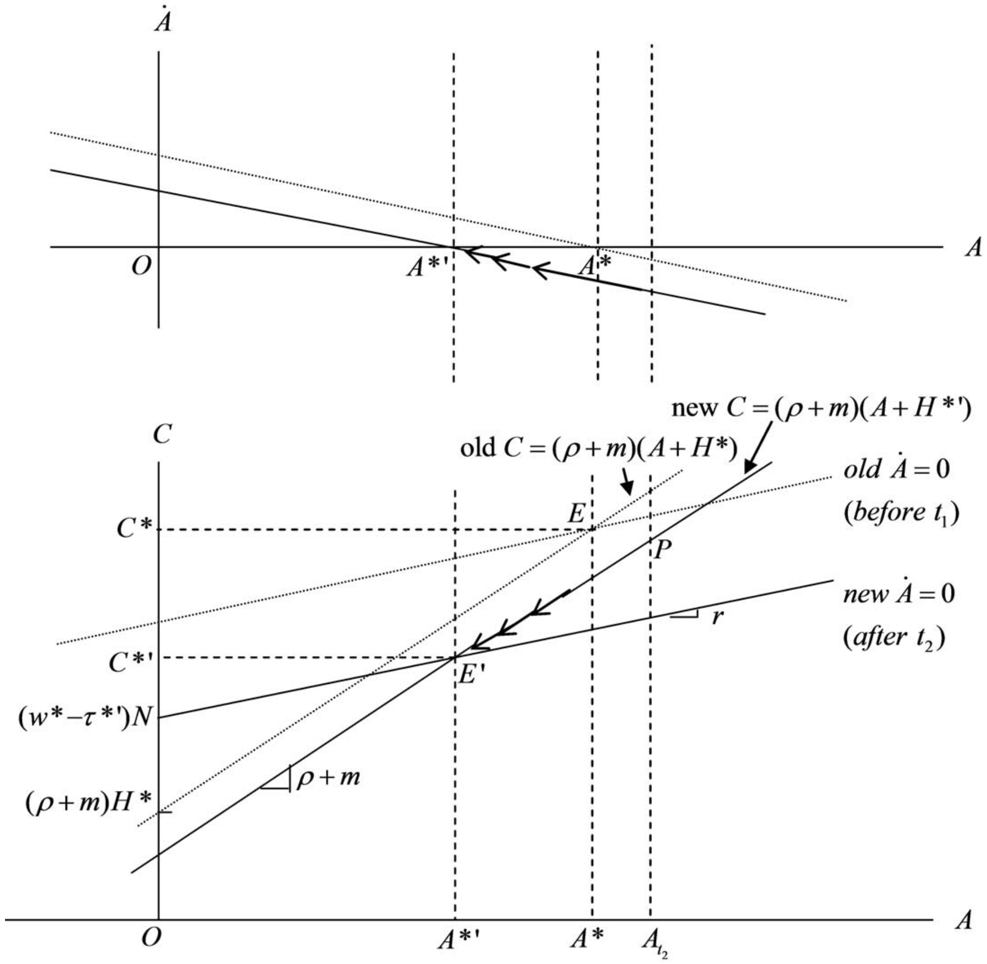

Figure 13.9: The adjustment after time t_2 showing the effect of a higher level of government debt.

because consumption responds less than one to one (in the opposite direction) when T is changed, cf. (13.42).

To find the exact long-run effect on national wealth of a rise in $\bar{B}$, in (13.35) replace B_0 by $\bar{B}$ and substitute into (13.39) to get

$$A^* = \frac{(r-\rho)(w^*N - r\bar{B} - \bar{G})}{(r+m)(\rho+m-r)}. \tag{13.44}$$

Inserting this into (13.43), we find the effect of public debt on national wealth in steady state to be

$$\frac{\partial A^{n*}}{\partial \bar{B}} = -\frac{(r-\rho)r}{(r+m)(\rho+m-r)} - 1. \tag{13.45}$$

This gives the size of the long-run effect on national wealth when a temporary tax cut causes a unit increase in long-run government debt. In our present medium impatience case, $r - m < \rho < r$ and so (13.45) implies $\partial A^{n*}/\partial \bar{B} < -1$.[17]

Very high impatience: $\rho > r$

Also this case with high impatience is a sub-case of (13.41). When $\rho > r$, (13.45) gives $-1 < \partial A^{n*}/\partial \bar{B} < 0$. This is because such an economy will have $0 < \partial A^*/\partial \bar{B} < 1$. In view of the high impatience, $A^* < 0$. That is, in the long run the SOE has *negative* private financial wealth reflecting that all physical capital in the country and some of the human wealth is essentially mortgaged to foreigners. This outcome is not plausible in practice. Owing to credit market imperfections there is likely to be difficulties of refinancing the debt in such a situation. In addition, politically motivated government intervention will presumably hinder such a development before national wealth is in any way close to zero.

Very low impatience: $\rho < r - m$

When $\rho < r - m$, an economically relevant steady state no longer exists since that would, by (13.40), require negative consumption. In the lower panel of Fig. 13.9 the slope of the $C = (\rho + m)(A + H^*)$ line will be smaller than that of the $\dot{A} = 0$ line and the two lines will never cross for a positive C.[18] With initial total wealth positive (i.e., $A_0 > -H^*$), the excess of r over $\rho + m$ results in sustained positive saving so as to keep A growing forever along the $C = (\rho + m)(A + H^*)$

[17] In the knife-edge case $\rho = r$, we get $A^* = 0$. In this case $\partial A^{n*}/\partial \bar{B} = -1$.

[18] In the upper panel of Fig. 13.9 the line representing $\dot{A}$ as a function of A will have positive slope. The stability condition (13.41) is no longer satisfied. There is still a "mathematical" steady-state value $A^* < 0$, but it can not be realized, because it requires negative consumption.

line. That is, the economy grows *large*. In the long run the interest rate in the world financial market can no longer be considered independent of this economy − the SOE framework ceases to fit.

As long as the country is still relatively small, however, we may use the model as an approximation. Though there is no steady state level of national wealth to focus at, we may still ask how the time path of national wealth, A_t^n, is affected by a rise in government debt caused by a temporary tax cut during the time interval $[t_1, t_2)$. We consider the situation after time t_2, where there is again a balanced government budget. For all $t \geq t_2$ we have $A_t^n = A_t - \bar{B}$, where $\bar{B} = B_{t_2}$ and, in analogy with (13.38),

$$A_t = (A_{t_2} - A^*)e^{(r-\rho-m)(t-t_2)} + A^*,$$

with A^* defined as in (13.44) (now a repelling state). For a given $A_{t_2} > -H^{*\prime}$ we find for $t > t_2$

$$\begin{aligned} \frac{\partial A_t^n}{\partial \bar{B}} &= \frac{\partial A_t}{\partial \bar{B}} - 1 = \left(1 - e^{(r-\rho-m)(t-t_2)}\right) \frac{\partial A^*}{\partial \bar{B}} - 1 \\ &= \left(1 - e^{(r-\rho-m)(t-t_2)}\right) \left(-\frac{(r-\rho)r}{(r+m)(\rho+m-r)}\right) - 1, \end{aligned} \quad (13.46)$$

by (13.44).[19] Since $\rho < r - m$, the right-hand side of (13.46) is less than -1 and over time rising in absolute value. In spite of the lower private saving triggered by the higher taxation after time t_2, private saving remains positive due to the low rate of impatience. Financial wealth is thus still rising and so is private income. But the lower saving out of a rising income implies more and more "forgone future income". This explains the rising crowding out envisaged by (13.46).

Current account deficits and foreign debt

Do persistent current account deficits in the balance of payments signify future borrowing problems and threatening bankruptcy? To address this question we need a few new variables.

Let NX_t denote net exports (exports minus imports). Then, the output-expenditure identity reads

$$Y_t = C_t + I_t + G_t + NX_t. \quad (13.47)$$

Net foreign assets are denoted A_t^f and equals minus net foreign debt, $-D_t =$

[19] The condition $A_{t_2} > -H^{*\prime}$ is needed for economic relevance since otherwise $C_{t_2} \leq 0$. The condition also ensures $A_{t_2} > A^*$, since $A^* < -H^{*\prime}$ when $\rho < r - m$.

$A_t - B_t - K_t$. Gross national income is $Y_t + rA_t^f = Y_t - rD_t$.[20] The current account surplus at time t is

$$\begin{aligned} CAS_t &= \dot{A}_t^f = \dot{A}_t - \dot{B}_t - \dot{K}_t = rA_t^f + NX_t \\ &= Y_t + rA_t^f - (C_t + I_t + G_t), \end{aligned} \tag{13.48}$$

by (13.47). The first line views CAS from the perspective of changes in assets and liabilities. The second line views it from an income-expenditure perspective, that is, the current account surplus is the excess of gross national income over and above home expenditure. Gross national saving, S_t, equals, by definition, gross national income minus the sum of private and public consumption, that is, $S_t = Y_t + rA_t^f - C_t - G_t$. Hence, the current account surplus can also be written as the excess of gross national saving over and above gross investment: $CAS_t = S_t - I_t$. Of course, the current account deficit is $CAD_t \equiv -CAS_t = I_t - S_t$.

In our SOE model above, with constant $r > 0$ and no economic growth, the capital stock is a constant, K^*. Then (13.47) gives net exports as a residual:

$$NX_t = F(K^*, N) - C_t - \delta K^* - \bar{G}, \tag{13.49}$$

where $C_t = (\rho + m)(A_t + H_t)$. We concentrate on the case where an economic steady state exists and is asymptotically stable, i.e., (13.41) holds. In the steady state being in force for $t < t_1$, $B_t = B_0$, $H_t = H^*$, and $A_t = A^*$, as given in (13.36) and (13.39), respectively. Thus, $A_t^f = A^* - B_0 - K^* \equiv A^{f*} \equiv -D^*$ so that $0 = \dot{A}_t^f = CAS_t \equiv -CAD_t$. Then, by (13.48),

$$NX_t = -rA^{f*} = rD^*. \tag{13.50}$$

This should also be the value of net exports we get from (13.49) in steady state. To check this, we consider

$$\begin{aligned} NX_t &= F(K^*, N) - C^* - \delta K^* - \bar{G} = F_K(K^*, N)K^* + F_L(K^*, N)N - C^* - \delta K^* - \bar{G} \\ &= (r + \delta)K^* + w^*N - C^* - \delta K^* - \bar{G}, \end{aligned}$$

where we have used Euler's theorem on a function homogeneous of degree one. Combining with (13.32) evaluated in steady state, we thus have

$$\begin{aligned} NX_t &= (r + \delta)K^* + w^*N - (rA^* + (w^* - \tau^*)N) - \delta K^* - \bar{G} \\ &= r(K^* - A^*) + \tau^*N - \bar{G} = r(K^* - A^* - B_0) = rD^*, \end{aligned}$$

where the third equality follows from the assumption of a balanced budget. Our accounting is thus coherent.

[20] In a more general setup also net foreign worker remittances, which we here ignore, should be added to GDP to calculate gross national income.

We see that permanent foreign debt is consistent with a steady state if net exports are sufficient to match the interest payments on the debt. That is, a steady state does not require trade balance, but a balanced *current account.* As we shall see in a moment, in an economy with economic *growth* not even the current account need be balanced. Before leaving the non-growing economy, however, a few remarks about the current account *out of steady state* are in place.

Emergence of twin deficits Consider again the fiscal easing regime ruling in the time interval $[t_1, t_2)$. The higher C_t resulting from the fiscal easing leads to a lower NX_t than before t_1, cf. (13.49). As a result, $CAD_t > 0$. So a current account deficit has emerged in response to the government budget deficit. This situation is known as the *twin deficits.* As we argued, the situation is not sustainable. Sooner or later, the incipient lack of solvency will manifest itself in difficulties with continued borrowing. Something must be changed.

From mere accounting we know that the current account deficit can also be written as the difference between aggregate net investment, I_t^n, and aggregate net saving, S_t^n. So

$$\begin{aligned} CAD_t &= I_t - S_t = I_t - \delta K_t - (S_t - \delta K_t) = I_t^n - S_t^n \\ &= I_t^n - (S_t^p + S_t^g) = I_t^n - S_t^p + \dot{B}_t, \end{aligned} \tag{13.51}$$

since public saving, S_t^g, equals $-\dot{B}_t$, the negative of the budget deficit. Now, starting from a balanced budget and balanced current account, whether a budget deficit tends to generate a current account deficit depends on how net investment and net private saving respond. In the present example we have $K_t = K^*$ and thereby $I_t^n = 0$ for all t. For $t < t_1$, also $S_t^p = rA^* + (w^* - \tau^*)N - C^* = 0$ and $\dot{B}_t = 0$. In the time interval $[t_1, t_2)$, we have $S_t^p > 0$ as well as $\dot{B}_t > 0$, but the budget deficit dominates and results in $CAD_t > 0$.

As before, suppose the government addresses the lack of fiscal sustainability by increasing taxation as of time t_2 so that the government budget is balanced for $t \geq t_2$. Then again $\dot{B}_t = 0$. Yet for a while $CAD_t > 0$ because now $S_t^p < 0$ as reflected in $\dot{A}_t < 0$, cf. Fig. 13.9. The deficit on the current account is, however, only temporary and not a signal of an impending default. It reflects that it takes time to complete the full downward adjustment of private consumption after the fiscal tightening.[21]

Let us consider a different scenario, namely one where the fiscal easing after time t_1 takes the form of a shift in government consumption to $\bar{G}' > \bar{G}$ without any change in taxation. Suppose the household sector expects that a fiscal tightening will not happen for a long time to come. Then, H_t and C_t are essentially unaffected, i.e., $C_t = C^*$ and $H_t = H^*$ as before t_1. So also A remains

[21] By construction of the model, households agents in the private sector are never insolvent.

at its steady-state value A^* from before t_1, given in (13.39). Owing to the absence of private saving, the government deficit must be fully financed by foreign borrowing. Indeed, by (13.51),

$$CAD_t = \dot{B}_t > 0$$

in this case. Here the two deficits exactly match each other. The situation is not sustainable, however. Government debt is mounting and if default is to be avoided, sooner or later fiscal policy must change.

It is the absence of Ricardian equivalence that suggests a positive relationship between budget and current account deficits. On the other hand, the course of events after t_2 in this example illustrates that a current account deficit *need not* coincide with a budget deficit. The empirical evidence on the relationship between budget and current account deficits is not entirely clear-cut. A cross-country regression analysis for 19 OECD countries with each country's data averaged over the 1981-86 period pointed to a positive relationship.[22] In fact, the attention to twin deficits derives from this period. Moreover, time series for the U.S. in the 1980s and first half of the 1990s also indicated a positive relationship. Nevertheless, other periods show no significant relationship. This mixed empirical evidence becomes more understandable when short-run mechanisms, with output determined from aggregate demand rather than supply, are taken into account.

The current account of a growing economy The above analysis ignored growth in GDP and therefore steady state required the current account to be balanced. It is different if we allow for economic growth. To see this, suppose there is Harrod-neutral technological progress at the constant rate g and that the labor force grows at the constant rate n. Then in steady state GDP grows at the rate $g+n$. From (13.48) follows, in analogy with the analysis of government debt in Chapter 6, that the law of movement of the foreign-debt/GDP ratio $d \equiv D/Y$ is

$$\dot{d} = (r - g - n)d - \frac{NX}{Y}. \tag{13.52}$$

A necessary condition for the SOE to remain solvent is that circumstances are such that the foreign-debt/GDP ratio does not tend to explode. For brevity, assume NX/Y remains equal to a constant, $\bar{x}$. Then the linear differential equation (13.52) has the solution

$$d_t = (d_0 - d^*)e^{(r-g-n)t} + d^*,$$

where $d^* = \bar{x}/(r - g - n)$. If $r > g + n > 0$, the SOE will have an exploding foreign-debt/GDP ratio and become insolvent vis-a-vis the rest of the world unless

[22] See Obstfeld and Rogoff (1996, pp. 144-45).

$\bar{x} \geq (r - g - n)d_0$. The right-hand-side of this inequality is an increasing function of the initial foreign debt and the growth-corrected interest rate.

Suppose $d_0 > 0$ and $\bar{x} = (r - g - n)d_0$. Then d remains positive and constant. The SOE has a permanent current account deficit in that foreign debt, D, is permanently increasing. But net exports continue to match the growth-corrected interest payments on the debt, which then grows at the same constant rate as GDP. The conclusion is that, contrary to the presumption arising from the case with no GDP growth and prevalent in the media, a country *can* have a permanent current account deficit without this being a sign of economic disease and mounting solvency problems. In this example the permanent current account deficit merely reflects that the country for some historical reason has an initial foreign debt and at the same time a rate of time preference such that only part of the interest payment is financed by net exports, the remaining part being financed by allowing the foreign debt to grow at the same speed as production.

The required net exports-income ratio, $(r - g - n)d_0$, measures the burden that the foreign debt imposes on the country. And if the foreign debt directly or indirectly is *public* debt, the additional problem of levying sufficient taxation to service the debt arises. If we go a little outside the model and allow credit market imperfections, the higher the net exports-income ratio the greater the likelihood that the debtors will face financial troubles. As in Section 6.4.1, a vicious self-fulfilling expectations spiral may arise.

A worrying feature of the U.S. economy is that its foreign debt has been growing since the middle of the 1980s accompanied by a permanent *trade deficit.* The *triple deficits* characterizing the U.S. economy in the new millennium (government budget deficit, current account deficit, and trade deficit) looks like an unsustainable state of affairs.[23]

The debt crisis in Latin America in the 1980s From the mid-1970s there was an almost worldwide slowdown in economic growth. In the early 1980s, the real interest rate for Latin American countries rose sharply and net lending to corporations and governments in Latin America fell severely, as shown in Fig. 13.10. The solid line in the figure indicates the London Inter-Bank Offered Rate (LIBOR) deflated by the rate of change in export unit prices; the LIBOR is the short-term interest rate that the international banks charge each other for unsecured loans in the London wholesale money market. Interest rates charged on bank loans to Latin American countries were typically variable and based on

[23]How long time the role of the US dollar as the world's principal currency reserve can postpone a substantial depreciation of the dollar is an open question.

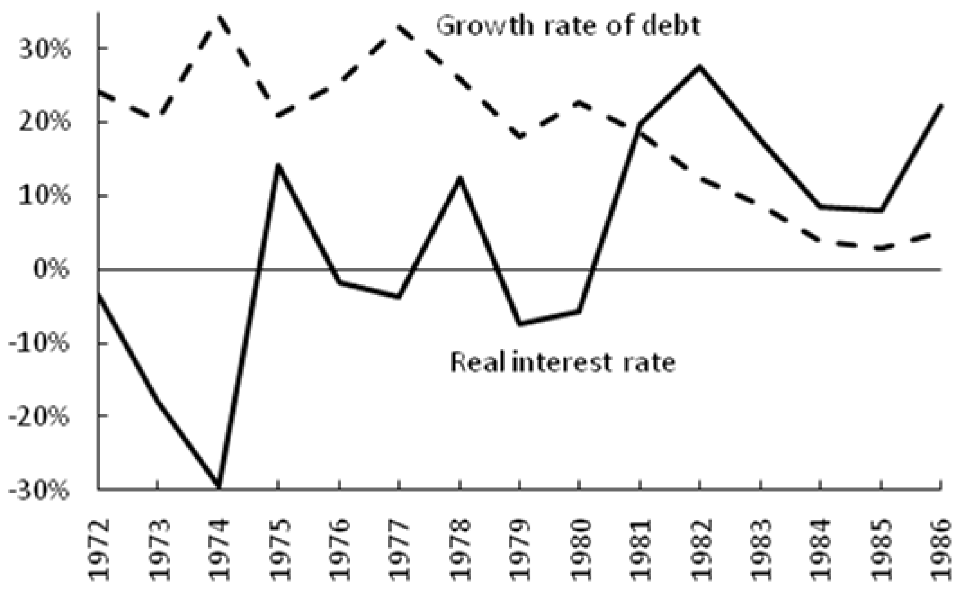

Figure 13.10

LIBOR.[24] A debt crisis ensued in the sense of mounting difficulties to refinance the debt. High interest rates and defaults resulted. Mexico suspended its payments in August 1982. By 1985, 15 countries were identified as requiring coordinated international assistance. The average debt-exports ratio (our d/x) peaked at 384 per cent in 1986 (Cline, 1995).

13.4 Government debt when taxes are distortionary*

So far we have, for simplicity, assumed that taxes are lump sum. Now we introduce a simple form of income taxation. We build on the same version of the Blanchard OLG model as was considered in Section 13.1. That is, the economy is closed, there is technological progress at the rate $g \geq 0$, and the population grows at the rate $n \geq 0$, whereas retirement is ignored (i.e., $\lambda = 0$). In addition to income taxation we bring in specific assumptions about government expenditure, namely that spending on goods and services as well as transfers grow at

[24] The correlation coefficient between the two variables in Fig. 13.10 is -0.615. The growth rate of total external debt is based on data for the following countries: Argentina, Bolivia, Brazil, Chile, Columbia, Costa Rica, Cuba, Dominican Republic, Ecuador, El Salvador, Guatemala, Haiti, Honduras, Mexico, Nicaragua, Panama, Paraguay, Peru, Uruguay, and Venezuela.

the rate $g+n$. The focus is on capital income taxation. Two main points of the analysis are that (a) capital income taxation results in lower capital intensity and consumption in the long run (if the economy is dynamically efficient); and (b) a higher level of government debt requires higher taxation and tends thereby to increase the excess burden of taxation.

Elements of the model

The household sector Assume there is a flat tax on the return on financial wealth at the rate τ_r. That is, an individual, born at time v and still alive at time $t \geq 0$, with financial wealth a_{vt} has to pay a tax equal to $\tau_r r_t a_{vt}$ per time unit, where τ_r is a given constant capital-income tax rate, $0 \leq \tau_r < 1$. The actuarial compensation is not taxed since it does not represent genuine income. There is symmetry in the sense that if $a_{vt} < 0$, then the tax acts as a subsidy (tax deductibility of interest payments). Labor income and transfers are taxed at a flat time-dependent rate, $\tau_{wt} < 1$. Only in steady state is the labor-income tax rate constant. Because labor supply is inelastic in the model, τ_{wt} acts like a lump-sum tax and is not of interest *per se.* Yet we include τ_{wt} in the analysis in order to have a simple tax instrument which can be adjusted to ensure a balanced budget when needed.

The dynamic accounting equation for the individual is

$$\dot{a}_{vt} = [(1-\tau_r)r_t + m]\, a_{vt} + (1-\tau_{wt})(w_t + x_t) - c_t, \qquad a_{v0} \text{ given},$$

where x_t is a lump-sum per-capita transfer. The No-Ponzi-Game condition, as seen from time $t_0 \geq v$, is

$$\lim_{t\to\infty} a_{vt} e^{-\int_{t_0}^{t}[(1-\tau_r)r_s + m]ds} \geq 0,$$

and the transversality condition requires that this holds with strict equality.

With logarithmic utility the Keynes-Ramsey rule takes the form

$$\frac{\dot{c}_{vt}}{c_{vt}} = (1-\tau_r)r_t + m - (\rho + m) = (1-\tau_r)r_t - \rho,$$

where $\rho \geq 0$ is the rate of time preference and $m > 0$ is the actuarial compensation, which equals the death rate. The consumption function is

$$c_{vt} = (\rho + m)(a_{vt} + h_t), \tag{13.53}$$

where

$$h_t = \int_t^{\infty} (1-\tau_{ws})(w_s + x_s) e^{-\int_t^s [(1-\tau_r)r_z + m]dz} ds. \tag{13.54}$$

At the aggregate level changes in financial wealth and consumption are:

$$
\begin{aligned}
\dot{A}_t &= (1-\tau_r)r_t A_t + (1-\tau_{wt})(w_t + x_t)N_t - C_t, \qquad \text{and} \\
\dot{C}_t &= \left[(1-\tau_r)r_t - \rho + n\right] C_t - \beta(\rho + m)A_t,
\end{aligned}
$$

respectively, where β is the birth rate.

Production The description of production follows the standard one-sector neoclassical competitive setup. The representative firm has a neoclassical production function, $Y_t = F(K_t, \mathcal{T}_t L_t)$, with constant returns to scale, where $\mathcal{T}_t$ (to be distinguished from the tax revenue T) is the exogenous technology level, assumed to grow at the constant rate $g \geq 0$. In view of profit maximization under perfect competition we have

$$
\begin{aligned}
\frac{\partial Y_t}{\partial K_t} &= f'(\tilde{k}_t) = r_t + \delta, \qquad \tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t), && (13.55) \\
\frac{\partial Y_t}{\partial L_t} &= \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] \mathcal{T}_t = w_t, && (13.56)
\end{aligned}
$$

where $\delta > 0$ is the constant capital depreciation rate and f is the production function in intensive form, given by $\tilde{y} \equiv Y/(\mathcal{T}L) = F(\tilde{k}, 1) \equiv f(\tilde{k})$, $f' > 0, f'' < 0$. We assume f satisfies the Inada conditions. In equilibrium, $L_t = N_t$, so that $\tilde{k}_t = K_t/(\mathcal{T}_t N_t)$, a pre-determined variable.

The government sector Government spending on goods and services, G, and transfers, X, grow at the same rate as the work force measured in efficiency units. Thus,

$$G_t = \gamma \mathcal{T}_t N_t, \qquad X_t = \chi \mathcal{T}_t N_t, \qquad \gamma, \chi > 0. \tag{13.57}$$

Gross tax revenue, $\tilde{T}_t$, is given by

$$\tilde{T}_t = \tau_r r_t A_t + \tau_{wt}(w_t + x_t)N_t. \tag{13.58}$$

Budget deficits are financed by bond issue whereby

$$
\begin{aligned}
\dot{B}_t &= r_t B_t + G_t + X_t - \tilde{T}_t && (13.59) \\
&= (1-\tau_r)r_t B_t + \gamma \mathcal{T}_t N_t + (1-\tau_{wt})\chi \mathcal{T}_t N_t - \tau_r r_t K_t - \tau_{wt} w_t N_t,
\end{aligned}
$$

where we have used (13.57) and the fact that in general equilibrium $A_t = K_t + B_t$. We assume parameters are such that in the long run the after-tax interest rate is higher than the output growth rate. Then government solvency requires the No-Ponzi-Game condition

$$\lim_{t\to\infty} B_t e^{-\int_0^t (1-\tau_r)r_s ds} \leq 0.$$

It is convenient to normalize the government debt by dividing with the effective labor force, $\mathcal{T}N$. Thus, we consider the ratio $\tilde{b}_t \equiv B_t/(\mathcal{T}_t N_t)$. By logarithmic differentiation w.r.t. t we find $\dot{\tilde{b}}_t/\tilde{b}_t = \dot{B}_t/B_t - (g+n)$, so that

$$\dot{\tilde{b}}_t = \frac{\dot{B}_t}{\mathcal{T}_t N_t} - (g+n)\tilde{b}_t = [(1-\tau_r)r_t - g - n]\,\tilde{b}_t + \gamma + (1-\tau_{wt})\chi - \tau_r r_t \tilde{k}_t - \tau_{wt}\tilde{w}_t,$$

where $\tilde{w}_t \equiv w_t/\mathcal{T}_t$. The tax τ_r redistributes income from the wealthy (here the old) to the poor (here the young), because the old have above-average financial wealth and the young have below-average wealth.

General equilibrium

Using that $n \equiv \beta - m$, we end up with three differential equations in $\tilde{k}$, $\tilde{c} \equiv C/(TN)$, and $\tilde{b}$:

$$\begin{aligned}
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \tilde{c}_t - \gamma - (\delta + g + \beta - m)\tilde{k}_t, && (13.60)\\
\dot{\tilde{c}}_t &= \Big[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - \rho - g\Big]\,\tilde{c}_t - \beta(\rho+m)(\tilde{k}_t + \tilde{b}_t), && (13.61)\\
\dot{\tilde{b}}_t &= \Big[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - g - (\beta - m)\Big]\,\tilde{b}_t + \gamma + (1-\tau_{wt})\chi \\
&\quad -\tau_r(f'(\tilde{k}_t) - \delta)\tilde{k}_t - \tau_{wt}\tilde{w}(\tilde{k}_t), && (13.62)
\end{aligned}$$

where $\tilde{w}(\tilde{k}_t) \equiv f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)$, cf. (13.56). Initial values of $\tilde{k}$ and $\tilde{b}$ are historically given and from the NPG condition of the government we get the terminal condition

$$\lim_{t\to\infty} \tilde{b}_t e^{-\int_0^t \left[(1-\tau_r)(f'(\tilde{k}_s)-\delta)-g-(\beta-m)\right]ds} = 0, \tag{13.63}$$

assuming that the NPG condition is not "over-satisfied".

Suppose that for $t \geq 0$ the growth-corrected budget deficit is "structurally balanced" in the sense that the growth-corrected debt is constant. Thus, $\tilde{b}_t = \tilde{b}_0$ for all $t \geq 0$. This requires that the labor income tax τ_{wt} is continually adjusted so that, from (13.62),

$$\tau_{wt} = \frac{1}{\chi + \tilde{w}(\tilde{k}_t)}\left\{\Big[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - g - (\beta - m)\Big]\,\tilde{b}_0 + \gamma + \chi - \tau_r(f'(\tilde{k}_t) - \delta)\tilde{k}_t\right\}. \tag{13.64}$$

Then (13.61) simplifies to

$$\dot{\tilde{c}}_t = \left[(1-\tau_r)(f'(\tilde{k}_t)-\delta)-\rho-g\right]\tilde{c}_t - \beta(\rho+m)(\tilde{k}_t+\tilde{b}_0),$$

which together with (13.60) constitutes an autonomous two-dimensional dynamic system. Only the capital income tax, τ_r, enters these dynamics. The labor income tax τ_{wt} does not. This is a trivial consequence of the model's simplifying assumption that labor supply is inelastic.

To construct the phase diagram for this system, note that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - \gamma - (\delta+g+\beta-m)\tilde{k}, \tag{13.65}$$

$$\dot{\tilde{c}} = 0 \text{ for } \tilde{c} = \frac{\beta(\rho+m)(\tilde{k}+\tilde{b}_0)}{(1-\tau_r)(f'(\tilde{k})-\delta)-\rho-g}. \tag{13.66}$$

There are two benchmark values of the effective capital-labor ratio, $\tilde{k}$. The first is the golden rule value, $\tilde{k}_{GR}$, given by $f'(\tilde{k}_{GR})-\delta = g+n$. The second is that value at which the denominator in (13.66) vanishes, that is, the value, $\bar{\tilde{k}}$, satisfying

$$(1-\tau_r)(f'(\bar{\tilde{k}})-\delta) = \rho + g.$$

The phase diagram is shown in Fig. 13.11. We assume $\tilde{b}_0 > 0$. But at the same time $\tilde{b}_0$ and γ are assumed to be "modest", given $\tilde{k}_0$, such that the economy initially is to the right of the totally unstable steady state close to the origin.

We impose the parameter restriction $\rho \geq n$, which implies $\bar{\tilde{k}} \leq k_{GR}$ for any $\tau_r \in [0,1)$, thus ensuring $\tilde{k}^* < k_{GR}$, in view of $\tilde{k}^* < \bar{\tilde{k}}$. That is,

$$f'(\tilde{k}^*) - \delta > f'(\bar{\tilde{k}}) - \delta = \frac{\rho+g}{1-\tau_r} \geq \frac{g+n}{1-\tau_r} \geq g+n.$$

It follows that (13.63) holds at the steady state, E.[25] At time 0 the economy will be where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the (stippled) saddle path. Over time the economy moves along this saddle path toward the steady state E with real interest rate equal to $r^* = f'(\tilde{k}^*) - \delta$. Further, in steady state the labor income tax rate is a constant,

$$\tau_w^* = \frac{\left[(1-\tau_r)(f'(\tilde{k}^*)-\delta)-g-n\right]\tilde{b}_0 + \gamma + \chi - \tau_r(f'(\tilde{k}^*)-\delta)\tilde{k}^*}{\chi + \tilde{w}(\tilde{k}^*)}, \tag{13.67}$$

[25] And so do the transversality conditions of the households. The argument is the same as in Appendix D of Chapter 12.

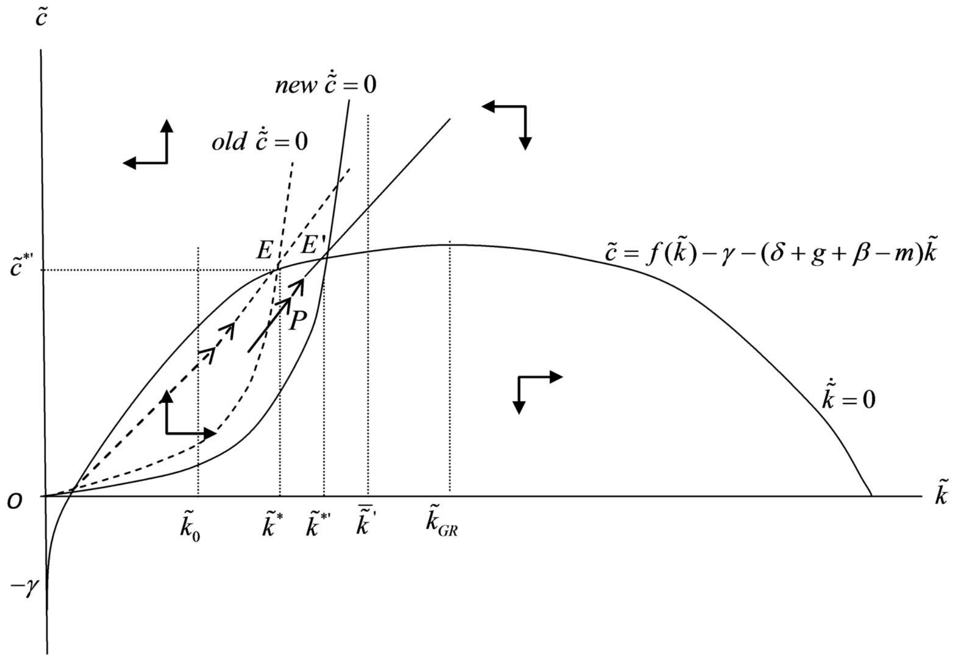

Figure 13.11: Phase diagram illustrating the effect of a fully financed reduction of capital income taxation.

from (13.64).

The capital income tax drives a wedge between the marginal transformation rate over time faced by the household, $(1-\tau_r)(f'(\tilde{k})-\delta)$, and that given by the production technology, $f'(\tilde{k})-\delta$. The implied efficiency loss is called the *excess burden* of the tax. A higher τ_r implies a greater wedge (higher excess burden) and for a given $\tilde{b}_0$, a lower $\tilde{k}^*$, cf. (13.66). Similarly, for a given τ_r, a higher level of debt, $\tilde{b}_0$, implies a lower $\tilde{k}^*$ and a higher r^* (and a corresponding adjustment of τ_w^*).[26] Finally, if for some reason (of a political nature, perhaps) τ_w^* is fixed, then a higher level of the debt may imply crowding out of $\tilde{k}^*$ for *two* reasons. First, there is the usual direct effect that higher debt decreases the scope for capital in households' portfolios. Second, there is the indirect effect, that higher debt may require a higher distortionary tax, τ_r, which further reduces capital accumulation and increases the excess burden.

We may reconsider the Ricardian equivalence issue from the perspective of both these effects. The Ricardian equivalence proposition says that when taxes are lump-sum, their timing does not affect aggregate consumption and saving. In the first section of this chapter we highlighted some of the reasons to doubt

[26] We can not say in what direction τ_w has to be adjusted. This is because it is theoretically ambiguous in what direction $(f'(\tilde{k}^*)-\delta)\tilde{k}^*$ moves when k^* goes down.

the validity of this proposition under "normal circumstances". Encompassing the fact that most taxes are *not* lump sum casts further doubt that debt neutrality should be a reliable guide for practical policy.

A fully financed reduction of capital income taxation

Now, suppose that until time t_1, the economy has been in its steady state E. Then, unexpectedly, the tax rate τ_r is *reduced* to a lower constant level, τ_r'. The tax rate is then expected by the public to remain at this lower level forever. The government budget remains "balanced" in the sense that taxation of labor income is immediately increased such that (13.64) holds for τ_r replaced by τ_r'. This shift in taxation policy does not affect the $\dot{\tilde{k}} = 0$ locus, but the $\dot{\tilde{c}} = 0$ locus is turned clockwise. At time t_1, when the shift in taxation policy occurs, the economy jumps to the point P and follows the new saddle path toward the new steady state with higher effective capital-labor ratio. (As noted at the end of the previous chapter, such adjustments may be quite slow.)

We see that the immediate effect on consumption is negative, whereas the long-run effect is positive (as long as everything takes place to the left of the golden rule capital intensity $\tilde{k}_{GR}$). The positive long-run effect on $\tilde{k}$ is due to the higher saving brought about by the initial fall in consumption. But what is the intuition behind this initial fall? Four effects are in play, a substitution effect, a pure income effect, a wealth effect, and a government budget effect. To understand these effects from a micro perspective, the intertemporal budget constraint as seen from time t_1 of an individual born at time $v \leq t_t$ is helpful:

$$\int_{t_1}^{\infty} c_{vt} e^{-\int_{t_1}^{t} [(1-\tau_r')r_s + m] ds} dt = a_{vt_1} + h_{t_1}. \tag{IBC}$$

The point of departure is that the *after-tax interest rate* immediately rises. As a result:

1) Future consumption becomes relatively cheaper as seen from time t_1. Hence there is a *negative substitution effect* on current consumption c_{vt_1}.

2) For given total wealth $a_{vt_1} + h_{t_1}$, it becomes possible to consume more at *any* time in the future (because the present discounted value of a given consumption plan has become smaller, see the left-hand side of (IBC)). This amounts to a *positive pure income effect* on current consumption.

3) At least for a while the after-tax interest rate, $(1 - \tau_r')r + m$, is higher than without the tax decrease. Everything else equal, this affects h_{t_1} negatively, which amounts to a *negative wealth effect.*

On top of these three "standard" effects comes the fact that:

4) At least initially, a rise in τ_w is necessitated by the lower capital income taxation if an unchanged $\tilde{b}$ is to be maintained, cf. (13.64). Everything else equal, this also affects h_{t_1} negatively and gives rise to a further *negative* effect on current consumption through what we may call the *government budget effect*.[27]

To sum up, the total effect on current individual consumption of a permanent decrease in the capital income tax rate and a concomitant rise in the tax on labor income and transfers consists of the following components:

$$\begin{aligned}&\text{substitution effect} + \text{pure income effect} + \text{wealth effect}\\&+ \text{effect through the change in the government budget} \;=\; \text{total effect.}\end{aligned}$$

From the consumption function $c_{vt} = (\rho + m)(a_{vt} + h_t)$, cf. (13.53), we see that the substitution and income effects exactly cancel each other out (due to the logarithmic specification of the utility function). This implies that the negative general equilibrium effect on current consumption, visible in the phase diagram, reflects the influence of the two remaining effects.

The conclusion is that whereas a tax on an inelastic factor (in this model labor) obviously does not affect its supply, a tax on capital or on capital income affects saving and thereby capital in the future. Yet such a tax may have intended effects on income *distribution*. The public finance literature studies, among other things, under what conditions such effects could be obtained by other means (see, e.g., Myles 1995).

13.5 Public debt policy

Main text for this section not yet available. See instead Elmendorf and Mankiw, Section 5 (Course Material).

13.6 Credibility problems due to time inconsistency

(incomplete)

When outcomes depend on expectations in the private sector, government policy may face a time-inconsistency problem.

[27] The proviso "everything else equal" both here and under 3) is due to the fact that at the aggregate level counteracting feedbacks in the form of higher future real wages and lower interest rates arise during the general equilibrium adjustment.

As an example consider the question: What is the stance taken by a government on negotiating with terrorists over the release of hostages? The official line, of course, is that the government will never negotiate. But ....

13.7 Literature notes

(incomplete)

Section 13.2 essentially builds on Blanchard and Fischer (1989).

For very readable surveys about how important – empirically – the departures from Ricardian equivalence are, see for example "Symposium on the Budget Deficit" in *Journal of Economic Perspectives*, vol. 3, 1989, Himarios (1995), and Elmendorf and Mankiw (1999).

In their analysis of 26 high public debt episodes in advanced economies 1800-2011 Reinhardt et al. (2012) find higher interest rate for 15 of the episodes. They find low economic growth in 23 of the episodes.

13.8 Appendix

A. A growth formula useful for debt arithmetic

Not yet available.

B. Long-run multipliers

We show here in detail how to calculate the long-run "crowding-out" effects of increases in government consumption and debt in the closed economy model of Section 13.2. In steady state we have $\dot{K}_t = \dot{C}_t = \dot{B}_t = \dot{T}_t = 0$, hence

$$
\begin{aligned}
F(K^*, N) - \delta K^* &= C^* + \bar{G}, && (13.68)\\
(F_K(K^*, N) - \delta - \rho)C^* &= m(\rho + m)(K^* + \bar{B}), && (13.69)\\
T^* &= (F_K(K^*, N) - \delta)\bar{B} + \bar{G}. && (13.70)
\end{aligned}
$$

We consider the level $\bar{B}$ of public debt as exogenous along with public consumption $\bar{G}$ and the labor force N. The tax revenue T^* in steady state is endogenous.

Assume (realistically) that $K^* + \bar{B} > 0$. Now, at zero order in the causal structure, (13.68) and (13.69) simultaneously determine K^* and C^* as implicit functions of $\bar{G}$ and $\bar{B}$, i.e., $K^* = K(\bar{G}, \bar{B})$ and $C^* = C(\bar{G}, \bar{B})$. Hereafter, (13.70) determines the required tax revenue T^* at first order as an implicit function of $\bar{G}$ and $\bar{B}$, i.e., $T^* = T(\bar{G}, \bar{B})$.

To calculate the partial derivatives of these implicit functions, insert $C^* = F(K^*, N) - \delta K^* - \bar{G}$ from (13.68) into (13.69) to get

$$(F_K^* - \delta - \rho)(F^* - \delta K^* - \bar{G}) = m(\rho + m)(K^* + B_0).$$

Next take the total differential on both sides:

$$(F_K^* - \delta - \rho)\left[(F_K^* - \delta)dK^* - d\bar{G}\right] + C^* F_{KK}^* dK^* = m(\rho + m)(dK^* + d\bar{B}), \quad \text{i.e.,}$$

$$\mathcal{D} \cdot dK^* = (F_K^* - \delta - \rho)d\bar{G} + m(\rho + m)d\bar{B}, \tag{13.71}$$

where

$$\mathcal{D} \equiv C^* F_{KK}^* + (F_K^* - \delta - \rho)(F_K^* - \delta) - m(\rho + m), \tag{13.72}$$

and the partial derivatives are evaluated in steady state.

We now show that in the interesting steady state we have $\mathcal{D} < 0$. As demonstrated in Section 13.2, normally there are *two* steady-state points in the (K, C) plane.[28] The lower steady-state point, that with $K = \tilde{K}^*$ in Fig. 13.2, is a "source" in the sense that all trajectories in its neighborhood points away from it. So the lower steady-state point is completely unstable. The upper steady-state point, that with $K = K^*$, is saddle-point stable. This is the interesting steady state (when $\bar{G}$ and $\bar{B}$ are of moderate size). In that state the $\dot{C} = 0$ locus crosses the $\dot{K} = 0$ locus from below. Hence

$$\begin{aligned}
\frac{\partial C}{\partial K}\Big|_{\dot{C}=0} &> F_K^* - \delta, \qquad \text{i.e.,} \\
m(\rho + m)\frac{F_K^* - \delta - \rho - (K^* + \bar{B})F_{KK}^*}{(F_K^* - \delta - \rho)^2} &> F_K^* - \delta \Rightarrow \\
m(\rho + m) - m(\rho + m)\frac{(K^* + \bar{B})}{r^* - \rho}F_{KK}^* &> (r^* - \rho)r^* \Rightarrow \\
m(\rho + m) - C^* F_{KK}^* &> (r^* - \rho)r^* \Rightarrow \\
0 &> C^* F_{KK}^* + (r^* - \rho)r^* - m(\rho + m) = \mathcal{D},
\end{aligned} \tag{13.73}$$

where the first implication arrow follows from $F_K^* = F_K(K^*, N) - \delta = r^*$, the second from (13.69), and the third by rearranging. A perhaps more useful formula[29] for $\mathcal{D}$ is obtained by noting that

$$(r^* - \rho)r^* - m(\rho + m) = r^{*2} + mr^* - mr^* - \rho r^* - m(\rho + m) = (r^* + m)(r^* - (\rho + m)).$$

[28] This is so, unless $\bar{G}$ and $\bar{B}$ are so large that there is only one (a knife-edge case) or no steady state with $K > 0$.

[29] More useful in the sense of being more in line with analogue formulas for a small open economy, cf. Section 13.3.

Hence, by (13.73),

$$\mathcal{D} = C^* F^*_{KK} - (r^* + m)(\rho + m - r^*) < 0.$$

So the implicit function $K^* = K(\bar{G}, \bar{B})$ has the partial derivatives, also called the long-run or steady-state multipliers,

$$K_{\bar{G}} = \frac{\partial K^*}{\partial \bar{G}} = \frac{r^* - \rho}{\mathcal{D}} < 0, \qquad (13.74)$$

$$K_{\bar{B}} = \frac{\partial K^*}{\partial \bar{B}} = \frac{m\,(\rho + m)}{\mathcal{D}} < 0, \qquad (13.75)$$

using (13.71) and $r^* = F_K(K^*, N) - \delta > \rho$. As to the effect on K^* of balanced changes in $\bar{G}$, it follows that $\Delta K^* \approx dK^* = (\partial K^*/\partial \bar{G})d\bar{G} = (r^* - \rho)d\bar{G}/\mathcal{D} < 0$ for $d\bar{G} > 0$. This gives the size of the long-run effect on the capital stock, when public consumption is increased by $d\bar{G}$ ($d\bar{G}$ "small"), and at the same time taxation is increased so as to balance the budget and leave public debt unchanged in the indefinite future.

As to the effect on K^* of higher public debt, it follows that $\Delta K^* \approx dK^* = (\partial K^*/\partial \bar{B})d\bar{B} = m\,(\rho + m)\,d\bar{B}/\mathcal{D} < 0$ for $d\bar{B} > 0$. This formula tells us the size of the long-run effect on the capital stock, when a tax cut implies, for some time, a budget deficit and thereby a cumulative increase, $d\bar{B}$, in public debt; afterwards the government increases taxation to balance the budget forever.[30] Similarly, $\Delta r^* \approx dr^* = F_{KK}(K^*, N)dK^* \approx F_{KK}(K^*, N) \cdot (\partial K^*/\partial \bar{B})d\bar{B} > 0$, for $d\bar{B} > 0$.

The long-run or steady-state multipliers associated with the implicit function $C^* = C(\bar{G}, \bar{B})$ are now found by implicit differentiation in (13.68) w.r.t. $\bar{G}$ and $\bar{B}$, respectively. We get $\partial C^*/\partial \bar{G} = (F_K(K^*, N) - \delta)\partial K^*/\partial \bar{G} - 1 < -1$ and $\partial C^*/\partial \bar{B} = (F_K(K^*, N) - \delta)\partial K^*/\partial \bar{B} < 0$.

Similarly, from (13.70) we get $\partial T^*/\partial \bar{G} = F_{KK}(K^*, N)(\partial K^*/\partial \bar{G}) \cdot \bar{B} + 1 > 1$ and $\partial T^*/\partial \bar{B} = F_{KK}(K^*, N) \cdot (\partial K^*/\partial \bar{B})\bar{B} + F_K(K^*, N) - \delta > 0$ (since $F_{KK} < 0$).

13.9 Exercises

13.1

[30] We assume that $t_2 - t_1$, hence $d\bar{B}$, is not so large as to not allow existence of a saddle-point stable steady state with $K > 0$ after t_2.

Chapter 14

Fixed capital investment and Tobin's q

The models considered so far (the OLG models as well as the representative agent models) have ignored capital adjustment costs. In the closed-economy version of the models aggregate investment is merely a reflection of aggregate saving and appears in a "passive" way as just the residual of national income after households have chosen their consumption. We can describe what is going on by telling a story in which firms just rent capital goods owned by the households and households save by purchasing additional capital goods. In these models only households solve intertemporal decision problems. Firms merely demand labor and capital services with a view to maximizing current profits. This may be a legitimate abstraction in some contexts within long-run analysis. In short- and medium-run analysis, however, the dynamics of fixed capital investment is important. So a more realistic approach is desirable.

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. One reason for this is that capital goods are often firm-specific, designed and adapted to the firm in which they are an integrated part. The capital goods are therefore generally worth more to the user than to others.

Tobin's *q-theory of investment* (after the American Nobel laureate James Tobin, 1918-2002) is an attempt to model these features. In this theory,

(a) *firms* make the *investment decisions* and *install* the purchased capital goods in their own businesses with the aim of maximizing discounted expected earnings in the future;

(b) there are certain *adjustment costs* associated with this investment: before acquiring new capital goods there are planning and design costs, and along with the implementation of the investment decisions there are costs of installation of the new equipment, costs of reorganizing the plant, costs of retraining workers to operate the new machines etc.;

(c) the adjustment costs are *strictly convex* so that marginal adjustment costs are increasing in the level of investment − think of constructing a plant in a month rather than a year.

The strict convexity of adjustment costs is the crucial constituent of the theory. It is that element which assigns investment decisions an *active* role in the model. There will be both a well-defined saving decision and a well-defined investment decision, separate from each other. Households decide the saving, firms the physical capital investment; households accumulate financial assets, firms accumulate physical capital. As a result, in a closed economy the current and expected future interest rates have to adjust for aggregate demand for goods (consumption plus investment) to match aggregate supply of goods. The role of interest rate changes is no longer to clear a rental market for capital goods.

To fix the terminology, from now on the different adjustment costs associated with investment will be subsumed under the term *capital installation costs*. When faced with strictly convex installation costs, the optimizing firm has to take the *future* into account, that is, firms' forward-looking *expectations* become important. To smooth out the adjustment costs, the firm will adjust its capital stock only *gradually* when new information arises. From an analytical point of view, we thereby avoid the counter-factual implication from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world financial market changes. Moreover, sluggishness in investment is exactly what the data show. Some empirical studies conclude that only a third of the difference between the current and the "desired" capital stock tends to be covered within a year (Clark 1979).

The q-theory of investment constitutes one approach to the explanation of this sluggishness in investment. Under certain conditions, to be described below, the theory gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

14.1 Convex capital installation costs

Let the technology of a single firm be given by

$$\tilde{Y} = F(K, L),$$

where $\tilde{Y}$, K, and L are "potential output" (to be explained), capital input, and labor input per time unit, respectively, while F is a concave neoclassical production function. So we allow decreasing as well as constant returns to scale (or a combination of locally CRS and locally DRS), whereas increasing returns to scale is ruled out. Until further notice technological change is ignored for simplicity. Time is continuous. The dating of the variables will not be explicit unless needed for clarity. The increase per time unit in the firm's capital stock is given by

$$\dot{K} = I - \delta K, \qquad \delta > 0, \tag{14.1}$$

where I is gross fixed capital investment per time unit and δ is the rate of wearing down of capital (physical capital depreciation). To fix ideas, we presume the realistic case with positive capital depreciation, but most of the results go through even for $\delta = 0$.

Let J denote the firm's capital installation costs (measured in units of output) per time unit. The installation costs imply that a part of the potential output, $\tilde{Y}$, is "used up" in transforming investment goods into installed capital (possibly simply forgone due to interruptions of production during the process of installation). Only $\tilde{Y} - J$ is output available for sale.

Assuming the price of investment goods is one (the same as that of output goods), then total investment outlay per time unit are $I + J$, i.e., the direct purchase price, $1 \cdot I$, plus the indirect cost, J, associated with installation. The q-theory of investment assumes that the installation cost is a strictly convex function of gross investment and a non-increasing function of the current capital stock. Thus,

$$J = G(I, K),$$

where the installation cost function, G, satisfies

$$G(0, K) = 0,\ G_I(0, K) = 0,\ G_{II}(I, K) > 0, \text{ and } G_K(I, K) \leq 0 \tag{14.2}$$

for all K and all (I, K), respectively. For fixed $K = \bar{K}$ the graph is as shown in Fig. 14.1. Also negative gross investment, i.e., sell off of capital equipment, involves costs (for dismantling, reorganization etc.). Therefore $G_I < 0$ for $I < 0$. The important assumption is that $G_{II} > 0$ (strict convexity in I), implying that the marginal installation cost is increasing in the level of gross investment. If the firm wants to accomplish a given installation project in only half the time, then

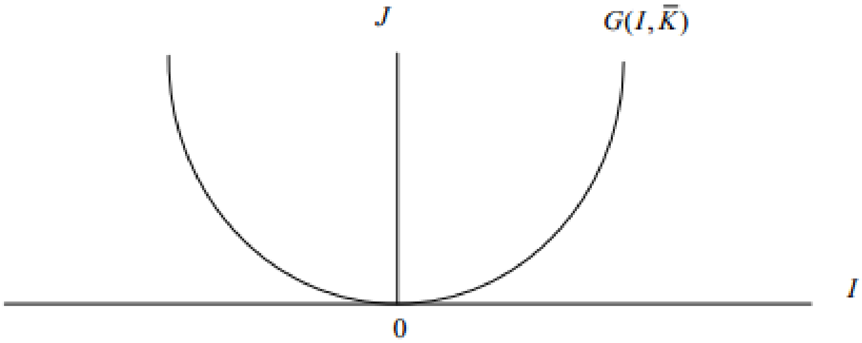

Figure 14.1: Installation costs as a function of gross investment when $K = \bar{K}$.

the installation costs are more than doubled (the risk of mistakes is larger, the problems with reorganizing work routines are larger etc.).

The strictly convex graph in Fig. 14.1 illustrates the essence of the matter. Assume the current capital stock in the firm is $\bar{K}$ and that the firm wants to increase it by a given amount $\overline{\Delta K}$. If the firm chooses the investment level $\bar{I} > 0$ per time unit in the time interval $[t, t + \Delta t)$, then, in view of (14.1), $\Delta K \approx (\bar{I} - \delta\bar{K})\Delta t$. So it takes $\Delta t \approx \overline{\Delta K}/(\bar{I} - \delta\bar{K})$ units of time to accomplish the desired increase $\overline{\Delta K}$. If, however, the firm slows down the adjustment and invests only half of $\bar{I}$ per time unit, then it takes approximately twice as long time to accomplish $\overline{\Delta K}$. Total costs of the two alternative courses of action are approximately $G(\bar{I}, \bar{K})\Delta t$ and $G(\frac{1}{2}\bar{I}, \bar{K})2\Delta t$, respectively (assuming, for simplicity, that $G_K(I, K) = 0$, and ignoring discounting). By drawing a few straight line segments in Fig. 14.1 the reader will be convinced that the last-mentioned cost is smaller than the first-mentioned due to strict convexity of installation costs (see Exercise 14.1). *Haste is waste.*

On the other hand, there are of course limits to how slow the adjustment to the desired capital stock should be. Slower adjustment means postponement of the potential benefits of a higher capital stock. So the firm faces a trade-off between fast adjustment to the desired capital stock and low adjustment costs.

In addition to the strict convexity of G with respect to I, (14.2) imposes the condition $G_K(I, K) \leq 0$. Indeed, it often seems realistic to assume that $G_K(I, K) < 0$ for $I \neq 0$. A given amount of investment may require more reorganization in a small firm than in a large firm (size here being measured by K). Owing to indivisibilities, when installing a new machine, a small firm has to stop production altogether, whereas a large firm can to some extent continue its production by shifting some workers to another production line. A further argument is that the more a firm has invested historically, the more experienced it is now concerning how to avoid large adjustment costs. So, for a given I today, the associated installation costs are lower, given a larger accumulated K.

14.1.1 The decision problem of the firm

In the absence of tax distortions, asymmetric information, and problems with enforceability of financial contracts, the Modigliani-Miller theorem (Modigliani and Miller, 1958) entails that the financial structure of the firm is both indeterminate and irrelevant for production decisions (see Appendix A). Although the conditions required for validity of this theorem are quite idealized, the q-theory of investment accepts them because they allow the analyst to concentrate on the production aspects in a first approach.

With the output good as unit of account, let the operating cash flow (the net payment stream to the firm before interest payments on debt, if any) at time t be denoted R_t (for "receipts"). Then

$$R_t \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \tag{14.3}$$

where the wage rate at time t is denoted w_t, and the market price of the investment good in terms of the output good is and remains 1. As mentioned, the installation cost $G(I_t, K_t)$ implies that a part of production, $F(K_t, L_t)$, is used up in transforming investment goods into installed capital. Only the difference $F(K_t, L_t) - G(I_t, K_t)$ is available for sale.

We ignore uncertainty and assume the firm is a price taker. The interest rate is r_t, which we assume to be positive, at least in the long run. The decision problem, as seen from time 0, is to choose a plan $(L_t, I_t)_{t=0}^{\infty}$ so as to maximize the firm's *market value*, i.e., the present value of the future stream of expected cash flows:

$$\max_{(L_t,I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t. (14.3) and} \tag{14.4}$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t), \tag{14.5}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given,} \tag{14.6}$$

$$K_t \geq 0 \text{ for all } t. \tag{14.7}$$

There is no specific terminal condition but we have posited the feasibility condition (14.7) saying that the firm can never have a negative capital stock.[1]

[1] It is assumed that w_t is a piecewise continuous function. At points of discontinuity (if any) in investment, we will consider investment to be a *right-continuous* function of time. That is, $I_{t_0} = \lim_{t \to t_0^+} I_t$. Likewise, at such points of discontinuity, by the "time derivative" of the corresponding state variable, K, we mean the *right-hand* time derivative, i.e., $\dot{K}_{t_0} = \lim_{t \to t_0^+}(K_t - K_{t_0})/(t - t_0)$. Mathematically, these conventions are inconsequential, but they help the intuition.

In the previous chapters the firm was described as solving a series of static profit maximization problems. Such a description is no longer valid, however, when there is dependence across time, as is the case here. When installation costs are present, current decisions depend on the expected future circumstances. The firm makes a plan for the whole future so as to maximize the value of the firm, which is what matters for the owners. This is the general neoclassical hypothesis about firms' behavior. As shown in Appendix A, when strictly convex installation costs or similar dependencies across time are absent, then value maximization is equivalent to solving a sequence of static profit maximization problems, and we are back in the previous chapters' description.

To solve the problem (14.4) − (14.7), where R_t is given by (14.3), we apply the Maximum Principle. The problem has two control variables, L and I, and one state variable, K. We set up the current-value Hamiltonian:

$$H(K, L, I, q, t) \equiv F(K, L) - wL - I - G(I, K) + q(I - \delta K), \tag{14.8}$$

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (14.6). For each $t \geq 0$ we maximize H w.r.t. the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w, \tag{14.9}$$

and $\partial H/\partial I = -1 - G_I(I, K) + q = 0$, i.e.,

$$1 + G_I(I, K) = q. \tag{14.10}$$

Next, we partially differentiate H w.r.t. the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (14.4):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - q\delta = rq - \dot{q}. \tag{14.11}$$

Then, the Maximum Principle says that for an interior optimal path (K_t, L_t, I_t) there exists an adjoint variable q, which is a continuous function of t, written q_t, such that for all $t \geq 0$ the conditions (14.9), (14.10), and (14.11) hold along the path. Moreover, it can be shown that the path will satisfy the "standard" infinite horizon transversality condition

$$\lim_{t\to\infty} K_t q_t e^{-\int_0^t r_s ds} = 0. \tag{14.12}$$

The optimality condition (14.9) is the usual employment condition equalizing the marginal productivity of labor to the real wage. In the present context with strictly convex capital installation costs, this condition attains a distinct role as labor will in the short run be the only variable input. Indeed, the firm's installed capital is in the short run a fixed production factor due to the strictly convex capital installation costs. So, effectively there are diminishing returns (equivalent to rising marginal costs) in the short run even though the production function might have CRS.

The left-hand side of (14.10) gives the total marginal investment cost at time t, that is, the sum of the purchase price of the investment good and the cost of its installation. So the left-hand side is the marginal cost, MC, of increasing the capital stock in the firm. Since (14.10) is a necessary condition for optimality, the right-hand side of (14.10) must then be the marginal benefit, MB, of increasing the capital stock. Hence, q_t must represent the value to the optimizing firm of having one more unit of (installed) capital at time t. To put it differently: the adjoint variable q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path.[2]

As to the interpretation of the differential equation (14.11), a condition for optimality must be that the firm acquires capital up to the point where the "marginal productivity of capital", $F_K - G_K$, equals the marginal "capital cost", $r_t q_t + (\delta q_t - \dot{q}_t)$. The first term in the latter expression represents interest costs and the second economic depreciation. In (14.11), the "marginal productivity of capital" appears as $F_K - G_K$. This expression takes into account the potential reduction, $-G_K$, of installation costs in the next instant brought about by the marginal unit of installed capital. The shadow price q_t appears as the "overall" price at which the firm can acquire the marginal unit of installed capital. At the margin, q_t can also be seen as the "overall" cost saving associated with reducing the investment by one unit. In this situation the firm recovers q_t by saving both on installation costs and the purchase in the investment goods market.

In accordance with this line of thought, by reordering in (14.11), we get the "no-arbitrage" condition

$$\frac{F_K(K, L) - G_K(I, K) - \delta q + \dot{q}}{q} = r, \tag{14.13}$$

saying that along the optimal path the rate of return on the marginal unit of installed capital must equal the interest rate.

The transversality condition (14.12) says that the present value of the capital stock "left over" at infinity must be zero. That is, the capital stock should not in the long run grow too fast, given the evolution of its discounted shadow price. In addition to necessity of (14.12) it can be shown[3] that the discounted shadow

[2] Recall that a *shadow price,* measured in some unit of account, of a good, from the point of view of the buyer, is the maximum number of units of account that he or she is willing to offer for one extra unit of the good.

[3] See Appendix B.

price itself in the far future must along an optimal path be asymptotically nil, i.e.,

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{14.14}$$

If along the optimal path, K_t grows without bound, then not only must (14.14) hold but, in view of (14.12), the discounted shadow price must in the long run approach zero *faster* than K_t grows. Intuitively, otherwise the firm would be "over-accumulating". The firm would gain by reducing the capital stock "left over" for eternity (which is like "money left on the table"). Reducing the ultimate investment and installation costs would raise the present value of the firm's expected cash flow.

In connection with (14.10) we claimed that q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path. A confirmation of this interpretation is obtained by solving the differential equation (14.11). Indeed, multiplying by $e^{-\int_0^t (r_s+\delta)ds}$ on both sides of (14.11), we get by integration and application of (14.14),[4]

$$q_t = \int_t^\infty \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\int_t^\tau (r_s+\delta)ds} d\tau. \tag{14.15}$$

The right-hand side of (14.15) is the present value, as seen from time t, of the expected future increases of the firm's cash-flow that would result if one extra unit of capital were installed at time t. Indeed, $F_K(K_\tau, L_\tau)$ is the direct contribution to output of one extra unit of capital, while $-G_K(I_\tau, K_\tau) \geq 0$ represents the potential reduction of installation costs in the next instant brought about by the marginal unit of installed capital. Note that the marginal future increases of cash-flow in (14.15) are discounted at a rate equal to the interest rate *plus* the capital depreciation rate. The reason is that from one extra unit of capital at time t there are only $e^{-\delta(\tau-t)}$ units left at time τ.

To concretize our interpretation of q_t as representing the value to the optimizing firm at time t of having one extra unit of installed capital, let us make a thought experiment. Assume that a extra units of installed capital at time t drops down from the sky. At time $\tau > t$ there are $a \cdot e^{-\delta(\tau-t)}$ units of these still in operation so that the stock of installed capital is

$$K'_\tau = K_\tau + a \cdot e^{-\delta(\tau-t)}, \tag{14.16}$$

where K_τ denotes the stock of installed capital as it would have been without this "injection". Now, in (14.3) replace t by τ and consider the optimizing firm's

[4] For details, see Appendix A.

cash-flow R_τ as a function of $(K_\tau, L_\tau, I_\tau, \tau, t, a)$. Taking the partial derivative of R_τ w.r.t. a at the point $(K_\tau, L_\tau, I_\tau, \tau, t, 0)$, we get

$$\frac{\partial R_\tau}{\partial a}_{|a=0} = [F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)]\, e^{-\delta(\tau - t)}. \tag{14.17}$$

Considering the value of the optimizing firm at time t as a function of installed capital, K_t, and t itself, we denote this function $V^*(K_t, t)$. Then at any point where V^* is differentiable, we have

$$\begin{aligned} \frac{\partial V^*(K_t, t)}{\partial K_t} &= \int_t^\infty \left(\frac{\partial R_\tau}{\partial a}_{|a=0} \right) e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty [F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)] e^{-\int_t^\tau (r_s + \delta) ds} d\tau = q_t \end{aligned} \tag{14.18}$$

when the firm moves along the optimal path. The second equality sign comes from (14.17) and the third is implied by (14.15). So the value of the adjoint variable, q, at time t equals the contribution to the firm's maximized value of a fictional marginal "injection" of installed capital at time t. This is just another way of saying that q_t represents the benefit to the firm of the marginal unit of installed capital along the optimal path.

This story facilitates the understanding that the control variables at any point in time should be chosen so that the Hamiltonian function is maximized. Thereby one maximizes the properly weighted sum of the current direct contribution to the criterion function and the indirect contribution, which is the benefit (as measured approximately by $q_t \Delta K_t$) of having a higher capital stock in the future.

As we know, the Maximum Principle gives only necessary conditions for an optimal path, not sufficient conditions. We use the principle as a tool for finding candidates for a solution. Having found in this way a candidate, one way to proceed is to check whether Mangasarian's sufficient conditions are satisfied. Given the transversality condition (14.12) and the non-negativity of the state variable, K, the only additional condition to check is whether the Hamiltonian function is jointly concave in the endogenous variables (here K, L, and I). If it is jointly concave in these variables, then the candidate *is* an optimal solution. Owing to concavity of $F(K, L)$, inspection of (14.8) reveals that the Hamiltonian function is jointly concave in (K, L, I) if $-G(I, K)$ is jointly concave in (I, K). This condition is equivalent to $G(I, K)$ being jointly convex in (I, K), an assumption allowed within the confines of (14.2); for example, $G(I, K) = (\frac{1}{2})\beta I^2/K$ as well as the simpler $G(I, K) = (\frac{1}{2})\beta I^2$ (where in both cases $\beta > 0$) will do. Thus, assuming joint convexity of $G(I, K)$, the first-order conditions and the transversality condition are not only necessary, but also sufficient for an optimal solution.

14.1.2 The implied investment function

From condition (14.10) we can derive an investment function. Rewriting (14.10), we have that an optimal path satisfies

$$G_I(I_t, K_t) = q_t - 1. \tag{14.19}$$

Combining this with the assumption (14.2) on the installation cost function, we see that

$$I_t \gtreqless 0 \text{ for } q_t \gtreqless 1, \text{ respectively,} \tag{14.20}$$

cf. Fig. 14.2.[5] By the implicit function theorem, in view of $G_{II} \neq 0$, (14.19) defines optimal investment, I_t, as an implicit function of the shadow price, q_t, and the state variable, K_t,

$$I_t = \mathcal{M}(q_t, K_t), \tag{14.21}$$

with partial derivatives

$$\frac{\partial I_t}{\partial q_t} = \frac{1}{G_{II}(\mathcal{M}(q_t, K_t), K_t)} > 0, \qquad \text{and} \qquad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(\mathcal{M}(q_t, K_t), K_t)}{G_{II}(\mathcal{M}(q_t, K_t), K_t)},$$

where the latter cannot be signed without further specification. In view of (14.20), $\mathrm{M}(1, K_t) = 0$.

It follows that optimal investment is an increasing function of the shadow price of installed capital. In view of (14.20), $\mathcal{M}(1, K) = 0$. Not surprisingly, the investment rule is: invest now, if and only if the value to the firm of the marginal unit of installed capital is larger than the price of the capital good (which is 1, excluding installation costs). At the same time, the rule says that, because of the convex installation costs, invest only up to the point where the marginal installation cost, $G_I(I_t, K_t)$, equals $q_t - 1$, cf. (14.19).

Condition (14.21) shows the remarkable information content that the shadow price q_t has. As soon as q_t is known (along with the current capital stock K_t), the firm can decide the optimal level of investment through knowledge of the installation cost function G alone (since, when G is known, so is in principle the inverse of G_I w.r.t. I, the investment function $\mathcal{M}$). All the information about the production function, input prices, and interest rates now and in the future that is relevant to the investment decision is summarized in one number, q_t. The form of the investment function, $\mathcal{M}$, depends only on the installation cost function G. These are very useful properties in theoretical and empirical analysis.

[5] From the assumptions made in (14.2), we only know that the graph of $G_I(I, \bar{K})$ is an upward-sloping curve going through the origin. Fig. 14.2 shows the special case where this curve happens to be linear.

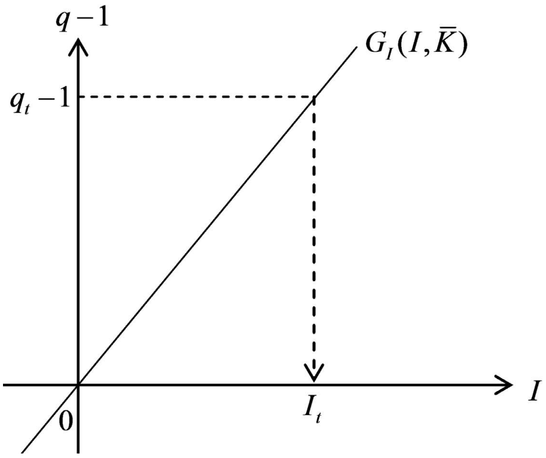

Figure 14.2: Marginal installation costs as a function of the gross investment level, I, for a given amount, $\bar{K}$, of installed capital. The optimal gross investment, I_t, when $q = q_t$ is indicated.

14.1.3 A not implausible special case

We now introduce the convenient case where the installation function G is homogeneous of degree one w.r.t. I and K so that we can, for $K > 0$, write

$$\begin{aligned} J &= G(I, K) = G(\frac{I}{K}, 1)K \equiv g(\frac{I}{K})K, \qquad \text{or} \qquad (14.22) \\ \frac{J}{K} &= g(\frac{I}{K}), \end{aligned}$$

where $g(\cdot)$ represents the installation cost-capital ratio and $g(0) \equiv G(0, 1) = 0$, by (14.2).

LEMMA 1 The function $g(\cdot)$ has the following properties:
(i) $g'(I/K) = G_I(I, K)$;
(ii) $g''(I/K) = G_{II}(I, K)K > 0$ for $K > 0$; and
(iii) $g(I/K) - g'(I/K)I/K = G_K(I, K) < 0$ for $I \neq 0$.

Proof. (i) $G_I = Kg'/K = g'$; (ii) $G_{II} = g''/K$; (iii) $G_K = \partial(g(I/K)K)/\partial K = g(I/K) - g'(I/K)I/K < 0$ for $I \neq 0$ since, in view of $g'' > 0$ and $g(0) = 0$, we have $g(x)/x < g'(x)$ for all $x \neq 0$. □

The graph of $g(I/K)$ is qualitatively the same as that in Fig. 14.1 (imagine we have $\bar{K} = 1$ in that graph). The installation cost relative to the existing capital stock is now a strictly convex function of the investment-capital ratio, I/K.

EXAMPLE 1 Let $J = G(I, K) = \frac{1}{2}\beta I^2/K$, where $\beta > 0$. Then G is homogeneous of degree one w.r.t. I and K and gives $J/K = \frac{1}{2}\beta(I/K)^2 \equiv g(I/K)$. □

A further important property of (14.22) is that the cash-flow function in (14.3) becomes homogeneous of degree one w.r.t. K, L, and I in the "normal" case where the production function has CRS. This has two implications. First, Hayashi's theorem applies (see below). Second, the q-theory can easily be incorporated into a model of economic growth.[6]

Does the hypothesis of linear homogeneity of the cash flow in K, L, and I make economic sense? According to the replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow. (Of course, owing to the possibility of indivisibilities, this reasoning does not take us all the way to linear homogeneity. Moreover, the argument ignores that also land is a necessary input. As discussed in Chapter 2, the empirical evidence on linear homogeneity is mixed.)

In view of (i) of Lemma 1, the linear homogeneity assumption for G allows us to write (14.19) as

$$g'(I/K) = q - 1. \tag{14.23}$$

This equation defines the investment-capital ratio, I/K , as an implicit function, m, of q :

$$\frac{I}{K} = m(q), \quad \text{where} \quad m(1) = 0 \quad \text{and} \quad m'(q) = \frac{1}{g''(m(q))} > 0, \tag{14.24}$$

by implicit differentiation in (14.23). In this case q encompasses all information that is of relevance to the decision about the investment-capital ratio.

In Example 1 above we have $g(I/K) = \frac{1}{2}\beta(I/K)^2$, in which case (14.23) gives $I/K = (q-1)/\beta$. So in this case we have $m(q) = q/\beta - 1/\beta$, a linear investment function, as illustrated in Fig. 14.3. The parameter β can be interpreted as the degree of sluggishness in the capital adjustment. The degree of sluggishness reflects the degree of convexity of installation costs.[7] Generally the graph of the investment function is positively sloped, but not necessarily linear. The interpretation of the stippled lines and q^* and n in Fig. 14.3 is as follows. Suppose the firm's employment grows at a constant rate n. Then a constant capital-labor ratio, K/L, requires $\dot{K}/K = n$, hence $I/K - \delta = m(q) - \delta = n$. The value of q satisfying this equation is denoted q^*.

[6] The relationship between the function g and other ways of formulating the theory is commented on in Appendix C.

[7] For a twice differentiable function, $f(x)$, with $f'(x) \neq 0$, we define the *degree of convexity* in the point x by $f''(x)/f'(x)$. So the degree of convexity of $g(I/K)$ is $g''/g' = (I/K)^{-1} = \beta(q-1)^{-1}$ and thereby we have $\beta = (q-1)g''/g'$. So, for given q, the degree of sluggishness is proportional to the degree of convexity of adjustment costs.

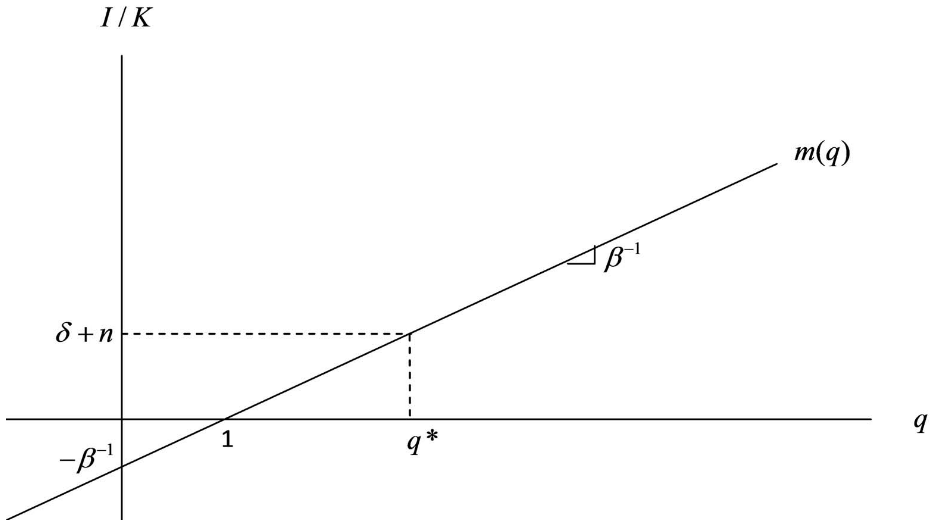

Figure 14.3: Optimal investment-capital ratio as a function of the shadow price of installed capital when $g(I/K) = \frac{1}{2}\beta(I/K)^2$.

To see how the shadow price q changes over time along the optimal path, we rearrange (14.11):

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + G_K(I_t, K_t). \tag{14.25}$$

Recall that $-G_K(I_t, K_t)$ indicates how much *lower* the installation costs are as a result of the marginal unit of installed capital. In the special case (14.22), we have, from Lemma 1,

$$G_K(I, K) = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} = g(m(q)) - (q-1)m(q),$$

using (14.24) and (14.23).

Inserting this into (14.25) gives

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.26}$$

This differential equation is very useful in macroeconomic analysis, as we will soon see, cf. Fig. 14.4 below.

In a macroeconomic context, for steady state to be achievable, gross investment must be large enough to match not only capital depreciation, but also growth in the labor input. Otherwise a constant capital-labor ratio can not be sustained. That is, the investment-capital ratio, I/K, must be equal to the sum of the depreciation rate and the growth rate of the labor force, i.e., $\delta + n$. The level of q which is required to motivate such an investment-capital ratio is called q^* in Fig. 14.3.

14.2 Marginal q and average q

Our q above, determining investment, should be distinguished from what is usually called Tobin's q or average q. In a more general context, let p_{It} denote the current purchase price (in terms of output units) per unit of the investment good (before installment). Then *Tobin's* q or *average* q, q_t^a, is defined as $q_t^a \equiv V_t/(p_{It}K_t)$, that is, Tobin's q is the ratio of the market value of the firm to the replacement value of the firm in the sense of the "reacquisition value of the capital goods before installment costs" (the top index "a" stands for "average"). In our simplified context we have $p_{It} \equiv 1$ (the price of the investment good is the same as that of the output good). Therefore Tobin's q can be written

$$q_t^a \equiv \frac{V_t}{K_t} = \frac{V^*(K_t, t)}{K_t}, \tag{14.27}$$

where the equality holds for an optimizing firm. Conceptually this is different from the firm's internal shadow price on capital, i.e., what we have denoted q_t in the previous sections. In the language of the q-theory of investment, this q_t is the *marginal* q, representing the value to the firm of one *extra* unit of installed capital relative to the price of un-installed capital equipment. The term marginal q is natural since along the optimal path, as a slight generalization of (14.18), we must have $q_t = (\partial V^*/\partial K_t)/p_{It}$. Letting q_t^m ("m" for "marginal") be an alternative symbol for this q_t, we have in our model above, where we consider the special case $p_{It} \equiv 1$,

$$q_t^m \equiv q_t = \frac{\partial V^*(K_t, t)}{\partial K_t}. \tag{14.28}$$

The two concepts, average q and marginal q, have not always been clearly distinguished in the literature. What is directly relevant to the investment decision is marginal q. Indeed, the analysis above showed that optimal investment is an increasing function of q^m. Further, the analysis showed that a "critical" value of q^m is 1 and that only if $q^m > 1$, is positive gross investment warranted.

The importance of q^a is that it can be measured empirically as the ratio of the sum of the share market value of the firm and its debt to the current acquisition value of its total capital before installment. Since q^m is much harder to measure than q^a, it is important to know the relationship between q^m and q^a. Fortunately, we have a simple theorem giving conditions under which $q^m = q^a$.

THEOREM (Hayashi, 1982) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K).[8] Then, along an optimal path we have:

[8] That is, in addition to (14.2), we assume $G_{KK} \geq 0$ and $G_{II}G_{KK} - G_{IK}^2 \geq 0$. The specification in Example 1 above satisfies this.

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. See Appendix D.

The assumption that the firm is a price taker may, of course, seem critical. The Hayashi theorem has been generalized, however. Also a monopolistic firm, facing a downward-sloping demand curve and setting its own price, may have a cash flow which is homogeneous of degree one in the three variables K, L, and I. If so, then the condition $q_t^m = q_t^a$ for all $t \geq 0$ still holds (Abel 1990). Abel and Eberly (1994) present further generalizations.

In any case, when q^m is approximately equal to (or just proportional to) q^a, the theory gives a remarkably simple operational investment function, $I = m(q^a)K$, cf. (14.24). At the macro level we interpret q^a as the market valuation of the firms relative to the replacement value of their total capital stock. This market valuation is an indicator of the expected future earnings potential of the firms. Under the conditions in (i) of the Hayashi theorem the market valuation also indicates the marginal earnings potential of the firms, hence, it becomes a determinant of their investment. This establishment of a relationship between the stock market and firms' aggregate investment is the basic point in Tobin (1969).

14.3 Applications

Capital installation costs in a closed economy

Allowing for convex capital installation costs in the economy has far-reaching implications for the causal structure of a model of a closed economy. Investment decisions attain an active role in the economy and forward-looking expectations become important for these decisions. Expected future market conditions and announced future changes in corporate taxes and depreciation allowance will affect firms' investment already today.

The essence of the matter is that current and expected future interest rates have to adjust for aggregate saving to equal aggregate investment, that is, for the output market to clear. Given full employment ($L_t = \bar{L}_t$), the output market clears when aggregate supply equals aggregate demand, i.e.,

$$F(K_t, \bar{L}_t) - G(I_t, K_t) \ (= \text{value added} \equiv GDP_t) = C_t + I_t,$$

where C_t is determined by the intertemporal utility maximization of the forward-looking households, and I_t is determined by the intertemporal value maximization of the forward-looking firms facing strictly convex installation costs. Like in the determination of C_t, current and expected future interest rates now also matter for the determination of I_t. This is the first time in this book where *clearing in the output market* is assigned an *active* role. In the earlier models investment was just a passive reflection of household saving. Desired investment was automatically equal to the residual of national income left over after consumption decisions had taken place. Nothing had to adjust to clear the output market, neither interest rates nor output. In contrast, in the present framework adjustments in interest rates and/or the output level are needed for the continuous clearing in the output market and these adjustments are decisive for the macroeconomic dynamics.

A related implication of the theory is that we have to discard the simple conception from our previous models that the real interest rate is the variable which adjusts so as to clear a rental market for capital goods. The interest rate will no longer be tied down by a requirement that such markets clear, and will, even under perfect competition, no longer in equilibrium equal the net marginal productivity of capital. This is seen for instance in the formula (14.13).

In actual economies there may of course exist "secondary markets" for used capital goods and markets for renting capital goods owned by others. In view of installation costs and similar, however, shifting capital goods from one plant to another is generally costly. Therefore the turnover in that kind of markets tends to be limited (with the exception of rental markets for cars, trucks, air planes, and similar). And, importantly for our theory, the effective capital cost per time unit for a firm that hire its capital goods, rather than buying them, will still consist not only of the simple rental rate (interest plus depreciation costs, $r + \delta$) but also costs associated with installation and now presumably also later dismantling.

In for instance Abel and Blanchard (1983), a Ramsey-style model integrating the q-theory of investment and Hayashi's theorem is presented. The authors study the two-dimensional general equilibrium dynamics resulting from the adjustment of current and expected future interest rates *needed for the output market to clear*. Adjustments of the whole structure of interest rates (the yield curve) take place and constitute the equilibrating mechanism in the output and asset markets.

By having output market equilibrium playing this role in the model, a first step is taken toward medium- and short-run macroeconomic theory. We take further steps in later chapters, by allowing imperfect competition and nominal price rigidities to enter the picture. Then the demand side gets an active role both in the determination of q (and thereby investment) and in the determination of aggregate output and employment. This is what Keynesian theory (old and new) deals with.

In the remainder of this chapter we will still assume perfect competition in all markets including the labor market. In this sense we will stay within the neoclassical framework (supply-dominated models) where, by instantaneous adjustment of the real wage, labor demand continuously matches labor supply. The next two subsections present simple examples of how Tobin's q-theory of investment can be integrated into the neoclassical framework. To avoid the complications arising from an endogenous interest rate, the focus is on a small open economy. In that context, households financial wealth is distinct from the market value of the capital stock, and the theoretical analysis is not dependent on Hayashi's theorem.

A small open economy with capital installation costs

By introducing convex capital installation costs in a model of a small open economy (SOE), we avoid the counterfactual outcome that the capital stock adjusts *instantaneously* when the interest rate in the world financial market changes. In the standard neoclassical growth model for a small open economy, without convex capital installation costs, a rise in the interest rate leads immediately to a complete adjustment of the capital stock so as to equalize the net marginal productivity of capital to the new higher interest rate. Moreover, in that model expected *future* changes in the interest rate or in corporate taxes and depreciation allowances do *not* trigger an investment response until these changes actually happen. In contrast, when convex installation costs are present, expected future changes tend to influence firms' investment already today.

We assume:

1. Perfect mobility across borders of goods and financial capital.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility across borders of labor.
4. Labor supply is inelastic and constant and there is no technological progress.
5. The capital installation cost function $G(I, K)$ is homogeneous of degree 1.

In this setting the SOE faces an exogenous interest rate, r, given from the world financial market. We assume r is a positive constant. The aggregate production function, $F(K_t, L_t)$, is neoclassical and concave as in the previous sections. Suppose markets are competitive. Let $\bar{L} > 0$ denote the constant labor supply. With profit maximizing firms and continuous clearing in the labor market we have, for all $t \geq 0$,

$$w_t = F_L(K_t, \bar{L}) \equiv w(K_t), \tag{14.29}$$

since $L_t = \bar{L}$. At any time t, K_t is predetermined in the sense that due to the convex installation costs, changes in K take time. Thus (14.29) *determines* the market real wage w_t.

To pin down the evolution of the economy, we now derive two coupled differential equations in K and q. By (14.24) we get

$$\dot{K}_t = I_t - \delta K_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{14.30}$$

As to the dynamics of q, we have (14.26). Since the capital installation cost function $G(I, K)$ is assumed to be homogeneous of degree 1, point (iii) of Lemma 1 applies and we can write (14.26) as

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.31}$$

As r and $\bar{L}$ are exogenous, the capital stock, K, and its shadow price, q, are the only endogenous variables in the differential equations (14.30) and (14.31). In addition, we have an initial condition for K and a necessary transversality condition involving q, namely

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0. \tag{14.32}$$

Fig. 14.4 shows the phase diagram for these two coupled differential equations. Let q^* be defined as the value of q satisfying the equation $m(q) = \delta$. Since $m' > 0$, q^* is unique. Suppressing for convenience the explicit time subscripts, we then have

$$\dot{K} = 0 \text{ for } m(q) = \delta, \text{ i.e., for } q = q^*.$$

As $\delta > 0$, we have $q^* > 1$. This is so because also mere reinvestment to offset capital depreciation requires an incentive, namely that the marginal value to the firm of replacing worn-out capital is larger than the purchase price of the investment good (since the installation cost must also be compensated). From (14.30) is seen that

$$\dot{K} \gtreqless 0 \text{ for } m(q) \gtreqless \delta, \text{ respectively, i.e., for } q \gtreqless q^*, \text{ respectively,}$$

cf. the horizontal arrows in Fig. 14.4.

From (14.31) we have

$$\dot{q} = 0 \text{ for } 0 = (r + \delta)q - F_K(K, \bar{L}) + g(m(q)) - (q - 1)m(q). \tag{14.33}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives

$$0 = (r + \delta)q^* - F_K(K, \bar{L}) + g(\delta) - (q^* - 1)\delta, \tag{14.34}$$

where the right-hand-side is increasing in K, in view of $F_{KK} < 0$. Hence, there exists at most one value of K such that the steady state condition (14.34) is

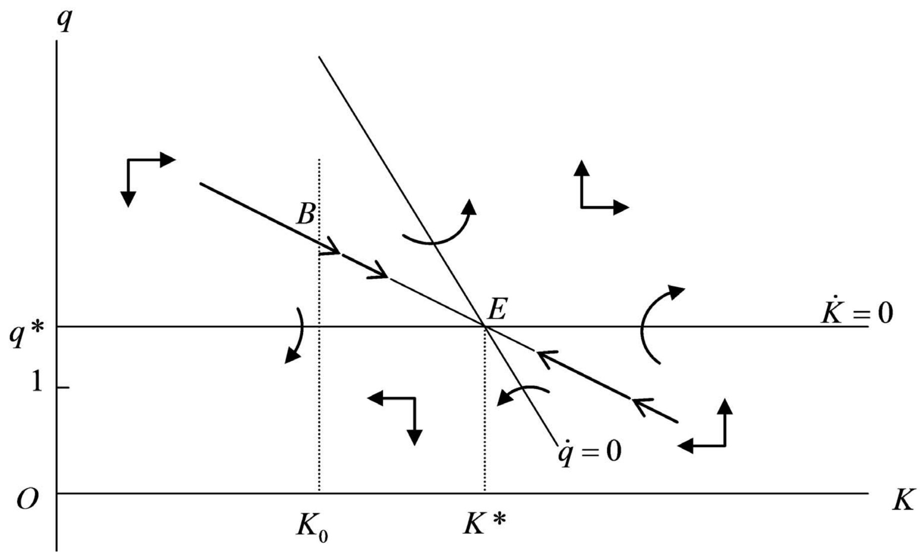

Figure 14.4: Phase diagram for investment dynamics in a small open economy (a case where $\delta > 0$).

satisfied;[9] this value is denoted K^*, corresponding to the steady state point E in Fig. 14.4. The question is now: what is the slope of the $\dot{q} = 0$ locus? In Appendix E it is shown that at least in a neighborhood of the steady state point E, this slope is negative in view of the assumption $r > 0$ and $F_{KK} < 0$. From (14.31) we see that

$$\dot{q} \lesseqgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $F_{KK}(K_t, \bar{L}) < 0$. The vertical arrows in Fig. 14.4 show these directions of movement.

Altogether the phase diagram shows that the steady state E is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, the steady state is saddle-point stable. At time 0 the economy will be at the point B in Fig. 14.4 where the vertical line $K = K_0$ crosses the saddle path. Then the economy will move along the saddle path towards the steady state. This solution satisfies the transversality condition (14.32) and is the unique solution to the model (for details, see Appendix F).

The effect of an unanticipated rise in the interest rate Suppose that until time 0 the economy has been in the steady state E in Fig. 14.4. Then,

[9] And assuming that F satisfies the Inada conditions, we are sure that such a value exists since (14.34) gives $F_K(K, \bar{L}) = rq^* + g(\delta) + \delta > 0$.

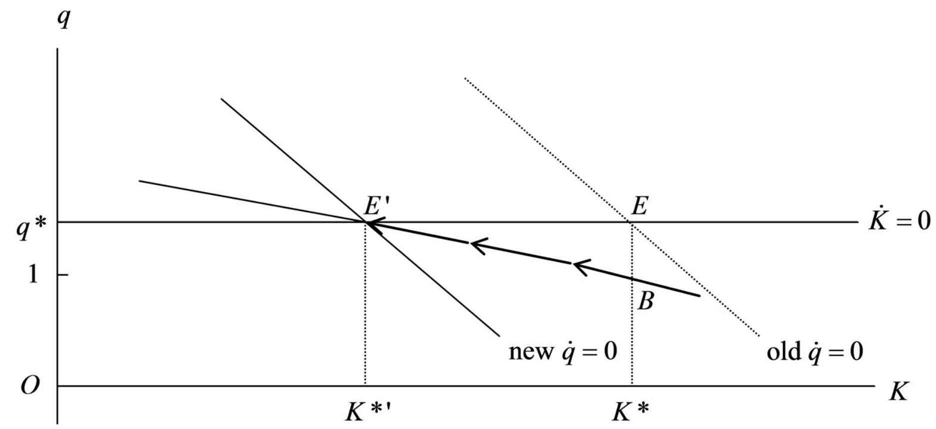

Figure 14.5: Phase portrait of an unanticipated rise in r (the case $\delta > 0$).

an unexpected shift in the interest rate occurs so that the new interest rate is a constant $r' > r$. We assume that the new interest rate is rightly expected to remain at this level forever. From (14.30) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. However, (14.33) implies that the $\dot{q} = 0$ locus and K^* shift to the left, in view of $F_{KK}(K, \bar{L}) < 0$.

Fig. 14.5 illustrates the situation for $t > 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 14.5. There is now a heavier discounting of the future benefits that the marginal unit of capital can provide. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually. This is where we see a crucial role of convex capital installation costs in an open economy. For now, the installation costs are the costs associated with disinvestment (dismantling and selling out of machines). If these convex costs were not present, we would get the same counterfactual prediction as from the previous open-economy models in this book, namely that the new steady state is attained immediately after the shift in the interest rate.

As the capital stock is diminished, the marginal productivity of capital rises and so does q. The economy moves along the new saddle path and approaches the new steady state E' as time goes by.

Suppose that for some reason such a decrease in the capital stock is not desirable from a social point of view. This could be because of positive external effects of capital and investment, e.g., a kind of "learning by doing". Then the government could decide to implement an investment subsidy $\sigma \in (0, 1)$ so that to attain an investment level I, purchasing the investment goods involves a cost of $(1 - \sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior, investment is increased again and the economy may in the long run end up at the old steady-state level of K (but the new q^* will be lower than the old).

In a similar way the effect of a depreciation allowance and a corporate tax can be studied.

A growing small open economy with capital installation costs*

The basic assumptions are the same as in the previous section except that now labor supply, $\bar{L}_t$, grows at the constant rate $n \geq 0$, while the technology level, T, grows at the constant rate $\gamma \geq 0$ (both rates exogenous and constant) and the production function is neoclassical with CRS. We assume that the world market real interest rate, r, is a constant and satisfies $r > \gamma + n$. Still assuming full employment, we have $L_t = \bar{L}_t = \bar{L}_0 e^{nt}$.

In this setting the production function on intensive form is useful:

$$\tilde{Y} = F(K, T\bar{L}) = F(\frac{K}{T\bar{L}}, 1)T\bar{L} \equiv f(\tilde{k})T\bar{L},$$

where $\tilde{k} \equiv K/(T\bar{L})$ and f satisfies $f' > 0$ and $f'' < 0$. Still assuming perfect competition, the market-clearing real wage at time t is determined as

$$w_t = F_2(K_t, T_t\bar{L}_t)T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t,$$

where both $\tilde{k}_t$ and T_t are predetermined. By log-differentiation of $\tilde{k} \equiv K/(T\bar{L})$ w.r.t. time we get $\dot{\tilde{k}}_t/\tilde{k}_t = \dot{K}_t/K_t - (\gamma + n)$. Substituting (14.30), we get

$$\dot{\tilde{k}}_t = \left[m(q_t) - (\delta + \gamma + n)\right]\tilde{k}_t. \tag{14.35}$$

The change in the shadow price of capital is now described by

$$\dot{q}_t = (r + \delta)q_t - f'(\tilde{k}_t) + g(m(q_t)) - (q_t - 1)m(q_t), \tag{14.36}$$

from (14.26). In addition, the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t q_t e^{-(r-\gamma-n)t} = 0, \tag{14.37}$$

must hold.

The differential equations (14.35) and (14.36) constitute our new dynamic system. Fig. 14.6 shows the phase diagram, which is qualitatively similar to that in Fig. 14.4. We have

$$\dot{\tilde{k}} = 0 \quad \text{for} \quad m(q) = \delta + \gamma + n, \text{ i.e., for } q = q^*,$$

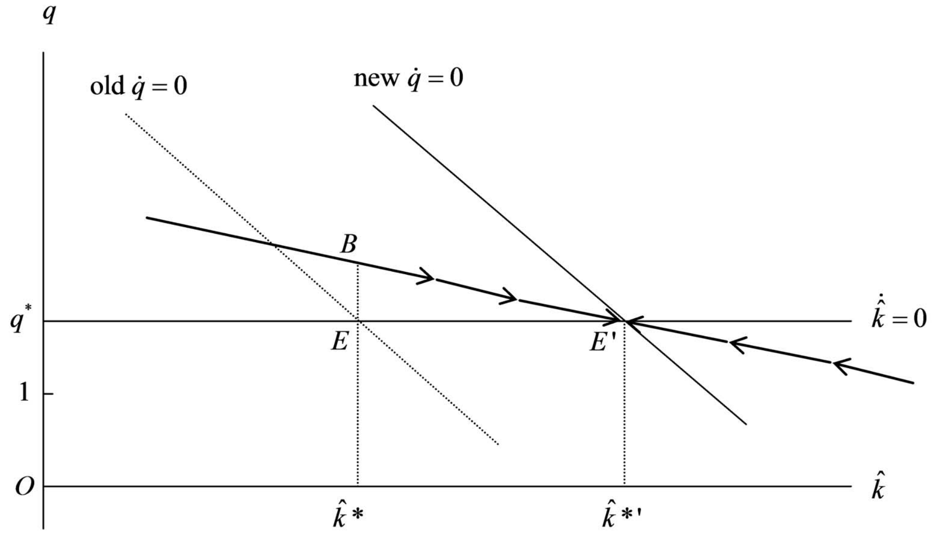

Figure 14.6: Phase portrait of an unanticipated fall in r (a growing economy with $\delta+\gamma+n \geq \gamma+n>0$).

where q^* now is defined by the requirement $m(q^*) = \delta+\gamma+n$. Notice, that when $\gamma+n>0$, we get a larger steady state value q^* than in the previous section. This is so because now a higher investment-capital ratio is required for a steady state to be possible. Moreover, the transversality condition (14.12) is satisfied in the steady state.

From (14.36) we see that $\dot{q}=0$ now requires

$$0 = (r+\delta)q - f'(\tilde{k}) + g(m(q)) - (q-1)m(q).$$

If, in addition $\dot{\tilde{k}} = 0$ (hence, $q=q^*$ and $m(q)=m(q^*)=\delta+\gamma+n$), this gives

$$0 = (r+\delta)q^* - f'(\tilde{k}) + g(\delta+\gamma+n) - (q^*-1)(\delta+\gamma+n).$$

Here, the right-hand-side is increasing in $\tilde{k}$ (in view of $f''(\tilde{k})<0$). Hence, the steady state value $\tilde{k}^*$ of the effective capital-labor ratio is unique, cf. the steady state point E in Fig. 14.6.

By the assumption $r>\gamma+n$ we have, at least in a neighborhood of E in Fig. 14.6, that the $\dot{q}=0$ locus is negatively sloped (see Appendix E).[10] Again the steady state is a saddle point, and the economy moves along the saddle path towards the steady state.

[10] In our perfect foresight model we in fact *have* to assume $r>\gamma+n$ for the firm's maximization problem to be well-defined. If instead $r \leq \gamma+n$, the market value of the representative firm would be infinite, and maximization would loose its meaning.

In Fig. 14.6 it is assumed that until time 0, the economy has been in the steady state E. Then, an unexpected shift in the interest rate to a *lower* constant level, r', takes place. The $\dot{q} = 0$ locus is shifted to the right, in view of $f'' < 0$. The shadow price, q, immediately jumps up to a level corresponding to the point B in Fig. 14.6. The economy moves along the new saddle path and approaches the new steady state E' with a higher effective capital-labor ratio as time goes by. In Exercise 14.2 the reader is asked to examine the analogue situation where an unanticipated downward shift in the rate of technological progress takes place.

14.4 Concluding remarks

Tobin's q-theory of investment gives a remarkably simple operational macroeconomic investment function, in which the variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

When faced with strictly convex installation costs, the firm has to take the *future* into account to invest optimally. Therefore, the firm's *expectations* become important. Owing to the strictly convex installation costs, the firm adjusts its capital stock only *gradually* when new information arises.

By incorporating these features, Tobin's q-theory helps explaining the sluggishness in investment we see in the empirical data. And the theory avoids the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world market changes. So the theory takes into account the time lags in capital adjustment in real life. Possibly, this feature can be abstracted from in long-run analysis and models of economic growth, but not in short- and medium-run analysis.

Many econometric tests of the q theory of investment have been made, often with critical implications. Movements in q^a, even taking account of changes in taxation, seemed capable of explaining only a minor fraction of the movements in investment. And the estimated equations relating fixed capital investment to q^a typically give strong auto-correlation in the residuals. Other variables, in particular availability of current corporate profits for internal financing, seem to have explanatory power independently of q^a (see Abel 1990, Chirinko 1993, Gilchrist and Himmelberg, 1995). So there is reason to be sceptical towards the notion that *all* information of relevance for the investment decision is reflected by the market valuation of firms. Or we might question the validity of the assumption in Hayashi's theorem (and its generalizations), that firms' cash flow tends to be homogeneous of degree one w.r.t. K, L, and I.

Further circumstances are likely to relax the link between q^a and investment. In the real world with many production sectors, physical capital is heterogeneous. If for example a sharp unexpected rise in the price of energy takes place, a firm with energy-intensive technology will loose in market value. At the same time it has an incentive to invest in energy-saving capital equipment. Hence, we might observe a fall in q^a at the same time as investment increases.

Imperfections in credit markets are ignored by the q-theory. Their presence further loosens the relationship between q^a and investment and may help explain the observed positive correlation between investment and corporate profits.

We might also question that capital installation costs really have the hypothesized *strictly convex* form. It is one thing that there are costs associated with installation, reorganizing and retraining etc., when new capital equipment is procured. But should we expect these costs to be strictly convex in the volume of investment? To think about this, let us for a moment ignore the role of the existing capital stock. Hence, we write total installation costs $J = G(I)$ with $G(0) = 0$. It does not seem problematic to assume $G'(I) > 0$ for $I > 0$. The question concerns the assumption $G''(I) > 0$. According to this assumption the average installation cost $G(I)/I$ must be increasing in I.[11] But against this speaks the fact that capital installation may involve indivisibilities, fixed costs, acquisition of new information etc. All these features tend to imply *decreasing* average costs. In any case, at least at the microeconomic level one should expect unevenness in the capital adjustment process rather than the above smooth adjustment.

Because of the mixed empirical success of the convex installation cost hypothesis other theoretical approaches that can account for sluggish and sometimes non-smooth and lumpy capital adjustment have been considered: uncertainty, investment irreversibility, indivisibility, or financial problems due to bankruptcy costs (Nickell 1978, Zeira 1987, Dixit and Pindyck 1994, Caballero 1999, Adda and Cooper 2003). These approaches notwithstanding, it turns out that the q-theory of investment has recently been somewhat rehabilitated from both a theoretical and an empirical point of view. At the theoretical level Wang and Wen (2010) show that financial frictions in the form of collateralized borrowing at the firm level can give rise to strictly convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. For large firms, unlikely to be much affected by financial frictions, Eberly et al. (2008) find that the theory does a good job in explaining investment behavior.

In any case, the q-theory of investment is in different versions widely used in short- and medium-run macroeconomics because of its simplicity and the ap-

[11] Indeed, for $I \neq 0$ we have $d[G(I)/I]/dI = [IG'(I) - G(I)]/I^2 > 0$, when G is strictly convex ($G'' > 0$) and $G(0) = 0$.

pealing link it establishes between asset markets and firms' investment. And the q-theory has also had an important role in studies of the housing market and the role of housing prices for household wealth and consumption, a theme to which we return in the next chapter.

14.5 Literature notes

A first sketch of the q-theory of investment is contained in Tobin (1969). Later advances of the theory took place through the contributions of Hayashi (1982) and Abel (1982), as surveyed in (1990).

Both the Ramsey model and the Blanchard OLG model for a closed market economy may be extended by adding strictly convex capital installation costs, see Abel and Blanchard (1983) and Lim and Weil (2003). Adding a public sector, such a framework is useful for the study of how different subsidies, taxes, and depreciation allowance schemes affect investment in physical capital as well as housing, see, e.g., Summers (1981), Abel and Blanchard (1983), and Dixit (1990).

Groth and Madsen (2016) study medium-term fluctuations in a closed economy, arising in a Tobin's q framework extended by sluggishness in real wage adjustments.

14.6 Appendix

A. When value maximization is - and is not - equivalent to continuous static profit maximization

For the idealized case where tax distortions, asymmetric information, and problems with enforceability of financial contracts are absent, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the market value (debt plus equity) does not depend on the level of the debt. So the financial structure of the firm is both indeterminate and irrelevant for production outcomes. Considering the firm described in Section 14.1, the implied separation of the financing decision from the production and investment decision can be exposed in the following way.

The Modigliani-Miller theorem in action Although the theorem allows for risk, we here ignore risk. Let the real debt of the firm be denoted B_t and the real dividends, X_t. We then have the accounting relationship

$$\dot{B}_t = X_t - \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t\right).$$

A positive X_t represents dividends in the usual meaning (payout to the owners of the firm), whereas a negative X_t can be interpreted as emission of new shares of stock. Since we assume perfect competition, the time path of w_t and r_t is exogenous to the firm.

Consider first the firm's combined financing and production-investment problem, which we call *Problem I.* Assume (realistically) that those who own the firm at time 0 want it to maximize its net worth, i.e., the present value of expected future dividends:

$$\begin{aligned}
\max_{(L_t,I_t,X_t)_{t=0}^{\infty}} \tilde{V}_0 &= \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t, \\
\dot{B}_t &= X_t - \left(F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t\right), \\
& \quad \text{where } B_0 \text{ is given}, \qquad (14.38) \\
\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} &\leq 0. \qquad \text{(NPG)}
\end{aligned}$$

The last constraint is the firm's No-Ponzi-Game condition, saying that a positive debt should in the long run at most grow at a rate which is *less* than the interest rate.

In Section 14.1 we considered another problem, namely a separate investment-production problem:

$$\begin{aligned}
\max_{(L_t,I_t)_{t=0}^{\infty}} V_0 &= \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.}, \\
R_t &\equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t.
\end{aligned}$$

Let this problem, where the financing aspects are ignored, be called *Problem II.* When considering the relationship between Problem I and Problem II, the following mathematical fact is useful.

LEMMA A1 Consider a continuous function $a(t)$ and a differentiable function $f(t)$. Then

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t)) e^{-\int_{t_0}^t a(s)ds} dt = f(t_1) e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0).$$

Proof. Integration by parts from time t_0 to time t_1 yields

$$\int_{t_0}^{t_1} f'(t) e^{-\int_{t_0}^t a(s)ds} dt = f(t) e^{-\int_{t_0}^t a(s)ds} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} f(t)a(t) e^{-\int_{t_0}^t a(s)ds} dt.$$

Hence,

$$\begin{aligned}&\int_{t_0}^{t_1}(f'(t)-a(t)f(t))e^{-\int_{t_0}^{t}a(s)ds}dt\\ =\;& f(t_1)e^{-\int_{t_0}^{t_1}a(s)ds}-f(t_0). \qquad \square\end{aligned}$$

CLAIM 1 If $(K_t^*, B_t^*, L_t^*, I_t^*, X_t^*)_{t=0}^{\infty}$ is a solution to Problem I, then $(K_t^*, L_t^*, I_t^*)_{t=0}^{\infty}$ is a solution to Problem II.

Proof. By (14.38) and the definition of R_t, $X_t = R_t + \dot{B}_t - r_t B_t$ so that

$$\tilde{V}_0 = \int_0^{\infty} X_t e^{-\int_0^t r_s ds}dt = V_0 + \int_0^{\infty}(\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds}dt. \tag{14.39}$$

In Lemma A1, let $f(t) = B_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty}\int_0^T(\dot{B}_t - r_t B_t)e^{-\int_0^t r_s ds}dt = \lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} - B_0 \leq -B_0,$$

where the weak inequality is due to (NPG). Substituting this into (14.39), we see that maximum of net worth $\tilde{V}_0$ is obtained by maximizing V_0 and ensuring $\lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} = 0$, in which case net worth equals ((maximized V_0) $- B_0$), where B_0 is given. So a plan that maximizes net worth of the firm must also maximize V_0 in Problem II. $\square$

In view of Claim 1, it does not matter for the firm's production and investment behavior whether the firm's investment is financed by issuing new debt or by issuing shares of stock. Moreover, if we assume investors do not care about whether they receive the firm's earnings in the form of dividends or valuation gains on the shares, the firm's dividend policy is also irrelevant. Hence, from now on we can concentrate on the investment-production problem, Problem II above.

The case with no capital installation costs Suppose the firm has no capital installation costs. Then the cash flow reduces to $R_t = F(K_t, L_t) - w_t L_t - I_t$.

CLAIM 2 When there are no capital installation costs, Problem II can be reduced to a series of static profit maximization problems.

Proof. Current (pure) profit is defined as

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t \equiv \Pi(K_t, L_t).$$

It follows that R_t can be written

$$R_t = F(K_t, L_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \Pi_t + (r_t + \delta)K_t - (\dot{K}_t + \delta K_t). \quad (14.40)$$

Hence,

$$V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + \int_0^\infty (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt. \quad (14.41)$$

The first integral on the right-hand side of this expression is independent of the second. Indeed, the firm can maximize the first integral by *renting* capital and labor, K_t and L_t, at the going factor prices, $r_t + \delta$ and w_t, respectively, such that $\Pi_t = \Pi(K_t, L_t)$ is maximized at each t. The factor costs are accounted for in the definition of Π_t.

The second integral on the right-hand side of (14.41) is the present value of net revenue from renting capital out to others. In Lemma A1, let $f(t) = K_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt = K_0 - \lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} = K_0, \quad (14.42)$$

where the last equality comes from the fact that maximization of V_0 requires maximization of the left-hand side of (14.42) which in turn, since K_0 is given, requires minimization of $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds}$. The latter expression is always non-negative and can be made zero by choosing any time path for K_t such that $\lim_{T\to\infty} K_T = 0$. (We may alternatively put it this way: it never pays the firm to accumulate costly capital so fast in the long run that $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} > 0$, that is, to maintain accumulation of capital at a rate equal to or higher than the interest rate.) Substituting (14.42) into (14.41), we get $V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + K_0$.

The conclusion is that, given K_0,[12] V_0 is maximized if and only if K_t and L_t are at each t chosen such that $\Pi_t = \Pi(K_t, L_t)$ is maximized. □

The case with strictly convex capital installation costs Now we reintroduce the capital installation cost function $G(I_t, K_t)$, satisfying in particular the condition $G_{II}(I, K) > 0$ for all (I, K). Then, as shown in the text, the firm adjusts to a change in its environment, say a downward shift in r, by a *gradual*

[12] Note that in the absence of capital installation costs, the historically given K_0 is no more "given" than the firm may instantly let it jump to a lower or higher level. In the first case the firm would immediately sell a bunch of its machines and in the latter case it would immediately buy a bunch of machines. Indeed, without convex capital installation costs nothing rules out jumps in the capital stock. But such jumps just reflect an immediate jump, in the opposite direction, in another asset item in the balance sheet and leave the maximized net worth of the firm unchanged.

adjustment of K, in this case upward, rather than attempting an instantaneous maximization of $\Pi(K_t, L_t)$. The latter would entail an instantaneous upward jump in K_t of size $\Delta K_t = a > 0$, requiring $I_t \cdot \Delta t = a$ for $\Delta t = 0$. This would require $I_t = \infty$, which implies $G(I_t, K_t) = \infty$, which may interpreted either as such a jump being impossible or at least so costly that no firm will pursue it.

Proof that q_t satisfies (14.15) along an interior optimal path Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t (r_s+\delta)ds}$, we get

$$[(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} = (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds}, \tag{14.43}$$

where $F_{Kt} \equiv F_K(K_t, L_t)$ and $G_{Kt} \equiv G_K(I_t, K_t)$. In Lemma A1, let $f(t) = q_t$, $a(t) = r_t + \delta$, $t_0 = 0$, $t_1 = T$. Then

$$\begin{aligned}\int_0^T [(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} dt &= q_0 - q_T e^{-\int_0^T (r_s+\delta)ds} \\ &= \int_0^T (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt,\end{aligned}$$

where the last equality comes from (14.43). Letting $T \to \infty$, we get

$$q_0 - \lim_{T\to\infty} q_T e^{-\int_0^T (r_s+\delta)ds} = q_0 = \int_0^\infty (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \tag{14.44}$$

where the first equality follows from the transversality condition (14.14), which we repeat here:

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{*}$$

Indeed, since $\delta \geq 0$, $\lim_{T\to\infty}(e^{-\int_0^T r_s ds} e^{-\delta T}) = 0$, when (*) holds. Initial time is arbitrary, and so we may replace 0 and t in (14.44) by t and τ, respectively. The conclusion is that (14.15) holds along an interior optimal path, given the transversality condition (*). A proof of necessity of the transversality condition (*) is given in Appendix B.[13]

B. Transversality conditions

In view of (14.44), a qualified conjecture is that the condition $\lim_{t\to\infty} q_t e^{-\int_0^t (r_s+\delta)ds} = 0$ is necessary for optimality. This is indeed true, since this condition follows from the stronger transversality condition (*) in Appendix A, the necessity of which along an optimal path we will now prove.

[13] An equivalent approach to derivation of (14.15) can be based on applying the transversality condition (*) to the general solution formula for linear inhomogeneous first-order differential equations. Indeed, the first-order condition (14.11) provides such a differential equation in q_t.

Proof of necessity of (14.14) As the transversality condition (14.14) is the same as (*) in Appendix A, from now we refer to (*).

Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t r_s ds}$, we have

$$(r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds} = (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}.$$

In Lemma A1, let $f(t) = q_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$. Then

$$\int_0^T (r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds}dt = q_0 - q_T e^{-\int_0^T r_s ds} = \int_0^T (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}dt.$$

Rearranging and letting $T \to \infty$, we see that

$$q_0 = \int_0^\infty (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}dt + \lim_{T\to\infty} q_T e^{-\int_0^T r_s ds}. \tag{14.45}$$

If, contrary to (*), $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} > 0$ along the optimal path, then (14.45) shows that the firm is over-investing. By reducing initial investment by one unit, the firm would save approximately $1 + G_I(I_0, K_0) = q_0$, by (14.10), which would be more than the present value of the stream of potential net gains coming from this marginal unit of installed capital (the first term on the right-hand side of (14.45)).

Suppose instead that $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} < 0$. Then, by a symmetric argument, the firm has under-invested initially.

Necessity of (14.12) In cases where along an optimal path, K_t remains bounded from above for $t \to \infty$, the transversality condition (14.12) is implied by (*). In cases where along an optimal path, K_t is not bounded from above for $t \to \infty$, the transversality condition (14.12) is stronger than (*). A proof of the necessity of (14.12) in this case can be based on Weitzman (2003) and Long and Shimomura (2003).

C. On different specifications of the q-theory

The simple relationship we have found between I and q can easily be generalized to the case where the purchase price on the investment good, p_{It}, is allowed to differ from 1 (its value in the main text) and the capital installation cost is $p_{It}G(I_t, K_t)$. In this case it is convenient to replace q in the Hamiltonian function by, say, λ. Then the first-order condition (14.10) becomes $p_{It} + p_{It}G_I(I_t, K_t) = \lambda_t$, implying

$$G_I(I_t, K_t) = \frac{\lambda_t}{p_{It}} - 1,$$

and we can proceed, defining as before q_t by $q_t \equiv \lambda_t/p_{It}$.

Sometimes in the literature installation costs, J, appear in a somewhat different form than in the above exposition. But applied to a model with economic growth this will result in installation costs that rise faster than output and ultimately swallow the total produce.

Abel and Blanchard (1983), followed by Barro and Sala-i-Martin (2004, p. 152-160), introduce a function, ϕ, representing capital installation costs *per unit of investment* as a function of the investment-capital ratio. That is, total installation cost is $J = \phi(I/K)I$, where $\phi(0) = 0, \phi' > 0$. This implies that $J/K = \phi(I/K)(I/K)$. The right-hand side of this equation may be called $g(I/K)$, and then we are back at the formulation in Section 14.1. Indeed, defining $x \equiv I/K$, we have installation costs per unit of capital equal to $g(x) = \phi(x)x$, and assuming $\phi(0) = 0, \phi' > 0$, it holds that

$$\begin{aligned} g(x) &= 0 \text{ for } x = 0,\ g(x) > 0 \text{ for } x \neq 0, \\ g'(x) &= \phi(x) + x\phi'(x) \gtreqless 0 \text{ for } x \gtreqless 0, \text{ respectively, and} \\ g''(x) &= 2\phi'(x) + x\phi''(x). \end{aligned}$$

Clearly, $g''(x)$ must be positive for the theory to work. But the assumptions $\phi(0) = 0, \phi' > 0$, and $\phi'' \geq 0$, imposed in p. 153 and again in p. 154 in Barro and Sala-i-Martin (2004), are *not* sufficient for this (as $x < 0$ is possible). Since in macroeconomics $x < 0$ is seldom, this is a minor point, however.

It is sometimes convenient to let the capital installation cost $G(I, K)$ appear, not as a reduction in output, but as a reduction in capital formation so that

$$\dot{K} = I - \delta K - G(I, K). \tag{14.46}$$

This approach is used in Hayashi (1982) and Heijdra and Ploeg (2002, p. 573 ff.; see also Uzawa (1969). For example, Heijdra and Ploeg write the rate of capital accumulation as $\dot{K}/K = \psi(I/K) - \delta$, where the "capital installation function" $\psi(I/K)$ has the properties $\psi' > 0$ and $\psi'' < 0$ and can be interpreted as $\psi(I/K) \equiv [I - G(I, K)]/K = I/K - g(I/K)$. The latter equality comes from assuming G is homogeneous of degree 1. To maintain the likely desired property that $\psi'(I/K) > 0$ (though small) even for large I/K, the G function should not be "too convex". That is, for instance $g(I/K) = (\beta/2)(I/K)^2$ would *not* do.

In one-sector models, as we usually consider in this text, this need not change anything of importance. In more general models the installation function approach (14.46) may have some analytical advantages; what gives the best fit empirically is an open question.

Finally, some analysts assume that installation costs are a strictly convex function of *net* investment, $I - \delta K$. This agrees well with intuition if mere replacement investment occurs in a smooth way not involving new technology, work interruption, and reorganization. To the extent capital investment involves indivisibilities and embodies new technology, it may seem more plausible to specify the installation costs as a convex function of *gross* investment.

D. Proof of Hayashi's theorem

For convenience we repeat:

THEOREM (Hayashi) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K). Then, along the optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. The value of the firm as seen from time t is

$$V_t = \int_t^\infty (F(K_\tau, L_\tau) - G(I_\tau, K_\tau) - w_\tau L_\tau - I_\tau)e^{-\int_t^\tau r_s ds} d\tau. \tag{14.47}$$

We introduce the functions

$$\begin{aligned} A &= A(K, L) \equiv F(K, L) - F_K(K, L)K - F_L(K, L)L, && (14.48) \\ B &= B(I, K) \equiv G_I(I, K)I + G_K(I, K)K - G(I, K). && (14.49) \end{aligned}$$

Then the cash-flow of the firm at time τ can be written

$$\begin{aligned} R_\tau &= F(K_\tau, L_\tau) - F_{L\tau} L_\tau - G(I_\tau, K_\tau) - I_\tau \\ &= A(K_\tau, L_\tau) + F_{K\tau} K_\tau + B(I_\tau, K_\tau) - G_{I\tau} I_\tau - G_{K\tau} K_\tau - I_\tau, \end{aligned}$$

where we have used first $F_{L_\tau} = w$ and then the definitions of A and B above. Consequently, when moving along the optimal path,

$$\begin{aligned} V_t &= V^*(K_t, t) = \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))\, e^{-\int_t^\tau r_s ds} d\tau && (14.50) \\ &\quad + \int_t^\infty [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau] e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))e^{-\int_t^\tau r_s ds} d\tau + q_t K_t, \end{aligned}$$

cf. Lemma D1 below. Isolating q_t, it follows that

$$q_t^m \equiv q_t = \frac{V_t}{K_t} - \frac{1}{K_t} \int_t^\infty [A(K_\tau, L_\tau) + B(I_\tau, K_\tau)]e^{-\int_t^\tau r_s ds} d\tau, \tag{14.51}$$

when moving along the optimal path.

Since F is concave and $F(0,0) = 0$, we have for all K and L, $A(K,L) \geq 0$ with equality sign, if and only if F is homogeneous of degree one. Similarly, since G is convex and $G(0,0) = 0$, we have for all I and K, $B(I,K) \geq 0$ with equality sign, if and only if G is homogeneous of degree one. Now the conclusions (i) and (ii) follow from (14.51) and the definition of q^a in (14.27). $\square$

LEMMA D1 The last integral on the right-hand side of (14.50) equals $q_t K_t$, when investment follows the optimal path.

Proof. We want to characterize a given optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$. Keeping t fixed and using z as our varying time variable, we have

$$(F_{Kz} - G_{Kz})K_z - (1 + G_{Iz})I_z = [(r_z + \delta)q_z - \dot{q}_z]K_z - (1 + G_{Iz})I_z$$

$$= [(r_z + \delta)q_z - \dot{q}_z]K_z - q_z(\dot{K}_z + \delta K_z) = r_z q_z K_z - (\dot{q}_z K_z + q_z \dot{K}_z) = r_z u_z - \dot{u}_z,$$

where we have used (14.11), (14.10), (14.6), and the definition $u_z \equiv q_z K_z$. We look at this as a differential equation: $\dot{u}_z - r_z u_z = \varphi_z$, where $\varphi_z \equiv -[(F_{Kz} - G_{Kz})K_z - (1 + G_{Iz})I_z]$ is considered as some given function of z. The solution of this linear differential equation is

$$u_z = u_t e^{\int_t^z r_s ds} + \int_t^z \varphi_\tau e^{\int_\tau^z r_s ds} d\tau,$$

implying, by multiplying through by $e^{-\int_t^z r_s ds}$, reordering, and inserting the definitions of u and φ,

$$\begin{aligned} & \int_t^z [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau] e^{-\int_t^\tau r_s ds} d\tau \\ = \ & q_t K_t - q_z K_z e^{-\int_t^z r_s ds} \to q_t K_t \quad \text{for} \quad z \to \infty, \end{aligned}$$

from the transversality condition (14.12) with t replaced by z and 0 replaced by t. $\square$

A different – and perhaps more illuminating – way of understanding (i) in Hayashi's theorem is the following.

Suppose F and G are homogeneous of degree one. Then $A = B = 0$, $G_I I + G_K K = G = g(I/K)K$, and $F_K = f'(k)$, where f is the production function in intensive form. Consider an optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$ and let $k_\tau \equiv K_\tau / L_\tau$ and $x_\tau \equiv I_\tau / K_\tau$ along this path which we now want to characterize. As the path is assumed optimal, from (14.47) follows

$$V_t = V^*(K_t, t) = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] K_\tau e^{-\int_t^\tau r_s ds} d\tau. \tag{14.52}$$

From $\dot{K}_t = (x_t - \delta)K_t$ follows $K_\tau = K_t e^{-\int_t^\tau (x_s - \delta)ds}$. Substituting this into (14.52) yields

$$V^*(K_t, t) = K_t \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau.$$

In view of (14.24), with t replaced by τ, the optimal investment *ratio* x_τ depends, for all τ, only on q_τ, not on K_τ, hence not on K_t. Therefore,

$$\partial V^*/\partial K_t = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau = V_t/K_t.$$

Hence, from (14.28) and (14.27), we conclude $q_t^m = q_t^a$.

Remark. We have assumed throughout that G is strictly convex in I. This does not imply that G is jointly strictly convex in (I, K). For example, the function $G(I, K) = I^2/K$ is strictly convex in I (since $G_{II} = 2/K > 0$). But at the same time this function has $B(I, K) = 0$ and is therefore homogeneous of degree one. Hence, it is not jointly strictly convex in (I, K).

E. The slope of the $\dot{q} = 0$ locus in the SOE case

First, we shall determine the sign of the slope of the $\dot{q} = 0$ locus in the case $g + n = 0$, considered in Fig. 14.4. Taking the total differential in (14.33) w.r.t. K and q gives

$$\begin{aligned} 0 &= -F_{KK}(K, \bar{L})dK + \{r + \delta + g'(m(q))m'(q) - [m(q) + (q-1)m'(q)]\} dq \\ &= -F_{KK}(K, \bar{L})dK + [r + \delta - m(q)] dq, \end{aligned}$$

since $g'(m(q)) = q - 1$, by (14.23) and (14.24). Therefore

$$\frac{dq}{dK}_{|\dot{q}=0} = \frac{F_{KK}(K, \bar{L})}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign dq/dK at all points along the $\dot{q} = 0$ locus. But in a neighborhood of the steady state we have $m(q) \approx \delta$, hence $r + \delta - m(q) \approx r > 0$. And since $F_{KK} < 0$, this implies that at least in a neighborhood of E in Fig. 14.4, the $\dot{q} = 0$ locus is negatively sloped.

Second, consider the case $g + n > 0$, illustrated in Fig. 14.6. Here we get in a similar way

$$\frac{dq}{d\tilde{k}}_{|\dot{q}=0} = \frac{f''(\tilde{k}^*)}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign $dq/d\tilde{k}$ at all points along the $\dot{q} = 0$ locus. But in a small neighborhood of the steady state, we have $m(q) \approx \delta + \gamma + n$, hence $r + \delta - m(q) \approx r - \gamma - n > 0$, where the inequality was assumed in the text. Since $f'' < 0$, then, at least in a small neighborhood of E in Fig. 14.6, the $\dot{q} = 0$ locus is negatively sloped, when $r > \gamma + n$.

F. The divergent paths

Text not yet available.

14.7 Exercises

14.1 *(induced sluggish capital adjustment).* Consider a firm with capital installation costs $J = G(I, K)$, satisfying

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0.$$

a) Can we from this conclude anything as to strict concavity or strict convexity of the function G? If yes, with respect to what argument or arguments?

b) For two values of K, $\underline{K}$ and $\bar{K}$, illustrate graphically the capital installation costs J in the (I, J) plane. Comment.

c) By drawing a few straight line segments in the diagram, illustrate that $G(\frac{1}{2}I, \bar{K})2 < G(I, \bar{K})$ for any given $I > 0$.

14.2 (see end of Section 14.3)

Chapter 15

Further applications of adjustment cost theory

In the previous chapter we studied how strictly convex capital installation costs affect firms' fixed capital investment and how changes in the world market interest rate affect aggregate fixed capital investment in a small open economy with perfect mobility of financial capital. In the first part of the present chapter this basic setup is extended by adding a third production factor, imported oil, and then considering the effects on the economy of an oil price shock. This includes the effects on households' aggregate consumption where the modeling of the household sector is based on the Blanchard OLG framework. The aim is not only to examine effects of an oil price shock *per se*, but also to set up a more complete accounting framework for an open economy than in earlier chapters. In the concluding remarks virtues of the OLG approach compared with the representative agent approach as modeling devices for open economies are discussed.

The strictly convex capital installation costs can be seen as an exemplification of the more general notion of strictly convex *adjustment costs*. This leads to the second part of the chapter where we apply the adjustment cost theory in an analysis of the housing market from a macroeconomic point of view. The idea is that like firms' fixed capital investment, residential construction can be seen as a time-consuming activity involving strictly convex adjustment costs.

15.1 Oil price shock in a small oil-importing economy

Our focus is here on medium- and long-run effects on a small open economy (abbreviated SOE) of a supply shock in the form of a shift in the world market price of some raw material or energy, that the SOE imports. The reader may think of any imported raw material of some importance. But for concreteness we consider the imported good to be oil. This is an interesting example because of its considerable weight in many countries' imports and because of the large and sudden changes that sometimes occur in the world market price of this natural resource. In 1973-74 the real price of oil almost tripled, and in 1979-80 more than a doubling of the real price of oil took place, see Fig. 15.1.

We assume:

1. Perfect mobility of goods and financial capital across borders.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility of labor across borders.
4. Labor supply is inelastic and constant.
5. There is no government sector and no technological progress.
6. The capital adjustment cost function $G(I, K)$ is homogeneous of degree one.
7. There is perfect competition in all markets.

Our SOE thus faces an exogenous real interest rate, r_t, given from the world financial market. For convenience, let $r_t = r$ for all $t \geq 0$, where r is a positive constant. Our analysis takes output to be supply-determined as if there is always full employment, that is, we ignore the short-term Keynesian demand effects of an oil price shock. Such effects would be due to the purchasing power of consumers being undermined by a sudden increase in the price of imported oil. We shall see that even without Keynesian effects, the overall effect of an adverse oil price shock is an economic contraction in both the short and the long run.

15.1.1 Three inputs: capital, labor, and raw material

The models in the previous chapters assumed that all output is produced in one sector using only capital and labor. We could also say that the earlier models implicitly assume that raw material is continuously produced at a low stage of production, but is in the next instant used up immediately at a higher stage of production. In effect, raw material need not be treated as a separate input.

When raw material is imported, we have to treat it as a separate input. The technology of the representative firm in the SOE is consequently given as a three-factor production function,

$$\tilde{Y}_t = F(K_t, L_t, M_t), \quad F_i > 0, F_{ii} < 0 \quad \text{for} \quad i = K, L, M, \tag{15.1}$$

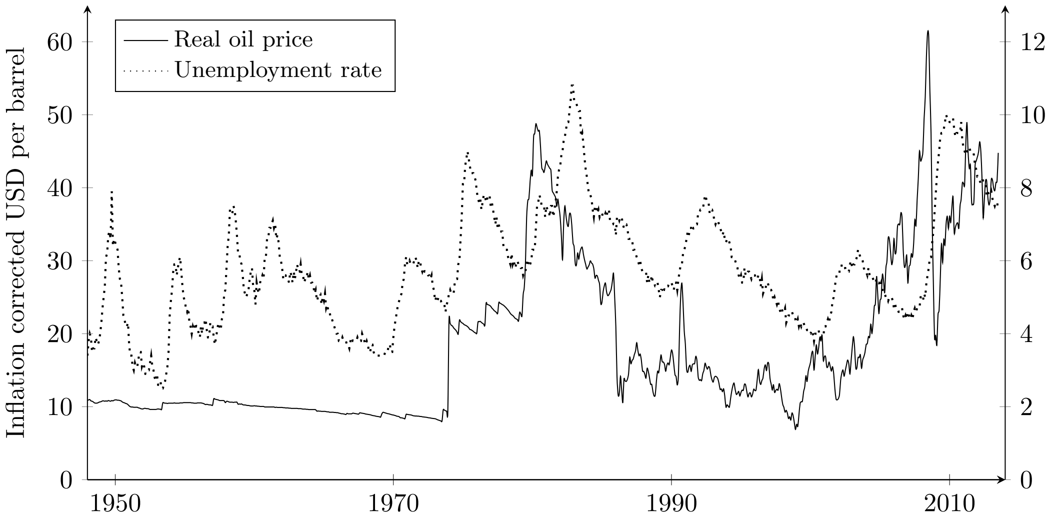

Figure 15.1: Real oil price per barrel and U.S. unemployment rate 1948-2013. Source: Bureau of Labor Statistics and Federal Reserve Bank of St. Louis.

where $\tilde{Y}$ is aggregate output gross of adjustment costs and physical capital depreciation, K is capital input, and L is labor input, whereas M is the input of the imported raw material, say oil, all measured per time unit.[1] The production function F is assumed neoclassical with CRS w.r.t. its three arguments. Thus, as usual there are positive, but diminishing marginal productivities of all three production factors. But in addition we shall need the assumption that the three inputs are *direct complements* in the sense that all the cross derivatives of F are positive:

$$F_{ij} > 0, \quad i \neq j. \tag{15.2}$$

In words: the marginal productivity of any of the production factors is greater, the more input there is of any of the other production factors.[2]

The increase per time unit in the firm's capital stock is given by

$$\dot{K}_t = I_t - \delta K_t, \qquad \qquad \delta \geq 0,$$

[1]As long as we have oil import in our mind, we should not primarily think of, for example, Denmark (even less so UK and Norway) as our case in point. Denmark has since 1996 been a net exporter of oil and natural gas. But most other European countries will fit as good examples.

[2]For a two-factor neoclassical production function with CRS we always have direct complementarity, i.e., $F_{12} > 0$. But with more than two production factors, direct complementarity for all pairs of production factors is not assured. Therefore, in general, direct complementarity is an additional assumption. However, the Cobb-Douglas function, $Y = K^{\alpha_1} L^{\alpha_2} M^{1-\alpha_1-\alpha_2}$, where $\alpha_i > 0$, $i = 1, 2$, and $\alpha_1 + \alpha_2 < 1$, automatically satisfies all the conditions in (15.2).

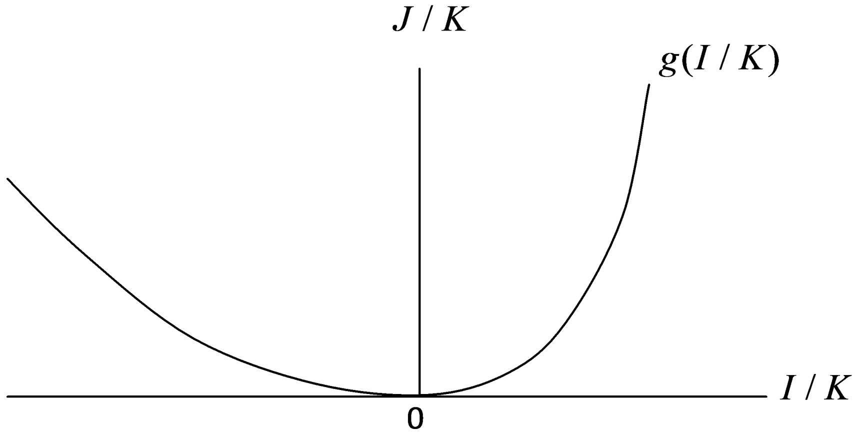

Figure 15.2: Capital adjustment cost per unit of installed capital.

where I is gross investment per time unit and δ is the rate of physical wearing-down of capital (physical depreciation in this model has to be distinguished from economic depreciation, cf. Section 15.1.3 below).

The firm faces strictly convex capital installation costs. Let these installation costs (measured in units of output) at time t be denoted J_t and assume they depend only on the level of investment and the existing capital stock; that is $J_t = G(I_t, K_t)$. The installation cost function G is assumed homogeneous of degree one so that we can write

$$J = G(I, K) = G(\frac{I}{K}, 1)K \equiv g(\frac{I}{K})K, \tag{15.3}$$

where the function g is *strictly convex* and satisfies

$$g(0) = 0, g'(0) = 0 \text{ and } g'' > 0. \tag{15.4}$$

The graph of g is shown in Fig. 15.2.

Gross domestic product (value added) at time t is

$$GDP_t \equiv \tilde{Y}_t - J_t - p_M M_t, \tag{15.5}$$

where p_M is the real price of oil, this price being exogenous to the SOE. For simplicity we assume that this price is a constant, but it may shift to another level (i.e., we use p_M as a shift parameter).

The decision problem of the firm

Let cash flow (before interest payments) at time t be denoted R_t. Then

$$R_t \equiv F(K_t, L_t, M_t) - g(\frac{I_t}{K_t})K_t - w_t L_t - p_M M_t - I_t, \tag{15.6}$$

where w_t is the real wage. The decision problem, as seen from time 0, is to choose a plan $(L_t, M_t, I_t)_{t=0}^{\infty}$ to maximize the market value of the firm,

$$V_0 = \int_0^{\infty} R_t e^{-rt} dt \quad \text{s.t. (15.6), and} \tag{15.7}$$

$$L_t \geq 0, M_t \geq 0, I_t \text{ free}, \qquad \text{(i.e., no restriction on } I_t) \tag{15.8}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 \text{ given}, \tag{15.9}$$

$$K_t \geq 0 \text{ for all } t. \tag{15.10}$$

To solve the problem, we use the Maximum Principle. The problem has three control variables, L, M, and I, and one state variable, K. We set up the current-value Hamiltonian:

$$\mathcal{H}(K, L, M, I, q, t) \equiv F(K, L, M) - g(\frac{I}{K})K - wL - p_M M - I + q(I - \delta K), \tag{15.11}$$

where q_t is the adjoint variable associated with the dynamic constraint (15.9). For each $t \geq 0$ we maximize the Hamiltonian w.r.t. the control variables: $\partial\mathcal{H}/\partial L = F_L(K, L, M) - w = 0$, i.e.,

$$F_L(K, L, M) = w; \tag{15.12}$$

$\partial\mathcal{H}/\partial M = F_M(K, L, M) - p_M = 0$, i.e.,

$$F_M(K, L, M) = p_M; \tag{15.13}$$

and $\partial\mathcal{H}/\partial I = -g'(\frac{I}{K}) - 1 + q = 0$, i.e.,

$$1 + g'(\frac{I}{K}) = q. \tag{15.14}$$

Next, we partially differentiate $\mathcal{H}$ w.r.t. the state variable, K, and set this derivative equal to $rq_t - \dot{q}_t$, since r is the discount rate in (15.7):

$$\frac{\partial\mathcal{H}}{\partial K} = F_K(K, L, M) - \frac{\partial\left[g(\frac{I}{K})K\right]}{\partial K} - \delta q = rq_t - \dot{q}. \tag{15.15}$$

The Maximum Principle now says that an interior optimal path (K_t, L_t, M_t, I_t) satisfies that there exists an adjoint variable q_t such that for all $t \geq 0$, the conditions (15.12), (15.13), (15.14), and (15.15) hold along the path, and the transversality condition,

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0, \tag{15.16}$$

is satisfied.

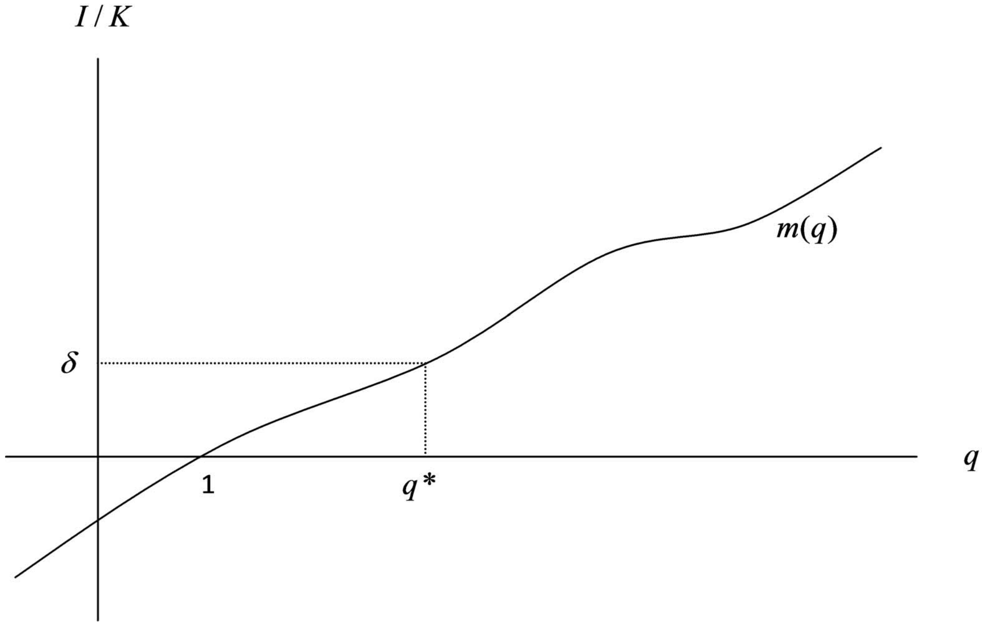

Figure 15.3

The only new optimality condition compared to the previous chapter is (15.13) which just says that optimality requires equalizing the marginal productivity of the imported input to its real price, p_M. By (15.14), the adjoint variable, q_t, can be interpreted as a *shadow price* (measured in current output units) of installed capital along the optimal path. That is, q_t represents the value to the firm of the marginal unit of installed capital at time t along the optimal path. The transversality condition says that the present value of the stock of installed capital "left over" at infinity must be vanishing.

The implied investment function

Since $g'' > 0$, the optimality condition (15.14) implicitly defines the optimal investment ratio, I/K, as a function of the shadow price q,

$$\frac{I_t}{K_t} = m(q_t), \quad \text{where } m(1) = 0 \text{ and } m' = 1/g'' > 0. \tag{15.17}$$

This in the investment function of the representative firm. An example is illustrated in Fig. 15.3.

To see what the optimality condition (15.15) implies, notice that

$$\begin{aligned} \frac{\partial \left[g(\frac{I}{K})K\right]}{\partial K} &= g(\frac{I}{K}) + Kg'(\frac{I}{K})\frac{-I}{K^2} = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} \\ &= g(m(q)) - g'(m(q))m(q) = g(m(q)) - (q-1)m(q) \end{aligned}$$

from (15.14) and (15.17). Insert this into (15.15) to get

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, L_t, M_t) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{15.18}$$

By reordering, this can be written as a no-arbitrage condition,

$$\frac{F_K(K_t, L_t, M_t) - [g(m(q_t)) - (q_t - 1)m(q_t)] - \delta q_t + \dot{q}_t}{q_t} = r, \tag{15.19}$$

saying that, the rate of return on the marginal unit of installed capital must equal the (real) interest rate.

To simplify the expression for the marginal productivity of capital in the differential equations (15.18), we shall now invoke some general equilibrium conditions.

15.1.2 General equilibrium and dynamics

We assume households' behavior is as described by a simple Blanchard OLG model. Yet, in the general equilibrium of the SOE, firms' choices are independent of households' consumption/saving behavior, the analysis of which we therefore postpone to Section 15.1.4.

Clearing in the labor market implies that employment, L_t, equals the exogenous constant labor supply, $\bar{L}$, for all $t \geq 0$. In view of the convex installation costs, K_t is given in the short run and changes only gradually. We now show that the demand for oil, the market clearing wage, and the marginal productivity of capital all can be written as functions of K_t and p_M (for fixed $\bar{L}$).

First, since $F_{MM} < 0$, the firm's optimality condition (15.13) determines oil demand, M_t, as an implicit function of K_t, p_M, and $\bar{L}$:

$$M_t = M(K_t, p_M), \qquad M_K = \frac{-F_{MK}}{F_{MM}} > 0, M_{p_M} = \frac{1}{F_{MM}} < 0, \tag{15.20}$$

where the exogenous constant $\bar{L}$ has been suppressed as an argument, for simplicity. The alleged signs on the partial derivatives are implied (see Appendix A) by the standard assumption $F_{MM} < 0$ and the assumption of direct complementarity: $F_{MK} > 0$.

Second, by inserting (15.20) and $L_t = \bar{L}$ in the optimality condition (15.12), we find an expression for the real wage,

$$w_t = F_L(K_t, \bar{L}, M(K_t, p_M)) \equiv w(K_t, p_M), \qquad w_K > 0, w_{p_M} < 0. \tag{15.21}$$

The alleged signs on the partial derivatives are implied (see Appendix A) by the direct complementarity assumptions $F_{LK} > 0$ and $F_{LM} > 0$.

Third, in view of (15.20) and $L_t = \bar{L}$ we can can simplify the expression for the marginal productivity of capital:

$$F_K(K_t, \bar{L}, M(K_t, p_M)) \equiv MPK(K_t, p_M), \quad MPK_K < 0, MPK_{p_M} < 0, \quad (15.22)$$

The label MPK for this function comes from "Marginal Productivity of K". The alleged sign on the first mentioned partial derivative is implied by F being neoclassical with non-increasing returns to scale combined with the input factors being complementary (see Appendix A). That $MPK_{p_M} < 0$ follows from $F_{MM} < 0$ and $F_{KM} > 0$.

Dynamics of the capital stock

We have thus established that even when the effect of increased K on oil input is taken into account, increased K implies lower marginal productivity of capital. By implication, the analysis of the dynamics of the capital stock is completely similar to that in Chapter 14.3. Indeed, inserting (15.22) into (15.18), we get

$$\dot{q}_t = (r + \delta)q_t - MPK(K_t, p_M) + g(m(q_t)) - m(q_t)(q_t - 1), \quad (15.23)$$

where we have applied Lemma 1 of Chapter 14.1.3. Since r and p_M are exogenous, this is a differential equation with the capital stock, K, and its shadow price, q, as the only endogenous variables. Another differential equation with these two variables being endogenous can be obtained by inserting (15.17) into (15.9) to get

$$\dot{K}_t = (m(q_t) - \delta)K_t. \quad (15.24)$$

Fig. 15.4 shows the phase diagram for these two coupled differential equations. We have (suppressing, for convenience, the explicit time subscripts)

$$\dot{K} = 0 \quad \text{for} \quad m(q) = \delta, \text{ i.e., for } q = q^*,$$

where q^* is defined by the requirement $m(q^*) = \delta$. Notice, that this implies $q^* > 1$ when $\delta > 0$. We see that

$$\dot{K} \gtrless 0 \text{ for } m(q) \gtrless \delta, \text{ respectively, i.e., for } q \gtrless q^*, \text{ respectively.}$$

This is illustrated by the horizontal arrows in Fig. 15.4.

From (15.23) we have $\dot{q} = 0$ for

$$0 = (r + \delta)q - MPK(K, p_M) + g(m(q)) - m(q)(q - 1). \quad (15.25)$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives $0 = (r + \delta)q^* - MPK(K, p_M) + g(\delta) - \delta(q^* - 1)$ or

$$rq^* = MPK(K, p_M) - g(\delta) - \delta, \quad (15.26)$$

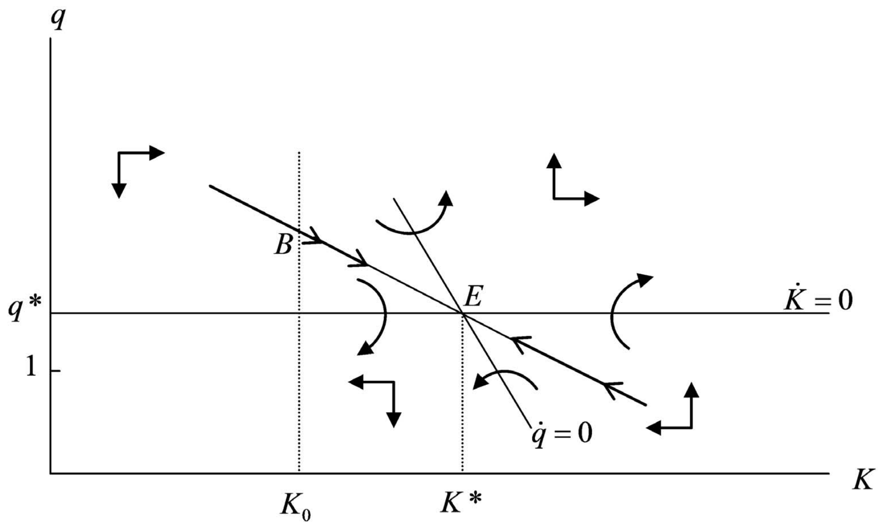

Figure 15.4

where the right-hand-side is decreasing in K, in view of $MPK_K < 0$ (see (15.22)). Hence, there exists at most one value of K such that the steady state condition (15.26) is satisfied.[3] This value is called K^*, corresponding to the steady state, point E, in Fig. 15.4.

As in Chapter 14.3, we end up with a phase diagram as in Fig. 15.4, where the steady state is saddle-point stable. The question now is: what is the slope of the $\dot{q} = 0$ locus? In the appendix of the previous chapter it was shown that at least in a neighborhood of the steady state point E this slope is negative, in view of $MPK_K < 0$ and the assumption $r > 0$. From (15.23) we see that

$$\dot{q} \lesseqgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $MPK_K < 0$. The vertical arrows in Fig. 15.4 show these directions of movement.

Altogether the phase diagram shows that the steady state, E, is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, this steady state is saddle-point stable. We can exclude the divergent paths by appealing to the representative firm's necessary transversality condition (15.16). Hence, a movement along the saddle path towards the steady state is the unique solution for the path of the capital stock and the shadow price of installed capital.

[3] Assuming that F satisfies the Inada conditions, such a value *does* exist.

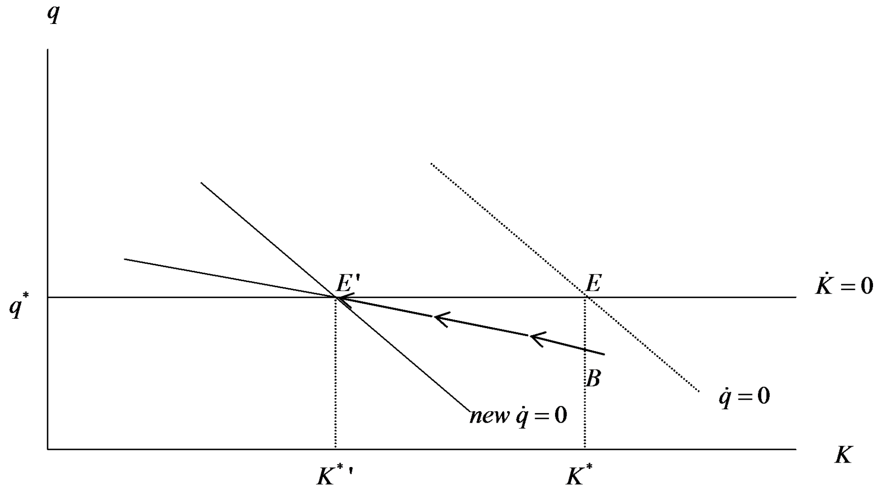

Figure 15.5

Effect of an oil price shock

Assume that until time 0, the economy has been in the steady state E in Fig. 15.4. Then, an unexpected shift in the world market price of oil occurs so that the new price is a constant $p'_M > p_M$ (and is expected to remain for ever at this level). From (15.24) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. But the $\dot{q} = 0$ locus shifts downward, in view of $MPK_{p_M} < 0$. Indeed, to offset the fall of MPK when p_M increases, a lower K is required, given q.

Fig. 15.5 illustrates the situation for $t \geq 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 15.5. This is because the cost of oil is now higher, reducing current and future optimal input of oil and therefore (by complementarity) reducing also the current and future marginal productivity of capital. As a result, the value to the firm of the marginal unit of capital is immediately diminished, implying a diminished incentive to invest. Hence, gross investment jumps to a lower level not sufficient to make up for the wearing-down of capital. The capital stock decreases gradually. But this implies increasing marginal productivity of capital, hence, increasing q, and the economy moves along the new saddle path and approaches the new steady state E*'* as time goes by.

This is where we see the crucial role of strictly convex capital installation costs. If these costs were not present, the model would lead to the counterfactual prediction that the new steady state would be attained instantaneously when the oil price shock occurs.

Notice, however, an important limitation of the theory. In a Keynesian short-run perspective, where firms solve a cost minimization problem for a given desired level of output (equal to the demand faced by the firms), the increase in the price of oil leads to less demand for oil, but *more* demand for capital equipment (a pure substitution effect). Hence, in the real world we may observe a fall in q^a (due to higher production costs) at the same time as investment increases, contrary to what the q-theory of investment predicts under perfect competition.[4]

The reader should recognize that to determine the investment dynamics of the SOE we did not need to consider the households' saving decision. Indeed, one of the convenient features of the SOE model is that it can be solved *recursively:* the total system can be decomposed into an investment subsystem, describing the dynamics of physical capital, and a saving subsystem, describing the dynamics of human wealth and financial wealth of households, H and A, respectively. Although the total system has five endogenous variables, K, q, H, A, and C, the dynamics of K and q are determined by (15.24) and (15.23) independently of the other variables. Thus, (15.24) and (15.23) constitute a *self-contained subsystem of zero order*. We shall soon see that, given the solution of this subsystem of zero order, the dynamics of H are determined in a subsystem of first order in the causal ordering, and, given this determination, the dynamics of A are determined in a subsystem of second order in the causal ordering. Finally, given the determination of H and A, the path of C is determined in a subsystem of third order.

Before turning to household behavior, however, some remarks on national income accounting for this open economy with capital installation costs may be useful.

15.1.3 National income accounting for an open economy with capital installation costs

We ignore the government sector, and therefore national wealth is identical to aggregate private financial wealth, which is here, as usual, called A. We have, by definition,

$$A = V + A_f,$$

[4]This is where we see the crucial role of strictly convex capital installation costs. If these costs were not present, the model would lead to the counterfactual prediction that the new steady state would be attained instantaneously when the oil price shock occurs.

Notice, however, an important limitation of the theory. In a Keynesian short-run perspective, where firms solve a cost minimization problem for a given desired level of output (equal to the demand faced by the firms), the increase in the price of oil leads to less demand for oil, but *more* demand for capital equipment (a pure substitution effect). Hence, in the real world we may observe a fall in q^a (due to higher production costs) at the same time as investment increases, contrary to what the q-theory of investment predicts under perfect competition.

where V is the market value of firms and A_f is net foreign assets (financial claims on the rest of the world).[5] Sometimes, it is more convenient to consider net foreign debt, say $D \equiv -A_f$, so that $A = V - D$. As usual, we define q_a (average q) as the ratio of the market value of firms to the replacement cost of the capital stock,

$$q_a \equiv \frac{V_t}{K_t}.$$

Hence, national wealth can be written

$$A = q_a K + A_f, \tag{15.27}$$

The current account surplus is

$$\dot{A}_f = NX + rA_f, \tag{15.28}$$

where NX is net export of goods and services, also called the trade surplus. We have

$$NX \equiv GDP - C - I = \tilde{Y} - J - p_M M - C - I, \tag{15.29}$$

by (15.5).

Now, look at the matter from the income side rather than the production side. Gross national income, also called gross national product, GNP, is generally defined as the gross income of inputs owned by residents of the home country, i.e., $GNP \equiv GDP + rA_f + wL_f$. Here $rA_f + wL_f$ is total net factor income earned in other countries by residents of the home country, the first term, rA_f, being net capital income from abroad, while the second term, wL_f, represents labor income earned in other countries by residents of the home country minus labor income earned in the home country by residents in the rest of the world. Our present model ignores mobility of labor so that $wL_f = 0$. Hence,

$$GNP = GDP + rA_f. \tag{15.30}$$

At the theoretical level net national product, NNP, is defined, following Hicks (1939), as that level of consumption which would leave financial wealth, A, unchanged. We shall see that this is equivalent to defining NNP as GNP minus *economic* depreciation, $\mathcal{D}$, that is,

$$NNP = GNP - \mathcal{D}. \tag{15.31}$$

We have

$$\mathcal{D} \equiv I - I_n, \tag{15.32}$$

[5] Housing wealth and land are ignored.

where I is domestic gross investment, whereas I_n is domestic net investment in the following *value* sense:[6]

$$I_n \equiv \frac{d(q_a K)}{dt} = q_a \dot{K} + \dot{q}_a K. \tag{15.33}$$

To check whether this is consistent with the Hicksian definition of NNP, insert (15.32) and (15.33) into (15.31) to get

$$\begin{aligned} NNP &= GNP - I + \frac{d(q_a K)}{dt} = GDP + rA_f - I + \frac{d(q_a K)}{dt} \quad \text{(by (15.30))} \\ &= C + NX + rA_f + \frac{d(q_a K)}{dt} \qquad \text{(by (15.29))} \\ &= C + \dot{A}_f + \frac{d(q_a K)}{dt} \qquad \text{(by (15.28))} \\ &= C + \dot{A}, \end{aligned} \tag{15.34}$$

where the last equality follows from (15.27)). This *is* consistent with the theoretical definition of NNP as the level of consumption which would leave financial wealth, A, unchanged.

From (15.34) we get

$$\dot{A} = NNP - C \equiv S_n, \tag{15.35}$$

where S_n is aggregate net saving. This is consistent with the standard definition of aggregate gross saving as $S \equiv GNP - C$, since

$$\begin{aligned} S_n &\equiv NNP - C = GNP - \mathcal{D} - C \qquad \text{(by (15.31))} \\ &\equiv S - \mathcal{D}. \end{aligned}$$

Observe also that

$$\begin{aligned} S &\equiv S_n + \mathcal{D} = GNP - C = GDP + rA_f - C \qquad \text{(by (15.30))} \\ &= rA_f + NX + I \qquad \text{(by (15.29))} \\ &= \dot{A}_f + I. \qquad \text{(by (15.28) and (15.32))} \end{aligned} \tag{15.36}$$

So we end up with the national accounting relationship that the current account surplus, $\dot{A}_f$, is the same as the excess of saving over domestic investment, $S - I$.

Finally, in a steady state with $\dot{A} = 0$ and $d(q_a K)/dt = 0$, (15.27) gives $\dot{A}_f = 0$. Hence, by (15.28), we have

$$NX = -rA_f \tag{15.37}$$

in the steady state, so that (in this model without economic growth) net exports exactly matches interest payments on net foreign debt, $-A_f$.

[6] Net investment in a *physical* sense is $\dot{K} = I - \delta K$, since δ is the rate of physical wearing-down of capital.

15.1.4 Household behavior and financial wealth

As already mentioned, households are described as in the simple Blanchard OLG framework with constant population, no retirement, no technical progress, and no government sector. Hence, aggregate consumption at time t is

$$C_t = (\rho + \mu)(A_t + H_t), \tag{15.38}$$

where $\rho \geq 0$ is the pure rate of time preference, and $\mu > 0$ is the mortality rate (here equal to the birth rate, since $n = 0$). Human wealth, H_t, is the present discounted value of future labor income of those people who are alive at time t, that is,

$$H_t = \int_t^\infty w_\tau \bar{L} e^{-(r+\mu)(\tau-t)} d\tau = \int_t^\infty w(K_\tau, p_M) \bar{L} e^{-(r+\mu)(\tau-t)} d\tau, \tag{15.39}$$

in view of (15.21). Inserting the solution for $(K_\tau)_{\tau=t}^\infty$, found above, (15.39) gives the solution for $(H_t)_{t=0}^\infty$. Notice, that whatever the initial value of K, we know from Section 15.1.2 above that $K_t \to K^*$ for $t \to \infty$. Applying this on (15.39) we see that, for $t \to \infty$,

$$H_t \to \int_t^\infty w(K^*, p_M) \bar{L} e^{-(r+\mu)(\tau-t)} d\tau = \frac{w(K^*, p_M)\bar{L}}{r + \mu} \equiv H^*. \tag{15.40}$$

In view of perfect competition and that the production function F and the capital installation cost function G are homogeneous of degree one, we know from Hayashi's theorem that "average q" = "marginal q", i.e., $q_a = q$ $(= \partial V^*/\partial K_t)$.[7] Therefore, by (15.27), national wealth can be written

$$A = qK + A_f. \tag{15.41}$$

Wealth and consumption dynamics

Observe that

$$\begin{aligned}
GDP &= \tilde{Y} - p_M M - J = F(K, \bar{L}, M) - F_M(K, \bar{L}, M)M - J \\
&\qquad\qquad\qquad\qquad \text{(by (15.5) and (15.13))} \\
&= F_K(K, \bar{L}, M)K + F_L(K, \bar{L}, M)\bar{L} - J \\
&= F_K(K, \bar{L}, M)K + w\bar{L} - J, \qquad \text{(by (15.12))}
\end{aligned} \tag{15.42}$$

where the second equality comes from Euler's Theorem applied to the CRS function $F(K, \bar{L}, M)$.

[7] See the previous chapter. Hayashi's theorem is valid also when, as here, there are three (or more) production factors.

From (15.35) we have

$$\begin{aligned}
\dot{A} &= S_n = NNP - C = GNP - \mathcal{D} - C && \text{(by (15.35) and (15.31))} \\
&= GDP + rA_f - \mathcal{D} - C && \text{(by (15.30))} \\
&= F_K(K,\bar{L},M)K + w\bar{L} - J - (I - I_n) + rA_f - C && \text{(by (15.42) and (15.32))} \\
&= F_K(K,\bar{L},M)K + w\bar{L} - J - (I - q\dot{K} - \dot{q}K) + rA_f - C
\end{aligned}$$

by (15.33) and the fact that q_a equals q. Continuing, we have

$$\begin{aligned}
\dot{A} &= F_K(K,\bar{L},M)K + w\bar{L} - J + (q-1)I - \delta qK + \dot{q}K + rA_f - C \text{ (by (15.9))} \\
&= \left[F_K(K,\bar{L},M) - g(m(q)) + m(q)(q-1) - \delta q + \dot{q}\right]K + rA_f + w\bar{L} - C \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(by (15.3) and (15.17))} \\
&= rqK + rA_f + w\bar{L} - C, \qquad \text{(by the no-arbitrage condition (15.19))} \\
&= rA + w\bar{L} - C. \qquad\qquad \text{(by (15.41))}.
\end{aligned}$$

Comparing this with (15.27) we see that in equilibrium, $NNP = r(qK+A_f)+w\bar{L}$. That is, national income is equal to the sum of income from financial wealth and income from labor, as expected. The rate of return on financial wealth is given from the world capital market, and the pay of labor is the market clearing real wage in the SOE.

Using (15.21) and (15.38), our differential equation for financial wealth can be written

$$\dot{A}_t = (r - \rho - \mu)A_t + w(K_t, p_M)\bar{L} - (\rho + \mu)H_t. \tag{15.43}$$

Since initial national wealth, A_0, is historically given, and the paths of K_t and H_t have already been determined, this differential equation determines uniquely the path of national wealth, A_t.

Suppose

$$\rho + \mu > r, \tag{15.44}$$

that is, we are *not* in the case of "very low impatience".[8] Then (15.43) implies stability of A_t so that, for $t \to \infty$,

$$A_t \to \frac{w(K^*, p_M)\bar{L} - (\rho+\mu)H^*}{\rho+\mu-r} = \frac{(r-\rho)H^*}{\rho+\mu-r} = \frac{(r-\rho)w(K^*,p_M)\bar{L}}{(\rho+\mu-r)(r+\mu)} \equiv A^*, \tag{15.45}$$

where we have used (15.40).

[8] Otherwise, i.e., if $\rho \leq r - p$, no steady state would exist (see (15.46) below)) and the SOE would grow *large* in the long run. Then the world market interest rate r could no longer be considered independent of what happens in this economy.

Finally, given the solution for H_t and A_t, (15.38) shows the solution for C_t. When the stability condition (15.44) holds, we have, for $t \to \infty$,

$$C_t \to (\rho+\mu)(A^* + H^*) = (\rho+\mu)\frac{\mu w(K^*, p_M)\bar{L}}{(\rho+\mu-r)(r+\mu)} \equiv C^*. \tag{15.46}$$

Given the stability condition (15.44), the steady-state value of national wealth in (15.45) is positive, if and only if $r - \mu < \rho < r$. This is the case of "medium impatience" where our SOE has a degree of impatience, ρ, that is *not* vastly different from that of the "average country" in the world economy.[9]

If on the other hand our SOE is *very* impatient ($\rho > r$), then, even supposing that initial national wealth is positive, so that interest income is positive, the economy consumes more than it earns so that net saving is negative and national wealth decreases over time. Indeed, we know from the Blanchard model that the change in aggregate consumption per time unit is given by

$$\dot{C}_t = (r-\rho)C_t - \mu(\rho+\mu)A_t,$$

so that, with $\rho > r$, we get $\dot{C}_t < 0$, at least as long as $A_t \geq 0$. The economy ends up with negative national wealth in the long run, as shown by (15.45). This entails a net foreign debt over and above the market value, q^*K^*, of the firms:

$$-A_f^* = q^*K^* - A^* = q^*K^* - \frac{(r-\rho)H^*}{\rho+\mu-r} > q^*K^*. \tag{15.47}$$

This is theoretically possible in view of the fact that the economy still has its human wealth, H, as a source of income. Indeed, as long as (15.44) holds, a steady state with $A^* + H^* > 0$ exists, as indicated by (15.46).

What (15.47) shows is that a very impatient country asymptotically mortgages all of its physical capital and part of its human capital. This is a counterfactual prediction, and below we return to the question why such an outcome is not likely to occur in practice.

Intertemporal interpretation of current account movements

Finally, the level of net exports is

$$\begin{aligned} NX &= \tilde{Y} - p_M M - J - I - C && \text{(by (15.29) and (15.5))} \\ &= F_K(K,\bar{L},M)K + w\bar{L} - J - I - C && \text{(by (15.42))} \\ &= \dot{A} - I_n - rA_f. && \text{(by the third row in the derivation of } \dot{A} \text{ above)} \end{aligned}$$

[9] If all countries can be described by the simple Blanchard model, then the interest rate r in the world market is somewhat larger than the pure rate of time preference of the "average country", cf. Chapter 12.

In steady state, $\dot{A} = 0 = I_n$, hence,

$$\begin{aligned} NX^* &= -rA_f^* & (15.48) \\ &= -r\left[q^*K^* - \frac{(r-\rho)w(K^*, p_M)\bar{L}}{(\rho+\mu-r)(r+\mu)}\right]. & \text{(from (15.47))} \end{aligned}$$

This determines the long-run level of net exports as being equal to the interest payments on net foreign debt, $D \equiv -A_f$, so that in the steady state, the current account deficit, $rD - NX$, is zero. As expected, the economy remains solvent. In fact, the consumption function (15.38) of the Blanchard model is derived under the constraint that solvency, through a NPG condition on the long-run path of financial wealth (or debt), is satisfied.

Whatever the size relation between ρ and r, it is not necessary for equilibrium that net foreign debt is zero in the long run. Necessary in this model, which is without economic growth, is that in the long run net foreign debt is constant, i.e., the current account is ultimately zero.

With economic growth, the SOE can have a permanent current account deficit and thus permanently increasing foreign D and yet remain solvent forever. What is in the long run needed for equilibrium, however, is that the foreign debt does not grow faster than GDP. As we saw in Chapter 13, this condition will be satisfied if net exports as a fraction of GDP are sufficient to cover the growth-corrected interest payments on the debt. (This analysis ignores that the scope for writing enforceable international credit contracts is somewhat limited and so, in practice, there is likely to be an upper bound on the debt-income ratio acceptable to the lenders. Such a bound is in fact apt to be operative well before the foreign debt moves beyond the value of the capital stock in the economy.)

Overall effect of an oil price shock

Returning to the model, without economic growth, analyzed in detail above, let us summarize. An oil price shock such that p_M shifts to a higher (constant) level implies a lower equilibrium real wage, $w_t = w(K_t, p_M)$, both on impact and in the longer run. The impact effect comes from lower input of oil, hence a lower marginal productivity of labor, cf. (15.21). This implies, on impact, a fall in H_t, see (15.39), and therefore also in C_t, see (15.38). In addition, as was shown in Section 15.1.2, K_t is gradually reduced over time and this decreases output and the marginal productivity of labor further. As a result the long-run values of H and A become lower than before, and so does the long-run value of C. Whether in the long run net foreign assets, A_f^*, and net exports, NX^*, become lower or not we cannot know, because the fall in national wealth, A^*, may, but need not, be larger than the fall in the capital stock, K^*.

To summarize: The overall effect of an adverse oil price shock is an economic contraction. If the model were extended by including short-term Keynesian demand effects, arising from the purchasing power of consumers being undermined by a sudden increase in the general price level, then the economic contraction may become more severe, leading to a pronounced recession.

Going further outside the model we could imagine that trade unions, by demanding compensation for price increases, resist the real wage decrease required for unchanged employment, when the oil price rises. As a result unemployment tends to go up. If in addition the wage-price spiral is accommodated by expansionary monetary policy, as after the first oil price shock in 1973-74, then simultaneous high inflation and low output may arise. This is exactly the phenomenon of *stagflation* that we saw in the aftermath of the first oil price shock.

15.1.5 General aspects of modeling a small open economy

Let us return to the case of a very impatient society ($\rho > r$) and focus on (15.45) and (15.47). If the mortality rate μ is very small, the model predicts that the country asymptotically mortgages, in addition to its physical capital, *all* its human capital. The long-run prospect could be a very low consumption level. The Ramsey model as well as the Barro model with an operative bequest motive, are examples of models with a very low μ since, effectively, they have $\mu = 0$. Hence, a Ramsey-style model for a small open economy (ignoring technical progress) with $\rho > r$ will satisfy the condition (15.44) and entail $A_t \to -H^*$, implying de-cumulation *forever*, that is, $C_t \to 0$, by (15.46. The fact that Ramsey-style models can predict such outcomes, is a warning that such models are in some contexts of limited value.

If, on the other hand, $\rho < r$, then the Ramsey model implies low consumption and high saving. Indeed, the country will forever accumulate financial claims on the rest of the world. This is because, in the Ramsey model the Keynes-Ramsey rule holds not only at the individual level, but also at the aggregate level. Eventually, the country becomes a large economy and begins to affect the world interest rate, contradicting the assumption that it is a small open economy.

To avoid these extreme outcomes, when applying the Ramsey model for studying a small open economy, one has to assume $\rho = r$. But this is an unwelcome *knife-edge* condition, a parameter restriction which is very unlikely to hold in reality.

It is otherwise with the Blanchard OLG model, where the generation replacement effect implies that the Keynes-Ramsey rule does *not* hold at the aggregate level. Therefore, the OLG model for a small open economy needs no knife-edge condition on parameters. The model works well whatever the size relation between ρ and r, as long as the stability condition (15.44) is satisfied. Or, to be more precise: the Blanchard model works well in the case $\rho < r$; in the opposite case, where $\rho > r$, the model works at least better than the Ramsey model, because it never implies that $C_t \to 0$ in the long run.

It should be admitted, however, that in the case of a very impatient country ($\rho > r$), even the OLG model implies a counterfactual prediction. What (15.47) tells us is that the impatient small open economy in a sense asymptotically mortgages all of its physical capital and part of its human capital. The OLG model predicts this will happen, *if* financial markets are perfect, and *if* the political sphere does not intervene. It certainly seems unlikely that an economic development, ending up with negative national wealth, is going to be observed in practice. There are two - complementary - explanations of this.

First, the international credit market is far from perfect. Because a full-scale supranational legal authority comparable with domestic courts is lacking, credit default risk in international lending is generally a more serious problem than in domestic lending. Physical capital can to some extent be used as a collateral on foreign loans, while human wealth is not suitable. Human wealth cannot be repossessed. This implies a constraint on the ability to borrow.[10] And lenders' risk perceptions depend on the level of debt.

Second, long before *all* the physical capital of an impatient country is mortgaged or have directly become owned by foreigners, the government presumably would intervene. In fear of losing national independence, it would use its political power to end the pawning of economic resources to foreigners.

This is a reminder, that we should not forget that the economic sphere of a society is just one side of the society. Politics as well as culture and religion are other sides. The economic outcome may be conditioned on these social factors, and the interaction of all these spheres determines the final outcome.

15.2 The housing market and residential construction

The housing market is from a macroeconomic point of view important for several reasons: a) housing makes up a substantial proportion of the consumption budget; b) housing wealth makes up a substantial part of private wealth of a major fraction

[10] We have been speaking as if domestic residents own the physical capital stock in the country, but have obtained part or all the financing of the stock by issuing bonds to foreigners. The results would not change if we allowed for foreign direct investment. Then foreigners would themselves own part of the physical capital rather than bonds. In such a context a similar constraint on foreign investment is likely to arise, since a foreigner can buy a factory or the shares issued by a firm, but it is difficult to buy someone else's stream of future labour income.

of the population; c) fluctuations in house prices and in construction activity are large and seem important for business cycles; and d) residential investment, whish typically is of magnitude 5 percent of GDP, and aggregate output are strongly positively correlated. The analysis will be based on a simple dynamic partial equilibrium model with rising marginal construction costs.

Let time be continuous. Let H_t denote the aggregate housing stock at time t and S_t the aggregate flow of housing services at time t. Ignoring heterogeneity, the housing *stock* can be measured in terms of m^2 floor area at a given point in time. For convenience we will talk about the stock as a certain number of houses of a standardized size. The supply of housing *services* at time t constitute a *flow*, thereby being measured *per time unit*, say per year: so and so many square meter-months are at the disposal for accommodation during the year. The two concepts are related through

$$S_t = \alpha H_t, \qquad \alpha > 0, \tag{15.49}$$

where we will treat α as a constant which depends only on the measurement unit for housing services. If these are measured in square meter-months, α equals the number of square meters of a "normal-sized" house times 12.

We ignore population growth and economy-wide technological progress.

15.2.1 The housing service market and the house market

There are two goods, houses and housing services, and therefore also two markets and two prices:

p_t = the (real) price of a "normal-sized'" house at time t,
R_t = the rental rate $\equiv$ the (real) price of housing services at time t.

The price R_t of housing services is known as the *rental rate* at the housing market. Buying a housing service means *renting* the apartment or the house for a certain period. Or, if we consider an owner-occupied house (or apartment), R_t is the imputed rental rate, that is, the owner's opportunity cost of occupying the house. The prices R_t and p_t are measured in *real* terms, or more precisely, they are deflated by the consumer price index. We assume perfect competition in both markets.

The market for housing services

In the short run the housing stock is historically given. Construction is time-consuming and houses cannot be imported. Owing to the long life of houses, investment in new houses per year tends to be a small proportion of the available housing stock (in advanced economies about 3 percent, say). So also the supply, S_t, of housing services is given in the short run.

Suppose the aggregate demand for housing services at time t is

$$S_t^d = D(R_t, A, PV(wL)), \qquad D_1 < 0, D_2 > 0, D_3 > 0, \tag{15.50}$$

where A is aggregate financial wealth and $PV(wL)$ is human wealth, i.e., the present discounted value of expected future labor income after tax for those alive. That demand depends negatively on the rental rate reflects that both the substitution effect and the income effect of a higher rental rate are negative. The wealth effect on housing demand of a higher rental rate is likely to be positive for owners and negative for tenants.[11]

The market for housing services is depicted in Fig. 15.6. We get a characterization of the equilibrium rental rate in the following way. In equilibrium at time t, $S_t^d = S_t$, that is,

$$D(R_t, A, PV(wL)) = \alpha H_t. \tag{15.51}$$

This equation determines R_t as an implicit function, $R_t = \tilde{R}(H_t,\ A,\ PV(wL))$, of H_t, A, and $PV(wL)$. By implicit differentiation in (15.51) we find the partial derivatives of this function, $\tilde{R}_H = \alpha/D_R < 0$, $\tilde{R}_A = -D_A/D_S > 0$, and $\tilde{R}_{PW} = -D_{PV}/D_R > 0$.

The supply of housing services is inelastic in the short run and the market clearing rental rate immediately moves up and down as the demand curve shifts rightward or leftward. But in our partial equilibrium framework, we will consider A and $PV(wL)$ as exogenous and constant. Hence we suppress these two variables as arguments in the functions and define $R(H_t) \equiv \tilde{R}(H_t, A, PV(wL))$, whereby

$$R_t = R(H_t), \qquad R' = \alpha/D_R < 0. \tag{15.52}$$

From now on our time unit will be one year and we define one unit of housing service per year to mean disposal of a house of standard size one year. By this, α in (15.49) equals 1.

[11] A simple microeconomic "rationale" behind the aggregate demand function (15.50) is obtained by assuming an instantaneous utility function $u(h_t, c_t) = \ln(h_t^{\gamma} c_t^{1-\gamma})$, where $0 < \gamma < 1$, and h_t is consumption of housing services at time t, whereas c_t is non-housing consumption. Then the share of housing expenditures in the total instantaneous consumption budget is a constant, γ. This is broadly in line with empirical evidence for the US (Davis and Heathcote, 2005). In turn, according to standard neoclassical theory, the total consumption budget will be an increasing function of total wealth of the household, cf. Chapter 9. Separation between the two components of wealth, A and $PV(wl)$, is relevant when credit markets are imperfect.

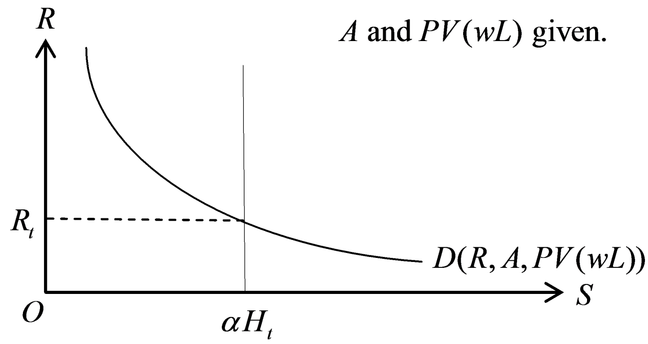

Figure 15.6: Supply and demand in the market for housing services at time t.

The market for existing houses

Because a house is a durable good with market value, it is an *asset.* This asset typically constitutes a substantial share of the wealth of a large fraction of the population, the house-owners. At the same time the supply of the asset can change only slowly.

Assume there is an exogenous and constant risk-free real interest rate $r > 0$. This is a standard assumption in partial equilibrium analysis. If the economy is a small open economy with perfect capital mobility, the exogeneity of r (if not constancy) is warranted even in general equilibrium analysis.

Considering the asset motive associated with housing, a series of aspects are central. We let houses depreciate physically at a constant rate $\delta > 0$. Suppose there is a constant tax rate $\tau_R \in [0, 1)$ applied to rental income (possibly imputed) after allowance for depreciation. In case of an owner-occupied house the owner must pay the tax $\tau_R(R_t - \delta p_t)$ out of the imputed income $(R_t - \delta p_t)$ per house per year. Assume further there is a constant property tax (real estate tax) $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose that a constant tax rate $\tau_r \in [0, 1)$ applies to interest income. There is symmetry in the sense that if you are a debtor and have negative interest income, then the tax acts as a rebate. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.[12]

Suppose there are no credit market imperfections, no transaction costs, and no uncertainty. Assume further that the user of housing services value these services

[12] Note, however, that if all capital income should be taxed at the same rate, capital gains should also be taxed at the rate τ_r, and τ_R should equal τ_r. In Denmark, in the early 2000s, the government replaced the rental value tax, τ_R, on owner-occupied houses by a lift in the property tax, τ_p. Since then, due to a *nominal* "tax freeze", τ_p has been gradually decreasing in real terms.

independently of whether he/she owns or rent. Under these circumstances the price of houses, p_t, will adjust so that the expected after-tax rate of return on owning a house equals the after-tax rate of return on a safe bond. We thus have the no-arbitrage condition

$$\frac{(1-\tau_R)(R(H_t)-\delta p_t)-\tau_p p_t+\dot{p}_t^e}{p_t}=(1-\tau_r)r, \tag{15.53}$$

where $\dot{p}_t^e$ denotes the expected capital gain per time unit (so far $\dot{p}_t^e$ is just a commonly held subjective expectation).

For given $\dot{p}_t^e$ we find the equilibrium price

$$p_t=\frac{(1-\tau_R)R(H_t)+\dot{p}_t^e}{(1-\tau_r)r+(1-\tau_R)\delta+\tau_p}.$$

Thus p_t depends on H_t, $\dot{p}_t^e$, r, and tax rates in the following way:

$$\begin{aligned}
\frac{\partial p_t}{\partial H_t} &= \frac{(1-\tau_R)R'(H_t)}{(1-\tau_r)r+(1-\tau_R)\delta+\tau_p}<0,\\
\frac{\partial p_t}{\partial \dot{p}_t^e} &= \frac{1}{(1-\tau_r)r+(1-\tau_R)\delta+\tau_p}>0,\\
\frac{\partial p_t}{\partial \tau_R} &= \frac{-\left[(1-\tau_r)r+\tau_p\right]R(H_t)+\delta\dot{p}_t^e}{\left[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p\right]^2}\lesseqqgtr 0 \text{ for } \dot{p}_t^e\lesseqqgtr\frac{\left[(1-\tau_r)r+\tau_p\right]R(H_t)}{\delta},\\
\frac{\partial p_t}{\partial \tau_p} &= -\frac{(1-\tau_R)R(H_t)+\dot{p}_t^e}{\left[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p\right]^2}<0,\\
\frac{\partial p_t}{\partial \tau_r} &= \frac{\left[(1-\tau_R)R(H_t)+\dot{p}_t^e\right]r}{\left[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p\right]^2}>0,\\
\frac{\partial p_t}{\partial r} &= -\frac{\left[(1-\tau_R)R(H_t)+\dot{p}_t^e\right](1-\tau_r)}{\left[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p\right]^2}<0,
\end{aligned}$$

where the sign of the last three derivatives are conditional on $\dot{p}_t^e$ being nonnegative or at least not "too negative".

Note that a higher expected increase in p_t, $\dot{p}_t^e$, implies a higher house price p_t. Over time this feeds back and may confirm and sustain the expectation, thus generating a further rise in p_t. Like other assets, a house is thus a good with the property that the expectation of price increases make buying more attractive and may become self-fulfilling if the expectation is generally held.

15.2.2 Residential construction

It takes time for the stock H_t to change. While manufacturing typically involves mass production of similar items, construction is generally done on location for a known client and within intricate legal requirements. It is time-consuming to design, contract, and execute the sequential steps involved in residential construction. Careful guidance and monitoring is needed. These features give rise to fixed costs (to management, architects etc.) and thereby rising marginal costs in the short run. Congestion and bottlenecks may easily arise.

The construction process

Assume the construction industry is competitive. At time t the *representative construction firm* produces B_t units of housing per time unit (B for "building"), thereby increasing the aggregate housing stock according to

$$\dot{H}_t = B_t - \delta H_t, \delta > 0. \tag{15.54}$$

The construction technology is described by a production function $\tilde{F}$:

$$B_t = \tilde{F}(K_t, L_t, \bar{M}; E_t) \equiv \bar{F}(F(K_t, L_t), \bar{M}; E_t) = \bar{F}(I_t, \bar{M}; E_t) \equiv T(I_t; E_t).$$

The last argument of $\tilde{F}$, E_t, is not a production factor but stands for construction experience acquired through accumulated learning in the construction industry. It determines the efficiency of the current technology. The three other arguments of $\tilde{F}$ represent input of capital, K_t, blue-collar labor, L_t, and "management labor", $\bar{M}$, which includes working hours of specialists like architects and lawyers. There are constant returns to scale with respect to these three production factors. We treat $\bar{M}$ as a fixed production factor even in the medium run. Hence the associated fixed cost (salaries) is, in real terms, constant for quite some time. We denote this fixed cost $\bar{f}$.

The remaining two production inputs, capital and blue-collar labor, produces components for residential construction − intermediate goods − in the amount $I_t = F(K_t, L_t)$ per time unit. The production function, F, is "nested" in the "global" production function, $\bar{F}$. Thus construction is modeled as if it makes up a two-stage process. First, capital and blue-collar labor produce intermediate goods for construction. Next, management accomplishes quality checks and "assembling" of these intermediate goods into new houses or at least final new components built into existing houses. The final output is measured in units corresponding to a standard house. This does not rule out that a large part of the output is really in the form of renovations, additions of a room etc.

We treat both blue-collar labor and capital as variable production factors in the short run and assume F has constant returns to scale. The intermediate goods are produced on a routine basis at minimum costs (convex *capital* adjustment costs, as in Chapter 14, are for simplicity ignored). Let the real cost per unit of I_t be denoted c. In our short-to-medium run perspective we treat c as a constant.

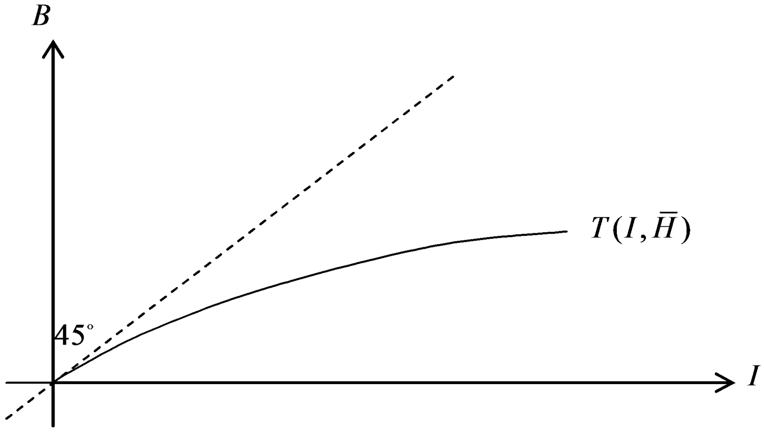

Figure 15.7: The number of new houses as a function of residential investment (for given $E = \bar{H}$).

The marginal productivity of I_t is decreasing in I_t. That is, keeping $\bar{M}$ fixed, the final output, B_t, has *diminishing* returns with respect to the level of *construction activity* per time unit as measured by the flow variable I_t. In the short run thus rising marginal costs obtains, "haste is waste".

To save notation, from now on, with the purpose of suppressing the constant argument $\bar{M}$, we introduce the production function T. Moreover, we suppress the explicit dating of the variables unless needed for clarity. To help intuition, we shall speak of the function T as a *transformation function.* This function is assumed to be strictly concave in I : the larger is I, the smaller is the rate at which a unit increase in I is transformed into new houses.

To summarize: the amount of new houses built per time unit is

$$\begin{aligned} B &= T(I, E), \quad \text{where} \\ T(0, E) &= 0, \quad T_I(0, E) = 1, \quad T_I > 0, \quad T_{II} < 0, \quad T_E \geq 0. \end{aligned} \tag{15.55}$$

A higher level of construction activity per time unit means that a larger fraction of I is "wasted" because of control, coordination, and communication difficulties. Hence $T_{II}(I, E) < 0$, i.e.,.T is strictly concave in I

The second argument in the transformation function is the construction experience, E. More experience means that the intermediate goods can be designed in a better way thus implying higher productivity of a given I than otherwise, hence $T_E \geq 0$.[13] As an indicator of cumulative experience it would be natural to use

[13]In a long-run perspective, the increasing scarcity of available land may hamper the productivity of the intermediate goods, for given I and E. This is ignored in our medium-run perspective. All the same, in the real world construction technology improves over time and the limited availability of land can to some extent be dealt with by building taller structures.

cumulative gross residential investment, $\int_{-\infty}^{t} B_s ds$, reflecting cumulative learning by doing. It is simpler, however, to use cumulative *net* residential investment, H_t. We thus assume that E_t is (approximately) proportional to H_t.[14] Normalizing the factor of proportionality to one, we have

$$E_t = H_t.$$

For fixed $E = \bar{H}$, Fig. 15.7 shows the graph of $T(I, \bar{H})$ in the $(I,\ B)$ plane. The assumptions $T_I(0, \bar{H}) = 1$ and $T_{II} < 0$ imply $T_I(I, \bar{H}) < 1$ for $I > 0$, as visualized in the figure. An example satisfying all the conditions in (15.55) is a CES function,[15]

$$T(I, H) = A(aI^{\beta} + (1-a)H^{\beta})^{1/\beta}, \quad \text{with } 0 < A < 1, 0 < a < 1, \text{ and } \beta < 0.$$

From the perspective of Tobin's q-theory of investment, we may let the "waste" be represented by a kind of adjustment cost function $G(I, H)$ akin to that considered in Chapter 14. Then $T(I, H) \equiv I - G(I, H)$. In Chapter 14 convex adjustment costs were associated with the installation of firms' fixed capital and acted as a reduction in the firms' output available for sale. In construction we may speak of analogue costs acting as a reduction in the productivity of the intermediate goods in the construction process. It is easily seen that, on the one hand, all the properties of G *required* in Chapter 14 when $I \geq 0$ are maintained. On the other hand, not all properties required of T in (15.55) need be satisfied in Tobin's q-theory (see Appendix B).

Profit maximization

The representative construction firm takes the current economy-wide experience $E = H$ as given. The gross revenue of the firm is pB and costs are cI. Given the market price p, the firm maximizes profit:

$$\begin{aligned} \max_I \Pi &= pB - cI \qquad \text{s.t.} \qquad B = T(I, H) \text{ and} \\ I &\geq 0. \end{aligned}$$

Inserting $B = T(I, H)$, we find that an interior solution satisfies

$$\frac{d\Pi}{dI} = pT_I(I, H) - c = 0, \text{ i.e., } \frac{p}{c}T_I(I, H) = 1. \tag{15.56}$$

[14] At least in an economic growth context, where H would almost never be decreasing, this approximation of the learning effect would not seem too coarse.

[15] As shown in the appendix to Chapter 4, by defining $T(I, H) = 0$ when $I = 0$ or $H = 0$, the domain of the CES function can be extended to include all $(I, H) \in \mathbb{R}_+^2$ also when $\beta < 0$, while maintaining continuity.

In view of $T_I(I,H) < 1$ for $I > 0$, the latter equation has a solution $I > 0$ only if $p > c$. For $p \leq c$, we get the corner solution $I = 0$. Naturally, when the current market price of houses is below marginal construction cost (which equals $c/(T_I(I,H) \geq c)$, no new houses will be built. This is a desired property of the model. On the other hand, when $p > c$, the construction firm will supply new houses up to the point where the rising marginal cost equals the current house price, p.[16]

A precise determination of optimal I is obtained the following way. For $p > c$, the first-order condition (15.56) defines construction activity, I, as an implicit function of p/c and H :

$$I = M\left(\frac{p}{c}, H\right), \quad \text{where } M(1,H) = 0. \tag{15.57}$$

By implicit differentiation with respect to p/c in (15.56), we find

$$M_{p/c} = \frac{\partial I}{\partial(p/c)} = \frac{-1}{(p/c)^2 T_{II}(I,H)} > 0,$$

where the argument I can be written as in (15.57).

Special case

From now on we assume the transformation function T is homogeneous of degree one. Thus, $B = T(I/H,1)H$. Then, by Euler's theorem, $T_I(I,H)$ is homogeneous of degree 0. So, with explicit timing of the time-dependent variables, (15.56) can be written

$$\frac{p}{c}T_I\left(\frac{I}{H},1\right) = 1.$$

This first-order condition defines I_t/H_t as an implicit function of p_t/c :

$$\frac{I}{H} = m\left(\frac{p}{c}\right), \quad \text{where } m(1) = 0. \tag{15.58}$$

By implicit differentiation with respect to p/c in the first-order condition we find

$$m' = \frac{-1}{(p/c)^2 T_{II}\left(I/H,1\right)} > 0,$$

where $I/H = m(p/c)$ can be inserted. A construction activity function m with this property is shown in Fig. 15.8, where $c = 1$.

[16] How to come from the transformation function $T(I,H)$ to the marginal cost schedule is detailed in Appendix C.

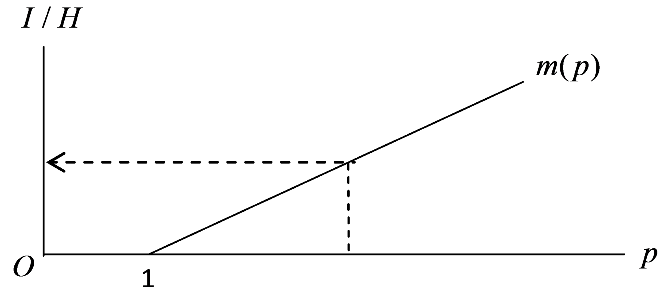

Figure 15.8: Construction activity (relative to the housing stock) as a function of the market price of houses ($c = 1$).

With explicit timing of the time-dependent variables, and letting b_t denote the flow of new houses relative to the stock of houses, we now have

$$b_t \equiv \frac{B_t}{H_t} = \frac{T(I_t, H_t)}{H_t} = T\left(\frac{I_t}{H_t}, 1\right) = T\left(m\left(\frac{p_t}{c}\right), 1\right) \equiv b\left(\frac{p_t}{c}\right), \tag{15.59}$$

where $b(1) = T(m(1), 1) = T(0, 1) = 0$, $b' = T_I m' > 0$.

Remark. Like Tobin's q, the house price p is the market value of a produced asset whose supply changes only slowly. As is the case for firms' fixed capital there are strictly convex stock adjustment costs, represented by the rising marginal construction costs. As a result the stock of houses does not change instantaneously if for instance p changes. But as shown by the above analysis, the flow variable, residential construction, responds to p in a way similar to the way firm's fixed-capital investment responds to Tobin's q according to the q theory. Recall that Tobin's q is defined as the economy-wide ratio $V/(p_I K)$, where V is the market value of the firms, p_I is a price index for investment goods, and K is the stock of physical capital. The analogue ratio in the housing sector is $V^{(H)}/(p_I \cdot H) \equiv p \cdot H/(p_I \cdot H)$ $= p/c$, in view of $p_I = c$. A higher p/c results in more construction activity. □

15.2.3 Equilibrium dynamics under perfect foresight

To determine the evolution over time in H and p, we derive two coupled differential equations in these two variables. When the transformation function T is homogeneous of degree one, we can in view of 15.59) write (15.54) as

$$\dot{H}_t = \left(b\left(\frac{p_t}{c}\right) - \delta\right) H_t, \tag{15.60}$$

where $b(1) = 0$ and $b' = T_I m' > 0$.

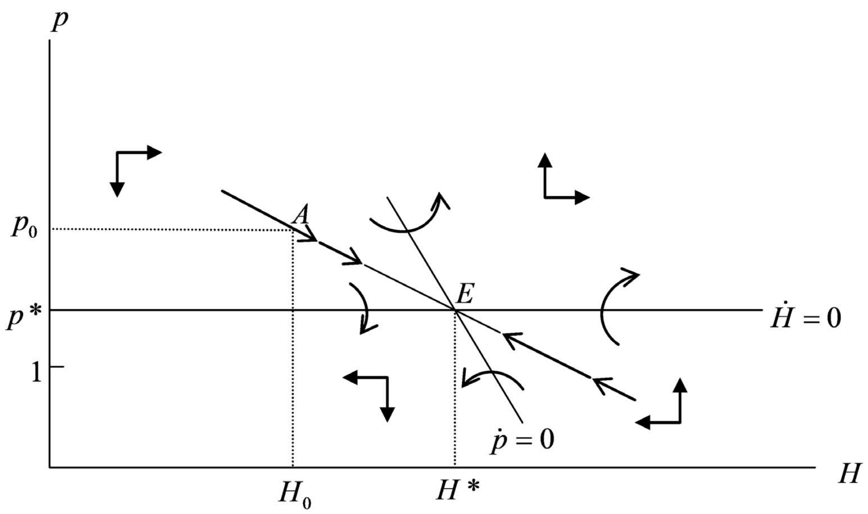

Figure 15.9: Phase diagram of aggregate construction activity ($c = 1$).

Assuming *perfect foresight*, we have $\dot{p}_t^e = \dot{p}_t$ for all t. Then we can write (15.53) on the standard form for a first-order differential equation:

$$\dot{p}_t = \left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right] p_t - (1-\tau_R)R(H_t), \tag{15.61}$$

where $R' < 0$. We have hereby obtained a dynamic system in H and p, the coupled differential equations (15.60) and (15.61). The corresponding phase diagram is shown in Fig. 15.9.

We have $\dot{H} = 0$ for $b(p/c) = \delta > 0$. The unique p satisfying this equation is the steady state value p^*. The $\dot{H} = 0$ locus is thus represented by the horizontal line segment $p = p^*$. The direction of movement for H is positive if $p > p^*$ and negative if $p < p^*$. Since $b(1) = 0$ and $b' > 0$, we have $p^* > c$.

We have $\dot{p} = 0$ for $p = (1-\tau_R)R(H)/\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right]$. Since $R'(H) < 0$, the $\dot{p} = 0$ locus has negative slope. The unique steady state value of H is denoted H^*. To the right of the $\dot{p} = 0$ locus, p is rising, and to the left p is falling. The directions of movement of H and p in the different regions of the phase plane are shown in Fig. 15.9.

The unique steady state is seen to be a saddle point with housing stock H^* and housing price p^*. The initial housing stock, H_0, is predetermined. Hence, at time $t = 0$, the economic system must be somewhere on the vertical line $H = H_0$. The question is whether there can be asset price bubbles in the system. An *asset price bubble* is present if the market value of the asset for some time systematically exceeds its *fundamental value* (the present value of the expected future services or dividends from the asset). Agents might be willing to buy at a price above the fundamental value if they expect the price will rise further in the future. The divergent paths ultimately moving North-East in the phase diagram are actually, by construction, consistent with the no-arbitrage condition and are thus candidates for asset price bubbles generated by self-fulfilling expectations. The fact that houses have clearly defined reproduction costs, however, implies a ceiling on the ultimate level of p since potential buyers of already existing houses have the alternative of initiating construction at "normal pace" of a new house at the cost p^*. Then, by backward induction, these explosive price paths will not, under rational expectations, get started in the first place. Given rational expectations, these paths - and therefore "rational bubbles" - can thus be ruled out.[17] This leaves us with the converging path as the unique solution to the model. At time 0 the residential construction sector will be at the point A in the diagram and then it will move along the saddle path and after some time the housing stock and the house price settle down at the steady state, E.

In this model (without economic growth) the steady-state price level, p^*, of houses equal the marginal building costs when building activity exactly matches the physical wearing down of houses so that the stock of houses is constant. Owing to the specific form of the positive relationship between building productivity and H, implied by the transformation function T being homogeneous of degree one in I and H, the marginal building costs are unchanged in the medium run even if r or one of the taxes change. The steady-state level of H is at the level required for the rental rate $R(H)$ to yield an after-tax rate of return on owning a house equal to $(1-\tau_r)r$. This level of H is H^*. The corresponding level of R is R^* $= R(H^*)$, which is that level at which the demand for housing services equals the steady-state supply, i.e., $D(R^*, A, PV(wl)) = S^* = H^*$.

Effect of a fall in the property tax

Suppose the residential construction sector has been in the steady state E in Fig. 15.10 until time t_1. Then there is an unanticipated downward shift in the property tax τ_p to a new constant level τ_p' rightly expected to last forever in the future. The resulting evolution of the system is shown in the figure. The new steady state is called E'. The new medium-run level of H is $H^{*\prime} > H^*$, because $R'(H) < 0$. On impact, p jumps up to the point where the vertical line $H = H^*$ crosses the new (downward-sloping) saddle path. The intuition is that the after-tax return on owning a house has been increased. Hence, by arbitrage the market price p rises to a level such that the after-tax rate of return on houses is as before, namely equal to $(1-\tau_r)r$. After t_1, owing to the high p relative to the unchanged building cost schedule, H increases gradually and p falls gradually (due to R

[17] In the last section we briefly return to the issue whether *other* kinds of housing price bubbles might arise.

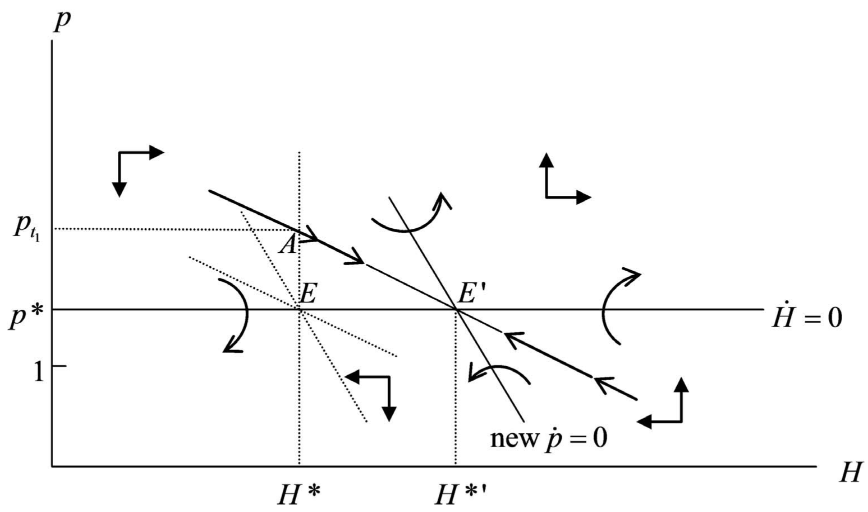

Figure 15.10: Response to a fall in the property tax ($c = 1$).

falling with the rising H). This continues until the new steady state is reached with unchanged p^*, but higher H.

The dichotomy between the short and medium run

There is a dichotomy between the price and quantity adjustment in the short and medium run:

1. In the *short* run, H, hence also the supply of housing services, is given. The rental rate R as well as the house price p immediately shifts up (down) if the demand for housing services shifts up (down).

2. In the medium run (i.e., without new disturbances), it is H that adjusts and does so gradually. The adjustment of H is in a direction indicated by the sign of the initial price difference, $p - p^*$, which in turn reflects the initial position of the demand curve in Fig. 15.6. On the other hand, the house price, p, converges towards the cost-determined level, p^*. This price level is constant as long as technical progress in the production of intermediate goods for construction follows the general trend in the economy.

15.2.4 Discussion

In many countries a part of the housing market is under some kind of rent control. Then there is, of course, rationing on the demand side of the housing market. It may still be possible to use the model in a modified version since the part of the housing market, which is *not* under regulation and therefore has a market determined price, p, usually includes the new building activity.

We have carried out partial equilibrium analysis in a simplified framework. Possible refinements of the analysis include considering household optimization with an explicit distinction between durable consumption (housing demand) and non-durable consumption and allowing uncertainty and credit market imperfections. Moreover, a general equilibrium approach would take into account the possible feedbacks on the financial wealth, A, from changes in H and possibly also p.[18] Allowing economic growth with rising wages in the model would also be preferable, so that a steady state with a *growing* housing stock can be considered (a growing housing stock at least in terms of quality-adjusted housing units). A more complete analysis would also include land prices and ground rent.

The issue of housing bubbles After a decade of sharply rising house prices, the US experienced between 2006 and 2009 a fall in house prices of about 30% (Shiller,), in Denmark about 20% (Economic Council, Fall 2011). We argued briefly that in the present model with rational expectations, housing bubbles can be ruled out. Let us here go a little more into detail about the concepts involved.

The question is whether the large volatility in house prices should be seen as reflecting the rise and burst of housing bubbles or just volatility of fundamentals. A *house price bubble* is present if the market price, p_t, of houses for some time systematically exceeds the *fundamental value,* that is, if $p_t > \hat{p}_t$, where $\hat{p}_t$ is the fundamental value (the present value of the expected future services or dividends from the asset). The latter can be found as the solution to the differential equation (15.61), assuming absence of housing price bubbles (see Appendix D).

Our model assumes rational expectations which in the absence of stochastic elements in the model amounts to perfect foresight. What we ruled out by referring to the well-defined reproduction costs of houses was that a *rational* deterministic asset price bubble could occur in the system. A *rational* asset price bubble is an asset price bubble that is consistent with the relevant no-arbitrage condition, here (15.53), when agents have model-consistent expectations. If stochastic elements are added to the model, a rational housing bubble (which would in this case be

[18] Feedbacks from changes in p are more intricate than one might imagine at first glance. In a representative agent model everybody is an average citizen and owns the house she lives in. Nobody is better off by a rise in house prices. In a model with heterogeneous agents, those who own more houses than they use themselves gain by a rise in house prices. And those in the opposite situation lose. Whether and how aggregate consumption is affected depends on differences in the marginal propensity to consume and on institutional circumstances concerning collaterals in credit markets. In two papers by Case, Quigley, and Shiller (2005, 2011) empirical evidence of a positive relationship between consumption and housing wealth in the US is furnished.

stochastic) can still be ruled out (the argument is similar to the one given for the deterministic case).

Including land and unique building sites with specific amenity values into the model will, however, make the argument against rational bubbles less compelling (see, e.g., Kocherlakota, 2011). Moreover, there are reasons to believe that in the real world, expectations are far from always rational. The behavioral finance literature has suggested alternative theories of speculative bubbles where market psychology (herding, fads, etc.) plays a key role. We postpone a more detailed discussion of asset price bubbles to Part VI.

15.3 Literature notes

(incomplete)

Poterba (1984).

Attanasio et al., 2009.

Buiter, Housing wealth isn't wealth, WP, London School of Economics, 20-07-2008.

The question of systematic bias in homebuyer's expectations in four U.S. metropolitan areas over the period 2003-2012 is studied in Case, Shiller, and Thompson (2012), based on questionnaire surveys. See also Cheng, Raina, and Xiong (2012).

Campbell and Cocco, 2007.

Mayer (2011) surveys theory and empirics about the cyclical movement of house prices.

The phenomenon that fast expansion may reduce efficiency when managerial capability is a fixed production factor is known as a *Penrose effect*, so named after a book from 1959 on management by the American economist Edith Penrose (1914-1996). Uzawa (1969) explores Penrose's ideas in different economic contexts. The construction process is sensitive to managerial capability which is a scarce resource in a construction boom.

15.4 Appendix

A. Complementary inputs

In Section 15.1.2 we claimed, without proof, certain properties of the oil demand function and the marginal productivities of capital and labor, respectively, in general equilibrium, given firms' profit maximization subject to a three-factor production function with inputs that exhibit direct complementarity. Here, we use the attributes of the production function F, including (15.2), and the first-order conditions of the representative firm, to derive the claimed signs of the partial derivatives of the functions $M(K, p_M)$, $w(K, p_M)$, and $MPK(K, p_M)$.

First, taking the total derivative w.r.t. K and M in (15.13) gives

$$F_{MK}dK + F_{MM}dM = dp_M.$$

Hence, $\partial M/\partial K = -F_{MK}/F_{MM} > 0$, and $\partial M/\partial p_M = 1/F_{MM} < 0$.

Second, taking the total derivative w.r.t. K and p_M in (15.12) gives

$$dw = F_{LK}dK + F_{LM}(M_K dK + M_{p_M}dp_M).$$

Hence, $\partial w/\partial K = F_{LK} + F_{LM}M_K > 0$, and $\partial w/\partial p_M = F_{LM}M_{p_M} < 0$.

Third, $\partial MPK/\partial p_M = F_{KM}M_{p_M} < 0$, since $F_{KM} > 0$ and $M_{p_M} < 0$. As to the sign of $\partial MPK/\partial K$, observe that

$$\begin{aligned} \partial MPK/\partial K &= F_{KK} + F_{KM}M_K = F_{KK} + F_{KM}(-F_{MK}/F_{MM}) \\ &= \frac{1}{F_{MM}}(F_{KK}F_{MM} - {F_{KM}}^2) < 0, \end{aligned}$$

where the inequality follows from $F_{MM} < 0$, if $F_{KK}F_{MM} - {F_{KM}}^2 > 0$. And the latter inequality does indeed hold. This follows from (15.62) in the lemma below.

Lemma. Let $f(x_1, x_2, x_3)$ be some arbitrary concave $\mathbb{C}^2$-function defined on $\mathbb{R}^3_+$. Assume $f_{ii} < 0$ for $i = 1, 2, 3$, and $f_{ij} > 0, i \neq j$. Then, concavity of f implies that

$$f_{ii}f_{jj} - {f_{ij}}^2 > 0 \quad \text{for } i \neq j. \tag{15.62}$$

Proof. By the general theorem on concave $\mathbb{C}^2$-functions (see Math Tools), f satisfies

$$f_{11} \leq 0, \qquad f_{11}f_{22} - {f_{12}}^2 \geq 0 \text{ and}$$

$$f_{11}(f_{22}f_{33} - {f_{23}}^2) - f_{12}(f_{21}f_{33} - f_{23}f_{31}) + f_{13}(f_{21}f_{32} - f_{22}f_{31}) \leq 0 \tag{15.63}$$

in the interior of $\mathbb{R}^3_+$. Combined with the stated assumptions on f, (15.63) implies (15.62) with $i = 2, j = 3$. In view of symmetry, the numbering of the arguments of f is arbitrary. So (15.62) also holds with $i = 1, j = 3$ as well as $i = 1, j = 2$. □

The lemma applies because F satisfies all the conditions imposed on f in the lemma. First, the direct complementarity condition $f_{ij} > 0, i \neq j$, is directly assumed in (15.2). Second, the condition $f_{ii} < 0$ for $i = 1, 2, 3$ is satisfied by F since, in view of F being neoclassical, the marginal productivities of F are diminishing. Finally, as F in addition to being neoclassical has non-increasing returns to scale, F is concave.

B. The transformation function and the adjustment cost function in Tobin's q-theory

As mentioned in Section 15.2.2 we may formulate the strictly concave transformation function $T(I, H)$ as being equal to $I - G(I, H)$, where the "waste" is represented by an adjustment cost function $G(I, H)$ familiar from Chapter 14. Then, on the one hand, all the properties of G *required* in Chapter 14.1 when $I \geq 0$ are maintained. On the other hand, not all properties required of T in (15.55) need be satisfied in Tobin's q-theory.

As to the first claim, note that when the function $T(I, H) \equiv I - G(I, H)$ has all the properties stated in (15.55), then the function G must, for $(I, H) \in \mathbb{R}_+^2$, satisfy:

$$\begin{aligned}
G(I, H) &= I - T(I, H), \\
G(0, H) &= 0 - T(0, H) = 0, \\
G_I(I, H) &= 1 - T_I(I, H) \geq 0, \text{ with } G_I \geq 0 \text{ for } I \geq 0, \text{ respectively,} \\
G_{II}(I, H) &= -T_{II}(I, H) > 0 \text{ for all } I \geq 0, \\
G_H(I, H) &= -T_H(I, H) \leq 0,
\end{aligned}$$

where the second line is implied by $T_I(0, H) = 1$ and $T_{II} < 0$. These conditions on G for $(I, H) \in \mathbb{R}_+^2$ are exactly those required in Chapter 14.1.

As to second claim, a requirement on the function T in (15.55) is that $T_I(0, H) = 1$ and $T_I(I, H) > 0$ for all $I \geq 0$ at the same time as $T_{II} < 0$. This requires that $0 < T_I(I, H) < 1$ for all $I > 0$. For $G(I, H) = I - T(I, H)$ to be consistent with this, we need that $0 < G_I < 1$ for all $I > 0$. So the G function should not be "too convex" in I. We would have to impose the condition that $\lim_{I \to \infty} G_{II} = 0$ holds with "sufficient" speed of convergence. Whereas for instance

$$G(I, H) = I - A(aI^\beta + (1 - a)H^\beta)^{1/\beta}, \quad \text{with } 0 < A < 1, 0 < a < 1, \text{ and } \beta < 0,$$

will do, a function like $G(I/H) = (\alpha/2)I^2/H$, $\alpha > 0$, will *not* do for large I. Nevertheless, the latter function satisfies all conditions required in Tobin's q-theory.

If for some reason one would like to use such a quadratic function to represent waste in construction, one could relax the in (15.55) required condition $T_I(I, H) > 0$ to hold only for I below some upper bound.

Finally, we observe that when $T(I, H) \equiv I - G(I, H)$, then, if the function G is homogeneous of degree k, so is the function T, and vice versa.

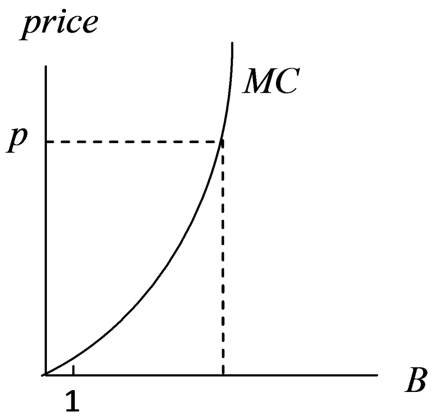

Figure 15.11: Marginal costs in house construction (housing stock given).

C. Interpreting construction behavior in a marginal cost perspective (Section 15.2.2)

We may look at the construction activity of the representative construction firm from the point of view of increasing marginal costs in the short run. First, let $\mathcal{TC}$ denote the total costs per time unit of the representative construction firm. We have $\mathcal{TC} = \bar{f} + \mathcal{TVC}$, where $\bar{f}$ is the fixed cost to management and $\mathcal{TVC}$ is the total variable cost associated with the construction of B $(= T(I, H))$ new houses per time unit, given the economy-wide stock H. All these costs are measured in real terms. We have $\mathcal{TVC} = cI$. The input of intermediates, I, required for building B new houses per time unit is an increasing function of B. Indeed, the equation

$$B = T(I, H), \qquad (*)$$

where $T_I > 0$, defines I as an implicit function of B and H, say $I = \varphi(B, H)$. By implicit differentiation in (*) we find

$$\varphi_B = \partial I/\partial B = 1/T_I(\varphi(B, H), H) > 1, \quad \text{when } I > 0$$

So $\mathcal{TVC} = cI = c\varphi(B, H)$, and short-run marginal cost is

$$\mathcal{MC}(c, B, H) = \frac{\partial \mathcal{TVC}}{\partial B} = c\varphi_B = \frac{c}{T_I(\varphi(B, H), H)} > c, \quad \text{when } I > 0. \qquad (**)$$

CLAIM
(i) The short-run marginal cost, $\mathcal{MC}$, of the representative construction firm is increasing in B.
(ii) The construction sector produces new houses up the point where $\mathcal{MC} = p$.
(iii) The cost of building one new house per time unit is approximately c.

Proof. (i) By (**) and (*),

$$\frac{\partial \mathcal{MC}}{\partial B} = \frac{-cT_{II}(\varphi(B,H),H)\varphi_B}{T_I(\varphi(B,H),H)^2} = \frac{-cT_{II}(\varphi(B,H),H)}{T_I(\varphi(B,H),H)^3} > 0,$$

since $T_I > 0$ and $T_{II} < 0$. (ii) Follows from (**) and the first-order condition (15.56) found in the text. (iii) The cost of building ΔB, when $B = 0$, is $\mathcal{MC}(c, \Delta B, H) \approx [c/T_I(0,H)] \cdot \Delta B = c\Delta B = c$ when $\Delta B = 1$, where we have used (**). $\square$

That it is profitable to produce new houses up the point where $\mathcal{MC} = p$ is illustrated in Fig. 15.11.

D. Solving the no-arbitrage equation for p_t in the absence of house price bubbles (Section 15.2.4)

By definition, if there are no housing bubbles, the market price of a house equals its *fundamental value*, i.e., the present value of expected (possibly imputed) after-tax rental income from owning the house. Denoting the fundamental value $\hat{p}_t$, we thus have

$$\begin{aligned} \hat{p}_t &= (1-\tau_R)\int_t^\infty R(H_s)e^{-(\tau_p+\delta)(s-t)}e^{\tau_R\delta(s-t)}e^{-(1-\tau_r)r(s-t)}ds, \qquad (15.64) \\ &= (1-\tau_R)\int_t^\infty R(H_s)e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](s-t)}ds, \end{aligned}$$

where the three discount rates appearing in the first line are, first, $\tau_p + \delta$, which reflects the rate of "leakage" from the investment in the house due to the property tax and wear and tear, second, $\tau_R\delta$, which reflects the tax allowance due to wear and tear, and, finally, $(1-\tau_r)r$, which is the usual opportunity cost discount. In the second row we have done an addition of the three discount rates so as to have just one discount factor easily comparable to the discount factor appearing below.

In Section 15.2.4 we claimed that in the absence of housing bubbles, the linear differential equation, (15.61), implied by the no-arbitrage equation (15.53) under perfect foresight, has a solution p_t equal to the fundamental value of the house, i.e., $p_t = \hat{p}_t$. To prove this, we write (15.61) on the standard form for a linear differential equation,

$$\dot{p}_t + ap_t = -(1-\tau_R)R(H_t), \qquad (15.65)$$

where

$$a \equiv -\left[(1-\tau_r)r + (1-\tau_R)\delta + \tau_p\right] < 0. \qquad (15.66)$$

The general solution to (15.65) is

$$p_t = \left(p_{t_0} - (1-\tau_R)\int_{t_0}^{t} R(H_s)e^{a(s-t_0)}ds\right)e^{-a(t-t_0)}.$$

Multiplying through by $e^{a(t-t_0)}$ gives

$$p_t e^{a(t-t_0)} = p_{t_0} - (1-\tau_R)\int_{t_0}^{t} R(H_s)e^{a(s-t_0)}ds.$$

Rearranging and letting $t \to \infty$, we get

$$p_{t_0} = (1-\tau_R)\int_{t_0}^{\infty} R(H_s)e^{a(s-t_0)}ds + \lim_{t\to\infty} p_t e^{a(t-t_0)}.$$

Inserting (15.66), replacing t by T and t_0 by t, and comparing with (15.64), we see that

$$p_t = \hat{p}_t + \lim_{T\to\infty} p_T e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](T-t)}.$$

The first term on the right-hand side is the fundamental value of the house at time t. The second term on the right-hand side thus amounts to the difference between the market price of the house and its fundamental value. By definition, this difference represents a bubble. In the absence of the bubble, the market price, p_t, therefore coincides with the fundamental value.

On the other hand, we see that a bubble being present requires that

$$\lim_{T\to\infty} p_T e^{-[(1-\tau_r)r+(1-\tau_R)\delta+\tau_p](T-t)} > 0.$$

In turn, this requires that the house price is explosive in the sense of ultimately growing at a rate not less than $(1-\tau_r)r + (1-\tau_R)\delta + \tau_p$. The candidate for a bubbly path ultimately moving North-East portrayed in Fig. 15.9 in fact has this property. Indeed, by (15.61), for such a path we have

$$\dot{p}_t/p_t = [(1-\tau_r)r + (1-\tau_R)\delta + \tau_p] - (1-\tau_R)R(H_t)/p_t \to (1-\tau_r)r + (1-\tau_R)\delta + \tau_p \text{ for } t \to \infty,$$

since $p_t \to \infty$ and $R'(H_t) < 0$.

15.5 Exercises

(15.61)

Advanced Macroeconomics. Special Note 2.

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Uncertainty, expectations, and asset price bubbles

This lecture note provides a framework for addressing themes where expectations in *uncertain* situations are important elements. Our previous models have not taken seriously the problem of uncertainty. Where agent's expectations about future variables were involved and these expectations were assumed to be model-consistent ("rational"), we only considered a special case: perfect foresight. Shocks were treated in a peculiar (almost self-contradictory) way: they might occur, but only as a complete surprise, a one-off event. Agents' expectations and actions never incorporated that new shocks could arrive.

We will now allow recurrent shocks to take place. The environment in which the economic agents act will be considered inherently uncertain. How can this be modeled and how can we solve the resultant models? Since it is easier to model uncertainty in discrete rather than continuous time, we examine uncertainty and expectations in a discrete time framework.

Our emphasis will be on the hypothesis that when facing uncertainty a predominant fraction of the economic agents form "rational expectations" in the sense of making probabilistic forecasts which coincide with the forecast calculated on the basis of the "relevant economic model".

1 Preliminaries

Let us first consider simple mechanistic expectation formation hypotheses that have been used to describe day-to-day expectations of people who do not think much about the probabilistic properties of their economic environment.

1.1 Simple expectation formation hypotheses

One simple supposition is that expectations change gradually to correct past expectation errors. Let P_t denote the general price level in period t and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$

the corresponding inflation rate. Further, let $\pi^e_{t-1,t}$ denote the "subjective expectation", formed in period $t-1$, of π_t, i.e., the inflation rate from period $t-1$ to period t. We may think of the "subjective expectation" as the expected value in a vaguely defined subjective conditional probability distribution.

The hypothesis of *adaptive expectations* (the AE hypothesis) says that the expectation is revised in proportion to the past expectation error,

$$\pi^e_{t-1,t} = \pi^e_{t-2,t-1} + \lambda(\pi_{t-1} - \pi^e_{t-2,t-1}), \; 0 < \lambda \leq 1, \tag{1}$$

where the parameter λ is called the adjustment speed. If $\lambda = 1$, the formula reduces to

$$\pi^e_{t-1,t} = \pi_{t-1}. \tag{2}$$

This limiting case is known as *static expectations* or *myopic expectations:* the subjective expectation is that the inflation rate will remain the same. As we shall see, *if* inflation follows a random walk, this subjective expectation is in fact the "rational expectation".

We may write (1) on the alternative form

$$\pi^e_{t-1,t} = \lambda\pi_{t-1} + (1-\lambda)\pi^e_{t-2,t-1}. \tag{3}$$

This says that the expected value concerning this period (period t) is a weighted average of the actual value for the last period and the expected value for the last period. By backward substitution we find

$$\begin{aligned}
\pi^e_{t-1,t} &= \lambda\pi_{t-1} + (1-\lambda)[\lambda\pi_{t-2} + (1-\lambda)\pi^e_{t-3,t-2}] \\
&= \lambda\pi_{t-1} + (1-\lambda)\lambda\pi_{t-2} + (1-\lambda)^2[\lambda\pi_{t-3} + (1-\lambda)\pi^e_{t-4,t-3}] \\
&= \lambda\sum_{i=1}^{n}(1-\lambda)^{i-1}\pi_{t-i} + (1-\lambda)^n\pi^e_{t-n-1,t-n}.
\end{aligned}$$

Since $(1-\lambda)^n \to 0$ for $n \to \infty$, we have (for $\pi^e_{t-n-1,t-n}$ bounded as $n \to \infty$),

$$\pi^e_{t-1,t} = \lambda\sum_{i=1}^{\infty}(1-\lambda)^{i-1}\pi_{t-i}. \tag{4}$$

Thus, according to the AE hypothesis with $0 < \lambda < 1$, the expected inflation rate is a weighted average of the historical inflation rates back in time. The weights are geometrically declining with increasing time distance from the current period. The weights sum to one: $\sum_{i=1}^{\infty}\lambda(1-\lambda)^{i-1} = \lambda(1-(1-\lambda))^{-1} = 1$.

The formula (4) can be generalized to the *general backward-looking expectations* formula,

$$\pi^e_{t-1,t} = \sum_{i=1}^{\infty} w_i \pi_{t-i}, \qquad \text{where } \sum_{i=1}^{\infty} w_i = 1. \tag{5}$$

If the weights w_i in (5) satisfy $w_i = \lambda(1-\lambda)^{i-1}$, $i = 1, 2, \ldots$, we get the AE formula (4). If the weights are

$$w_1 = 1 + \beta, \; w_2 = -\beta, \; w_i = 0 \text{ for } i = 3, 4, \ldots,$$

we get

$$\pi^e_{t-1,t} = (1+\beta)\pi_{t-1} - \beta\pi_{t-2} = \pi_{t-1} + \beta(\pi_{t-1} - \pi_{t-2}). \tag{6}$$

This is called the hypothesis of *extrapolative expectations* and says:

$$\begin{aligned}
&\text{if } \beta > 0, \text{ then the recent direction of change in } \pi \text{ is expected to continue;}\\
&\text{if } \beta < 0, \text{ then the recent direction of change in } \pi \text{ is expected to be reversed;}\\
&\text{if } \beta = 0, \text{ then expectations are static as in (2).}
\end{aligned}$$

As hinted, there *are* cases where for instance myopic expectations *are* "rational" (in a sense to be defined below). Exercise 1 provides an example. But in many cases purely backward-looking formulas are too rigid, too mechanistic. They will often lead to systematic expectation errors to one side or the other. It seems implausible that people should not then respond to their experience and revise their expectations formula. When expectations are about things that really matter for them, people are likely to listen to professional forecasters who build their forecasting on statistical or econometric *models.* Such models are based on a formal probabilistic framework, take the interaction between different variables into account, and incorporate new information about future possible events.

A basic distinction is that between two ways in which expectations, whatever their nature, may enter a macroeconomic model.

1.2 Two model types

We first recapitulate a few concepts from statistics. A sequence $\{X_t\}$ of random variables indexed by time is called a *stochastic process.* A stochastic process $\{X_t\}$ is called *white noise* if for all t, X_t has zero expected value, constant variance, and zero covariance across time.[1] A stochastic process $\{X_t\}$ is called a *first-order autoregressive process,* abbreviated AR(1), if $X_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_t$, where β_0 and β_1 are constants, and $\{\varepsilon_t\}$ is white noise. If $|\beta_1| < 1$, then $\{X_t\}$ is called a *stationary* AR(1) process. A stochastic process $\{X_t\}$ is called a *random walk* if $X_t = X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is white noise.

1.2.1 Model type A: models with past expectations of current endogenous variables

Suppose a given macroeconomic model can be reduced to two equations, the first being

$$Y_t = a\, Y^e_{t-1,t} + c\, X_t, \qquad t = 0, 1, 2, ..., \tag{7}$$

where Y_t is some endogenous variable (not necessarily GDP), a and c are given constant coefficients, and X_t is an exogenous random variable which follows some specified stochastic process.

In line with the notation from Section 1, $Y^e_{t-1,t}$ is the subjective expectation formed in period $t-1$, of the value of the variable Y in period t. The economic agents are in simple models assumed to have the same expectations. Or, at least there is a dominating expectation, $Y^e_{t-1,t}$, in the society. What the equation (7) claims is that the endogenous variable, Y_t, depends, in the specified linear way, on the "generally held" expectation of Y_t, formed in the previous period. It is convenient to think of the outcome Y_t as being the aggregate result of agents' decisions and market mechanisms, the decisions being made at discrete points in time $\ldots, t-2, t-1, t, \ldots$, immediately after the uncertainty concerning the period in question is resolved.

The second equation specifies how the subjective expectation is formed. To fix ideas, let us assume myopic expectations,

$$Y^e_{t-1,t} = Y_{t-1}, \tag{8}$$

as in (2) above. A *solution* to the model is a stochastic process for Y_t such that (7) holds, given the expectation formation (8) and the stochastic process which X_t follows.

EXAMPLE 1 *(imported raw materials and the domestic price level)* Let the endogenous variable in (7) represent the domestic price level (the consumer price index) P_t, and let

[1] The expression white noise derives from electrotechnics. In electrotechnical systems signals will often be subject to noise. If this noise is arbitrary and has no dominating frequence, it looks like white light. The various colours correspond to a certain wave length, but white light is light which has all frequences (no dominating frequence).

X_t be the price level of imported raw materials. Suppose the price level is determined through a markup on unit costs,

$$P_t = (\lambda W_t + \eta X_t)(1+\mu), \qquad 0 < \lambda < \frac{1}{1+\mu}, \tag{*}$$

where W_t is the nominal wage level in period $t = 0, 1, 2, \ldots$, and λ and η are positive technical coefficients representing the assumed constant labor and raw materials requirements, respectively, per unit of output; μ is a constant markup. Assume further that workers in period $t-1$ negotiate next period's wage level, W_t, so as to achieve, in expected value, a certain target real wage which we normalize to 1, i.e.,

$$\frac{W_t}{P^e_{t-1,t}} = 1.$$

Inserting into (*), we have

$$P_t = a\ P^e_{t-1,t} + c\ X_t, \qquad 0 < a = \lambda(1+\mu) < 1, 0 < c = \eta(1+\mu). \tag{9}$$

Suppose $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a positive constant and $\{\varepsilon_t\}$ is white noise. Assuming myopic expectations,

$$P^e_{t-1,t} = P_{t-1}, \tag{10}$$

the solution for the evolution of the price level is

$$P_t = a\ P_{t-1} + c(\bar{x} + \varepsilon_t), \qquad t = 0, 1, 2, \ldots.$$

Without shocks, and starting from an arbitrary $P_{-1} > 0$, the time path of the price level would be $P_t = (P_{-1} - P^*)a^{t+1} + P^*$, where $P^* = c\bar{x}/(1-\alpha)$. Shocks to the price of imported raw materials result in transitory deviations from P^*. But as the shocks are only temporary and $|a| < 1$, the domestic price level gradually returns towards the constant level P^*. The intervening changes in wage demands in response to the changes in the price level changes prolong the time it takes to return to P^* in the absence of new shocks. □

Equation (7) can also be interpreted as a vector equation (such that Y_t and $Y^e_{t-1,t}$ are n-vectors, a is an $n \times n$ matrix, c an $n \times m$ matrix, and X an m-vector). The crucial feature is that the endogenous variables dated t *only* depend on previous expectations of date-t values of these variables and on the exogenous variables.

Models with past expectations of current endogenous variables will serve as our point of reference when introducing the concept of rational expectations below.

1.2.2 Model type B: models with forward-looking expectations

Another way in which agents' expectations may enter is exemplified by

$$Y_t = a\ Y^e_{t,t+1} + c\ X_t, \qquad a \neq 1, \qquad t = 0, 1, 2, ... \tag{11}$$

Here $Y^e_{t,t+1}$ is the subjective expectation, formed in period t, of the value of Y in period $t+1$. Example: the equity price today depends on what the equity price is expected to be tomorrow. Or more generally: the current expectation of a future value of an endogenous variable influences the current value of this variable. We name this the case of *forward-looking expectations*. (In "everyday language" also $Y^e_{t-1,t}$ in model type 1 can be said to be a forward-looking variable as seen from period $t-1$. But the dividing line between the two model types, (7) and (11), is whether *current* expectations of future values of the endogenous variables do or do not influence the current values of these.)

The complete model with forward-looking expectations will include an additional equation, specifying how the subjective expectation, $Y^e_{t,t+1}$, is formed. We might again impose the myopic expectations hypothesis, now taking this form:

$$Y^e_{t,t+1} = Y_t. \tag{12}$$

A *solution* to the model is a stochastic process for Y_t satisfying (11), given the stochastic process followed by X_t and given the specified expectation formation (12) and perhaps some additional restrictions in the form of boundary conditions. In the present case, where expectations are myopic, the solution is

$$Y_t = aY_t + cX_t = \frac{cX_t}{1-a}.$$

The case of forward-looking expectations is important in connection with many topics in macroeconomics, including firms' investment, evolution of asset prices, issues of asset price bubbles, etc. This case will be dealt with in sections 3 and 4 below.

In passing we note that in both model type 1 and model type 2, it is the mean (in the subjective probability distribution) of the random variable(s) that enters. This is typical of simple macroeconomic models which often ignore other measures such as the median, mode, or higher-order moments. The latter, say the variance of X_t, may be included in more advanced models where for instance behavior towards risk is important.

1.2.3 The concept of a model-consistent expectation

The concepts of a *rational expectation* and *model-consistent expectation* are closely related, but not the same. We start with the latter.

Let there be given a stochastic model of type A, as represented by (7) combined with some given expectation formation (8), say. We put ourselves in the position of the investigator or model builder and ask what the *model-consistent expectation* of the endogenous variable Y_t is as seen from period $t-1$. It is the mathematical *conditional expectation* that can be calculated on the basis of the model and available relevant data revealed up to and including period $t-1$. Let us denote this expectation

$$E(Y_t|I_{t-1}), \tag{13}$$

where E is the expectation operator and I_{t-1} denotes the information available at time $t-1$. We think of period $t-1$ as the half-open time interval $[t-1, t)$ and imagine that the uncertainty concerning the exogenous random variable X_{t-1} is resolved at time $t-1$. So I_{t-1} includes knowledge of X_{t-1} and thereby, via the model, also of Y_{t-1}.

The information I_{t-1} may comprise knowledge of the realized values of X and Y up until and including period $t-1$. Instead of (13) we could, for instance, write

$$E(Y_t|Y_{t-1} = y_{t-1}, \ldots, Y_{t-n} = y_{t-n}; X_{t-1} = x_{t-1}, \ldots, X_{t-n} = x_{t-n}).$$

Here information (some of which may be redundant) goes back to a given initial period, say period 0, in which case n equals t. Alternatively, perhaps information goes back to "ancient times", possibly represented by $n = \infty$. Anyway, as time proceeds, in general more and more realizations of the exogenous and endogenous variables become known and in *this* sense the information I_{t-1} *expands* with rising t. The information I_{t-1} may also be interpreted as "partial lack of uncertainty", so that an "increasing amount of information" and "reduced uncertainty" are seen as two sides of the same thing. The "reduced uncertainty" lies in the fact that the space of *possible* time paths $\{(X_t, Y_t)\}_{t-n}^{t+T}$ as of time t *shrinks* as time proceeds (T denotes the time horizon as seen from time t).[2] Indeed, this space shrinks precisely because more and more realizations of the variables take place (more information appears) and thereby rule out an increasing subset of paths that were earlier possible.[3]

In Example 1, as long as the subjective expectation is the myopic expectation (10), the model-consistent expectation is

$$E(P_t|I_{t-1}) = a\ P_{t-1} + c\bar{x}.$$

[2] By "possible" is meant "ex ante feasible according to a given model".

[3] We refer to I_{t-1} as the "available information" rather than the "information set" which is an alternative term used in the literature. The latter term is tricky because, as we have just exemplified, it is ambiguous what is meant by a "larger information set". Moreover, the term "information set" has different meanings in different branches of economics, hence we are hesitant to use it. More about subtleties relating to "information" in Appendix B, dealing with mathematical conditional expectations in general.

Inserting the investigator's estimated values of the coefficients a and c, the investigator's forecast of P_t is obtained.

2 The rational expectations hypothesis

Unsatisfied with mechanistic formulas like those of Section 1, the American economist John F. Muth (1961) introduced a radically different approach, the hypothesis of *rational expectations.* Muth stated the hypothesis the following way:

> I should like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory. At the risk of confusing this purely descriptive hypothesis with a pronouncement as to what firms ought to do, we call such expectations 'rational' (Muth 1961).

Muth applied this hypothesis to simple microeconomic problems. The hypothesis was subsequently extended and applied to general equilibrium theory and macroeconomics by what since the early 1970s became known as the New Classical Macroeconomics school. Nobel laureate Robert E. Lucas from the University of Chicago lead the way by a series of papers starting with Lucas (1972) and Lucas (1973). Assuming rational expectations in a model instead of, for instance, adaptive expectations may radically change the dynamics as well as the impact of economic policy.

2.1 The concept of rational expectations

Assuming the economic agents have *rational expectations* (RE) is to assume that their subjective expectation equals the model-consistent expectation, that is, the mathematical conditional expectation that can be calculated on the basis of the model and available relevant information about the exogenous stochastic variables. In connection with the model ingredient (7), assuming the agents have rational expectations thus means that

$$Y_{t-1,t}^{e} = E(Y_t|I_{t-1}), \tag{14}$$

i.e., agents' subjective conditional expectation coincides with the "objective" or "true" conditional expectation, given the model (7).

Together, the equations (7) and (14) constitute a simple *rational expectations model* (henceforth an RE model). We may write the model in compact form as

$$Y_t = aE(Y_t|I_{t-1}) + c\ X_t, \qquad t = 0, 1, 2, ... \tag{15}$$

The assumption of rational expectations thus relies on idealized conditions.

2.2 Solving an RE model of type A

To solve the model means to find the stochastic process followed by Y_t, given the stochastic process followed by the exogenous variable X_t. For a linear RE model with past expectations of current endogenous variables, the solution procedure is the following.

1. By substitution, reduce the RE model (or the relevant part of the model) into a form like (15) expressing the endogenous variable in period t in terms of its past expectation and the exogenous variable(s). (The case with multiple endogenous variables is treated similarly.)

2. Take the conditional expectation on both sides of the equation and solve for the conditional expectation of the endogenous variable.

3. Insert into the "reduced form" attained at 1.

In practice there is often a fourth step, namely to express *other* endogenous variables in the model in terms of those found in step 3. Let us see how the procedure works by way of the following example.

EXAMPLE 2 We modify Example 1 by replacing myopic expectations by rational expectations, i.e., (10) is replaced by $P^e_{t-1,t} = E(P_t|I_{t-1})$. We still assume $X_t = \bar{x} + \varepsilon_t$. Now "available information" includes that the subjective expectations are rational expectations. Step 1:

$$P_t = aE(P_t|I_{t-1}) + c\ X_t, \qquad 0 < \alpha < 1, c > 0. \tag{16}$$

Step 2: $E(P_t|I_{t-1}) = aE(P_t|I_{t-1}) + c\bar{x}$, implying

$$E(P_t\,|I_{t-1}) = c\frac{\bar{x}}{1-a}.$$

Step 3: Insert into (16) to get

$$P_t = c\frac{a\bar{x}}{1-a} + c(\bar{x} + \varepsilon_t).$$

This is the solution of the model in the sense of a specification of the stochastic process followed by P_t.

To compare with myopic expectations, suppose the event $\varepsilon_t \neq 0$ is relatively seldom and that at $t = 0, 1, ..., t_0 - 1$, it so happens that $\varepsilon_t = 0$, hence $P_t = c\bar{x}/(1-a) \equiv P^*$. Then, at $t = t_0$, $\varepsilon_{t_0} > 0$, so that $P_{t_0} = P^* + c\varepsilon_{t_0} > P^*$. But for $t = t_0 + 1, t_0 + 2, ..., t_0 + n$ there is again a sequence of periods with $\varepsilon_t = 0$. Then, under RE, domestic price level returns to P^* already in period $t_0 + 1$.

With myopic expectations, combined with $P_{-1} = P^*$, say, the positive shock to import prices at $t = t_0$ will imply $P_{t_0} = aP^* + c(\bar{x} + \varepsilon_{t_0}) = P^* + c\varepsilon_{t_0}$, $P_{t_0+1} = a(P^* + c\varepsilon_t) + c\bar{x}$ $= P^* + ac\varepsilon_t$, $P_{t_0+i} = P^* + a^i c\varepsilon_t$ for $i = 1, 2, ..., n$. After t_0 there is a systematic positive forecast error. This is because the mechanical expectation does not consider how the economy really functions. □

Returning to the general form (15), without specifying the process $\{X_t\}$, the second step gives

$$E(Y_t\,|I_{t-1}) = c\frac{E(X_t\,|I_{t-1})}{1-a}, \tag{17}$$

when $a \neq 1$.[4] Then, in the third step we get

$$Y_t = c\frac{aE(X_t\,|I_{t-1}) + (1-a)X_t}{1-a} = c\frac{X_t - a(X_t - E(X_t\,|I_{t-1}))}{1-a}. \tag{18}$$

For instance, let X_t follow the process $X_t = \bar{x} + \rho X_{t-1} + \varepsilon_t$, where $0 < \rho < 1$ and ε_t has zero expected value, given all observed past values of X and Y. Then (18) yields the solution

$$Y_t = c\frac{X_t - a\varepsilon_t}{1-a} = c\frac{\bar{x} + \rho X_{t-1} + (1-a)\varepsilon_t}{1-a}, \qquad t = 0, 1, 2, ...$$

In Exercise 2 you are asked to solve a simple Keynesian model of this form and compare the solution under rational expectations with the solution under static expectations.

Rational expectations should be viewed as a simplifying assumption that at best offers an approximation. *First*, the assumption entails essentially that the economic agents share one and the same understanding about how the economic system functions (and in this chapter they also share one and the same information, I_{t-1}). This is already a big mouthful. *Second*, this perception is assumed to *comply with the model* of the informed economic specialist. *Third*, this model is supposed to be the *true* model of the economic

[4] If $a = 1$, the model (15) is inconsistent unless $E(X_t\,|I_{t-1})) = 0$ in which case there are multiple solutions. Indeed, for any number $k \in (-\infty, +\infty)$, the process $Y_t = k + cX_t$ solves the model when $E(X_t\,|I_{t-1}) = 0$.

process, including the true parameter values as well as the true stochastic process which X_t follows. Indeed, by equalizing $Y^e_{t-1,t}$ with the true conditional expectation, $E(Y_t|I_{t-1})$, and not at most some econometric estimate of this, it is presumed that agents know the true values of the parameters a and c in the data-generating process which the model is supposed to mimic. In practice it is not possible to attain such precise knowledge, at least not unless the considered economic system has reached some kind of steady state and no structural changes occur (a condition which is hardly ever satisfied in macroeconomics).

Nevertheless, a model based on the rational expectations hypothesis can in many contexts be seen as a useful cultivation of a theoretical research question. The results that emerge cannot be due to *systematic* expectation errors from the economic agents' side. In this sense the assumption of rational expectations makes up a theoretically interesting *benchmark case*.

We shall stick to the term "rational expectation" because it is standard. The term can easily be misunderstood, however. Usually, in economists' terminology "rational" refers to behavior based on optimization subject to the constraints faced by the agent. So one might think that the RE hypothesis stipulates that economic agents try to get the most out of a situation with limited information, contemplating the benefits and costs of gathering more information and using adequate statistical estimation methods. But this is a misunderstanding. The RE hypothesis presumes that the true model is already known to the agents. The "rationality" refers to taking this assumed knowledge fully into account in the chosen actions.

2.3 The forecast error*

Let the forecast of some variable Y one period ahead be denoted $Y^e_{t-1,t}$. Suppose the forecast is determined by some given function, f , of realizations of Y and X up to and including period $t-1$, that is, $Y^e_{t-1,t} = f(y_{t-1},\, y_{t-2}, ..., \, x_{t-1},\, x_{t-2}, ...)$. Such a function is known as a *forecast function*. It might for instance be one of the mechanistic forecasting principles in Section 1. At the other extreme the forecast function might, at least theoretically, coincide with the a model-consistent conditional expectation. In the latter case it is a *model-consistent forecast function* and we can write

$$
\begin{aligned}
& f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...) = E(Y_t \,| I_{t-1}) && (19) \\
= \; & E(Y_t \,| Y_{t-1} = y_{t-1}, Y_{t-2} = y_{t-2}, ..., x_{t-1} = x_{t-1}, x_{t-2} = x_{t-2}, ...)\,.
\end{aligned}
$$

The *forecast error* is the difference between the actually occurring future value, Y_t, of a variable and the forecasted value. So, for a given forecast, $Y^e_{t-1,t}$, the *forecast error* is $e_t \equiv Y_t - Y^e_{t-1,t}$ and is itself a stochastic variable.

If the forecast function in (19) complies with the true data-generating process (a big "if"), then the implied forecasts would have several ideal properties:

(a) the forecast error would have zero mean;

(b) the forecast error would be uncorrelated with any of the variables in the information I_{t-1} and therefore also with its own past values; and

(c) the expected squared forecast error would be minimized.

To see these properties, note that the model-consistent forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1})$. From this follows that $E(e_t \,|I_{t-1}) = 0$, cf. (a). Also the unconditional expectation is nil, i.e., $E(e_t) = 0$. This is because $E(E(e_t \,|I_{t-1})) = E(0) = 0$ at the same time as $E(E(e_t \,|I_{t-1})) = E(e_t)$, by the *law of iterated expectations* from statistics saying that the unconditional expectation of the conditional expectation of a stochastic variable Z is given by the unconditional expectation of Z, cf. Appendix B. Considering the specific model (7), the model-consistent-forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1}) = c(X_t - E(X_t \,|I_{t-1}))$, by (17) and (18). An ex post error ($e_t \neq 0$) thus emerges if and only if the realization of the exogenous variable deviates from its conditional expectation as seen from the previous period.

As to property (b), for $i = 1, 2, ...$, let s_{t-i} be some variable value belonging to the information I_{t-i}. Then, property (b) is the claim that the (unconditional) covariance between e_t and s_{t-i} is zero, i.e., $\text{Cov}(e_t s_{t-i}) = 0$, for $i = 1, 2, ...$ This follows from the *orthogonality property* of model-consistent expectations (see Appendix C). In particular, with $s_{t-i} = e_{t-i}$, we get $\text{Cov}(e_t e_{t-i}) = 0$, i.e., the forecast errors exhibit *lack of serial correlation.* If the covariance were not zero, it would be possible to improve the forecast by incorporating the correlation into the forecast. In other words, under the assumption of rational expectations economic agents have no more to learn from past forecast errors. As remarked above, the RE hypothesis precisely refers to a fictional situation where learning has been completed and underlying mechanisms do not change.

Finally, a desirable property of a forecast function $f(\cdot)$ is that it maximizes "accuracy", i.e., minimizes an appropriate loss function. A popular loss function, L, in this context is the expected squared forecast error conditional on the information I_{t-1},

$$L = E((Y_t - f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...))^2 \,|I_{t-1}) \,.$$

Assuming $Y_t, Y_{t-1}, ..., X_{t-1}, X_{t-2}, ...$ are jointly normally distributed, then the solution to the problem of minimizing L is to set $f(\cdot)$ equal to the conditional expectation $E(Y_t\,|I_{t-1})$ based on the data-generating model as in (19).[5] This is what property (c) refers to.

EXAMPLE 3 Let $Y_t = aE(Y_t\,|I_{t-1}) + cX_t$, with $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise with variance σ^2. Then (18) applies, so that

$$Y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t, \qquad t = 0, 1, ...,$$

with variance $c^2\sigma^2$. The model-consistent forecast error is $e_t = Y_t - E(Y_t\,|I_{t-1}) = c\varepsilon_t$ with conditional expectation equal to $E(c\varepsilon_t\,|I_{t-1}) = 0$. This forecast error itself is white noise and is therefore uncorrelated with the information on which the forecast is based. □

It is worth emphasizing that the "true" conditional expectation usually can not be known − neither to the economic agents nor to the investigator. At best there can be a reasonable estimate, probably somewhat different across the agents because of differences in information and conceptions of how the economic system functions. A deeper model of expectations would give an account of the mechanisms through which agents *learn* about the economic environment. An important ingredient here would be how agents contemplate the costs and potential gains associated with further information search needed to reduce systematic expectation errors where possible. This contemplation is intricate because information search often means entering unknown territory. Moreover, for a significant subset of the agents the costs may be prohibitive. A further complicating factor involved in learning is that when the agents have obtained *some* knowledge about the statistical properties of the economic variables, the resulting behavior of the agents may *change* these statistical properties. The rational expectations hypothesis sets these problems aside. It is simply assumed that the structure of the economy remains unchanged and that the learning process has been completed.

2.3.1 Perfect foresight as a special case

The notion of *perfect foresight* corresponds to the limiting case where the variance of the exogenous variable(s) is zero so that with probability one, $X_t = E(X_t\,|I_{t-1})$ for all t. Then we have a non-stochastic model where rational expectations imply that agents' ex post forecast error with respect to Y_t is zero.[6] To put it differently: rational expectations in a

[5] For proof, see Pesaran (1987). Under the restriction of only *linear* forecast functions, property (c) holds even without the joint normality assumption, see Sargent (1979).

[6] Here we disregard zero probability events.

non-stochastic model is equivalent to perfect foresight. Note, however, that perfect foresight necessitates the exogenous variable X_t to be known in advance. Real-world situations are usually not like that. If we want our model to take this into account, the model ought to be formulated in an explicit stochastic framework. And assumptions should be stated about how the economic agents respond to the uncertainty. The rational expectations assumption is one approach to the problem and has been much applied in macroeconomics in recent decades, perhaps due to lack of compelling tractable alternatives.

3 Models with rational forward-looking expectations

We here turn to models where current expectations of a future value of an endogenous variable have an influence on the current value of this variable, that is, the case exemplified by equation (11). At the same time we introduce two simplifications in the notation. First, instead of using capital letters to denote the stochastic variables (as we did above and is common in mathematical statistics), we follow the tradition in macroeconomics to often use lower case letters. So a lower case letter may from now on represent a stochastic variable *or* a specific value of this variable, depending on the context.

An equation like (11) will now read $y_t = a\ y^e_{t,t+1} + c\ x_t$. Under rational expectations it takes the form $y_t = aE(y_{t+1}\,|I_t) + c\ x_t$, $t = 0, 1, 2, \ldots$. Second, from now on we write this equation as

$$y_t = aE_t y_{t+1} + c\ x_t, \ldots t = 0, 1, 2, \ldots,\ \ a \neq 0. \tag{20}$$

That is, the expected value of a stochastic variable, z_{t+i}, conditional on the information I_t, will be denoted $E_t z_{t+i}$.

A stochastic difference equation of the form (20) is called a linear *expectation difference equation of first order* with constant coefficient a.[7] A *solution* is a specified stochastic process $\{y_t\}$ which satisfies (20), given the stochastic process followed by x_t. In the economic applications usually no initial value, y_0, is given. On the contrary, the interpretation is that y_t depends, for all t, on expectations about the future.[8] So y_t can be a *jump variable* that can immediately shift its value in response to the emergence of new information about the future x's. For example, a share price may immediately jump to a

[7] To keep things simple, we let the coefficients a and c be constants, but a generalization to time-dependent coefficients is straightforward.

[8] The reason we say "depends on" is that it would be inaccurate to say that y_t is *determined* (in a one-way-sense) by expectations about the future. Rather there is *mutual dependence.* In view of y_t being an element in the information I_t, the expectation of y_{t+1} in (20) may depend on y_t just as much as y_t depends on the expectation of y_{t+1}.

new value when the accounts of the firm become publicly known (often even before, due to sudden rumors).

Due to the lack of an initial condition for y_t, there can easily be infinitely many processes for y_t satisfying our expectation difference equation. We have an infinite forward-looking "regress", where a variable's value today depends on its expected value tomorrow, this value depending on the expected value the day after tomorrow and so on. Then usually there are infinitely many expected sequences which can be self-fulfilling in the sense that if only the agents expect a particular sequence, then the aggregate outcome of their behavior will be that the sequence is realized. It "bites its own tail" so to speak. Yet, when an equation like (20) is part of a larger model, there will often (but not always) be conditions that allow us to select *one* of the many solutions to (20) as the only *economically* relevant one. For example, an economy-wide transversality condition or another general equilibrium condition may rule out divergent solutions and leave a unique convergent solution as the final solution.

We assume $a \neq 0$, since otherwise (20) itself is already the unique solution. It turns out that the set of solutions to (20) takes a different form depending on whether $|a| < 1$ or $|a| > 1$:

The case $|a| < 1$. In general, there is a unique *fundamental solution* and infinitely many explosive solutions ("bubble solutions").

The case $|a| > 1$. In general, there is no fundamental solution but infinitely many non-explosive solutions. (The case $|a| = 1$ resembles this.)

In the case $|a| < 1$, the expected future has modest influence on the present. Here we will concentrate on this case, since it is the case most frequently appearing in macroeconomic models with rational expectations.

4 Solutions when $|a| < 1$

Various solution methods are available. *Repeated forward substitution* is the most easily understood method.

4.1 Repeated forward substitution

Repeated forward substitution consists of the following steps. We first shift (20) one period ahead:

$$y_{t+1} = a\ E_{t+1}y_{t+2} + c\ x_{t+1}.$$

Then we take the conditional expectation on both sides to get

$$E_t y_{t+1} = a\ E_t(E_{t+1}y_{t+2}) + c\ E_t x_{t+1} = a\ E_t y_{t+2} + c\ E_t x_{t+1}, \tag{21}$$

where the second equality sign is due to the *law of iterated expectations*, which says that

$$E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}. \tag{22}$$

see Box 1. Inserting (21) into (20) then gives

$$y_t = a^2 E_t y_{t+2} + ac\ E_t x_{t+1} + c\ x_t. \tag{23}$$

The procedure is repeated by forwarding (20) two periods ahead; then taking the conditional expectation and inserting into (23), we get

$$y_t = a^3 E_t y_{t+3} + a^2 c\ E_t x_{t+2} + ac\ E_t x_{t+1} + c\ x_t.$$

We continue in this way and the general form (for $n = 0, 1, 2, ...$) becomes

$$\begin{aligned} y_{t+n} &= a\ E_{t+n}(y_{t+n+1}) + c\ x_{t+n}, \\ E_t y_{t+n} &= a\ E_t y_{t+n+1} + c\ E_t x_{t+n}, \\ y_t &= a^{n+1} E_t y_{t+n+1} + c x_t + c \sum_{i=1}^{n} a^i E_t x_{t+i}. \end{aligned} \tag{24}$$

Box 1. The law of iterated expectations

The method of repeated forward substitution is based on the law of iterated expectations which says that $E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}$, as in (22). The logic is the following. Events in period $t+1$ are stochastic as seen from period t and so $E_{t+1}y_{t+2}$ (the expectation conditional on these events) is a stochastic variable. Then the law of iterated expectations says that the conditional expectation of this stochastic variable as seen from period t is the same as the conditional expectation of y_{t+2} itself as seen from period t. So, given that expectations are rational, then an earlier expectation of a later expectation of y is just the earlier expectation of y. Put differently: my best forecast today of how I am going to forecast tomorrow a share price the day after tomorrow, will be the same as my best forecast today of the share price the day after tomorrow. If beforehand we have good reasons to expect that we will revise our expectations upward, say, when next period's additional information arrives, the original expectation would be biased, hence not rational.[9]

[9] A formal account of conditional expectations and the law of iterated expectations is given in Appendix

4.2 The fundamental solution

PROPOSITION 1 Consider the expectation difference equation (20), where $a \neq 0$. If

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \text{ exists,} \tag{25}$$

then

$$y_t = c\sum_{i=0}^{\infty} a^i E_t x_{t+i} = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i} \equiv y_t^*, \qquad t = 0, 1, 2, ..., \tag{26}$$

is a solution to the equation.

Proof Assume (25). Then the formula (26) is meaningful. In view of (24), it satisfies (20) if and only if $\lim_{n\to\infty} a^{n+1} E_t y_{t+n+1} = 0$. Hence, it is enough to show that the process (26) satisfies this latter condition.

In (26), replace t by $t+n+1$ to get $y_{t+n+1} = c\sum_{i=0}^{\infty} a^i E_{t+n+1} x_{t+n+1+i}$. Using the law of iterated expectations, this yields

$$\begin{aligned} E_t y_{t+n+1} &= c\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} \qquad \text{so that} \\ a^{n+1} E_t y_{t+n+1} &= c\ a^{n+1} \sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} = c \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}. \end{aligned}$$

It remains to show that $\lim_{n\to\infty} \sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = 0$. From the identity

$$\sum_{j=1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{n} a^j E_t x_{t+j} + \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}$$

follows

$$\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{n} a^j E_t x_{t+j}.$$

Letting $n \to \infty$, this gives

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{\infty} a^j E_t x_{t+j} = 0,$$

which was to be proved. □

The solution (26) is called the *fundamental solution* of (20), often marked by an asterisk *. The fundamental solution is (for $c \neq 0$) defined only when the condition (25)

B.

holds. In general this condition requires that $|a| < 1$. In addition, (25) requires that the absolute value of the expectation of the exogenous variable does not increase "too fast". More precisely, the requirement is that $|E_t x_{t+i}|$, when $i \to \infty$, has a growth factor less than $|a|^{-1}$. As an example, let $0 < a < 1$ and $g > 0$, and suppose that $E_t x_{t+i} > 0$ for $i = 0, 1, 2, ...$, and that $1 + g$ is an upper bound for the growth factor of $E_t x_{t+i}$. Then

$$E_t x_{t+i} \leq (1+g) E_t x_{t+i-1} \leq (1+g)^i E_t x_t = (1+g)^i x_t.$$

Multiplying by a^i, we get $a^i E_t x_{t+i} \leq a^i (1+g)^i x_t$. By summing from $i = 1$ to n,

$$\sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \sum_{i=1}^{n} \left[a(1+g)\right]^i.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \lim_{n\to\infty} \sum_{i=1}^{n} \left[a(1+g)\right]^i = x_t \frac{a(1+g)}{1-a(1+g)} < \infty,$$

if $1 + g < a^{-1}$, using the sum rule for an infinite geometric series.

As noted in the proof of Proposition 1, the fundamental solution, (26), has the property that

$$\lim_{n\to\infty} a^n E_t y_{t+n} = 0. \tag{27}$$

That is, the expected value of y is not "explosive": its absolute value has a growth factor less than $|a|^{-1}$. Given $|a| < 1$, the fundamental solution is the only solution of (20) with this property. Indeed, it is seen from (24) that whenever (27) holds, (26) must also hold. In Example 1 below, y_t is interpreted as the market price of a share and x_t as dividends. Then the fundamental solution gives the share price as the present value of the expected future flow of dividends.

EXAMPLE 1 *(the fundamental value of an equity share)* Consider arbitrage between shares of stock and a riskless asset paying the constant rate of return $r > 0$. Let period t be the current period. Let p_{t+i} be the market price (in real terms, say) of the share at the beginning of period $t + i$ and d_{t+i} the dividend paid out at the end of that period, $t + i$, $i = 0, 1, 2, ...$ As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, ...$ An investor who buys n_t shares at time t (the beginning of period t) thus invests $V_t \equiv p_t n_t$ units of account at time t. At the end of the period the gross return comes out as the known dividend $d_t n_t$ and the potential sales value of the shares at the beginning of next period. This is unlike standard *accounting* and *finance* notation in discrete time, where

V_t would be the end-of-period-t market value of the stock of shares that begins to yield dividends in period $t + 1$.[10]

Suppose investors have rational expectations and care only about expected return. Then the no-arbitrage condition reads

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r > 0. \tag{28}$$

This can be written

$$p_t = \frac{1}{1+r} E_t p_{t+1} + \frac{1}{1+r} d_t, \tag{29}$$

which is of the same form as (20) with $a = c = 1/(1 + r) \in (0, 1)$. Assuming dividends do not grow "too fast", we find the fundamental solution, denoted p_t^*, as

$$p_t^* = \frac{1}{1+r} d_t + \frac{1}{1+r} \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} E_t d_{t+i} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} E_t d_{t+i}. \tag{30}$$

The fundamental solution is simply the present value of expected future dividends.

If the dividend process is $d_{t+1} = d_t + \varepsilon_{t+1}$, where ε_{t+1} is white noise, then the dividend process is known as a *random walk* and $E_t d_{t+i} = d_t$ for $i = 1, 2, \ldots$. Thus $p_t^* = d_t/r$, by the sum rule for an infinite geometric series. In this case the fundamental value is thus itself a random walk. More generally, the dividend process could be a *martingale,* that is, a sequence of stochastic variables with the property that the expected value next period exists and equals the current actual value, i.e., $E_t d_{t+1} = d_t$; but in a martingale, $\varepsilon_{t+1} \equiv d_{t+1} - d_t$ need not be white noise; it is enough that $E_t \varepsilon_{t+1} = 0$.[11] Given the constant required return r, we still have $p_t^* = d_t/r$. So the fundamental value itself is in this case a martingale. □

In finance theory the present value of the expected future flow of dividends on an equity share is referred to as the *fundamental value* of the share. It is by analogy with this that the general designation *fundamental solution* has been introduced for solutions

[10] Our use of p_t for the (real) price of a share bought at the beginning of period t is not inconsistent with our use, in earlier chapters, of P_t to denote the nominal price per unit of consumption in period t, but paid for at the *end* of the period. At the beginning of period t, after the uncertainty pertaining to period t has been resolved and available information thereby been updated, a consumer-investor will decide both the investment and the consumption flow for the period. But only the investment expence, p_t, is disbursed immediately.

It is convenient to think of the course of actions such that receipt of the previous period's dividend, d_{t-1}, and payment for that period's consumption, at the price P_{t-1}, occur right before period t begins and the new information arrives. Indeed, the resolution of uncertainty at discrete points in time motivates a *distinction* between "end of" period $t - 1$ and "beginning of" period t, where the new information has just arrived.

[11] A random walk is thus a special case of a martingale.

of form (26). We could also think of p_t as the market price of a house rented out and d_t as the rent. Or p_t could be the market price of an oil well and d_t the revenue (net of extraction costs) from the extracted oil in period t.

4.3 Bubble solutions

Other than the fundamental solution, the expectation difference equation (20) has infinitely many *bubble solutions*. In view of $|a| < 1$, these are characterized by violating the condition (27). That is, they are solutions whose expected value explodes over time.

It is convenient to first consider the *homogenous* expectation equation associated with (20). This is defined as the equation emerging when setting $c = 0$ in (20):

$$y_t = aE_t y_{t+1}. \tag{31}$$

Every stochastic process $\{b_t\}$ of the form

$$b_{t+1} = a^{-1}b_t + u_{t+1}, \quad \text{where } E_t u_{t+1} = 0, \tag{32}$$

has the property that

$$b_t = aE_t b_{t+1}, \tag{33}$$

and is thus a solution to (31). The "disturbance" u_{t+1} represents "new information" which may be related to movements in "fundamentals", x_{t+1}. But it does not have to. In fact, u_{t+1} may be related to conditions that *per se* have no economic relevance whatsoever.

For ease of notation, from now on we just write b_t even if we think of the whole process $\{b_t\}$ rather than the value taken by b in the specific period t. The meaning should be clear from the context. A solution to (31) is referred to as a *homogenous solution* associated with (20). Let b_t be a given homogenous solution and let K be an arbitrary constant. Then $B_t = Kb_t$ is also a homogenous solution (try it out for yourself). Conversely, any homogenous solution b_t associated with (20) can be written in the form (32). To see this, let b_t be a given homogenous solution, that is, $b_t = aE_t b_{t+1}$. Let $u_{t+1} = b_{t+1} - E_t b_{t+1}$. Then

$$b_{t+1} = E_t b_{t+1} + u_{t+1} = a^{-1}b_t + u_{t+1},$$

where $E_t u_{t+1} = E_t b_{t+1} - E_t b_{t+1} = 0$. Thus, b_t is of the form (32).

For convenience we here repeat our original expectation difference equation (20) and name it (*):

$$y_t = aE_t y_{t+1} + c\, x_t, \dots t = 0, 1, 2, \dots, \ a \neq 0. \tag{*}$$

PROPOSITION 2 Consider the expectation difference equation (*), where $a \neq 0$. Let $\tilde{y}_t$ be a particular solution to the equation. Then:

(i) every stochastic process of the form

$$y_t = \tilde{y}_t + b_t, \tag{34}$$

where b_t satisfies (32), is a solution to (*);

(ii) every solution to (*) can be written in the form (34) with b_t being an appropriately chosen homogenous solution associated with (*).

Proof. Let some particular solution $\tilde{y}_t$ be given. (i) Consider $y_t = \tilde{y}_t + b_t$, where b_t satisfies (32). Since $\tilde{y}_t$ satisfies (*), we have $y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + b_t$. Consequently, by (31),

$$y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + a\ E_t b_{t+1} = a\ E_t(\tilde{y}_{t+1} + b_{t+1}) + c\ x_t = a\ E_t y_{t+1} + c\ x_t,$$

saying that (34) satisfies (*). (ii) Let Y_t be an arbitrary solution to (*). Define $b_t = Y_t - \tilde{y}_t$. Then we have

$$\begin{aligned} b_t &= Y_t - \tilde{y}_t = aE_tY_{t+1} + cx_t - (aE_t\tilde{y}_{t+1} + cx_t) \\ &= aE_t(Y_{t+1} - \tilde{y}_{t+1}) = aE_t b_{t+1}, \end{aligned}$$

where the second equality follows from the fact that both Y_t and $\tilde{y}_t$ are solutions to (*). This shows that b_t is a solution to the homogenous equation (31) associated with (*). Since $Y_t = \tilde{y}_t + b_t$, the proposition is hereby proved. □

Proposition 2 holds for any $a \neq 0$. In case the fundamental solution (26) exists and $|a| < 1$, it is convenient to choose this solution as the particular solution in (34). Thus, referring to the right-hand side of (26) as y_t^*, we can use the particular form,

$$y_t = y_t^* + b_t. \tag{35}$$

When the component b_t is different from zero, the solution (35) is called a *bubble solution* and b_t is called the *bubble component*. In the typical economic interpretation the bubble component shows up only because it is expected to show up next period, cf. (33). The name bubble springs from the fact that the expected value of b_t, conditional on the information available in period t, explodes over time when $|a| < 1$. To see this, as an example, let $0 < a < 1$. Then, from (31), by repeated forward substitution we get

$$b_t = a\ E_t(aE_{t+1}b_{t+2}) = a^2E_tb_{t+2} = ... = a^iE_tb_{t+i}, \quad i = 1, 2, ...$$

It follows that $E_t b_{t+i} = a^{-i} b_t$, and from this follows that the bubble, for t going to infinity, is unbounded in expected value:

$$\lim_{i\to\infty} E_t b_{t+i} = \begin{cases} \infty, \text{ if } b_t > 0 \\ -\infty, \text{ if } b_t < 0 \end{cases}. \tag{36}$$

Indeed, the absolute value of $E_t b_{t+i}$ will for rising i grow *geometrically* towards infinity with a growth factor equal to $1/a > 1$.

Let us consider a special case of (*) that allows a simple graphical illustration of both the fundamental solution and some bubble solutions.

4.3.1 When x_t has constant mean

Suppose the stochastic process x_t (the "fundamentals") takes the form $x_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise. Then

$$y_t = a \; E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad 0 < |a| < 1. \tag{37}$$

The fundamental solution is

$$y_t^* = c \; x_t + c \sum_{i=1}^{\infty} a^i \bar{x} = c\bar{x} + c\varepsilon_t + c\frac{a\bar{x}}{1-a} = \frac{c\bar{x}}{1-a} + c\varepsilon_t.$$

Referring to (i) of Proposition 2,

$$y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t + b_t \tag{38}$$

is thus also a solution of (37) if b_t is of the form (32).

It may be instructive to consider the case where all stochastic features are eliminated. So we assume $u_t \equiv \varepsilon_t \equiv 0$. Then we have a model with perfect foresight; the solution (38) simplifies to

$$y_t = \frac{c\bar{x}}{1-a} + b_0 a^{-t}, \tag{39}$$

where we have used repeated *backward* substitution in (32). By setting $t = 0$ we see that $y_0 - \frac{c\bar{x}}{1-a} = b_0$. Inserting this into (39) gives

$$y_t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})a^{-t}. \tag{40}$$

In Fig. 1 we have drawn three trajectories for the case $0 < a < 1$, $c > 0$. Trajectory I has $y_0 = c\bar{x}/(1-a)$ and represents the fundamental solution. Trajectory II, with $y_0 > c\bar{x}/(1-a)$, and trajectory III, with $y_0 < c\bar{x}/(1-a)$, are bubble solutions. Since we have

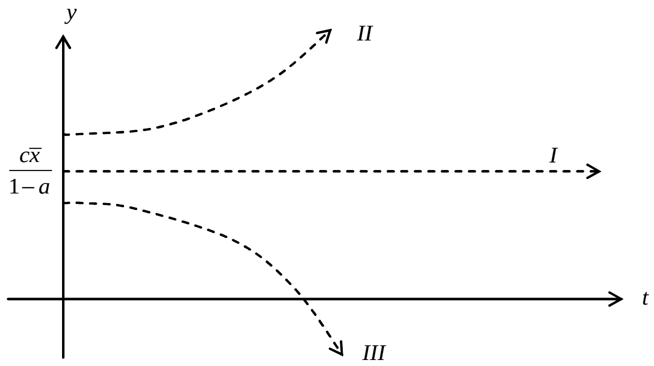

Figure 1: Deterministic bubbles (the case $0 < a < 1$, $c > 0$, and $x_t = \bar{x}$).

imposed no boundary condition apriori, one y_0 is as good as any other. The interpretation is that there are infinitely many trajectories with the property that if only the economic agents expect the economy will follow that particular trajectory, the aggregate outcome of their behavior will be that this trajectory is realized. This is the potential indeterminacy arising when y_t is not a predetermined variable. However, as alluded to above, in a complete economic model there will often be restrictions on the endogenous variable(s) not visible in the basic expectation difference equation(s), here (37). It may be that the economic meaning of y_t precludes negative values (a share certificate would be an example). In that case no-one can rationally expect a path such as III in Fig. 1. Or perhaps, for some reason, there is an upper bound on y_t (think of the full-employment ceiling for output in a situation where the "natural" growth factor for output is smaller than a^{-1}). Then no one can rationally expect a trajectory like II in the figure.

To sum up: in order for a solution of a first-order linear expectation difference equation with constant coefficient a, where $|a| < 1$, to differ from the fundamental solution, the solution must have the form (35) where b_t has the form described in (32). This provides a clue as to what asset price bubbles might look like.

4.3.2 Asset price bubbles

A stylized fact of stock markets is that stock price indices are quite volatile on a month-to-month, year-to-year, and especially decade-to-decade scale, cf. Fig. 2. There are different views about how these swings should be understood. According to the *Efficient Market Hypothesis* the swings just reflect unpredictable changes in the "fundamentals", that is, changes in the present value of rationally expected future dividends. This is for instance the view of Nobel laureate Eugene Fama (1970, 2003) from University of Chicago.

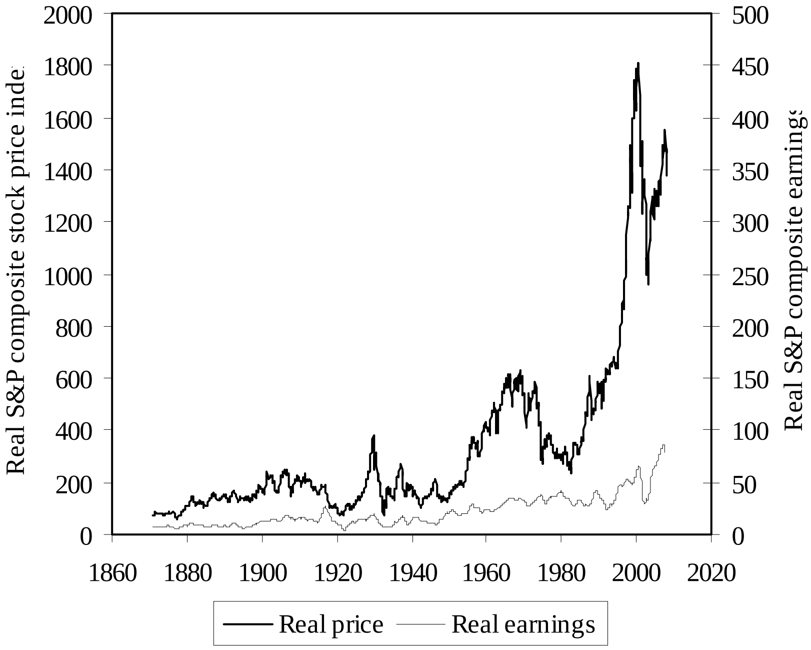

Figure 2: Monthly real S&P composite stock prices from January 1871 to January 2008 (left) and monthly real S&P composite earnings from January 1871 to September 2007 (right). Source: http://www.econ.yale.edu/~shiller/data.htm.

In contrast, Nobel laureate Robert Shiller (1981, 2003, 2005) from Yale University, and others, have pointed to the phenomenon of *excess volatility*. The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Although in no way a verification, graphs like those in Fig. 2 and Fig. 3 are suggestive. Fig. 2 shows the monthly real Standard and Poors (S&P) composite stock prices and real S&P composite earnings for the period 1871-2008. The unusually large increase in real stock prices since the mid-90's, which ended with the collapse in 2000, is known as the "dot-com bubble". Fig. 3 shows, on a monthly basis, the ratio of real S&P stock prices to an average of the previous ten years' real S&P earnings along with the long-term real interest rate. It is seen that this ratio reached an all-time high in 2000, by many observers considered as "the year the dot-com bubble burst".

Shiller's interpretation of the large stock market swings is that they are due to fads, herding, and shifts in fashions and "animal spirits" (the latter being a notion from Keynes).

A third possible source of large stock market swings was pointed out by Blanchard (1979) and Blanchard and Watson (1982). They argued that bubble phenomena need not

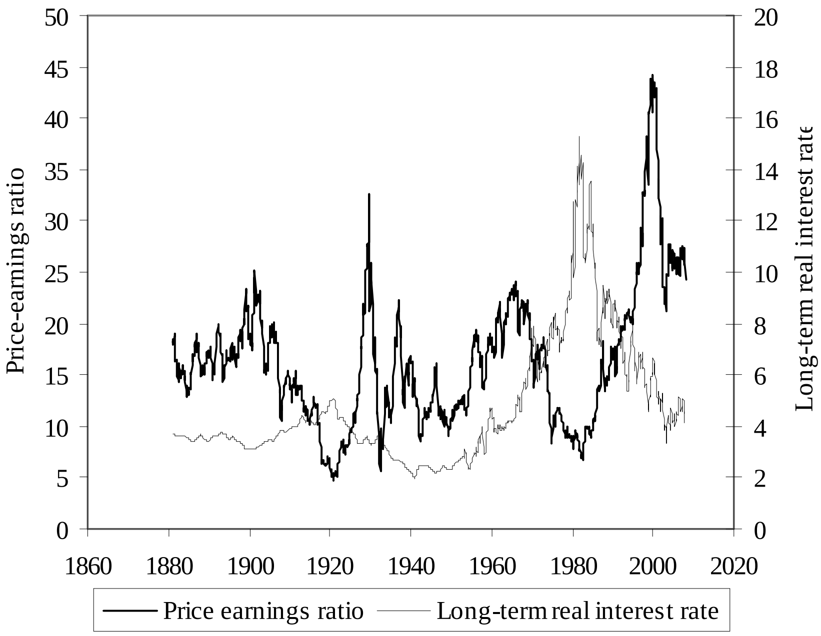

Figure 3: S&P price-earnings ratio and long-term real interest rates from January 1881 to January 2008. The earnings are calculated as a moving average over the preceding ten years. The long-term real interest rate is the 10-year Treasury rate from 1953 and government bond yields from Sidney Homer, "A History of Interest Rates" from before 1953. Source: http://www.econ.yale.edu/~shiller/data.htm.

be due to irrational behavior and non-rational expectations. This lead to the theory of *rational bubbles* – the idea that excess volatility can be explained as speculative bubbles arising from self-fulfilling *rational* expectations.

Consider an asset which yields either dividends or services in production or consumption in every period in the future. The fundamental value of the asset is, at the theoretical level, defined as the present value of the expected future flow of dividends or services.[12] An *asset price bubble* is then defined as a systematic positive deviation of the market price, p_t, of the asset from its fundamental value, p_t^*:

$$p_t = p_t^* + b_t. \tag{41}$$

An asset price bubble, $p_t - p_t^*$, that emerges in a setting where the no-arbitrage condition (28) holds under rational expectations, is called a *rational bubble.* It emerges only because there is in the market a self-fulfilling belief that it will appreciate at a rate high enough to warrant the overcharge involved.

[12] In practice there are many ambiguities involved in this definition of the fundamental value because it relates to a future which is often essentially unknown.

EXAMPLE 2 *(an ever-expanding rational bubble)* Consider again an equity share for which the no-arbitrage condition is

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r > 0. \tag{42}$$

As in Example 1, the implied expectation difference equation is $p_t = aE_t p_{t+1} + cd_t$, with $a = c = 1/(1+r) \in (0,1)$. Let the price of the share at time t be $p_t = p_t^* + b_t$, where p_t^* is the fundamental value and $b_t > 0$ a bubble component following the deterministic process, $b_{t+1} = (1+r)b_t$, $b_0 > 0$, so that $b_t = b_0(1+r)^t$. This is called a *deterministic rational bubble*. The sum $p_t^* + b_t$ will satisfy the no-arbitrage condition (42) just as much as p_t^* itself, because we just add something which equals the discounted value of itself one period later.

Agents may be ready to pay a price over and above the fundamental value (whether or not they know the "true" fundamental value) if they expect they can sell at a sufficiently higher price later; trading with such motivation is called *speculative behavior*. If generally held and lasting for some time, this expectation may be self-fulfilling. Note that (42) implies that the asset price ultimately grows at the rate r. Indeed, let $d_t = d_0(1+\gamma)^t$, $\gamma < r$ (if $r \leq \gamma$, the asset price would be infinite). By the rule of the sum of an infinite geometric series, we then have $p_t^* = d_t/(r-\gamma)$, showing that the fundamental value grows at the rate γ. Consequently, $p_t/b_t = (p_t^* + b_t)/b_t = p_t^*/b_t + 1 \rightarrow 1$, as $\gamma < r$. It follows that the asset price in the long run grows at the same rate as the bubble, the rate r.

We are not acquainted with *ever*-expanding incidents of that caliber in real world situations, however. A deterministic rational bubble thus appears implausible. □

In some contexts it may not matter whether or not we think of the "rational" market participants as actually knowing the probability distribution of the "fundamentals", hence knowing p_t^* (by "fundamentals" is meant any information relating to the future dividend or service capacity of an asset: a firm's technology, resources, market conditions etc.). All the same, it seems common to imply such a high level of information in the term "rational bubbles". Unless otherwise indicated, we shall let this implication be understood.

While a deterministic rational bubble was found implausible, let us now consider an example of a *stochastic* rational bubble which sooner or later *bursts*.

EXAMPLE 3 *(a bursting bubble)* Once again we consider the no-arbitrage condition is (42) where for simplicity we still assume the required rate of return is constant, though possibly including a risk premium. Following Blanchard (1979), we assume that the market price, p_t, of the share contains a stochastic bubble of the following form:

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases} \tag{43}$$

where $t = 0, 1, 2, ...$ and $b_0 > 0$. In addition we may assume that $q_t = f(p_t^*, b_t)$, $f_{p^*} \geq 0$, $f_b \leq 0$. If $f_{p^*} > 0$, the probability that the bubble persists at least one period ahead is higher the greater the fundamental value has become. If $f_b < 0$, the probability that the bubble persists at least one period ahead is less, the greater the bubble has already become. In this way the probability of a crash becomes greater and greater as the share price comes further and further away from fundamentals. As a compensation, the longer time the bubble has lasted, the higher is the expected growth rate of the bubble in the absence of a collapse.

This bubble satisfies the criterion for a rational bubble. Indeed, (43) implies

$$E_t\ b_{t+1} = (\frac{1+r}{q_{t+1}} b_t) q_{t+1} + 0 \cdot (1 - q_{t+1}) = (1+r) b_t.$$

This is of the form (32) with $a^{-1} = 1 + r$, and the bubble is therefore a stochastic rational bubble. The stochastic component is $u_{t+1} = b_{t+1} - E_t b_{t+1} = b_{t+1} - (1+r) b_t$ and has conditional expectation equal to zero. Although u_{t+1} must have zero conditional expectation, it need not be white noise (it can for instance have varying variance). $\square$

As this example illustrates, a stochastic rational bubble does not have the implausible ever-expanding form of a deterministic rational bubble. Yet, under certain conditions even stochastic rational bubbles can be ruled out or at least be judged implausible. The next section reviews some cases.

4.4 When rational bubbles in asset prices can or can not be ruled out

We concentrate on assets whose services are valued independently of the price.[13] Let p_t be the market price and p_t^* the fundamental value of the asset as of time t. Even if the asset yields services rather than dividends, we think of p_t^* as in principle the same for all agents. This is because a user who, in a given period, values the service flow of the asset relatively low can hire it out to the one who values it highest (the one with the highest willingness to pay). Until further notice we assume p_t^* known to the market participants.

[13] This is in contrast to assets that serve as means of payment.

4.4.1 Partial equilibrium arguments

The principle of reasoning to be used is called *backward induction:* If we know something about an asset price in the future, we can conclude something about the asset price today.

(a) Assets which can be freely disposed of ("free disposal") Can a rational asset price bubble be *negative*? The answer is no. The logic can be illustrated on the basis of Example 2 above. For simplicity, let the dividend be the same constant $d > 0$ for all $t = 0, 1, 2, ...$ Then, from the formula (40) we have

$$p_t - p^* = (p_0 - p^*)(1 + r)^t,$$

where $r > 0$ and $p^* = d/r$. Suppose there is a negative bubble in period 0, i.e., $p_0 - p^* < 0$. In period 1, since $1 + r > 1$, the bubble is greater in absolute value. The downward movement of p_t continues and sooner or later p_t is negative. The intuition is that the low p_0 in period 0 implies a high dividend-price ratio. Hence a negative capital gain ($p_{t+1} - p_t < 0$) is needed for the no-arbitrage condition (42) to hold. Thereby $p_1 < p_0$, and so on. After some time, $p_t < 0$.

But in a market with self-interested rational agents, an object which can be freely disposed of can never have a negative price. A negative price means that the "seller" has to *pay* to dispose of the object. Nobody will do that if the object can just be thrown away. An asset which can be freely disposed of (share certificates for instance) can therefore never have a negative price. We conclude that a *negative* rational bubble can not be consistent with rational expectations. Similarly, with a stochastic dividend, a negative rational bubble would imply that in expected value the share price becomes negative at some point in time, cf. (36). Again, rational expectations rule this out.

Hence, if we imagine that for a short moment $p_t < p_t^*$, then everyone will want to *buy* the asset and hold it forever, which by own use or by hiring out will imply a discounted value equal to p_t^*. There is thus excess demand until p_t has risen to p_t^*.

When a negative rational bubble can be ruled out, then, if at the first date of trading of the asset there were no positive bubble, neither can a positive bubble arise later. Let us make this precise:

PROPOSITION 3 Assume free disposal of a given asset. Then, if a rational bubble in the asset price is present today, it must be positive and must have been present also yesterday and so on back to the first date of trading the asset. And if a rational bubble bursts, it will not restart later.

Proof As argued above, in view of free disposal, a negative rational bubble in the asset price can be ruled out. It follows that $b_t = p_t - p_t^* \geq 0$ for $t = 0, 1, 2, ...$, where $t = 0$ is the first date of trading the asset. That is, any rational bubble in the asset price must be a positive bubble. We now show by contradiction that if, for an arbitrary $t = 1, 2, ...$, it holds that $b_t > 0$, then $b_{t-1} > 0$. Let $b_t > 0$. Then, if $b_{t-1} = 0$, we have $E_{t-1}b_t = E_{t-1}u_t$ $= 0$ (from (32) with t replaced by $t-1$), implying, since $b_t < 0$ is not possible, that $b_t = 0$ with probability *one* as seen from period $t-1$. Ignoring zero probability events, this rules out $b_t > 0$ and we have arrived at a contradiction. Thus $b_{t-1} > 0$. Replacing t by $t-1$ and so on backward in time, we end up with $b_0 > 0$. This reasoning also implies that if a bubble bursts in period t, it can not restart in period $t+1$, nor, by extension, in any subsequent period. □

This proposition (due to Diba and Grossman, 1988) claims that a rational bubble in an asset price must have been there since trading of the asset began. Yet such a conclusion is not without ambiguities. If new information about radically new technology comes up at some point in time, is a share in the firm then the same asset as before? In a legal sense the firm is the same, but is the asset also the same? Even if an earlier bubble has crashed, cannot a new rational bubble arise later in case of an utterly new situation?

These ambiguities reflect the difficulty involved in the concepts of rational expectations and rational bubbles when we are dealing with uncertainties about future developments of the economy. The market's evaluation of many assets of macroeconomic importance, not the least shares in firms, depends on vague beliefs about future preferences, technologies, and societal circumstances. The fundamental value can not be determined in any objective way. There is no well-defined probability distribution over the potential future outcomes. *Fundamental uncertainty*, also called *Knightian uncertainty*,[14] is present.

(b) Bonds with finite maturity The finite maturity ensures that the value of the bond is given at some finite future date. Therefore, if there were a positive bubble in the market price of the bond, no rational investor would buy just before that date. Anticipating this, no one would buy the date before, and so on. Consequently, nobody will buy in the first place. By this backward-induction argument follows that a positive bubble cannot get started. And since there also is "free disposal", *all* rational bubbles can be precluded.

From now on we take as given that negative rational bubbles are ruled out. So, the

[14] After the Chicago of University economist Frank Knight who in his book, *Risk, Uncertainty, and Profit* (1921), coined the important distinction between *measurable risk* and *unmeasurable uncertainty.*

discussion is about whether *positive* rational asset price bubbles may exist or not.

(c) Assets whose supply is elastic Real capital goods (including buildings) can be reproduced and have clearly defined costs of reproduction. This precludes rational bubbles on this kind of assets, since a potential buyer can avoid the overcharge by producing instead. Notice, however, that building sites with a specific amenity value and apartments in attractive quarters of a city are not easily reproducible. Therefore, rational bubbles on such assets are more difficult to rule out.

Here are a few intuitive remarks about bubbles on shares of stock in an established firm. An argument against a rational bubble might be that if there were a bubble, the firm would tend to exploit it by issuing more shares. But thereby market participants mistrust is raised and may pull market evaluation back to the fundamental value. On the other hand, the firm might anticipate this adverse response from the market. So the firm chooses instead to "fool" the market by steady financing behavior, calmly enjoying its solid equity and continuing as if no bubble were present. It is therefore not obvious that this kind of argument can rule out rational bubbles on shares of stock.

(d) Assets for which there exists a "backstop-technology" For some articles of trade there exists substitutes in elastic supply which will be demanded if the price of the article becomes sufficiently high. Such a substitute is called a "backstop-technology". For example oil and other fossil fuels will, when their prices become sufficiently high, be subject to intense competition from substitutes (renewable energy sources). This precludes an unbounded bubble process in the price of oil.

On account of the arguments (c) and (d), it seems more difficult to rule out rational bubbles when it comes to assets which are not reproducible or substitutable, let alone assets whose fundamentals are difficult to ascertain. For some assets the fundamentals are not easily ascertained. Examples are paintings of past great artists, rare stamps, diamonds, gold etc. Also new firms that introduce completely novel products and technologies are potential candidates. Think of the proliferation of radio broadcasting in the 1920s before the wall Street crash in 1929 and the internet in the 1990s before the dotcom bubble burst in 2000.

What these situations allow for may not be termed rational bubbles, if by definition this concept requires a well-defined fundamental. Then we may think of a broader class of real-world bubbly phenomena driven by self-reinforcing expectations.

4.4.2 Adding general equilibrium arguments

The above considerations are of a partial equilibrium nature. On top of this, *general equilibrium* arguments can be put forward to limit the possibility of rational bubbles. We may briefly give a flavour of two such general equilibrium arguments. We still consider assets whose services are valued independently of the price and which, as in (a) above, can be freely disposed of. A house, a machine, or a share in a firm yields a service in consumption or production or in the form of a dividend stream. Since such an asset has an intrinsic value, p_t^*, equal to the present value of the flow of services, one might believe that positive rational bubbles on such assets can be ruled out in general equilibrium. As we shall see, this is indeed true for an economy with a finite number of "neoclassical" households (to be defined below), but not necessarily in an overlapping generations model. Yet even there, rational bubbles can under certain conditions be ruled out.

(e) An economy with a finite number of infinitely-lived households Assume that the economy consists of a finite number of infinitely-lived agents − here called households − indexed $i = 1, 2, ..., N$. The households are "neoclassical" in the sense that they save only with a view to future consumption.

Under free disposal in point (a) we saw that $p_t < p_t^*$ can not be an equilibrium. We now consider the case of a positive bubble, i.e., $p_t > p_t^*$. All owners of the bubble asset who are users will in this case prefer to *sell* and then *rent*; this would imply excess supply and could thus not be an equilibrium. Hence, we turn to households that are not users, but speculators. Assuming "short selling" is legal, speculators may pursue "short selling", that is, they first rent the asset (for a contracted interval of time) and immediately sell it at p_t. This results in excess supply and so the asset price falls towards p_t^*. Within the contracted interval of time the speculators buy the asset back and return it to the original owners in accordance with the loan accord. So $p_t > p_t^*$ can not be an equilibrium.

Even ruling out "short selling" (which *is* sometimes outright forbidden), we can exclude positive bubbles in the present setup with a finite number of households. To assume that owners who are not users would want to hold the bubble asset forever as a permanent investment will contradict that these owners are "neoclassical". Indeed, their transversality condition would be violated because the value of their wealth would grow at a rate asymptotically equal to the rate of interest. This would allow them to increase their consumption now without decreasing it later and without violating their No-Ponzi-Game condition.

We have to instead imagine that the "neoclassical" households who own the bubble asset, hold it against future sale. This could on the face of it seem rational enough if there were some probability that not only would the bubble continue to exist, but it would also grow so that the return would be at least as high as that yielded on an alternative investment. Owners holding the asset in the expectation of a capital gain, will thus plan to sell at some later point in time. Let t_i be the point in time where household i wishes to sell and let

$$T = \max\left[t_1, t_2, ..., t_N\right].$$

Then nobody will plan to hold the asset after T. The household speculator, i, having $t_i = T$ will thus not have anyone to sell to (other than people who will only pay p_T^*). Anticipating this, no-one would buy or hold the asset the period before, and so on. So no-one will want to buy or hold the asset in the first place.

The conclusion is that $p_t > p_t^*$ cannot be a rational expectations equilibrium in a setup with a finite number of "neoclassical" households, as in, for instance, the Ramsey model or the Barro dynasty model.

The same line of reasoning does not, however, go through in an overlapping generations model where *new* households − that is, new traders − enter the economy every period.

(f) An economy with interest rate above the output growth rate In an overlapping generations (OLG) model with an infinite sequence of new decision makers, rational bubbles are under certain conditions theoretically possible. The argument is that with $N \to \infty$, T as defined above is not bounded. Although this unboundedness is a necessary condition for rational bubbles, it is not sufficient, however.

To see why, let us return to the arbitrage examples 1, 2, and 3 where we have $a^{-1} = 1 + r$ so that a hypothetical rational bubble has the form $b_{t+1} = (1 + r)b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. So in expected value the hypothetical bubble is growing at a rate equal to the interest rate, r. If at the same time r is higher than the long-run output growth rate, the value of the expanding bubble asset would sooner or later be larger than GDP and aggregate saving would not suffice to back its continued growth. Agents with rational expectations anticipate this and so the bubble never gets started.

This point is valid when the interest rate in the OLG economy is higher than the growth rate of the economy − which is normally considered the realistic case. Yet, the opposite case *is* possible and in that situation it is less easy to rule out rational asset price bubbles. This is also the case in situations with imperfect credit markets. It turns out that the presence of segmented financial markets or externalities that create a wedge between private and social returns on productive investment may increase the scope for rational bubbles (Blanchard, 2008).

4.5 Conclusion

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and mis-specified fundamentals. Rational bubbles can also have more complicated forms than the bursting bubble in Example 3 above. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles.

Whatever the possible limits to the plausibility of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for a variety of "behavioral asset price bubbles", i.e., bubbles arising through particular psychological mechanisms. This would take us to *behavioral finance* theory. The reader is referred to, e.g., Shiller (2003).

For surveys on the theory of rational bubbles and econometric bubble tests, see Salge (1997) and Gürkaynak (2008). For discussions of famous historical bubble episodes, see the symposium in *Journal of Economic Perspectives* 4, No. 2, 1990, and Shiller (2005).

5 Appendix

A. The log-linear specification

In many macroeconomic models with rational expectations the equations are specified as log-linear, that is, as being linear in the logarithms of the variables. If Y, X, and Z are the original positive stochastic variables, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, a log-linear relationship between Y, X, and Z is a relation of the form

$$y = \alpha + \beta x + \gamma z, \tag{44}$$

where α, β, and γ are constants. The motivation for assuming log-linearity can be:

(a) Linearity is convenient because of the simple rule for the expected value of a sum: $E(\alpha + \beta x + \gamma z) = \alpha + \beta E(x) + \gamma E(z)$, where E is the expectation operator. Indeed, for a non-linear function, $f(x, z)$, we generally have $E(f(x, z)) \neq f(E(x), E(z))$.

(b) Linearity in logs may often seem a more realistic assumption than linearity in anything else.

(c) In time series models a logarithmic transformation of the variables followed by formation of first differences can be the road to eliminating a trend in the mean and variance.

As to point (b) we state the following:

CLAIM To assume linearity in logs is equivalent to assuming constant elasticities.

Proof Let the positive variables Y, X and Z be related by $Y = F(X,\ Z)$, where F is a continuous function with continuous partial derivatives. Taking the differential on both sides of $\ln\ Y = \ln F(X,Z)$, we get

$$\begin{aligned} d\ln Y &= \frac{1}{F(X,Z)}\frac{\partial F}{\partial X}dX + \frac{1}{F(X,Z)}\frac{\partial F}{\partial Z}dZ \qquad (45) \\ &= \frac{X}{Y}\frac{\partial Y}{\partial X}\frac{dX}{X} + \frac{Z}{Y}\frac{\partial Y}{\partial Z}\frac{dZ}{Z} = \eta_{YX}\frac{dX}{X} + \eta_{YZ}\frac{dZ}{Z} = \eta_{YX}d\ln X + \eta_{YZ}d\ln Z, \end{aligned}$$

where η_{YX} and η_{YZ} are the partial elasticities of Y w.r.t. X and Z, respectively. Thus, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, gives

$$dy = \eta_{YX}dx + \eta_{YZ}dz. \qquad (46)$$

Assuming constant elasticities amounts to putting $\eta_{YX} = \beta$ and $\eta_{YZ} = \gamma$, where β and γ are constants. Then we can write (46) as $dy = \beta dx + \gamma dz$. By integration, we get (44) where α is now an arbitrary integration constant. Hereby we have shown that constant elasticities imply a log-linear relationship between the variables.

Now, let us instead start by assuming the log-linear relationship (44). Then,

$$\frac{\partial y}{\partial x} = \beta, \frac{\partial y}{\partial z} = \gamma. \qquad (47)$$

But (44), together with the definitions of y, x and z, implies that

$$Y = e^{\alpha+\beta x+\gamma z} = e^{\alpha+\beta\ln X+\gamma\ln Z},$$

from which follows that

$$\frac{\partial Y}{\partial X} = Y\beta\frac{1}{X} \text{ so that } \eta_{YX} \equiv \frac{X}{Y}\frac{\partial Y}{\partial X} = \beta,$$

and

$$\frac{\partial Y}{\partial Z} = Y\gamma\frac{1}{Z} \text{ so that } \eta_{YZ} \equiv \frac{Z}{Y}\frac{\partial Y}{\partial Z} = \gamma.$$

That is, the partial elasticities are constant. $\square$

So, when the variables are in logs, then the coefficients in the linear expressions are the elasticities. Note, however, that the interest rate is normally an exception. It is often regarded as more realistic to let the interest rate itself and not its logarithm enter linearly. Then the associated coefficient indicates the *semi-elasticity* with respect to the interest rate.

B. Conditional expectations and the law of iterated expectations

The mathematical conditional expectation is a weighted sum of the possible values of the stochastic variable with weights equal to the corresponding conditional probabilities.

Let Y and X be two *discrete* stochastic variables with joint probability function $j(y,x)$ and marginal probability functions $f(y)$ and $g(x)$, respectively. If the conditional probability function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have $h(y\,|x_0) = j(y,x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$, denoted $E(Y|X = x_0)$, is then

$$E(Y|X = x_0) = \sum_y y\frac{j(y,x_0)}{g(x_0)}, \tag{48}$$

where the summation is over all the possible values of y.

This conditional expectation is a function of x_0. Since x_0 is just one possible value of the stochastic variable X, we interpret the conditional expectation itself as a stochastic variable and write it as $E(Y|X)$. Generally, for a function of the discrete stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \sum_x k(x)g(x).$$

When we here let the conditional expectation $E(Y|X)$ play the role of $k(X)$ and sum over all x for which $g(x) > 0$, we get

$$\begin{aligned} E(E(Y|X)) &= \sum_x E(Y|x)g(x) = \sum_x \left(\sum_y y\frac{j(y,x)}{g(x)}\right) g(x) \qquad \text{(by (48))} \\ &= \sum_y y\left(\sum_x j(y,x)\right) = \sum_y yf(y) = E(Y). \end{aligned}$$

This result is a manifestation of the *law of iterated expectations:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

Now consider the case where Y and X are *continuous* stochastic variables with joint probability *density* function $j(y,x)$ and marginal density functions $f(y)$ and $g(x)$, respectively. If the conditional density function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have $h(y\,|x_0) = j(y,x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$ is

$$E(Y|X = x_0) = \int_{-\infty}^{\infty} y\frac{j(y,x_0)}{g(x_0)}dy, \tag{49}$$

where we have assumed that the range of Y is $(-\infty,\infty)$. Again, we may view the conditional expectation itself as a stochastic variable and write it as $E(Y|X)$. Generally, for a function of the continuous stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \int_R k(x)g(x)dx,$$

where R stands for the range of X. When we let the conditional expectation $E(Y|X)$ play the role of $k(X)$, we get

$$\begin{aligned} E(E(Y|X)) &= \int_R E(Y|x)g(x)dx = \int_R \left(\int_{-\infty}^{\infty} y\frac{j(y,x)}{g(x)}dy\right) g(x)dx \text{ (by (49))} \\ &= \int_{-\infty}^{\infty} y \left(\int_R j(y,x)dx\right) dy = \int_{-\infty}^{\infty} yf(y)dy = E(Y). \end{aligned} \tag{50}$$

This shows us the *law of iterated expectations* in action for continuous stochastic variables*:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

EXAMPLE Let the two stochastic variables, X and Y, follow a two-dimensional normal distribution. Then, from mathematical statistics we know that the conditional expectation of Y given X satisfies

$$E(Y|X) = E(Y) + \frac{\mathrm{Cov}(Y,X)}{\mathrm{Var}(X)}(X - E(X)).$$

Taking expectations on both sides gives

$$E(E(Y|X)) = E(Y) + \frac{\mathrm{Cov}(Y,X)}{\mathrm{Var}(X)}(E(X) - E(X)) = E(Y). \qquad \square$$

We may also express the law of iterated expectations in terms of subsets of the original outcome space for a stochastic variable. Let the event $\mathcal{A}$ be a subset of the outcome space for Y and let $\mathcal{B}$ be a subset of $\mathcal{A}$. Then the law of iterated expectations takes the form

$$E(E(Y|\mathcal{B})|\mathcal{A}) = E(Y|\mathcal{A}). \tag{51}$$

That is, when $\mathcal{B} \subseteq \mathcal{A}$, the expectation, conditional on $\mathcal{A}$, of the expectation of Y, conditional on $\mathcal{B}$, is the same as the expectation, conditional on $\mathcal{A}$, of Y.

Often we consider a dynamic context where expectations are conditional on dated information I_{t-i} $(i = 1, 2, ...)$. By a, so far, "informal analogy" with (50) we then write the law of iterated expectations this way:

$$E(E(Y_t|I_{t-i})) = E(Y_t), \qquad \text{for } i = 1, 2, ... \tag{52}$$

In words: the unconditional expectation of the conditional expectation of Y_t, given the information up to time $t - i$ equals the unconditional expectation of Y_t. Similarly, by a, so far, "informal analogy" with (51) we may write

$$E(E(Y_{t+2}|I_{t+1})|I_t) = E(Y_{t+2}|I_t). \tag{53}$$

That is, the expectation today of the expectation tomorrow, when more may be known, of a variable the day after tomorrow is the same as the expectation today of the variable the day after tomorrow. Intuitively: you ask a stockbroker in which direction she expects to revise her expectations upon the arrival of more information. If the broker answers "upward", say, then another broker is recommended.

The notation used in the transition from (51) to (53) might seem problematic, though. That is why we talk of "informal analogy". The sets $\mathcal{A}$ and $\mathcal{B}$ are subsets of the outcome space and $\mathcal{B} \subseteq \mathcal{A}$. In contrast, the "information" or "information content" represented by our symbol I_t will, for the uninitiated, inevitably be understood in a meaning *not* fitting the inclusion $I_{t+1} \subseteq I_t$. Intuitively "information" dictates the opposite inclusion, namely as a set which *expands* over time – more and more "information" (like "knowledge" or "available data") is revealed as time proceeds.

It is possible, however, to interpret the information I_t from another angle so as to make the notation in (53) fully comply with that in (51). Let the outcome space Ω denote the set of ex ante possible[15] sequences $\{(Y_t, X_t)\}_{t=t_0}^{T}$, where Y_t and X_t are vectors of date-t endogenous and exogenous stochastic variables, respectively, and where T is the time horizon, possibly $T = \infty$. For $t \in \{t_0, t_0 + 1, \ldots, T\}$, let the subset $\Omega_t \subseteq \Omega$ be defined as the of time t still possible sequences $\{(Y_s, X_s)\}_{s=t}^{T}$. Now, as time proceeds, more and more realizations occur, that is, more and more of the ex ante random states, (Y_t, X_t), become historical data, (y_t, x_t). Hence, as time proceeds, the subset Ω_t *shrinks* in the sense that $\Omega_{t+1} \subseteq \Omega_t$. The increasing amount of information and the "reduced uncertainty" can thus

[15] By "possible" is meant "feasible according to a given model".

be seen as two sides of the same thing. Interpreting I_t this way, i.e., as "partial lack of uncertainty", the expression (53) means the same thing as

$$E(E(Y_{t+2}|\Omega_{t+1})|\Omega_t) = E(Y_{t+2}|\Omega_t).$$

This is in complete harmony with (51).

C. Properties of the model-consistent forecast

As in the text of Section 24.2.2, let e_t denote the model-consistent forecast error $Y_t - E(Y_t|I_{t-1})$. Then, if S_{t-1} represents information contained in I_{t-1},

$$\begin{aligned} E(e_t\,|S_{t-1}) &= E(Y_t - E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1}) - E(E(Y_t\,|I_{t-1})\,|S_{t-1}) \\ &= E(Y_t\,|S_{t-1}) - E(Y_t\,|S_{t-1}) = 0, \end{aligned} \tag{54}$$

where we have used that $E(E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1})$, by the law of iterated expectations. With $S_{t-1} = I_{t-1}$ we have, as a special case,

$$\begin{aligned} E(e_t\,|I_{t-1}) &= 0, \quad \text{as well as} \qquad (55) \\ E(e_t) &= E(Y_t - E(Y_t\,|I_{t-1})) = E(Y_t) - E(E(Y_t\,|I_{t-1})) = 0, \end{aligned}$$

in view of (52) with $i = 1$. This proves property (a) in Section 24.2.3.

As to property (b) in Section 24.2.2, for $i = 1, 2, ...,$ let s_{t-i} be an arbitrary variable value belonging to the information I_{t-i}. Then, $E(e_t s_{t-i}\,|I_{t-i}) = s_{t-i}E(e_t\,|I_{t-i}) = 0$, by (54) with $S_{t-1} = I_{t-i}$ (since I_{t-i} is contained in I_{t-1}). Thus, by the principle (52),

$$E(e_t s_{t-i}) = E\left(E(e_t s_{t-i}\,|I_{t-i})\right) = E(0) = 0 \quad \text{for } i = 1, 2, ... \tag{56}$$

This result is known as the *orthogonality property* of model-consistent expectations (two stochastic variables Z and V are said to be *orthogonal* if $E(ZV) = 0$). From the general formula for the (unconditional) covariance follows

$$\mathrm{Cov}(e_t s_{t-i}) = E(e_t s_{t-i}) - E(e_t)E(s_{t-i}) = 0 - 0 = 0, \quad \text{for } i = 1, 2, ...,$$

by (55) and (56). In particular, with $s_{t-i} = e_{t-i}$, we get $\mathrm{Cov}(e_t e_{t-i}) = 0$. This proves that model-consistent forecast errors exhibit *lack of serial correlation.*

6 Exercises

1. Let $\{X_t\}$ be a stochastic process in discrete time. Suppose $Y_t = X_t + e_t$ and $X_t = X_{t-1} + \varepsilon_t$, where e_t and ε_t are white noise.

a) Is $\{X_t\}$ a random walk? Why or why not?

b) Is $\{Y_t\}$ a random walk? Why or why not?

c) Calculate the rational expectation of X_t conditional on all relevant information up to and including period $t-1$.

d) What is the rational expectation of Y_t conditional on all relevant information up to and including period $t-1$?

e) Compare with the subjective expectation of Y_t based on the adaptive expectations formula with adjustment speed equal to one.

2. Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$
\begin{aligned}
Y_t &= D_t = C_t + \bar{I} + G_t, && (1)\\
C_t &= \alpha + \beta Y^e_{t-1,t}, \qquad \alpha > 0,\ 0 < \beta < 1, && (2)\\
G_t &= (1-\rho)\bar{G} + \rho G_{t-1} + \varepsilon_t, \qquad \bar{G} > 0,\ 0 < \rho < 1, && (3)
\end{aligned}
$$

where the endogenous variables are Y_t = output (= income), D_t = aggregate demand, C_t = consumption, and $Y^e_{t-1,t}$ = expected output (income) in period t as seen from period $t-1$, while G_t, which stands for government spending on goods and services, is considered exogenous as is ε_t, which is white noise. Finally, investment, $\bar{I}$, and the parameters α, β, ρ, and $\bar{G}$ are given positive constants.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier (partial derivative of Y_t) with respect to a change in G_{t-1} and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in G_{t-1} and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

3. Consider arbitrage between equity shares and a riskless asset paying the constant rate of return $r > 0$. Let p_t denote the price at the beginning of period t of a share that at the end of period t yields the dividend d_t. As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, \ldots$. Suppose agents have rational expectations and care only about expected return (risk neutrality).

a) Write down the no-arbitrage condition.

Suppose dividends follow the process $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is a positive constant and ε_t is white noise, observable in period t, but not known in advance.

b) Find the fundamental solution for p_t and let it be denoted p_t^*. *Hint:* given $y_t = aE_t y_{t+1} + c\ x_t$, the fundamental solution is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Suppose someone claims that the share price follows the process

$$p_t = p_t^* + b_t,$$

with a given $b_0 > 0$ and, for $t = 0, 1, 2, \ldots$,

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases}$$

where $q_t = f(b_t)$, $f' < 0$.

c) What is an asset price bubble and what is a rational asset price bubble?

d) Can the described b_t process be a rational asset price bubble? *Hint:* a bubble component associated with the inhomogenous equation $y_t = aE_t y_{t+1} + c\ x_t$ is a solution, different from zero, to the homogeneous equation, $y_t = aE_t y_{t+1}$.

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Chapter 16

Money in macroeconomics

Money buys goods and goods buy money; but goods do not buy goods.

—Robert W. Clower (1967).

Up to now we have put monetary issues aside. The implicit assumption has been that the exchange of goods and services in the market economy can be carried out without friction as mere intra- or intertemporal barter. This is, of course, not realistic. At best it can provide an acceptable approximation to reality for a limited set of macroeconomic issues, primarily long-run issues. We now turn to models in which there is a demand for money. We thus turn to *monetary theory*, that is, the study of causes and consequences of the fact that a large part of the exchange of goods and services in the real world is mediated through the use of money.

16.1 What is money?

16.1.1 The concept of money

In economics *money* is defined as an asset (a store of value) which functions as a generally accepted medium of exchange, i.e., it can be used directly to buy *any* good or service offered for sale in the economy. Bitcoins may also be a medium of exchange, but are not *generally* accepted and are therefore not money. A note of IOU (a bill of exchange) may be a medium of exchange, but is not *generally* accepted and is therefore not money. The extent to which an IOU is acceptable in exchange depends on a series of circumstances like reliable information about the issuer as well as the general state of the economy.

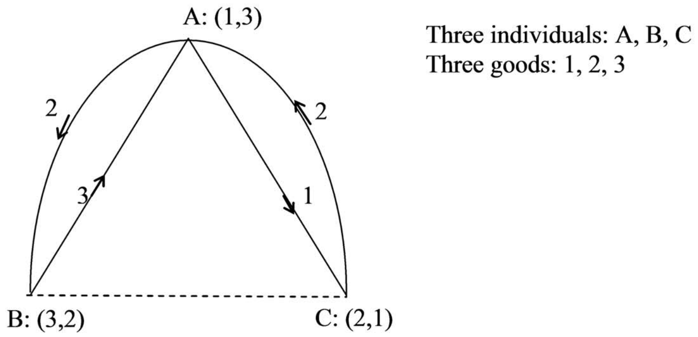

Figure 16.1: No direct exchange possible. A medium of exchange, here good 2, solves the problem (details in text).

The term *means of payment* include money as well as assets that are at least *often* accepted as a medium of exchange, for instance an IOU. Or, if one lives near the border to a neighboring country with another currency, then this foreign currency can often be used locally as a medium of exchange. Hence, equivalently with the above definition of money we may say that *money* is a *generally accepted means of payment.*

An asset's *liquidity* is the *ease* with which the asset can be *converted into money or be used directly for making payments.* So liquidity should be conceived as a matter of degree. An asset has a higher or lower degree of liquidity depending on the extent to which it can "near-immediately, conveniently, and cheaply" be exchanged for money or itself be used as a means of payment. An asset is *fully liquid* if it can be used instantly, unconditionally, and without any extra costs or restrictions to make payments. So money can be characterized as a *fully liquid asset.*

Where to draw the line between "money" and "non-money assets" depends on what is appropriate for the problem at hand. In the list below of different monetary aggregates (Section 16.2), it is M_1 that comes closest to the conception of the "money supply" in economics, at least traditionally. Defined as currency in circulation plus demand deposits held by the non-bank public in commercial banks, M_1 embraces all under "normal circumstances" (for instance beyond financial crises) fully liquid assets in the hands of the non-bank public.

The reason that a market economy uses money is that money facilitates trade enormously, thereby reducing transaction costs. Money helps an economy to avoid the need for a "double coincidence of wants". The classical way of illustrating this is by the *exchange triangle* in Fig. 16.1. The individuals A, B, and C are endowed with one unit of the goods 1, 3, and 2, respectively. But A, B, and C want to consume 3, 2, and 1, respectively. Thus, no direct exchange is possible between two individuals each wanting to consume the other's good. There is a *lack of double coincidence of wants*. The problem can be solved by indirect exchange where A exchanges good 1 for good 2 with C and then, in the next step, uses good 2 in an exchange for good 3 with B. Here good 2 serves as a medium of exchange. If good 2 becomes widely used and accepted as a medium of exchange, it is money. Extending the example to a situation with n goods, we have that exchange without money (i.e., barter) requires $n(n-1)/2$ markets ("trading spots"). Exchange with money requires only n markets.

16.1.2 Historical remarks

In the past, ordinary commodities, such as seashells, rice, cocoa, precious metals etc., served as money. That is, commodities that were easily divisible, handy to carry, immutable, and involved low costs of storage and transportation could end up being used as money. This form of money is called *commodity money*. Applying ordinary goods as a medium of exchange is costly, however, because these goods have alternative uses. A more efficient way to trade is by using currency, i.e., coins and notes in circulation with little or no intrinsic value, or pieces of paper, checks, representing claims on such currency. In spite of no intrinsic value, these media of exchange may be generally accepted means of payment. Regulation by a central authority (the state or the central bank) has been of key importance in bringing about this transition into the payment system of the historic present.

This form of money with no intrinsic value is traditionally called *paper money*. By having these paper moneys - in modern times rather electronic entries in banks' deposit accounts - circulating and the real goods moving only once, from initial producer to final consumer, the trading costs in terms of time and effort are minimized.

In the industrialized countries paper monies were in the last third of the nineteenth century and until the outbreak of the First World War *backed* through the gold standard. And under the Bretton-Woods agreement, covering 1947-71, the currencies of the developed Western countries outside the United States were convertible into US dollars at a fixed exchange rate (or rather an exchange rate which is adjustable only under specific circumstances). And US dollar reserves of these countries were (in principle) convertible into gold by the United States at a fixed price (though in practice with some discouragement from the United States).

This indirect gold-exchange standard broke down in 1971-73, and nowadays money in most countries is *unbacked* paper money, including electronic entries in bank accounts. This feature of modern money makes its valuation very different from that of other assets. A piece of paper money in a modern payments system has no worth at all to an individual unless she *expects* other economic agents to value it in the next instant. There is an *inherent circularity* in the acceptance of paper money. Hence the viability of a paper money system is very much dependent on adequate juridical institutions as well as confidence in the ability and willingness of the government and central bank to conduct policies that sustain the purchasing power of the currency. One elementary juridical institution is that of "legal tender", a status which is conferred to currency, also called cash: coins and central bank notes. An example is the law that a money debt can always be settled by cash and a tax always be paid in cash. Money whose market value derives entirely from its legal tender status is called *fiat money* (because the value exists through "fiat", a ruler's declaration). In view of the absence of intrinsic value or even a link to an intrinsic value as under the gold standard, maintaining the exchange value of fiat money over time, that is, avoiding high or fluctuating inflation, is one of the central tasks of monetary policy.

16.1.3 The functions of money

The following three functions are sometimes considered to be the definitional characteristics of money:

1. It is a generally accepted medium of exchange.

2. It is a store of value.

3. It serves as a unit of account in which prices are quoted and books kept (the *numeraire*).

One can argue, however, that the last function is on a different footing compared to the two others. Thus, we should make a distinction between the functions that money *necessarily* performs, according to our definition above, and additional functions that money *usually* performs. Property 1 and 2 certainly belong to the essential characteristics of money. By its role as a device for making transactions, money helps an economy to avoid the need for a double coincidence of wants. In order to perform this role, money *must* be a store of value, i.e., a device that maintains value over time. The reason that people are willing to exchange their goods for pieces of paper or electronic signals from plastic cards or smart phones to bank accounts is exactly that these pieces of paper or electronic signals can later be used to purchase other goods. As a store of value, however, money is *dominated* by other stores of value such as bonds and equity shares that pay a higher rate of return. When nevertheless there is a demand for money, it is due to the *liquidity* of this store of value, that is, its service as a generally accepted medium of exchange.

Property 3, however, is not an indispensable function of money as we have defined it. Though the money unit is usually used as the unit of account in which prices are quoted, this function of money is conceptually distinct from the other two functions and has sometimes been distinct in practice. During times of high inflation, foreign currency has been used as a unit of account, whereas the local money continued to be used as the medium of exchange. During the German hyperinflation of 1922-23 US dollars were the unit of account used in parts of the economy, whereas the mark was the medium of exchange; and during the Russian hyperinflation in the middle of the 1990s again US dollars were often the unit of account, but the rouble was still the medium of exchange.

This is not to say that it is of little importance that money *usually* serves as numeraire. Indeed, this function of money plays an important role for the short-run macroeconomic effects of changes in the money supply. These effects are due to *nominal rigidities*, that is, the fact that prices, usually denominated in money, of most goods and services generally adjust only sluggishly (in contrast to prices of financial assets traded in centralized auction markets).

16.2 The money supply

The money supply is the total amount of money available in an economy at a particular point in time (a stock). As noted above, where to draw the line between assets that should be counted as money and those that should not, depends on the context.

16.2.1 Different measures of the money stock

Usually the money stock in an economy is measured as one of the following alternative so-called *monetary aggregates*:

- M_0, i.e., the *monetary base*, alternatively called *base money, central bank money,* or *high-powered money.* The monetary base is defined as fully liquid claims on the central bank held by the private sector, that is, currency (coins and notes) in circulation plus *bank reserves.* The latter consist of demand deposits held by the commercial banks in the central bank plus currency in the "vaults" of these banks.[1] This monetary aggregate is under

[1] The commercial banks are usually part of the private sector, and by law it is generally only the commercial banks that are allowed to have demand deposits in the central bank − the "banks' bank".

the direct control of the central bank and is changed through *open-market operations*, that is, through the central bank trading bonds, usually short-term government bonds, with the private sector. But clearly the monetary base is an imperfect measure of the liquidity in the private sector.

- M_1, defined as *currency in circulation* plus *demand deposits* held by the non-bank general public *in commercial banks.* Currency "in circulation" is currency held by the general public (households and non-bank firms). The demand deposits are also called *checking accounts* because they are deposits on which checks can be written and payment cards (debit cards) and mobile payment be used. M_1 does not include currency held by commercial banks and demand deposits held by commercial banks in the central bank. But currency in circulation, (usually the *major* part of M_0) *is* included in M_1. Most importantly, the commercial banks use a portion of the funds received from depositors to make interest-bearing loans. Through this *bank lending* and the resulting increase in the borrowers' deposit accounts, M_1 generally becomes substantially larger than M_0.

  The measure M_1 is one measure intended to reflect the quantity of assets serving as media of exchange in the hands of the non-bank part of the private sector. Broader measures of the money stock include:

- $M_2 = M_1$ plus savings accounts and small-denomination time deposits (say below € 100,000) that can easily be converted into a checkable account, although with a penalty. These claims are not instantly liquid, but they are close to.

- $M_3 = M_2$ plus large-denomination time-deposits (say above € 100,000).[2]

As we move down the list, the liquidity of the added assets decreases, while their interest yield increases. Currency is of course fully liquid and earns zero interest. Along with currency, the demand deposits in the commercial banks are normally fully liquid, at least as long as they are guaranteed by a governmental deposit insurance (although normally only up to a certain maximum per account). The interest earned on these demand deposits is usually low or even nil (at least for "small" depositors) and is often ignored in simple theoretical models. When in macroeconomic texts the term "money supply" is used, traditionally M_1, but more recently perhaps M_2, is meant. Yet, part of M_2 is not directly usable as a means of payment and therefore not money in the strict sense.[3]

[2] In casual notation, $M_1 \subset M_2 \subset M_3$, but $M_0 \nsubseteq M_1$ since only a part of M_0 belongs to M_1.

[3] It has been proposed to weight near-money aggregates, for instance M_2 or M_3, by their degree of liquidity and then entering them in a total measure of the money supply (see Barnett, 1980, and Spindt, 1985).

A related and theoretically important, simple classification of money types is the following:

1. *Outside money* = money that on net is an asset of the private sector.
2. *Inside money* = money that is not net wealth of the private sector.

Clearly M_0 is *outside money*. Most money in modern economies is *inside money*, however. Demand deposits at the commercial banks are an example of inside money. These deposits are an asset to their holders, but a liability of the banks.

Payment cards versus credit cards Does it make sense to include the amounts that people are allowed to charge by using their *credit cards* in the concept of "broad money"? No, this would imply *double counting*. Actually *you* do not pay when you use a credit card at the store. It is the company issuing the credit card that pays to the store (shortly after you made your purchases). You postpone your payment until you receive your monthly bill from the credit card company. That is, the credit card company does the payment for you and gives credit *to you*. It is otherwise with a *payment card* (debet card) where the amount for which you buy is instantly charged your electronic account in the bank.

16.2.2 The money multiplier

Bank lending is the channel through which the monetary base expands to an effective money supply, the "money stock", considerably larger than the monetary base. The excess of the deposits of the non-bank part of the private sector over *bank reserves* ("vault cash" and demand deposits in the central bank) is lent out in the form of bank loans or used to buy government or corporate bonds. The non-bank public then deposits a fraction of these loans on checking accounts. Next, the banks lend out a fraction of these and so on. This process is named the *money multiplier process*. And the ratio of the "money stock", measured as M_1, say, to the monetary base is called the *money multiplier*.

Let

$$\begin{aligned}
CUR &= \text{ currency in circulation (= held by the non-bank general public)}, \\
DEP &= \text{ demand deposits held by the non-bank general public}, \\
\frac{CUR}{DEP} &= cd, \text{ the desired currency-deposit ratio}, \\
RES &= \text{ bank reserves } = \text{ currency held by the commercial banks} \\
& \qquad \text{("vault cash") plus their demand deposits in the central bank}, \\
\frac{RES}{DEP} &= rd, \text{ the desired reserve-deposit ratio} \geq \text{ the required reserve-deposit ratio.}
\end{aligned}$$

Notice that the currency-deposit ratio, cd, is chosen by the non-bank public, whereas the reserve-deposit ratio, rd, refers to the behavior of commercial banks. In many countries there is a minimum reserve-deposit ratio required by law to ensure a minimum liquidity buffer to forestall "bank runs" (situations where many depositors, fearing that their bank will be unable to repay their deposits in full and on time, simultaneously try to withdraw their deposits). On top of the minimum reserve-deposit ratio the banks may hold "excess reserves" depending on their assessment of their lending risks and need for liquidity.

We may express the money multiplier in terms of cd and rd. First, note that

$$M_1 = CUR + DEP = (cd + 1)DEP, \tag{16.1}$$

where DEP is related to the monetary base, $M_{0,}$ through

$$M_0 = CUR + RES = cdDEP + rdDEP = (cd + rd)DEP.$$

Then, substituting into (16.1) gives

$$M_1 = \frac{cd + 1}{cd + rd} M_0 = mmM_0, \tag{16.2}$$

where $mm = (cd + 1)/(cd + rd)$ is the *money multiplier*.

As a not unrealistic example consider $cd \approx 0.7$ and $rd \approx 0.07$. Then we get $mm \approx 2.2$. When broader measures of money supply are considered, then, of course, a larger money multiplier arises. It should be kept in mind that both cd and rd, and therefore also mm, are neither constant nor exogenous from the point of view of monetary models. They are highly endogenous and depend on many things, including the degree of liquidity, risk, and expected returns on alternative assets − from the banks' perspective as well as the customers'. In the longer run, cd and rd are affected by the evolution of payment technologies.

To some extent it is therefore a matter of simple identities and not particularly informative, when we say that, given M_0 and the currency-deposit ratio, the money supply is smaller, the larger is the reserve-deposit ratio. Similarly, since the latter ratio is usually considerably smaller than one, the money supply is also smaller the larger is the currency-deposit ratio. Nevertheless, the money multiplier turns out to be fairly stable under "normal circumstances". But not always. During 1929-33, in the early part of the Great Depression, the money multiplier in the US fell sharply. Although M_0 increased by 15% during the four-year period, liquidity (M_1) declined by 27%.[4] Depositors became nervous about their bank's health and began to withdraw their deposits (thereby increasing cd) and this forced the banks to hold more reserves (thereby increasing rd). There is

[4] Blanchard (2003).

general agreement that this banking panic contributed to the depression and the ensuing deflation.

There is another way of interpreting the money multiplier. By definition of cd, we have $CUR = cdDEP$. Let cm denote the non-bank public's desired *currency-money ratio*, i.e., $cm = CUR/M_1$. Suppose cm is a constant. Then

$$CUR = cmM_1 = cm(cd+1)DEP. \qquad \text{(by (16.1))}$$

It follows that $cm = cd/(cd+1)$ and $1 - cm = 1/(cd+1)$. Combining this with (16.2) yields

$$M_1 = \frac{1}{\frac{cd}{cd+1} + rd\frac{1}{cd+1}}M_0 = \frac{M_0}{cm + rd(1-cm)} = \frac{1}{1-(1-rd)(1-cm)}M_0 = mmM_0. \tag{16.3}$$

The way the central bank is able to control the monetary base is through *open-market operations*. In *outright open-market operations* the central bank trades short-term government bonds with the banks. When the central bank buys a government bond from a bank, it takes over a loan from the bank to the government, thereby in effect increasing the monetary base as if lending to the bank. The aim may be to sustain a desired level of M_1 or a desired level of the short-term interest rate or, in an open economy, a desired exchange rate vis-a-vis other currencies. The central bank may alternatively increase the monetary base through a *repurchase agreement*, which is another form of open-market operation, see below.

To obtain a perhaps more intuitive understanding of the money multiplier and the way commercial banks "create money", let us take a *dynamic perspective*. Suppose the central bank increases M_0 by the amount ΔM_0 through purchasing bonds in the market. This is the first round. The seller of the bonds deposits the fraction $1 - cm$ of the proceeds on a checking account in her bank and keeps the rest as cash. The bank keeps the fraction rd of $(1 - cm)\Delta M_0$ as reserves and provides bank loans or buys bonds in the market with the rest. This is the second round. Thus, in the first round money supply is increased by ΔM_0. In the second round it is further increased by $(1 - rd)(1 - cm)\Delta M_0$. In the third round further by $(1 - rd)^2(1 - cm)^2\Delta M_0$, etc.[5] In the end, the total increase in money supply is

$$\begin{aligned} \Delta M_1 &= \Delta M_0 + (1-rd)(1-cm)\Delta M_0 + (1-rd)^2(1-cm)^2\Delta M_0 + ... \\ &= \frac{1}{1-(1-rd)(1-cm)}\Delta M_0 = mm\Delta M_0. \end{aligned}$$

[5]For simplicity, we assume here that cm and rd are constant.

The second last equality comes from the rule for the sum of an infinite geometric series with quotient in absolute value less than one. The conclusion is that the money supply is increased by mm times the increase in the monetary base.

16.3 Money demand

Explaining in a precise way how paper money gets purchasing power and how holding money - the "money demand" - is determined, is a complicated task and not our endeavour here. Suffice it to say that:

- In the presence of sequential trades and the absence of complete information and complete markets, there is a need for a generally accepted medium of exchange − *money.*
- The money demand, by which we usually mean the quantity of money willingly held by the non-bank public, should be seen as part of a broader *portfolio decision* by which economic agents allocate their financial wealth to different existing assets, including money, and liabilities. The portfolio decision involves a balanced consideration of *after-tax expected return*, *risk*, and *liquidity.*

Money is demanded primarily because of its liquidity service in transactions. Money holding therefore depends on the *amount of transactions* households and firms plan to carry out with money in the near future. Money holding also depends on the *need for flexibility* in spending when there is *uncertainty*: it is appropriate to have ready liquidity in case favorable shopping opportunities should turn up and to have a buffer in case of ill-foreseen adverse events. Keynes (1936, p. 170 ff.) also emphasized the *speculative motive*, i.e., the liquidity demand induced when speculators believe they will know "better than the market" that a fall in the price of bonds, equity shares, or foreign currency will happen very soon.

Generally money earns no interest at all or at least less interest than other assets. Therefore money holding involves a trade-off between the need for liquidity and the wish for interest yield.

The incorporation of a to some extent micro-founded money demand in macro-models is often based on some kind of short-cut:

- The *cash-in-advance constraint* (also called the *Clower constraint*).[6] Generally, households' purchases of nondurable consumption goods are in every

[6] After the American monetary theorist Robert Clower (1967). A better name for the constraint would be "money-in-advance constraint", since by "cash" is usually just meant currency.

short period paid for by money held at the beginning of the period. With the cash-in-advance constraint it is simply postulated that to be able to carry out most transactions, you *must* hold money in advance. In continuous time models the household holds a stock of money which is an increasing function of the desired level of consumption per time unit and a decreasing function of the opportunity cost of holding money.

- The *shopping-costs* approach. Here the liquidity services of money are modelled as reducing shopping time or other kinds of non-pecuniary or pecuniary shopping costs. The shopping time needed to purchase a given level of consumption, c_t, is decreasing in real money holdings and increasing in c_t.

- The *money-in-the-utility function* approach. Here, the indirect utility that money provides through reducing non-pecuniary as well as pecuniary transaction costs is modelled as if the economic agents obtain utility directly from holding money. This will be our approach in the next chapter.

- The *money-in-the-production-function* approach. Here money facilitates the firms' transactions, making the provision of the necessary inputs easier. After all, typically around a third of the aggregate money stock is held by firms.

16.4 What is then the "money market"?

In macroeconomic theory, by the "money market" is usually meant an imaginary market place where at any moment the available aggregate stock of money (supply) "meets" the aggregate desired money holding (demand). Equilibrium in this market is presented by an equation saying that the supply equals the demand in the sense of the amount of money willingly held by the general public. Note that we talk about supply and demand in terms of *stocks* (amounts at a given point in time), not flows. To be specific, let the money supply in focus be money in the sense of M_1 (currency in circulation plus demand deposits in the banks) and let P denote the general price level in the economy (say the GDP deflator). At the demand side, let the aggregate demand for real money balances be represented by the function $L(Y, i)$, where $L_Y > 0$ and $L_i < 0$ (L for "liquidity demand"). The level of aggregate economic activity, Y, enters as an argument because it is an (approximate) indicator of the volume of transactions in the near future for which money is needed. The second argument, i, in the liquidity demand function is some index for the *short-term nominal interest rate* which reflects the opportunity cost of holding money instead of interest-bearing short-term financial claims that are close substitutes to money, i.e., have relatively high but not full liquidity. We may think of interest-bearing time-deposits that are easily convertible into money, although at a penalty. Or i could be the interest rate on short-term government bonds ("treasury bills") or the interbank rate, see below.[7]

So money market equilibrium is present if

$$M_1 = PL(Y, i). \tag{16.4}$$

One of the issues in monetary theory is to account for how this stock equilibrium is brought about. During the history of economic thought there has been different views about which of the variables M_1, P, Y, and i is the equilibrating variable such that the available stock of money becomes willingly held by the agents. Presuming that the central bank somehow controls M_1, classical (pre-Keynesian) monetary theory has P as the equilibrating variable. In Keynes' monetary theory (now mainstream), however, it is i which has this role while the general price level for goods and services is considered sticky in the short run. It will be the bond price, and hence i, which responds.− and establishes the equilibrium (16.4) very fast. Popular specifications of the function L include $L(Y, i) = Y^{\alpha} i^{-\beta}$ (constant elasticity of money demand with respect to i) and $L(Y, i) = Y^{\alpha} e^{-\beta i}$ (constant semi-elasticity of money demand with respect to i), where α and β are positive constants.

One may alternatively think of the "money market" in a more narrow sense, however. We may translate (16.4) into a description of demand and supply for *base money* (currency plus bank reserves in the central bank):

$$M_0 = \frac{P}{mm} L(Y, i), \tag{16.5}$$

where mm is the money multiplier. The right-hand side of this equation reflects that the demand for M_1 via the actions of commercial banks is transformed into a demand for base money.[8] If the general public wants to hold more money, the demand for bank loans rises and when granted, deposits expand. Then the banks try to increase their reserves to maintain the required (or in any case desired) reserve-deposit ratio. A bank that finds it has too little reserves will want to borrow reserves from other banks in what is known as the *interbank market,* often on a day-to-day basis. But the immediate situation is one of excess demand for

[7] To simplify, in (16.4) we assume that none of the components in the monetary aggregate considered earns interest. In practice demand deposits may earn a small nominal interest. In this case, i would indicate the excess of the short-term interest rate over this rate.

[8] Although the money multiplier tends to depend positively on i as well as other interest rates, this aspect is unimportant for the "qualitative" discussion below and is ignored in the notation in (16.5).

bank reserves, and if its supply is not increased by the central bank, the interest rate in the interbank market, the *interbank rate,* rises. Then the interest rates on other short-term financial assets (short-term government bonds, time-deposits accounts, commercial paper, etc.) tend to move in the same direction because all these assets compete with each other. Assets offering higher-than-average rate of return will attract funds from assets offering lower-than-average rate of return, thereby roughly averaging out.

The "narrow" money market considered in (16.5) is a compact description of what is in the financial market statistics, and by practitioners, often called the "money market". This is the collective name for markets where trade in short-term debt-instruments (time to maturity less than one year). The agents trading in these markets include the central bank, the commercial banks, mortgage credit institutions, and sometimes additional financial institutions. From a logical point of view a more appropriate collective name than "money market" would be "short-term bond market" or "near-money market". This would be in line with the usual way we in economics use the term "market", namely as a "place" where a certain type of goods or assets are traded *for* money. Moreover, speaking of a "short-term bond market" corresponds well to the standard collective name for the markets for financial assets with maturity of *more* than one year, namely the "capital market" (where "capital" is synonymous with longer-term bonds and equity).

Anyway, in this book we maintain the standard term "money market" for the abstract market place where the aggregate supply of money "meets" the aggregate demand for money. As to what kind of money is in focus, "narrow" or "broad", further specification is always to be added.

Monetary policy and open-market operations

In recent decades the short-term interest rate has become the *main* monetary policy tool of the leading central banks in the world. In recent decades leading central banks, for instance both the Federal Reserve System of the US, the Fed, and the European Central Bank, the ECB, has increasingly focused on the short-term nominal interest rate as their policy tool. These central banks *announce* a *target level* for the chosen *policy interest rate* and then adjusts the monetary base through open-market operations such that the policy interest rate ends up very close to the announced target.

To understand the mechanism let us first imagine that the policy interest rate is the annualized interest rate, i, on one-month government bonds. Suppose the payoff is 1 euro at the *maturity date* and that there is no payoff between the *issue date* and the maturity date. Let p be the market price (in euros) of the bond at the issue date. The implicit monthly interest rate, x, is then the solution to the equation $(1+x)^{-1} \cdot 1 = p$, i.e.,

$$x = p^{-1} - 1.$$

Translated into an annual interest rate, this amounts to $i = (1+x)^{12} - 1 = p^{-12} - 1$ per year. With $p = 0.9975$, we get $i = 0.03049$ per year.[9]

Now, suppose the central bank finds that the current i is too high and therefore enters the market to buy a substantial amount of these bonds from the private sector. To find sellers, a higher price of the bonds must be offered. The bond price, p, is thus driven up, and the rate i thereby lowered − until the available stocks of bonds and money in their new proportion are willingly held. In practice this adjustment of p, and hence i, to a new equilibrium level takes place fast. If the resulting i is not as low as the target value, the central bank continues its buying bonds for base money until it is.

In fact, for both the Fed and the ECB the chosen policy interest rate for which they announce a target value is not short-term government bonds but the interbank rate. Even so, the procedure to obtain that rate is still some form of open-market operations where government bonds are traded with the private sector. Because of the competition between different short-term assets in the financial markets, other interest rates, *including the interbank rate*, will also be affected in a downward direction. The central bank continues its trading until the injection of base money has brought the interbank rate down to the announced target level. The interbank market is in the US called the Federal Funds market, and the interest rate.in this market is called the *Federal Funds Rate.*[10] Similarly, the ECB announces a certain target value for its quite similar policy rate called EONIA (Euro Overnight Index Average).

The aim of controlling the policy rate may be to stimulate or dampen the general level of economic activity, and the purpose of this may be controlling inflation, unemployment, or the foreign exchange rate. In this context what really matters is the interest rate households and firms have to pay when they borrow, the "bank lending rate"or the "corporate bond rate". Because of the higher risk involved, these rates tend to exceed the interbank rate by a substantial amount, known as the *interest spread.* In a financial crisis this spread may soar.

Repurchase agreement and repo rate Nowadays a lot of open market operations are carried out in the form of *repurchase agreements, repos* in brief. The

[9]With continuous compounding we have $p = e^{-i/12}$ so that $i = 12 \ln p^{-1} = 0.03004$ when $p = 0.9975$.

[10]In spite of its name, the Federal Funds Rate is not an interest rate charged by the U.S. central bank but a weighted average of the short-term interest rates commercial banks in the U.S. charge each other on overnight loans on an uncollateral basis.

central bank announces a specific interest rate at which it is willing to buy short-term government bonds from a commercial bank with the agreement that the bank buys back the bonds after a week, say, at a pre-agreed price such that the implied annualized interest rate on this loan equals the announced interest rate, the *repo rate*. This government bond serves as a collateral in the sense that if the bank defaults, the central bank has the bond.

In a *reverse repo* the buy and buy back roles of the two parties are reversed.

The zero lower bound on the nominal interest rate In recessions, when the central bank attempts to stimulate aggregate demand by lowering the policy rate, i, it may reach a point where no further lowering is possible no matter how much money supply is increased. This point is attained when $i = 0$, the "zero lower bound". Agents would prefer holding cash at zero interest rather than short-term bonds at negative interest. That is, the "=" in the equilibrium condition (16.4), or its equivalent, (16.5), should be replaced by "$\geq$" or, equivalently, $L(Y, i)$ should at $i = 0$ be interpreted as a "set-valued function". Strictly speaking the lower bound is slightly below zero because the alternative to holding bonds is holding cash which gives zero interest but involves costs of storing, insuring, and transporting.

Monetary policy and the implications of the zero lower bound (or rather the slightly less than zero lower bound) are explored later in this book.

16.5 Key questions in monetary theory and policy

Some of the central questions in monetary theory and policy are:

1. How, and through what channels, do changes in the *level* of the money supply (in the M_0 sense, say), or the *growth rate* of the money supply affect (a) the real variables in the economy (resource allocation), and (b) the price level and the rate of inflation?

2. How do the effects of money supply movements depend on whether they occur through open-market operations or through the financing of budget deficits?

3. How do the effects depend on the state of the economy with respect to capacity utilization?

4. How can monetary policy be designed to stabilize the purchasing power of money and optimize the liquidity services to the inhabitants?

5. How can monetary policy be designed to stabilize the economy and dampen business cycle fluctuations?
6. Do rational expectations rule out persistent real effects of changes in the money supply?
7. Is hyperinflation always the result of an immense growth in the money supply or can hyperinflation be generated by self-fulfilling expectations?
8. What kind of regulation of commercial banks is conducive to a smooth functioning of the credit system and reduced risk of a financial crisis?

As an approach to some of these issues, we will in the next chapter consider a neoclassical monetary model by Sidrauski (1967). In this model money enters as a separate argument in the utility function. The model has been applied to the study of long-run aspects like the issues 1, 4, and 7 above. The model is less appropriate, however, for short- and medium-run issues such as 3, 5, and 8 in the list.

16.6 Literature notes

In the *Arrow-Debreu model*, the basic microeconomic general equilibrium model, there is assumed to exist a *complete set of markets*. That is, there is a market for each "contingent commodity", by which is meant that there are as many markets as there are possible combinations of physical characteristics of goods, dates of delivery, and "states of nature" that may prevail. In such an fictional world any agent knows for sure the consequences of the choices made. All trades can be made once for all and there will thus be no need for any money holding (Arrow and Hahn, 1971).

For the case of incomplete markets, Kiyotaki and Wright (JPE, 1989) and Trejos and Wright (JMCB, 1993) develop a microeconomic theory of how intrinsically valueless notes can obtain the role as a generally accepted means of exchange.

For a detailed account of the different ways of modelling money demand in macroeconomics, the reader is referred to, e.g., Walsh (2003). Concerning "money in the production function", see Mankiw and Summers (1986).

16.7 Exercises

Chapter 19

The theory of effective demand

> In essence, the "Keynesian revolution" was a shift of emphasis from one type of short-run equilibrium to another type as providing the appropriate theory for actual unemployment situations.
>
> −Edmund Malinvaud (1977), p. 29.

In this and the following chapters the focus is shifted from long-run macroeconomics to short-run macroeconomics. The long-run models concentrated on factors of importance for the economic evolution over a time horizon of at least 10-15 years. With such a horizon the supply side (think of capital accumulation, population growth, and technological progress) is the primary determinant of cumulative changes in output and consumption − the trend. Mainstream macroeconomists see the demand side and monetary factors as of key importance for the *fluctuations* of output and employment about the trend. In a long-run perspective these fluctuations are of only secondary quantitative importance. The preceding chapters have chiefly ignored them. But within shorter horizons, fluctuations are the focal point and this brings the demand-side, monetary factors, market imperfections, nominal rigidities, and expectation errors to the fore. The present and subsequent chapters deal with the role of these short- and medium-run factors for the failure of the laissez-faire market economy to ensure full employment and for the possibilities of active macro policies as a means to improve outcomes.

This chapter introduces building blocks of Keynesian theory of the short run. By "Keynesian theory" we mean a macroeconomic framework that (a) aims at understanding "what determines the actual employment of the available resources",[1] including understanding why mass unemployment arises from time to time, and (b) in this endeavor ascribes a primary role to aggregate demand. Whether a particular building block in this framework comes from Keynes himself, post-war Keynesians, or "new" Keynesians of some sort is not our concern.

[1] Keynes, 1936, p. 4.

We present the basic concept of *effective demand* and compare with the pre-Keynesian (Walrasian) macroeconomic theory which did not distinguish systematically between ex ante demand and supply on the one hand and actual transactions on the other. Attention to this distinction leads to a refutation of *Say's law*, the doctrine that "supply creates its own demand". Next we present some microfoundation for the notion of *nominal price stickiness.* In particular the *menu cost theory* is discussed. We also address the conception of "abundant capacity" as the prevailing state of affairs in an industrialized market economy.

19.1 Stylized facts about the short run

The idea that prices of most goods and services are sticky in the short run rests on the empirical observation that in the short run firms in the manufacturing and service industries typically let output do the adjustment to changes in demand while keeping prices unchanged. In industrialized societies firms are able to do that because under "normal circumstances" there is "abundant production capacity" available in the economy. Three of the most salient short-run features that arise from macroeconomic time series analysis of industrialized market economies are the following (cf. Blanchard and Fischer, 1989, Christiano et al., 1999):

1) Shifts in aggregate demand (induced by sudden changes in the state of confidence, exports, fiscal or monetary policy, or other events) are largely accommodated by changes in quantities rather than changes in nominal prices − *nominal price insensitivity.*

2) Even large movements in quantities are often associated with little or no movement in relative prices − *real price insensitivity.* The real wage, for instance, exhibits such insensitivity in the short run.

3) Nominal prices are *sensitive* to general changes in *input costs.*

These stylized facts pertain to final goods and services. It is not the case that *all* nominal prices in the economy are in the short run insensitive vis-a-vis demand changes. One must distinguish between production of most final goods and services on the one hand and production of primary foodstuff and raw materials on the other. This leads to the associated distinction between "cost-determined" and "demand- determined" prices.

Final goods and services are typically differentiated goods (imperfect substitutes). Their production takes place under conditions of imperfect competition. As a result of existing reserves of production capacity, generally speaking, the *production is elastic w.r.t. demand.* A shift in demand tends to be met by a change in production rather than price. The price changes which do occur are mostly a response to general changes in costs of production. Hence the name "cost-determined" prices.

For *primary foodstuff* and many *raw materials* the situation is different. To increase the supply of most agricultural products requires considerable time. This is also true (though not to the same extent) with respect to mining of raw materials as well as extraction and transport of crude oil. When production is *inelastic* w.r.t. demand in the short run, an increase in demand results in a diminution of stocks and a rise in price. Hence the name "demand-determined prices". The price rise may be enhanced by a speculative element: temporary hoarding in the expectation of further price increases. The price of oil and coffee − two of the most traded commodities in the world market − fluctuate a lot. Through the channel of *costs* the changes in these demand-determined prices spill over to the prices of final goods. Housing construction is time consuming and is also an area where, apart from regulation, demand-determined prices is the rule in the short run.

In industrialized economies manufacturing and services are the main sectors, and the general price level is typically regarded as cost-determined rather than demand determined. Two further aspects are important. First, many wages and prices are set in nominal terms by *price setting agents* like craft unions and firms operating in imperfectly competitive output markets. Second, these wages and prices are in general deliberately kept unchanged for some time even if changes in the environment of the agent occurs; this aspect, possibly due to pecuniary or non-pecuniary costs of changing prices, is known as *nominal price stickiness.* Both aspects have vast consequences for the functioning of the economy as a whole compared with a regime of perfect competition and flexible prices.

Note that *price insensitivity* just refers to the sheer observation of absence of price change in spite of changes in the "environment" − as in the context of facts 1) and 2) above. *Price stickiness* refers to more, namely that prices do not move quickly enough to clear the market in the short run. While price stickiness is in principle a matter of degree, the term includes the limiting case where prices are entirely "fixed" over the period considered − the case of *price rigidity.*

19.2 A simple short-run model

The simple model presented below is close to what Paul Krugman named the *World's Smallest Macroeconomic Model.*[2] The model is crude but nevertheless useful in at least three ways:

[2]Krugman (1999). Krugman tells he learned the model back in 1975 from Robert Hall. As presented here there is an inspiration from Barro and Grossman (1971).

- the model demonstrates the fundamental difference in the *functioning* of an economy with fully flexible prices and one with sticky prices;
- by addressing spillovers across markets, the model is a suitable point of departure for a definition of the Keynesian concept of *effective demand;*
- the model displays the logic behind the Keynesian *refutation* of *Say's law.*

19.2.1 Elements of the model

We consider a monetary closed economy which produces a consumption good. There are three sectors in the economy, a production sector, a household sector, and a public sector with a consolidated government/central bank. Time is discrete. There is a *current period*, of length a month or a quarter of a year, say, and "the future", compressing the next period and onward. Labor is the only input in production. To simplify notation, the model presents its story as if there is just one representative household and one representative firm owned by the household, but the reader should of course imagine that there are numerous agents, all alike, of each kind.

The production function has CRS,

$$Y = AN, \qquad A > 0, \tag{19.1}$$

where Y is aggregate output of a consumption good which is perishable and therefore cannot be stored, A is a technology parameter and N is aggregate employment in the current period. In short- and medium-run macroeconomics the tradition is to use N to denote labor input ("number of hours"), while L is typically used for liquidity demand, i.e., money demand. We follow this tradition.

The price of the consumption good in terms of money, i.e., the *nominal* price, is P. The wage rate in terms of money, the *nominal* wage, is W. We assume that the representative firm maximizes profit, taking these current prices as given. The nominal profit, possibly nil, is

$$\Pi = PY - WN. \tag{19.2}$$

There is free exit from the production sector in the sense that the representative firm can decide to produce nothing. Hence, an equilibrium with positive production requires that profits are non-negative.

The representative household supplies labor inelastically in the amount $\bar{N}$ and receives the profit obtained by the firm, if any. The household demands the consumption good in the amount C^d in the current period (since we want to allow cases of non-market clearing, we distinguish between consumption *demand,*

C^d, and realized consumption, C. Current income not consumed is saved for the future. The output good cannot be stored and there is no loan market. Or we might say that there will exist no interest rate at which intertemporal exchange could be active (neither in the form of a loan market or a market for ownership rights to the firms' profits, if any, in the future). This is because all households are alike, and firms have no use for funding. The only asset on hand for saving is *fiat money* in the form of currency in circulation. Until further notice the money stock is constant.

The preferences of the household are given by the utility function,

$$U = \ln C^d + \beta \ln \frac{\hat{M}}{P^e}, \qquad 0 < \beta < 1, \tag{19.3}$$

where $\hat{M}$ is the amount of money transferred to "the future", and P^e is the expected future price level. The utility discount factor β (equal to $(1+\rho)^{-1}$ if ρ is the utility discount rate) reflects "patience".

Consider the household's choice problem. Facing P and W and expecting that the future price level will be P^e, the household chooses C^d and $\hat{M}$ to maximize U s.t.

$$PC^d + \hat{M} = M + WN + \Pi \equiv B, \qquad N \leq N^s = \bar{N}. \tag{19.4}$$

Here, $M > 0$ is the stock of money held at the beginning of the current period and is predetermined. The actual employment is denoted N and equals the minimum of the amount of employment offered by the representative firm and the labor supply $\bar{N}$ (the principle of voluntary trade). The sum of initial financial wealth, M, and nominal income, $WN + \Pi$, constitutes the budget, B.[3] Payments occur at the end of the period. Nominal financial wealth at the beginning of the next period is $\hat{M} = M + WN + \Pi - PC^d$, i.e., the sum of initial financial wealth and planned saving where the latter equals $WN + \Pi - PC^d$. The benefit obtained by transferring $\hat{M}$ depends on the expected purchasing power of $\hat{M}$, hence it is $\hat{M}/P^e$ that enters the utility function. (Presumably, the household has expectations about real labor and profit income also in the future. But these future incomes are assumed given. So there is no role for changed expectations about the future.)

Substituting $\hat{M} = B - PC^d$ into (19.3), we get the first-order condition

$$\frac{dU}{dC^d} = \frac{1}{C^d} + \beta \frac{P^e}{B - PC^d}(-\frac{P}{P^e}) = 0,$$

[3] As time is discrete, expressions like $M + WN + \Pi$ are legitimate. Although it is meaningless to add a stock and a flow (since they have different denominations), the sum $M + WN + \Pi$ should be interpreted as $M + (WN + \Pi)\Delta t$, where Δt is the period length. With the latter being the time unit, we have $\Delta t = 1$.

which gives

$$PC^d = \frac{1}{1+\beta}B. \tag{19.5}$$

We see that the marginal (= average) propensity to consume is $(1+\beta)^{-1}$, hence inversely related to the patience parameter β. The planned stock of money to be held at the end of the period is

$$\hat{M} = (1 - \frac{1}{1+\beta})B = \frac{\beta}{1+\beta}B.$$

So, the expected price level, P^e, in the future does not affect the demands, C^d and $\hat{M}$. This is a special feature caused by the additive-logarithmic specification of the utility function in (19.3). Indeed, with this specification the substitution and income effects on current consumption of a change in the expected real gross rate of return, $(1/P^e)/(1/P)$, on saving exactly offset each other. And there is no wealth effect on current consumption from a change in the expected rate of return because there is no channel for (or interest in) intertemporal transfer of purchasing power.

Inserting (19.4) and (19.2) into (19.5) gives

$$C^d = \frac{B}{P(1+\beta)} = \frac{M + WN + \Pi}{P(1+\beta)} = \frac{\frac{M}{P} + Y}{1+\beta}, \tag{19.6}$$

In our simple model output demand is the same as the consumption demand C^d. So *clearing* in the output market, in the sense of equality between demand and actual output, requires $C^d = Y$. So, *if* this clearing condition holds, substituting into (19.6) gives the relationship

$$Y = \frac{M}{\beta P}. \tag{19.7}$$

This is only a *relationship* between Y and P, not a solution for any of them since both are endogenous variables so far. Moreover, the relationship is *conditional on clearing* in the output market.

We have assumed that agents take prices as given when making their demand and supply decisions. But we have said nothing about whether nominal prices are flexible or rigid as seen from the perspective of the system as a whole.

19.2.2 The case of competitive markets with fully flexible W and P

What Keynes called "classical economics" is nowadays also often called "Walrasian macroeconomics" (sometime just "pre-Keynesian macroeconomics"). In this theoretical tradition both wages and prices are assumed fully flexible and all markets perfectly competitive.

Firms' ex ante output supply conditional on a hypothetical wage-price pair (W, P) and the corresponding labor demand will be denoted Y^s and N^d, respectively. As we know from microeconomics, the pair (Y^s, N^d) need not be unique, it can easily be a "set-valued function" of (W, P). Moreover, with constant returns to scale in the production function, the range of this function may for certain pairs (W, P) include (∞, ∞).

The distinguishing feature of the Walrasian approach is that wages and prices are assumed fully flexible. Both W and P are thought to adjust immediately so as to clear the labor market and the output market like in a centralized auction market. Clearing in the labor market requires that W and P are adjusted so that actual employment, N, equals labor supply, N^s, which is here inelastic at the given level $\bar{N}$. So

$$N = N^s = \bar{N} = N^d, \tag{19.8}$$

where the last equality indicates that this employment level is willingly demanded by the firms.

We have assumed a constant-returns-to-scale production function (19.1). Hence, the clearing condition (19.8) requires that firms have zero profit. In turn, by (19.1) and (19.2), zero profit requires that the real wage equals labor productivity:

$$\frac{W}{P} = A. \tag{19.9}$$

With clearing in the labor market, output must equal full-employment output,

$$Y = A\bar{N} \equiv Y^f = Y^s, \tag{19.10}$$

where the superscript "f" stands for "full employment", and where the last equality indicates that this level of output is willingly supplied by the firms. For this level of output to match the demand, C^d, coming from the households, the price level must be

$$P = \frac{M}{\beta Y^f} \equiv P^c, \tag{19.11}$$

in view of (19.7) with $Y = Y^f$. This price level is the *classical equilibrium price*, hence the superscript "c". Substituting into (19.9) gives the *classical equilibrium wage*

$$W = AP^c \equiv W^c. \tag{19.12}$$

For general equilibrium we also need that the desired money holding at the end of the period equals the available money stock. By *Walras' law* this equality follows automatically from the household's *Walrasian* budget constraint and clearing in the output and labor markets. To see this, note that the *Walrasian* budget constraint is a *special case* of the budget constraint (19.4), namely the case

$$PC^d + \hat{M} = M + WN^s + \Pi^c, \tag{19.13}$$

where Π^c is the notional profit associated with the hypothetical production plan (Y^s, N^d), i.e.,

$$\Pi^c \equiv PY^s - WN^d. \tag{19.14}$$

The Walrasian budget constraint thus *imposes* replacement of the term for *actual* employment, N, with the households' desired labor supply, $N^s(= \bar{N})$. It also *imposes* replacement of the term for *actual* profit, Π, with the hypothetical profit Π^c ("c" for "classical") calculated on the basis of the firms' aggregate production plan (Y^s, N^d).

Now, let the Walrasian auctioneer announce an arbitrary price vector $(W, P, 1)$, with $W > 0$, $P > 0$, and 1 being the price of the numeraire, money. Then the values of excess demands add up to

$$\begin{aligned}
& W(N^d - N^s) + P(C^d - Y^s) + \hat{M} - M \\
= \ & WN^d - PY^s + PC^d + \hat{M} - M - WN^s \quad \text{(by rearranging)} \\
= \ & WN^d - PY^s + \Pi^c \quad \text{(by (19.13))} \\
= \ & WN^d - PY^s + \Pi^c \equiv 0. \qquad \text{(from definition of } \Pi^c \text{ in (19.14))}
\end{aligned}$$

This exemplifies *Walras' law,* saying that with Walrasian budget constraints the aggregate value of excess demands is *identically* zero. Walras' law reflects that when households satisfy their Walrasian budget constraint, then as an arithmetic necessity the economy as a whole has to satisfy an aggregate budget constraint for the period in question. It follows that the equilibrium condition $\hat{M} = M$ is ensured as soon as there is clearing in the output and labor markets. And more generally: if there are n markets and $n-1$ of these clear, so does the n'th market.

Consequently, when $(W, P) = (W^c, P^c)$, all markets clear in this flexwage-flexprice economy with perfect competition and a representative household with the "endowment"-pair $(M, \bar{N})$. Such a state of affairs is known as a *classical* or *Walrasian equilibrium.*[4] A key feature is expressed by (19.8) and (19.10): output and employment are *supply-determined*, i.e., determined by the supply of production factors, here labor.

The intuitive mechanism behind this equilibrium is the following adjustment process. Imagine that in an ultra-short sub-period $W/P - A \neq 0$. In case $W/P - A > 0$ (< 0), there will be excess supply (demand) in the labor market. This drives

[4] To underline its one-period nature, it may be called a Walrasian *short-run* or a Walrasian *temporary equilibrium.*

W down (up). Only when $W/P = A$ and full employment obtains, can the system be at rest. Next imagine that $P - P^c \neq 0$. In case $P - P^c > 0$ (< 0), there is excess supply (demand) in the output market. This drives P down (up). Again, only when $P = P^c$ *and* $W/P = A$ (whereby $W = W^c$), so that the output market clears under full employment, will the system be at rest.

This adjustment process is fictional, however. Outside equilibrium the Walrasian supplies and demands, which supposedly drive the adjustment, are artificial constructs. Being functions only of initial resources and price signals, the Walrasian supplies and demands are mutually inconsistent outside equilibrium and can therefore not tell what quantities will be traded during an adjustment process. The story needs a considerable refinement unless one is willing to let the mythical "Walrasian auctioneer" enter the scene and bring about adjustment toward the equilibrium prices without allowing trade until these prices are found.

Anyway, assuming that Walrasian equilibrium has been attained, by *comparative statics* based on (19.11) and (19.12) we see that in the classical regime: (a) P and W are proportional to M; (b) output is at the unchanged full-employment level whatever the level of M. This is the *neutrality of money* result of classical macroeconomics.

The neutrality result also holds when we consider an actual change in the money stock at the beginning of the period. Suppose the government/central bank decides a lump-sum transfer to the households in the total amount $\Delta M > 0$ at the beginning of the period. There is no taxation, and so this implies a government budget deficit which is thus fully financed by money issue.[5] So (19.4) is replaced by

$$PC^d + \hat{M} = M + \Delta M + W\bar{N} + \Pi^c. \tag{19.15}$$

If we replace M in the previous formulas by $M' \equiv M + \Delta M$, we see that money neutrality still holds. As *saving* is income minus consumption, there is now positive nominal private saving of size $S^p = \Delta M + W\bar{N} + \Pi^c - PC^d = M' - M = \Delta M$. On the other hand the government dissaves, in that its saving is $S^g = -\Delta M$, where ΔM is the government budget deficit. So national saving is and remains $S \equiv S^p + S^g = 0$ (it *must* be nil since there are no durable produced goods).

[5] Within the model this is in fact the only way to increase the money stock. As money is the only asset in the economy, a change in the money stock can not be brought about through open-market operations where the central bank buys or sells another financial asset. So $\Delta M > 0$ represents a combination of fiscal policy (the transfers) and monetary policy (the financing of the transfers by money).

19.2.3 The case of imperfect competition and W and P fixed in the short run

In standard Keynesian macroeconomics nominal wages are considered predetermined in the short run, fixed in advance by wage bargaining between workers and employers (or workers' unions and employers' unions). Those who end up unemployed in the period do not try to – or are not able to – undercut those employed, at least not in the current period.

Likewise, nominal prices are set in advance by firms facing downward-sloping demand curves. It is understood that there is a large spectrum of differentiated products, and Y and C are composites of these. This heterogeneity ought of course be visible in the model – and it will become so in Section 19.3. But at this point the model takes an easy way out and ignores the involved aggregation issue.

Let W in the current period be given at the level $\bar{W}$. Because firms have market power, the profit-maximizing price involves a mark-up on marginal cost, $\bar{W}N/Y = \bar{W}/A$ (which is also the average cost). We assume that the price setting occurs under circumstances where the chosen mark-up becomes a *constant* $\mu > 0$, so that

$$P = (1+\mu)\frac{\bar{W}}{A} \equiv \bar{P}. \tag{19.16}$$

While $\bar{W}$ is considered exogenous (not determined within the model), $\bar{P}$ is endogenously determined by the given $\bar{W}$, A, and μ. There are barriers to entry in the short run.

Because of the fixed wage and price, the distinction between *ex ante* (also called *planned* or *intended*) demands and supplies and the *ex post* carried out purchases and sales are now even more important than before. This is because the different markets may now also *ex post* feature excess demand or excess supply (to be defined more precisely below). According to the principle that no agent can be forced to trade more than desired, the actual amount traded in a market must equal the minimum of demand and supply. So in the output market and the labor market the actual quantities traded will be

$$Y = \min(Y^d, Y^s) \quad \text{and} \tag{19.17}$$

$$N = \min(N^d, N^s), \tag{19.18}$$

respectively, where the superscripts "d" and "s" are now used for demand and supply in a *new* meaning to be defined below. This principle, that the short side of the market determines the traded quantity, is known as the *short-side rule.* The other side of the market is said to be *quantity rationed* or just *rationed* if there is discrepancy between Y^d and Y^s. In view of the produced good being non-storable, intended inventory investment is ruled out. Hence, the firms try to avoid producing more than can be sold. In (19.17) we have thus identified the traded quantity with the produced quantity, Y.

But what exactly do we mean by "demand" and "supply" in this context where market clearing is not guaranteed? We mean what is appropriately called the *effective demand* and the *effective supply* ("effective" in the meaning of "operative" in the market, though possibly frustrated in view of the short-side rule). To make these concepts clear, we need first to define an agent's *effective* budget constraint:

DEFINITION 1 An agent's (typically a household's) *effective budget constraint* is the budget constraint conditional on the perceived price and quantity signals from the markets.

It is the last part, "and quantity signals from the markets", which is not included in the concept of a Walrasian budget constraint. The perceived quantity signals are in the present context a) the *actual* employment constraint faced by the household and b) the profit expected to be received from the firms and determined by their *actual* production and sales. So the household's effective budget constraint is given by (19.4). In contrast, the Walrasian budget constraint is not conditional on quantity signals from the markets but only on the "endowment" $(M, \bar{N})$ and the perceived price signals and profit.

DEFINITION 2 An agent's *effective demand* in a given market is the amount the agent *bids for* in the market, conditional on the perceived price and quantity signals that constrains the bidding.

By "bids for" is meant that the agent is both *able* to buy the amount in question and *wishes* to buy it, given the effective budget constraint. Summing over all potential buyers, we get the *aggregate effective demand* in the market.

DEFINITION 3 An agent's *effective supply* in a given market is the amount the agent *offers for sale* in the market, conditional on perceived price and quantity signals that constrains the offer.

By "offers for sale" is meant that the agent is both *able* to bring that amount to the market and *wishes* to sell it, given the set of opportunities available. Summing over all potential sellers, we get the *aggregate effective supply* in the market.

When $P = \bar{P}$, the aggregate effective output demand, Y^d, is the same as households' consumption demand given by (19.6) with $P = \bar{P}$, i.e.,

$$Y^d = C^d = \frac{\frac{M}{\bar{P}} + Y}{1 + \beta}. \tag{19.19}$$

In view of the inelastic labor supply, households' aggregate effective labor supply is simply

$$N^s = \bar{N}.$$

Firms' aggregate effective output supply is

$$Y^s = Y^f \equiv A\bar{N}. \tag{19.20}$$

Indeed, in the aggregate the firms are *not able* to bring more to the market than full-employment output , Y^f. And every individual firm is not able to bring to the market than what can be produced by "its share" of the labor force. On the other hand, because of the constant marginal costs, every unit sold at the preset price adds to profit. The firms are therefore happy to satisfy any output demand forthcoming $-$ which is in practice testified by a lot of sales promotion.

Firms' aggregate effective demand for labor is constrained by the perceived output demand, Y^d, because the firm would loose by employing more labor. Thus,

$$N^d = \frac{Y^d}{A}. \tag{19.21}$$

By the short-side rule (19.17), combined with (19.20), follows that actual aggregate output (equal to the quantity traded) is

$$Y = \min(Y^d, Y^f) \leqq Y^f.$$

So the following three mutually exclusive cases exhaust the possibilities regarding aggregate output:

$$\begin{aligned} Y &= Y^d < Y^f \quad \text{(the Keynesian regime),} \\ Y &= Y^f < Y^d \quad \text{(the repressed inflation regime),} \\ Y &= Y^d = Y^f \quad \text{(the border case).} \end{aligned}$$

The Keynesian regime: $Y = Y^d < Y^f$.

In this regime we can substitute $Y = Y^d$ into (19.19) and solve for Y:

$$Y = Y^d = \frac{M}{\beta\bar{P}} \equiv Y^k < Y^f \equiv \frac{M}{\beta P^c} = Y^s. \tag{19.22}$$

where we have denoted the resulting output Y^k (the superscript "k" for "Keynesian"). The inequality in (19.22) is required by the definition of the Keynesian regime, and the identity comes from (19.11). Necessary and sufficient for the inequality is that $\bar{P} > P^c \equiv W^c/A$. In view of (19.16), the economy is thus in the Keynesian regime if and only if

$$\bar{W} > W^c/(1+\mu). \tag{19.23}$$

Since $Y < Y^s$ in this regime, we may say there is "excess supply" in the output market or, with a perhaps better term, there is a "buyers' market" situation (sale less than desired). The reservation regarding the term "excess supply" is due to the fact that we should not forget that $Y - Y^s < 0$ is a completely voluntary state of affairs on the part of the price-setting firms.

From (19.1) and the short-side rule now follows that actual employment will be

$$N = N^d = \frac{Y}{A} = \frac{M}{A\beta\bar{P}} < \bar{N} = N^s. \tag{19.24}$$

Also the labor market is thus characterized by "excess supply" or a "buyers' market" situation. Profits are $\Pi = \bar{P}Y - \bar{W}N = (1 - \bar{W}/(\bar{P}A))\bar{P}Y = (1-(1+\mu)^{-1})\beta^{-1}M > 0$, where we have used, first, $Y = AN$, then the price setting rule (19.16), and finally (19.22).

This solution for (Y, N) is known as a *Keynesian equilibrium* for the current period. It is named an *equilibrium* because the system is "at rest" in the following sense: (a) agents do the best they can given the constraints (which include the preset prices and the quantities offered by the other side of the market); and (b) the chosen actions are *mutually compatible* (purchases and sales match). The term equilibrium is here not used in the Walrasian sense of market clearing through instantaneous price adjustment but in the sense of a *Nash equilibrium* conditional on perceived price and quantity signals. To underline its temporary character, the equilibrium may be called a Keynesian *short-run* (or *temporary*) equilibrium. The flavor of the equilibrium is *Keynesian* in the sense that there is unemployment and at the same time it is aggregate demand in the output market, not the real wage, which is the binding constraint on the employment level. A higher propensity to consume (lower discount factor β) results in higher aggregate demand, Y^d, and thereby a higher equilibrium output, Y^k. In contrast, a lower real wage due to either a higher mark-up, μ, or a lower marginal (= average) labor productivity, A, does *not* result in a higher Y^k. On the contrary, Y^k becomes *lower*, and the causal chain behind this goes via a higher $\bar{P}$, cf. (19.16) and (19.22). In fact, the given real wage, $\bar{W}/\bar{P} = A/(1+\mu)$, is consistent with unemployment as well as full employment, see below. It is the sticky nominal *price* at an excessive level, caused by a sticky nominal *wage* at an "excessive" level, that makes unemployment prevail through a too low aggregate demand, Y^d. A lower nominal wage would imply a lower $\bar{P}$ and thereby, for a given M, stimulate Y^d and thus raise Y^k.

In brief, the Keynesian regime leads to an equilibrium where output as well as employment are *demand-determined*.

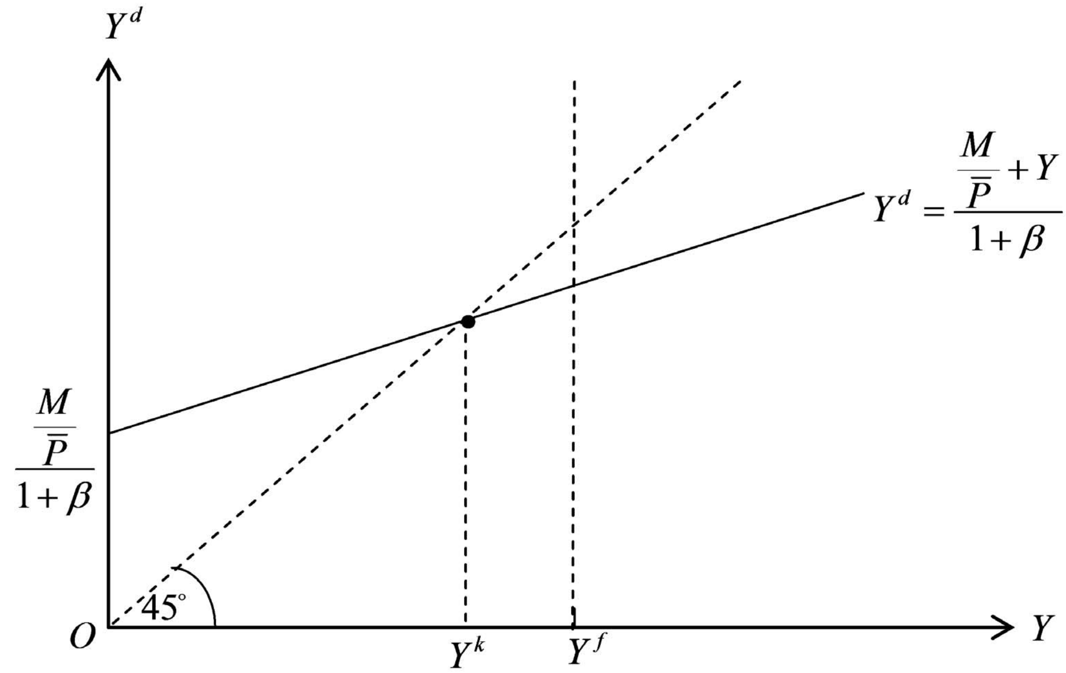

Figure 19.1: The Keynesian regime ($\bar{P} = (1+\mu)\bar{W}/A$, $\bar{W} < W^c/(1+\mu)$, M, and Y^f given).

The "Keynesian cross" and effective demand The situation is illustrated by the "Keynesian cross" in the (Y, Y^d) plane shown in Fig. 19.1, where $Y^d = C^d = (1+\beta)^{-1}(M/\bar{P} + Y)$. We see the vicious circle: Output is below the full-employment level because of low consumption demand; and consumption demand is low because of the low employment. The economy is in a *unemployment trap*. Even though at Y^k we have $\Pi > 0$ and there are constant returns to scale, the individual firm has no incentive to increase production because the firm already produces as much as it rightly perceives it can sell at its preferred price. We also see that here money is *not neutral*. For a given $W = \bar{W}$, and thereby a given $P = \bar{P}$, a higher M results in higher output and higher employment.

Although the microeconomic background we have alluded to is a specific "market power story" (one with differentiated goods and downward sloping demand curves), the Keynesian cross in Fig. 19.1 may turn up also for other microeconomic settings. The key point is the fixed $\bar{P} > P^c$ and fixed $\bar{W} < A\bar{P}$.

The fundamental difference between the Walrasian and the present framework is that the latter allows trade outside Walrasian equilibrium. In that situation the households' consumption demand depends *not* on how much labor the households would *prefer* to sell at the going wage, but on how much they are *able* to sell, that is, on a *quantity signal* received from the labor market. Indeed, it is the *actual* employment, N, that enters the operative budget constraint, (19.4), not the desired employment as in classical or Walrasian theory.

The repressed-inflation regime: $Y = Y^f < Y^d$.

This regime represents the "opposite" case of the Keynesian regime and arises if and only if the opposite of (19.23) holds, namely

$$\bar{W} < W^c/(1+\mu).$$

In view of (19.16), this inequality is equivalent to $\bar{P} < W^c/A \equiv P^c$. Hence $M/(\beta\bar{P}) > M/(\beta P^c) = Y^f = A\bar{N}$. In spite of the high output demand, the shortage of labor hinders the firms to produce more than Y^f. With $Y = Y^f$, output demand, which in this model is always the same as consumption demand, C^d, is, from (19.6),

$$Y^d = \frac{\frac{M}{P} + Y^f}{1+\beta} > Y = Y^s = Y^f. \tag{19.25}$$

As before, effective output supply, Y^s, equals full-employment output, Y^f.

The new element here in that firms perceive a demand level in excess of Y^f. As the real-wage level does not deter profitable production, firms would thus prefer to employ people up to the point where output demand is satisfied. But in view of the short side rule for the labor market, actual employment will be

$$N = N^s = \bar{N} < N^d = \frac{Y^d}{A}.$$

So there is excess demand in both the output market and the labor market. Presumably, these excess demands generate pressure for wage and price increases. By assumption, these potential wage and price increases do not materialize until possibly the next period. So we have a *repressed-inflation equilibrium* $(Y, N) = (Y^f, \bar{N})$, although possibly short-lived.

Fig. 19.2 illustrates the repressed-inflation regime. In the language of the microeconomic theory of quantity rationing, consumers are quantity rationed in the goods market, as realized consumption $= Y = Y^f < Y^d =$ consumption demand. Firms are quantity rationed in the labor market, as $N < N^d$. This is the background for the parlance that in the repressed inflation regime, output and employment are not demand-determined but *supply-determined*. Both the output market and the labor market are *sellers' markets* (purchases less than desired). Presumably, the repressed inflation regime will not last long unless there are wage and price controls imposed by the government, as for instance may be the case for an economy in a war situation.[6]

[6] As another example of repressed inflation (simultaneous excess demand for consumption

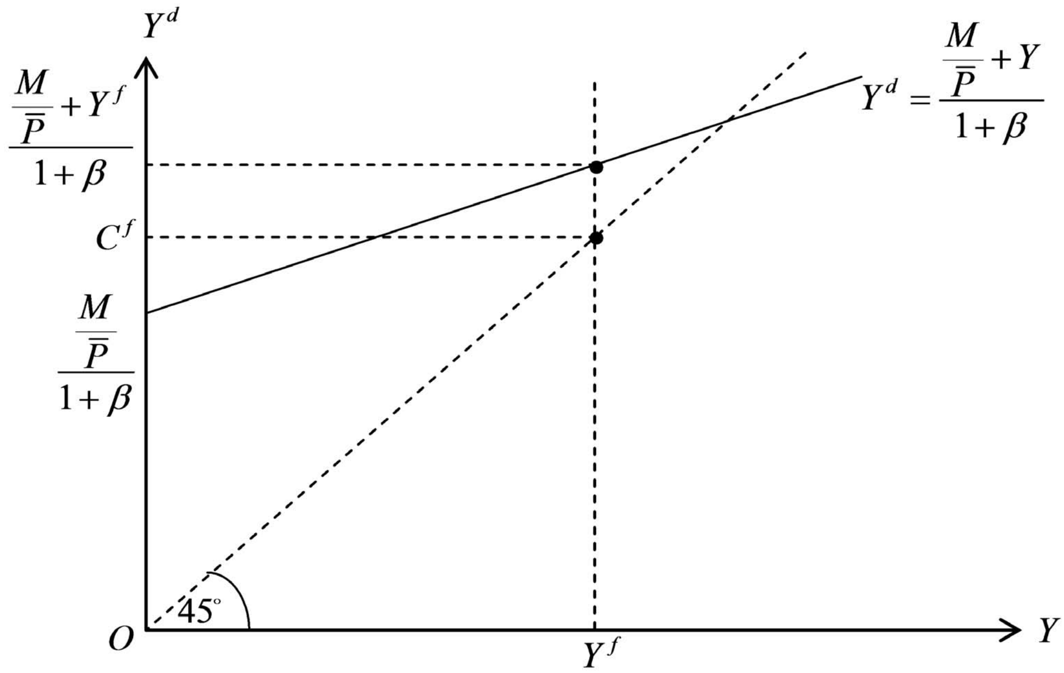

Figure 19.2: The repressed-inflation regime ($\bar{P} = (1 + \mu)\bar{W}/A$, $\bar{W} > W^c/(1 + \mu)$, M, and Y^f given).

The border case between the two regimes: $Y = Y^d = Y^f$.

This case arises if and only if $\bar{W} = W^c/(1+\mu)$, which is in turn equivalent to $\bar{P} = (1+\mu)\bar{W}/A = W^c/A \equiv P^c \equiv M/(\beta Y^f)$. No market has quantity rationing and we may speak of both the output market and the labor market as *balanced markets*.

There are two differences compared with the classical equilibrium, however. The first is that due to market power, there is a wedge between the real wage and the marginal productivity of labor. In the present context, though, where labor supply is inelastic, this does not imply inefficiency but only a higher profit/wage-income ratio than under perfect competition (where the profit/wage-income ratio is zero). The second difference compared with the classical equilibrium is that due to price stickiness, the impact of shifts in exogenous variables will be different. For instance a lower M will here result in unemployment, while in the classical model it will just lower P and W and not affect employment.

In terms of effective demands and supplies Walras' law does not hold

As we saw above, with *Walrasian* budget constraints, the aggregate value of excess demands in the given period is zero for any given price vector, $(W, P, 1)$, with $W > 0$ and $P > 0$. In contrast, with *effective* budget constraints, effective demands and supplies, and the short-side rule, this is no longer so. To see this, consider a pair (W, P) where $W < PA$ and $P \neq P^c \equiv M/(\beta Y^f)$. Such a pair leads to either the Keynesian regime or the repressed-inflation regime. The pair *may*, but need not, equal one of the pairs $(\bar{W}, \bar{P})$ considered above in Fig. 19.1 or 19.2 (we say "need not", because the particular μ-markup relationship between W and P is not needed). We have, *first*, that in both the Keynesian and the repressed-inflation regime, effective output supply equals full-employment output,

$$Y^s = Y^f. \tag{19.26}$$

The intuition is that in view of $W < PA$, the representative firm *wishes* to satisfy any output demand forthcoming but it is only able to do so up to the point of where the availability of workers becomes a binding constraint.

Second, the aggregate value of excess effective demands is, for the considered

goods and labor) we may refer to Eastern Europe before the dissolution of the Soviet Union in 1991. In response to severe and long-lasting rationing in the consumption goods markets, households tended to decrease their labor supply (Kornai, 1979). This example illustrates that if labor supply is elastic, the *effective* labor supply may be less than the Walrasian labor supply due to spillovers from the output market.

price vector $(W, P, 1)$, equal to

$$\begin{aligned}
& W(N^d - N^s) + P(C^d - Y^s) + \hat{M} - M \\
= \;& W(N^d - \bar{N}) + PC^d + \hat{M} - M - PY^f \\
= \;& W(N^d - \bar{N}) + WN + \Pi - PY^f \quad \text{(by (19.4))} \\
= \;& W(N^d - \bar{N}) + PY - PY^f \quad \text{(by (19.2))} \\
= \;& W(N^d - \bar{N}) + P(Y - Y^f) \begin{cases} < 0 \text{ if } P > M/(\beta Y^f), \text{ and} \\ > 0 \text{ if } P < M/(\beta Y^f) \text{ and } W < PA. \end{cases}
\end{aligned} \tag{19.27}$$

The aggregate value of excess effective demands is thus not identically zero. As expected, it is negative in a Keynesian equilibrium and positive in a repressed-inflation equilibrium.[7] The reason that Walras' law does not apply to effective demands and supplies is that outside Walrasian equilibrium some of these demands and supplies are not realized in the final transactions.

This takes us to Keynes' refutation of Say's law and thereby what Keynes and others regarded as the core of his theory.

Say's law and its refutation

The classical principle "supply creates its own demand" (or "income is automatically spent on products") is named Say's law after the French economist and business man Jean-Baptiste Say (1767-1832). In line with other classical economists like David Ricardo and John Stuart Mill, Say maintained that although mismatch between demand and production can occur, it can only occur in the form of excess production in some industries at the same time as there is excess demand in other industries.[8] General overproduction is impossible. Or, by a classical catchphrase:

> Every offer to sell a good implies a demand for some other good.

By "good" is here meant a produced good rather than just any traded article, including for instance money. Otherwise Say's law would be a platitude (a simple implication of the definition of trade). So, interpreting "good" to mean a produced good, let us evaluate Say's law from the point of view of the result (19.27). We first subtract $W(N^d - N^s) = W(N^d - \bar{N})$ on both sides of (19.27), then insert (19.26) and rearrange to get

$$P(C^d - Y) + \hat{M} - M = 0, \tag{19.28}$$

[7] At the same time, (19.27) together with the general equations $N^d = \bar{N}$ and $Y^s = Y^f$, shows that we have $\hat{M} = M$ in a Keynesian equilibrium (where $Y = C^d$) and $\hat{M} < M$ in a repressed-inflation equilibrium (where $Y = Y^f$).

[8] There were two dissidents at this point, Thomas Malthus (1766–1834) and Karl Marx (1818–1883), two writers that were otherwise not much aggreeing.

for any $P > 0$. Consider the case $W < AP$. In this situation every unit produced and sold is profitable. So any Y in the interval $0 < Y \leq Y^f$ is profitable from the supply side angle. Assume further that $P = \bar{P} > P^c \equiv M/(\beta Y^f)$. This is the case shown in Fig. 19.1. The figure illustrates that aggregate demand *is* rising with aggregate production. So far so well for Say's law. We also see that if aggregate production is in the interval $0 < Y < Y^k$, then C^d $(= Y^d) > Y$. This amounts to excess demand for goods and in effect, by (19.28), excess supply of money. Still, Say's law is not contradicted. But if instead aggregate production is in the interval $Y^k < Y \leq Y^f$, then C^d $(= Y^d) < Y$; now there is *general overproduction.* Supply no longer creates its own demand. There is a general shortfall of demand. By (19.28), the other side of the coin is that when $C^d < Y$, then $\hat{M} > M$, which means excess demand for money. People try to hoard money rather than spend on goods. Both the Great Depression in the 1930s and the Great Recession 2008- can be seen in this light.[9]

The refutation of Say's law does not depend on the market power and constant markup aspects we have adhered to above. All that is needed for the argument is that the agents are price takers within the period. Moreover, the refutation does not hinge on *money* being the asset available for transferring purchasing power from one period to the next. We may imagine an economy where M represents *land* available in limited supply. As land is also a non-produced store of value, the above analysis goes through − with one exception, though. This is that ΔM in (19.15) can no longer be interpreted as a policy choice. Instead, a positive ΔM could be due to discovery of new land.

We conclude that general overproduction is possible, and Say's law is thereby refuted. It might be objected that our "aggregate reply" to Say's law is not to the point since Say had a disaggregate structure with many industries in mind. Considering explicitly a multiplicity of production sectors makes no essential difference, however, as the following example will show.

Many industries* Suppose there is still one labor market, but m industries with production function $y_i = An_i$, where y_i and n_i are output and employment in industry i, respectively, $i = 1, 2, \ldots, m$. Let the preferences of the representative household be given by

$$U = \sum_i \gamma_i \ln c_i + \beta \ln \frac{\hat{M}}{P^e}, \qquad \gamma_i > 0, i = 1, 2, \ldots, m, \quad 0 < \beta < 1.$$

[9] Paul Krugman stated it this way: "When everyone is trying to accumulate cash at the same time, which is what happened worldwide after the collapse of Lehman Brothers, the result is an end to demand [for output], which produces a severe recession" (Krugman, 2009).

In analogy with (19.4), the budget constraint is

$$\sum_i P_i c_i + \hat{M} = B \equiv M + W \sum_i n_i + \sum_i \Pi_i = M + \sum_i P_i y_i,$$

where the last equality comes from

$$\Pi_i = P_i y_i - W n_i.$$

Utility maximization gives $P_i c_i = \gamma_i B/(1+\beta)$.

As a special case, consider $\gamma_i = 1/m$ and $P_i = P$, $i = 1, 2, \ldots, m$. Then

$$c_i = \frac{B/m}{(1+\beta)P}, \tag{19.29}$$

and

$$B = M + P \sum_i y_i \equiv M + PY.$$

Substituting into (19.29), we thus find demand for consumption good i as

$$c_i = \frac{\frac{M/m}{P} + Y/m}{1+\beta} \equiv y^d, \quad \text{for all } i.$$

Let $P > \min\left[W/A, M/(\beta Y^f)\right]$, where $Y^f \equiv A\bar{N}$. It follows that every unit produced and sold is profitable and that

$$my^d = \frac{\frac{M}{P} + Y}{1+\beta} \leq \frac{\frac{M}{P} + Y^f}{1+\beta} < Y^f,$$

where the weak inequality comes from $Y \leq Y^f$ (always) and the strict inequality from $P > M/(\beta Y^f)$.

Now, suppose good 1 is brought to the market in the amount y_1, where $y^d < y_1 < Y^f/m$. Industry 1 thus experiences a shortfall of demand. Will there in turn necessarily be another industry experiencing excess demand? No. To see this, consider the case $y^d < y_i < Y^f/m$ for all i. All these supplies are profitable from a supply side point of view, and enough labor is available. Indeed, by construction the resource allocation is such that

$$my^d < \sum y_i \equiv Y \leq m\bar{y} < Y^f, \tag{19.30}$$

where $\bar{y} = \max\left[y_1, \ldots, y_m\right] < Y^f/m$. This is a situation where people try to save (hoard money) rather than spend all income on produced goods. It is an example of *general overproduction,* thus falsifying Say's law.

In the special case where all $y_i = Y/m$, the situation for each single industry can be illustrated by a diagram as that in Fig. 19.1. Just replace Y^d, Y, Y^k, Y^f, and M in Fig. 19.1 by y^d, Y/m, $Y^k/m \equiv M/(m\beta P)$, Y^f/m, and M/m, respectively.

19.2.4 Short-run adjustment dynamics*

We now return to the aggregate setup. Apart from the border case of balanced markets, we have considered two kinds of "fix-price equilibria", *repressed inflation* and *Keynesian equilibrium.* Many economists consider nominal wages and prices to be less sticky upwards than downwards. So a repressed inflation regime is typically regarded as having little durability (unless there are wage and price controls imposed by a government). It is otherwise with the Keynesian equilibrium. A way of thinking about this is the following.

Suppose that up to the current period full-employment equilibrium has applied: $Y = Y^d = M/(\beta\bar{P}) = Y^f$ and $\bar{P} = (1+\mu)\bar{W}/A = W^c/A \equiv P^c \equiv M/(\beta Y^f)$. Then, for some external reason, at the start of the current period a *rise* in the patience parameter occurs, from β to β', so that the new propensity to save is $\beta^{'}/(1 + \beta') > \beta/(1 + \beta)$. We may interpret this as "precautionary saving" in response to a sudden fall in the general "state of confidence".

Let our "period" be divided into n sub-periods, indexed $i = 0, 1, 2, \ldots, n - 1$, of length $1/n$, where n is "large". At least within the first of these sub-periods, the preset $\bar{W}$ and $\bar{P}$ are maintained and firms produce without having yet realized that aggregate demand will be lower than in the previous period. After a while firms realize that sales do not keep track with production.

There are basically two kinds of reaction to this situation. One is that wages and prices are maintained throughout all the sub-periods, while production is gradually scaled down to the Keynesian equilibrium $Y^k = M/(\beta'\bar{P})$. Another is that wages and prices adjust downward so as to soon reestablish full-employment equilibrium. Let us take each case at a time.

Wage and price stay fixed: Sheer quantity adjustment For simplicity we have assumed that the produced goods are perishable. So unsold goods represent a complete loss. If firms fully understand the functioning of the economy and have model-consistent expectations, they will adjust production per time unit down to the level Y^k as fast as possible. Suppose instead that firms have naive adaptive expectations of the form

$$C^e_{i-1,i} = C_{i-1}, \qquad i = 0, 1, 2, ..., n.$$

This means that the "subjective" expectation, formed in sub-period $i - 1$, of demand next sub-period is that it will equal the demand in sub-period $i - 1$. Let the time-lag between the decision to produce and the observation of the demand correspond to the length of the sub-periods. It is profitable to satisfy demand, hence actual output in sub-period i will be

$$Y_i = C^e_{i-1,i} = C^d_{i-1} = \frac{M/\bar{P}}{1+\beta'} + \frac{Y_{i-1}}{1+\beta'},$$

in analogy with (19.19). This is a linear first-order difference equation in Y_i, with constant coefficients. The solution is (see Math Tools)

$$Y_i = (Y_0 - Y^{*\prime}) \left(\frac{1}{1+\beta'} \right)^i + Y^{*\prime}, \qquad Y^{*\prime} = \frac{M}{\beta' \bar{P}} = Y^k < Y^f. \qquad (19.31)$$

Suppose $\beta' = 0.9$, say. Then actual production, Y_i, converges fast towards the steady-state value Y^k. When $Y = Y^k$, the system is at rest. Fig. 19.x illustrates. Although there is excess supply in the labor market and therefore some downward pressure on wages, the Keynesian presumption is that the workers's side in the labor market generally withstand the pressure.[10]

Fig. 19.x about here (not yet available).

The process (19.31) also applies "in the opposite direction". Suppose, starting from the Keynesian equilibrium $Y = M/(\beta' \bar{P})$, a *reduction* in the patience parameter β' occurs, such that $M/(\beta' \bar{P})$ increases, but still satisfies $M/(\beta' \bar{P}) < Y^f$. Then the initial condition in (19.31) is $Y_0 < Y^{*\prime}$, and the greater propensity to consume leads to an upward quantity adjustment.

Downward wage and price adjustment Several of Keynes' contemporaries, among them A. C. Pigou, maintained that the Keynesian state of affairs with $Y = Y^k < Y^f$ could only be very temporary. Pigou's argument was that a fall in the price level would take place and lead to higher purchasing power of M. The implied stimulation of aggregate demand would bring the economy back to full employment. This hypothetically equilibrating mechanism is known as the "real balance effect" or the "Pigou effect" (after Pigou, 1943).

Does the argument go through? To answer this, we imagine that the time interval between different rounds of wage and price setting is as short as our sub-periods. We imagine the time interval between households' decision making to be equally short. Given the fixed markup μ, an initial fall in the preset $\bar{W}$ is needed to trigger a fall in the preset $\bar{P}$. The new *classical* equilibrium price and wage levels will be

$$P^{c\prime} = \frac{M}{\beta' Y^f} \text{ and } W^{c\prime} = AP^{c\prime}.$$

Both will thus be lower than the original ones − by the same factor as the patience parameter has risen, i.e., the factor β'/β. In line with "classical" thinking, assume that soon after the rise in the propensity to save, the incipient unemployment

[10] Possible explanations of downward wage stickiness are discussed in Chapter 24.

prompts wage setters to reduce $\bar{W}$ and thereby price setters to reduce $\bar{P}$. Let both $\bar{W}$ and $\bar{P}$ after a few rounds be reduced by the factor β'/β. Denoting the resulting wage and price $\bar{W}'$ and $\bar{P}'$, respectively, we then have

$$\bar{W}' = \frac{W^{c\prime}}{1+\mu}, \qquad \bar{P}' = (1+\mu)\frac{\bar{W}'}{A} = \frac{W^{c\prime}}{A} \equiv P^{c\prime} \equiv \frac{M}{\beta' Y^f}.$$

Seemingly, this restores aggregate demand at the full-employment level $Y^d = M/(\beta'\bar{P}') = Y^f$.

While this "classical" adjustment is conceivable in the abstract, Keynesians question its practical relevance for several reasons:

1. Empirically, it seems to be particularly in the downward direction that nominal wages are sticky. And without an initial fall in the nominal wage, the downward wage-price spiral does not get started.

2. If downward wage-price spiral does get started, the implied deflation increases the implicit real interest rate, $(P_t - P_{t+1})/P_{t+1}$. In a more elaborate modeling of consumption and investment, this would tend to dampen aggregate demand rather than the opposite.

3. Additional points, when going a little outside the present simple model, are:

   (a) the monetary base is in reality only a small fraction of financial wealth, and so the real balance effect can not be very powerful unless the fall in the price level is drastic;

   (b) many firms and households have nominal debt, the real value of which would rise, thereby potentially leading to bankruptcies and a worsening of the confidence crisis, thus counteracting a return to full employment.

A clarifying remark. In this context we should be aware that there are two kinds of "price flexibility" to be distinguished: "imperfect" versus "perfect" (or "full") price flexibility. The first kind relates to a *gradual* price process, for instance generated by a wage-price spiral as at item 2 above. The latter kind relates to *instantaneous* and complete price adjustment as with a Walrasian auctioneer. It is the first kind that may be destabilizing rather than the opposite.

19.2.5 Digging deeper

As it stands the above theoretical framework has many limitations. The remainder of this chapter gives an introduction to how the following three problems have been dealt with in the literature:

(i) Price setting should be explicitly modeled, and in this connection there should be an explanation of price stickiness.

(ii) It should be made clear how to come from the existence of many differentiated goods and markets with imperfect competition to aggregate output and income which in turn constitute the environment conditioning the individual agents' actions.

(iii) The analysis has ignored that capital equipment is in practice an additional factor constraining production.

In subsequent chapters we consider additional problems:

(iv) Also wage setting should be explicitly modeled, and in this connection there should be an explanation of wage stickiness.

(v) At least one additional financial asset, an interest-bearing asset, should enter. This will open up for intertemporal trade and for clarifying the primary function of money as a medium of exchange rather than as a store of value.

(vi) The model should include forward-looking decision making and endogenous expectations.

(vii) The model should be truly dynamic with gradual wage and price changes depending on the market conditions and expectations. This should lead to an explanation why wages and prices do not tend to find their market clearing levels relatively fast.

The next section deals with point (i) and (ii), and Section 19.4 with point (iii).

19.3 Price setting and menu costs

The classical theory of perfectly flexible wages and prices and neutrality of money treats wages and prices as if they were prices on assets traded in centralized auction markets. In contrast, the Keynesian conception is that the general price level is a weighted average of millions of individual prices set − and sooner or later reset − in an asynchronous way by price setters in a multitude of markets and localities.

What we need to understand the determination of prices and their sometimes slow response to changed circumstances, is a theory of how agents set prices and decide when to change them and by how much. This brings the objectives and constraints of agents with market power into the picture. So *imperfect competition* becomes a key ingredient of the theory.

19.3.1 Imperfect competition with price setters

Suppose the market structure is one with *monopolistic competition*:

1. There is a given "large" number, m, of firms and equally many (horizontally) differentiated products.
2. Each firm supplies its own differentiated product on which it has a monopoly and which is an imperfect substitute for the other products.
3. A price change by one firm has only a negligible effect on the demand faced by any other firm.

Another way of stating property 3 is to say that firms are "small" so that each good constitutes only a small fraction of the sales in the overall market system. Each firm faces a perceived downward-sloping demand curve and chooses a price which maximizes the firm's expected profit, thus implying a mark-up on marginal costs. There is no perceivable reaction from the firm's (imperfect) competitors. So the monopolistic competition setup abstracts from strategic interaction between the firms and is thereby different from oligopoly.

With respect to assets, so far our framework corresponds to the World's Smallest Macroeconomic Model of Section 19.2 in the sense that there are no commercial banks and no other non-human assets than fiat money.

Price setting firms

In the short run there is a given large number, m, of firms and equally many (horizontally) differentiated products. Firm i has the production function $y_i = An_i^{\alpha}$, where n_i is labor input (raw materials and physical capital ignored).[11] For notational convenience we imagine measurement units are such that $A = 1$. Thereby,

$$n_i = y_i^{1/\alpha}, \quad 0 < \alpha \leq 1, \quad i = 1, 2, \ldots, m, \quad m \text{ “large”}. \tag{19.32}$$

To extend the perspective compared with Section 19.2, the possibility of rising marginal costs ($\alpha < 1$) is now included.

The demand constraint faced by the firm ex ante is perceived by the firm to be

$$y_i = \left(\frac{P_i}{P}\right)^{-\eta} \frac{Y^e}{m} \equiv D(\frac{P_i}{P}, \frac{Y^e}{m}), \qquad \eta > 1, \tag{19.33}$$

[11] The following can be seen as an application of the more general framework with price-setting firms outlined at the end of Chapter 2.

where P_i is the price set by the firm and fixed for some time, P is the "general price level" (taken as given by firm i because it is "small" enough for its price to have any noticeable effect on P), Y^e/m is the expectation (for simplicity the same for all firms) of the position of the demand curve, and η is the (absolute) price elasticity of demand (assumed greater than one since otherwise there is no finite profit maximizing price).[12] The firms' expectation of the position of the demand curve reflects their expectation, Y^e, of the general level of demand in the economy.

Let firm i choose P_i at the end of the previous period with a view to maximization of expected nominal profit in the current period:

$$\max_{P_i} \Pi_i = P_i y_i - W n_i \quad \text{s.t. } (19.33),$$

where W is the going nominal wage, taken as given by the firm. We may substitute (19.32) and the constraint (19.33) into the profit function to get an unconstrained maximization problem which is then solved for P_i. The more intuitive approach, however, is to apply the rule that the profit maximizing quantity of a monopolist (in the standard case with non-decreasing marginal cost) is the quantity at which marginal revenue equals marginal cost,

$$MR_i = MC_i = \frac{W}{\alpha} y_i^{\frac{1}{\alpha}-1}. \tag{19.34}$$

Total revenue is $TR_i = P_i(y_i)y_i$, where $P_i(y_i)$ is the price at which expected sales is y_i units. So

$$\begin{aligned} MR_i &= \frac{dTR_i}{dy_i} = P_i(y_i) + y_i P_i'(y_i) = P_i(y_i)(1 + \frac{y_i P_i'(y_i)}{P_i(y_i)}) \qquad (19.35)\\ &= P_i(y_i)(1 - \frac{1}{\eta}) = \left(\frac{y_i}{Y^e/m}\right)^{-1/\eta} P \frac{\eta - 1}{\eta}, \end{aligned}$$

where we have inserted $P_i(y_i) = (y_i/(Y^e/m))^{-1/\eta} P$, which follows from (19.33). Inserting this into (19.34), the unique solution for y_i is the profit maximizing quantity, given Y^e and P. We denote this planned individual output level y_i^e.

The associated price is

$$\bar{P}_i = P_i(y_i^e) = \frac{\eta}{\eta - 1} \frac{W}{\alpha} (y_i^e)^{\frac{1}{\alpha}-1} \equiv (1 + \mu)\frac{W}{\alpha}(y_i^e)^{\frac{1}{\alpha}-1}, \tag{19.36}$$

[12] Chapter 20 goes deeper and gives an account of the class of consumer preferences that underlie the constancy of this price elasticity. That chapter also presents a precise definition of the "general price level".

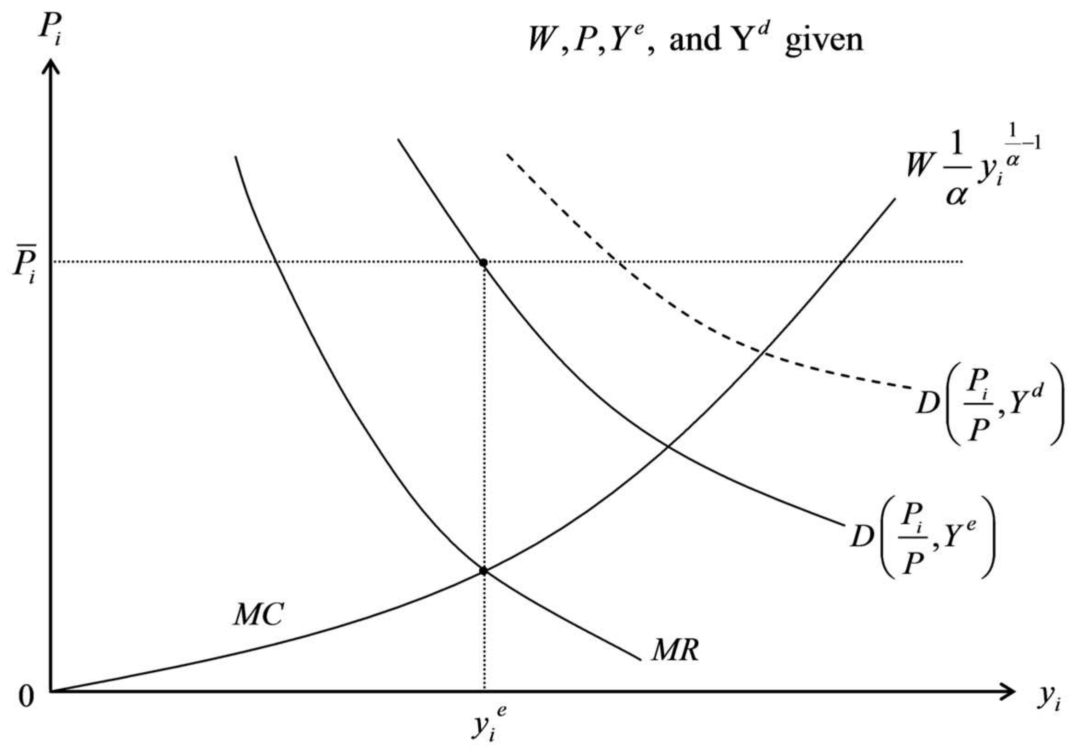

Figure 19.3: Firm i's price choice under the expectation that the general demand level will be Y^ϵ (the case $\alpha < 1$). The demand curve for a higher general demand level, Y^d, is also shown.

where the second equality comes from (19.35) inserted into (19.34), and μ is the mark-up on marginal cost at output level y_i^e, that is, $1 + \mu = \eta(\eta - 1)^{-1}$ $= 1 + (\eta - 1)^{-1}$.

This outcome is illustrated in Fig. 19.3 for the case $\alpha < 1$ (decreasing returns to scale). For fixed Y^e and P, the perceived demand curve faced by firm i is shown as the solid downward-sloping curve $D(P_i/P, Y^e/m)$ to which corresponds the marginal revenue curve, MR. For fixed W, the marginal costs faced by the firm are shown as the upward-sloping marginal cost curve, MC. It is assumed that firm i knows W in advance. The price $\bar{P}_i$ is set in accordance with the rule $MR = MC$.

Because of the symmetric setup, all firms end up choosing the same price, which therefore becomes the general price level, i.e., $\bar{P}_i = P$, $i = 1, 2, \ldots, m$. So all firms' planned level of sales equals the expected average real spending per consumption good, i.e., $y_i^e = Y^e/m \equiv y^e$, $i = 1, 2, \ldots, m$.

In case actual aggregate demand, Y^d, turns out as expected, firm i's actual output, y_i, equals the planned level, y^e. As this holds for all i, we have in this case

$$Y \equiv \frac{\sum_i \bar{P}_i y_i}{P} = \sum_i y_i = \sum_i y^e = Y^e = Y^d. \tag{19.37}$$

In some new-Keynesian models the labor market is described in an analogue way with heterogeneous labor organized in craft unions and monopolistic competition between these. To avoid complicating the exposition, however, we here treat labor as homogeneous. And until further notice we will simply *assume* that at the going wage there is enough labor available to carry out the desired production. We shall consider the question: If aggregate demand in the current period turns out different from expected, what will the firms do: change the price or output or both? To fix ideas we will concentrate on the case where the wage level, W, is unchanged. In that case the answer will be that "only output will be adjusted" if one of the following conditions is present:

(a) The marginal cost curve is horizontal and the price elasticity of demand is constant.

(b) The perceived cost of price adjustment exceeds the potential benefit.

That point (a) is sufficient for "only output will be adjusted" (as long as W is unchanged) follows from (19.36) with $\alpha = 1$. With rising marginal costs ($\alpha < 1$), however, the presence of sufficient price adjustment costs becomes decisive.

19.3.2 Price adjustment costs

The literature has modelled price adjustment costs in two different ways. *Menu costs* refer to the case where there are *fixed costs* of changing price. Another case considered in the literature is the case of *strictly convex adjustment costs*, where the marginal price adjustment cost is increasing in the size of the price change.

As to *menu costs*, the most obvious examples are costs associated with:

1. remarking commodities with new price labels,

2. changing price lists ("menu cards") and catalogues.

But "menu costs" should be interpreted in a broader sense, including pecuniary as well as non-pecuniary costs associated with:

3. information-gathering and recomputing optimal prices,

4. conveying rapidly the new directives to the sales force,

5. the risk of offending customers by frequent price changes (whether these are upward or downward),

6. search for new customers willing to pay a higher price,

7. renegotiation of contracts.

Menu costs induce firms to change prices less often than if no such costs were present. And some of the points mentioned in the list above, in particular point 6 and 7, may be relevant also in labor markets.

The menu cost theory provides the more popular explanation of nominal price stickiness. Another explanation rests on the presumption of *strictly convex price adjustment costs*. In this theory the cost for firm i of changing price is assumed to be $k_{it} = \xi_i(P_{it} - P_{it-1})^2$, $\xi_i > 0$. Under this assumption the firm is induced to avoid *large* price changes, which means that it tends to make frequent, but small price adjustments. This theory is related to the customer market theory. Customers search less frequently than they purchase. A large upward price change may be provocative to customers and lead them to do search in the market, thereby perhaps becoming aware of attractive offers from other stores. The implied "kinked demand curve" can explain that firms are reluctant to suddenly increase their price.[13]

Below we describe the role of the first kind of price adjustment costs, menu costs, in more detail.

The menu cost theory

The menu cost theory originated almost simultaneously in Akerlof and Yellen (1985a, 1985b) and Mankiw (1985). It makes up the predominant microfoundation for the presumption that nominal prices and wages tend to be sticky in the short run vis-a-vis demand changes. For simplicity, we will concentrate on product prices and downplay the intertemporal aspects of price-setting.

The key theoretical insight of the menu cost theory is that even *small* menu costs can be enough to prevent firms from changing their price vis-a-vis demand changes. This is because the opportunity cost of not changing price is only of *second order*, that is, "small", which is a reflection of the *envelope theorem*; hence the potential benefit of changing price can easily be smaller than the cost of changing price. Yet, owing to imperfect competition (price > MC), the effect on aggregate output, employment, and welfare of not changing prices is of *first order*, i.e., "large". Let us spell this out in detail.

[13] For details in a macro context, see McDonald (1990).

As in the World's Smallest Macroeconomic Model, suppose the aggregate demand is proportional to the real money stock:

$$Y^d = \frac{M}{\beta P}, \tag{19.38}$$

where $\beta \in (0, 1)$ is a parameter reflecting consumers' patience. Consider now firm i, $i = 1, 2, \ldots, m$, contemplating its pricing policy. With actual aggregate demand as given by (19.38) inserted into (19.33), the nominal profit as a function of the chosen price, P_i, becomes

$$\begin{aligned} \Pi_i &= P_i y_i - W y_i^{1/\alpha} = P_i \left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P} - W \left(\left(\frac{P_i}{P}\right)^{-\eta} \frac{M}{m\beta P}\right)^{1/\alpha} \\ &\equiv \Pi(P_i, P, W, M). \end{aligned} \tag{19.39}$$

Suppose that, initially, $P_i = \bar{P}_i$, where $\bar{P}_i$ is the unique price that maximizes Π_i, given P, W, and M. By (19.36) with $y_i^e = M/(m\beta P)$, we have

$$\bar{P}_i = (1 + \mu)\frac{W}{\alpha}\left(\frac{M}{m\beta P}\right)^{\frac{1}{\alpha}-1}. \tag{19.40}$$

In our simplifying setup there is complete symmetry across the firms so that the profit maximizing price is in fact the same for all firms. Nevertheless we maintain the subscript i on the profit-maximizing price since the logic of the menu cost theory is valid independently of this symmetry. We let $\bar{\Pi}_i$ denote firm i's maximized profit, i.e.,

$$\bar{\Pi}_i = \Pi(\bar{P}_i, P, W, M),$$

as illustrated in Fig. 19.4.

In view of the constant price elasticity of demand, η, and hence a constant markup, μ, if marginal costs are constant ($\alpha = 1$), then the profit-maximizing price is unaffected by a change in aggregate demand, cf. (19.40). This is the well-known case where, owing to constancy of marginal costs, The challenging case in a Keynesian context is the case with rising marginal costs. So let us assume that $\alpha < 1$. In this situation, by (19.40), a higher M, for unchanged P and W, will imply higher $\bar{P}_i$ as also illustrated in Fig. 19.4.

Given the price $P_i = \bar{P}_i$, set in advance, suppose that, at the beginning of the period, an unanticipated, fully money-financed lump-sum transfer payment to the households takes place so that M in (19.38) is replaced by $M' = M + \Delta M$, where $\Delta M > 0$. Suppose further that both W and P remain unchanged, that is, no other price setter responds by changing price. Let $\bar{P}_i'$ denote the new price which under these conditions would be profit maximizing for firm i in the absence

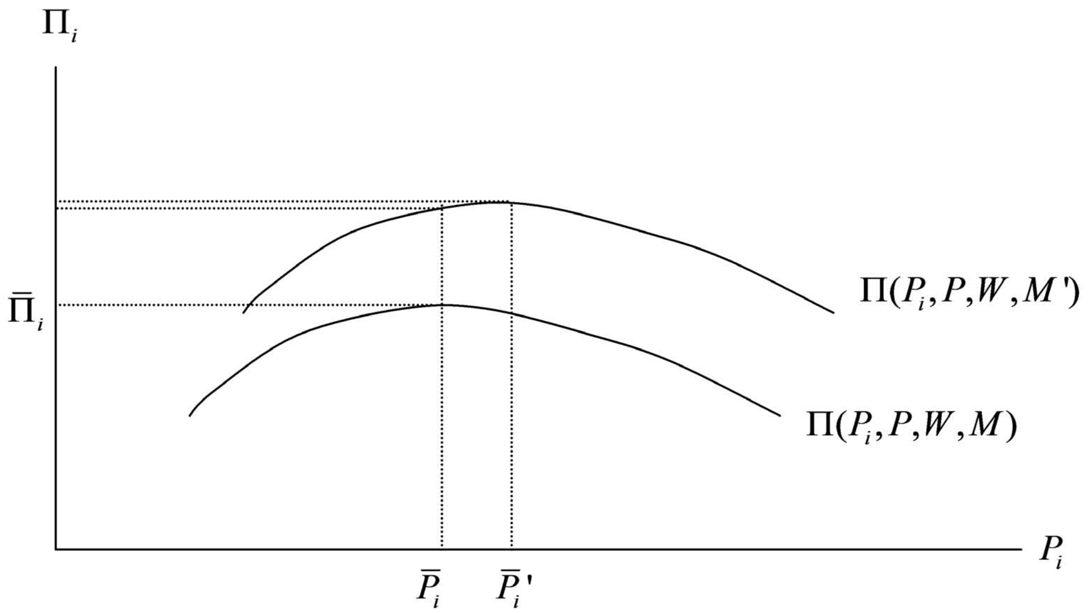

Figure 19.4: The profit curve is flat at the top ($\alpha < 1$, P and W are fixed, $M' > M$).

of menu costs. Fig. 19.4 illustrates. Will firm i have an incentive to change its price to $\bar{P}_i'$? Not necessarily. The menu cost may exceed the opportunity cost associated with not changing price. This opportunity cost to firm i tends to be small. Indeed, considering the marginal effect on Π of the higher M, we have

$$\begin{aligned}\frac{d\Pi}{dM}(\bar{P}_i, P, W, M) &= \frac{\partial \Pi}{\partial P_i}(\bar{P}_i, P, W, M)\frac{\partial P_i}{\partial M} + \frac{\partial \Pi}{\partial M}(\bar{P}_i, P, W, M) \\ &= 0 + \frac{\partial \Pi}{\partial M}(\bar{P}_i, P, W, M).\end{aligned} \tag{19.41}$$

The first term on the right-hand side of (20.36) vanishes at the profit maximum because $\partial\Pi/\partial P_i = 0$ at the point $(\bar{P}_i, P, W, M)$..The profit curve is flat at the profit-maximizing price $\bar{P}_i$. Moreover, since our thought experiment is one where P and W remain unchanged, there is no indirect effect of the rise in M via P or W. Thus, only the direct effect through the fourth argument of the profit function is left. And this effect is independent of a marginal change in the chosen price. This result reflects the *envelope theorem*: in an interior optimum, the total derivative of a maximized function w.r.t. a parameter equals the partial derivative w.r.t. that parameter.[14]

The relevant parameter here is the aggregate money stock, M. As Fig. 19.4 visualizes, the effect of a small change in M on the profit is approximately the

[14]For a general statement, see Math Tools.

same (to a first order) whether or not the firm adjusts its price. In fact, owing to the envelope theorem, for an infinitesimal change in M, the profit of firm i is not affected at all by a marginal change in its price.

For a *finite* change in M this is so only approximately. First, (19.39) shows that the entire profit curve is shifted up, cf. Fig. 19.4. Second, from (19.40) follows that there *will* be a discernible rise in the profit-maximizing price, in Fig. 19.4 from $\bar{P}_i$ to $\bar{P}_i'$. So the new top of the profit curve is north-east of the old. It follows that by not changing price a potential profit gain is left unexploited. Still, if the rise in M in not "too large", the slope of the profit curve at the old price $\bar{P}_i$ may still be small enough to be dominated by the menu cost.

Given a change in M of size $\Delta M > 0$, the opportunity cost of not changing price can be shown to be of "second order", i.e., proportional to $(\Delta M/M)^2$.[15] This is a "very small" number, when $|\Delta M/M|$ is just "small". Therefore, in view of the menu cost, say c, it may be advantageous not to change price. Indeed, the net gain ($= c -$ opportunity cost) by *not* changing price may easily be positive. Suppose this is so for firm i, given that the other firms do not change price. Since each individual firm is in the same situation as long as the other firms have not changed price, the outcome that no firm changes its price is an equilibrium. As in this equilibrium there is no change in the general price level, there will be a higher output level than without the rise in M.

The reference to changes in the money stock, M, in this discussion should not be misunderstood. It is not as a medium of exchange or similar that M has a role in the model, but as the sole constituent of non-human wealth. The increase in M does not reflect an open market purchase of bonds by the central bank, but a money-financed government budget deficit created by transfers to the households without any taxes in the opposite direction. This amounts to a combined monetary-fiscal stimulus to the economy, an example of "helicopter money", cf. Chapter 17.

Doesn't W respond?

The considerations above presuppose that workers or workers' unions do not immediately increase their wage demands in response to the increased demand for labor. This assumption can be rationalized in two different ways. One way is to assume that also the labor market is characterized by monopolistic competition between craft unions, each of which supplies its specific type of labor. If there are menu costs associated with changing the wage claim and they are not too small, the same envelope theorem logic as above applies and so, theoretically, an

[15] Appendix A shows this by taking a second-order Taylor approximation of the opportunity cost.

increase in labor demand need not in the short run have any effect on the wage claims.

There is an alternative way of rationalizing absence of an immediate upward wage pressure. This alternative way is more apt in the present context since we have treated labor as homogeneous, implying that there is no basis for existence of many different craft unions.[16] Instead, let us here assume that *involuntary unemployment* is present. This means that there are people around without a job although they are as qualified as the employed workers and are ready and willing to take a job at the going wage or even a lower wage.[17] Such a state of affairs is in fact what several labor market theories tell us we should expect to see often. In both efficiency wage theory, social norms and fairness theory, insider-outsider theory, and bargaining theory, there is scope for a wage level above the individual reservation wage (see Chapter 24). Presence of involuntary unemployment implies that employment can change with negligible effect on the wage level in the short run. In combination with little price sensitivity to output and employment changes, this observation also offers a rationalization of stylized fact no. 2 in the list of Section 19.1 saying that *relative* prices, including the real wage, exhibit little sensitivity to changes in the corresponding quantities, here employment.

19.3.3 Menu costs in action

Under these conditions even *small* menu costs can be enough to prevent firms from changing their price in response to a change in demand. At the same time even small menu costs can have *sizeable* effects on aggregate output, employment, and social welfare. To understand this latter point, note that under monopolistic competition neither output, employment, or social welfare are maximized in the initial equilibrium. Therefore the envelope theorem does not apply to these variables.

This line of reasoning is illustrated in Fig. 19.5. There are two differences compared with Fig. 19.3. First, aggregate demand is now specified as in (19.38).

[16] Even with heterogenous labor, the craft union explanation runs into an empirical problem in the form of a "too low" wage elasticity of labor supply according to the microeconometric evidence. We come back to this issue in Chapter 20.3.

[17] In case firms have considerable hiring costs (announcing, contracting, and training) these add to the full cost of employing people. Typically there is then an initial try-out period with a comparatively low introductory wage rate. The criterion for being involuntarily unemployed is then whether the person in question is willing to take a job under similar conditions as those who currently got a job.

Although the term "involuntary" may provoke moral sentiment, this definition of *involuntary unemployment* should be understood as purely technical, referring to something that can in principle be measured by observation.

Second, along the vertical axis we have set off the *relative* price, P_i/P, so that marginal revenue, $\mathcal{MR}$, as well as marginal costs, $\mathcal{MC}$, are indicated in real terms, i.e., $\mathcal{MR} = MR/P$ and $\mathcal{MC} = MC/P$. For fixed M/P, the demand curve faced by firm i is shown as the solid downward-sloping curve $D(P_i/P, M/(m\beta P))$ to which corresponds the real marginal revenue curve, $\mathcal{MR}$. For fixed W/P, the real marginal costs faced by the firm are shown as the upward-sloping real marginal cost curve, $\mathcal{MC}$ (recall that we consider the case $\alpha < 1$).

If firms have rational (model consistent) expectations and know M and W in advance, we have $Y^e = M/(\beta P)$. The price chosen by firm i in advance, given this expectation, is then the price $\bar{P}_i$ shown in Fig. 19.3. As the chosen price will be the same for all firms, the relative price, P_i/P, equals 1 for all i. Equilibrium output for every firm will then be $M/(m\beta P)$, as indicated in the figure. If the actual money stock turned out to be higher than expected, say $M' = \lambda M$, $\lambda > 1$, and there were no price and wage adjustment costs *and* if wages were also multiplied by the factor λ, prices would be multiplied by the same factor and the real money stock, production, and employment be unchanged.

With menu costs, however, it is possible that prices and wages do not change. The menu cost may make it advantageous for each single firm not to change price. Then, the higher nominal money stock translates into a higher *real money* stock and the demand curve is shifted to the right, as indicated by the stippled demand curve in Fig. 19.5. As long as $\bar{P}_i/P > \mathcal{MC}$ still holds, each firm is willing to deliver the extra output corresponding to the higher demand. The extra profit obtainable this way is marked as the hatched area in Fig. 19.5. Firms in the other production lines are in the same situation and also willing to raise output. As a result, aggregate employment is on the point of increasing. The only thing that could hold back a higher employment is a concomitant rise in W in response to the higher demand for labor. Assuming presence of involuntary unemployment in the labor market hinders this, the tendency to higher employment is realized, and firm i's production ends up at y_i' in Fig. 19.5, while the price $\bar{P}_i$ is maintained. The other firms act similarly and the final outcome is higher aggregate consumption and higher welfare.

Thus, the effects on aggregate output, employment, and social welfare of not changing price can be substantial; they are of "first order", namely proportional to $|\Delta M/M|$, as implied by the aggregate demand formula (19.38).

In the real world, nominal aggregate demand (here proportional to the money stock) fluctuates up and down around some expected level. Sometimes the welfare effects of menu costs will be positive, sometimes negative. Hence, *on average* the welfare effects tend to cancel out to a first order. This does not affect the basic point of the menu cost theory, however, which is that changes in aggregate nominal demand can have first-order real effects (in the same direction) because

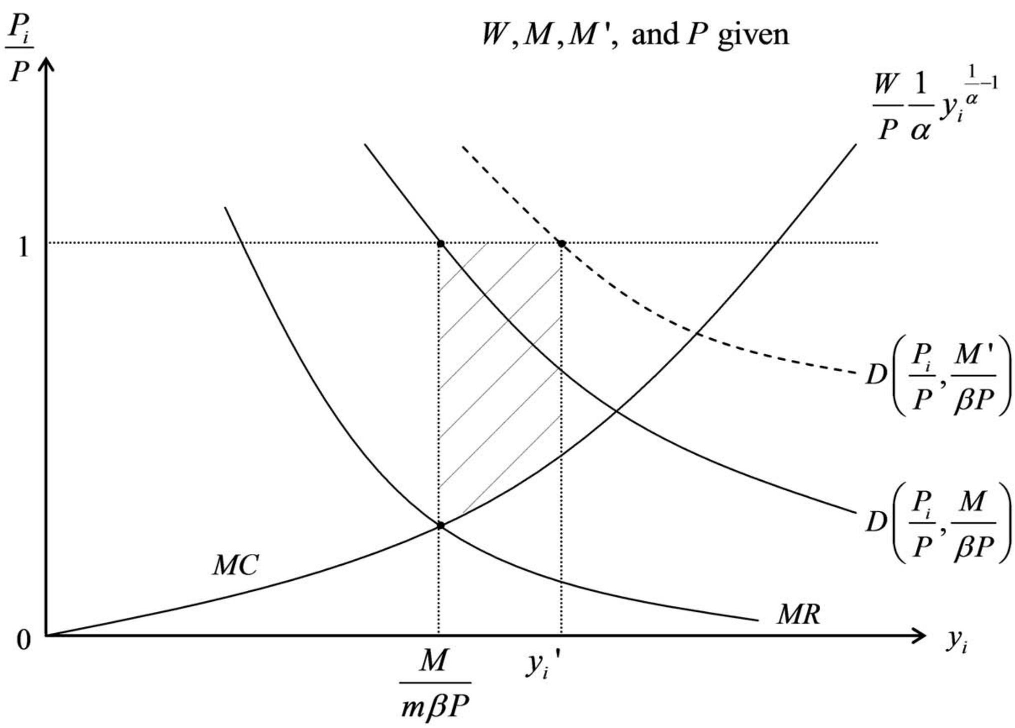

Figure 19.5: The impact in general equilibrium of a shift to $M' > M$ when menu costs are binding (the case $\alpha < 1$).

the opportunity cost by not changing price is only of second order.[18]

A reservation As presented here, the menu-cost story is not entirely convincing. The rather static nature of the setup is a drawback. For instance, the setup gives no clear answer to the question whether it is a change from a given past price, $\bar{P}_{i,t-1}$, to the preset price, $\bar{P}_{i,t}$, for the current period that is costly or a change from the preset price, $\bar{P}_{i,t}$, for the current period to another price within a sub-period of that period.

More importantly, considering a sequence of periods, there would in any period be *some* prices that are not at their ex ante "ideal" level. The firms in question are then not at the flat part of their profit curve. The menu cost necessary to prevent price adjustment will then be higher for these firms, thus making it more demanding for menu costs to be decisive. Moreover, in an intertemporal perspective it is the present value of the expected stream of future gains and costs that matter rather than instantaneous gains and costs. An aspect of a complete dynamic modeling is also that ongoing inflation would have to be taken into account. In modern times where money is paper money (or electronic money), there is usually an underlying upward trend in the general input price level. To maintain profitability, the individual producers will therefore surely *have* to adjust their prices from time to time. The decision about *when* and *how much* to change price will be made with a view to maximizing the *present value* of the expected future cash flow taking the expected menu costs into account.[19]

The key point from static menu-cost theory, based on the envelope theorem, is not necessarily destroyed by the dynamics of price-setting. It becomes less cogent, however.

The rule of the minimum

For a preset price, $\bar{P}_i$, it is beneficial for the firm to satisfy demand as long as the corresponding output level is within the area where nominal marginal cost is below the price. Returning to Fig. 19.3, actual aggregate demand is given as Y^d. Let actual demand faced by firm i be denoted y_i^d, so that $y_i^d = D(\bar{P}_i/P, Y^d/m)$. Compare this demand to y_i^c, defined as the production level at which $MC = \bar{P}_i$ (assuming $\alpha < 1$). This is the production level known as the *Walrasian* or *classical* or *competitive supply* by firm i (the superscript "c" stands for "classical" or "competitive"). It indicates the output that would prevail under perfect competition, given W, $P_i = \bar{P}_i$, and the assumption of rising marginal costs

[18] Sustained increases in aggregate demand are likely to lead to capacity investment by the existing firms or entry of new firms supplying substitutes.

[19] As to contributions within this dynamic perspective, see Literature notes.

($\alpha < 1$). As this desired output level is a function of only W and $\bar{P}_i$, we write it $y_i^c = y^c(W, \bar{P}_i)$. In the case of constant MC, i.e., $\alpha = 1$, we interpret $y^c(W, \bar{P}_i)$ as $+\infty$.

As long as $y_i^d < y_i^c$, and enough labor is available, actual output will be $y_i = y_i^d$. If $y_i^d > y_i^c$, however, the firm will prefer to produce only y_i^c. Producing beyond this level would entail a loss since marginal cost would be above the price. Presupposing enough labor is available, the rule is therefore that given the demand D and the classical supply y_i^c, actual production is the minimum of the two, that is,

$$y_i = \min\left[D(\frac{\bar{P}_i}{P}, \frac{Y^d}{m}), y^c(W, \bar{P}_i)\right].$$

Given the production function $y_i = n_i^\alpha$, $i = 1, 2, \ldots, m$, the corresponding *effective labor demand* by firm i is

$$n_i^d = y_i^{1/\alpha}.$$

The *aggregate effective labor demand* is $N^d = \sum_{i=1}^m n_i^d$. Let the aggregate effective supply of labor be a given constant, $\bar{N}$, and assume that in the short run, $\bar{N}/m$ workers are available to each firm. Then the *effective supply* of firm i is $\min\left[y^c(W, \bar{P}_i), (\bar{N}/m)^\alpha\right]$. Actual output of firm i will be

$$y_i = \min\left[D(\frac{\bar{P}_i}{P}, \frac{Y^d}{m}), y^c(W, \bar{P}_i), (\frac{\bar{N}}{m})^\alpha\right], \tag{19.42}$$

that is, the minimum of effective demand, classical supply, and output at full employment in "product line i". This rule is known as the *rule of the minimum.*[20]

If at the given wage and price level, labor supply is the binding constraint in most of the product lines, *repressed inflation*, as defined in Section 19.2, prevails with *excess demand* for labor and goods.

Keynesian versus classical unemployment In the opposite case, where labor is abundant, two alternative kinds of unemployment may prevail. If the demand, D, is the binding constraint in most of the product lines, what is known as *Keynesian unemployment* prevails. This is a situation where both the typical output market and the labor market are in a state of "buyers' market" (sale less than preferred).

[20]Note that this rule determines production of the single firm. It is related to, but not identical to the *short-side rule,* which we encountered in Section 19.2. This is the "voluntary trade" principle saying that the actual quantity traded in a market is the minimum of effective demand and effective supply in the market.

The alternative possibility is that the classical supply, y^c, is the binding constraint in most of the product lines. In this case what is known as *classical unemployment* will prevail. This is a situation with downward pressure on the wage level and upward pressure on the price level. The huge unemployment during the Great Depression in the interwar period was by most economists of the time diagnosed as a momentary phenomenon caused by a "too high real wage". Keynes and a few like-minded disagreed. It is in this context that the quote by the outstanding French economist Edmond Malinvaud (1923-) at the front page of this chapter should be seen.

In the *World's Smallest Macroeconomic Model* of Section 19.2 classical unemployment can not occur because of constant marginal costs combined with a positive mark-up μ. The limiting case $\alpha = 1$ in our present disaggregate model also leads to $MC = W$, a constant. Thus (19.42) gives $y_i = D(\bar{P}_i/P, Y^d/m) < y^c(W, \bar{P}_i) = +\infty$ for all i. All the goods markets are demand-constrained and any unemployment is thereby Keynesian. Note also that because of constant MC combined with the constant mark-up, no menu costs are needed to maintain that the output level rather than prices respond to changes in aggregate demand, Y^d.

Although constant MC within certain limits may be an acceptable assumption, an additional factor potentially constraining production in the short run is the capital equipment of the firm. Hitherto this factor has not been visible. Or we might say that the case of rising MC ($\alpha < 1$) can be interpreted as reflecting that in practice labor is not the only production factor. This motivates the next section.

19.4 Abundant capacity

One of the stylized facts listed in Section 19.1 is that under "normal circumstances" a majority of firms in an industrialized economy respond to short-run shifts in aggregate demand by adjusting production rather than price. Key elements in the explanation of this phenomenon have been sketched: (a) the distinction between *Walrasian* and *effective* demand and supply; (b) price setting agents in markets with imperfect competition; (c) the "envelope argument" that the potential benefit of adjusting the price can easily be smaller than the cost of adjusting; and (d) because prices are generally above marginal costs, firms are willing to adjust production when aggregate demand shocks occur.

This leads us to the problem whether quantity adjustment is in the main *realizable* in the short run. Under the assumption that involuntary unemployment is present, lack of workers will not be an impediment. But the production capacity of firms depends also on their command of capital equipment. To throw light on this aspect we now let the production function have two inputs, capital and labor.

19.4.1 Putty-clay technology

Suppose firm i has the production function

$$y_i = f(k_i, n_i), \tag{19.43}$$

where k_i is the installed capital stock and n_i the labor input, $i = 1, 2, \ldots, m$. (At the disaggregate level we use small letters for the variables. So, contrary to earlier chapters, k_i is here not the capital-labor ratio, but simply the capital stock in firm i.) Because of strictly convex installation costs, k_i is given in the short run. Raw materials and energy are ignored. So in the short run the capital costs are fixed but labor costs variable since n_i can be varied.

Realistic short-run analysis makes a distinction between the "ex ante" and the "ex post" production function. By *ex ante* is here meant the point in time where the decision about investment, whether in plant or equipment, is to be made. We imagine that in making this decision, a wide range of production techniques (input-output combinations) is available as represented by the function f in (19.43), hence called the ex ante production function. The decision will be made with a forward-looking perspective. Construction and installation are time consuming and to some extent irreversible.

By *ex post* is meant "when construction and installation are finished and the capital is ready for use". In this situation, the substitutability between capital and labor tends to be limited. Our long-run models in previous chapters implicitly ignored this aspect by assuming that substitutability between capital and labor is the same ex ante and ex post. In reality, however, when a machine has been designed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machine per time unit. In the terminology of Section 2.5 of Chapter 2, technologies tend in a short-run perspective to be "putty-clay" rather than "putty-putty".

An example: suppose the production function f in (19.43) is a neoclassical production function. This is our ex ante production function. Ex post, this function no longer describes the choice opportunities for firm i. These are instead given by a Leontief production function with CRS:

$$y_i = \min(Au_i\bar{k}_i, Bn_i), \qquad A > 0, B > 0, \tag{19.44}$$

where A and B are given technical coefficients, $\bar{k}_i$ is the size of the installed capital (now a fixed factor) and u_i its utilization rate ($0 \leq u_i \leq 1$), $i = 1, \ldots, m$.[21] Presumably the firms would have acquired their capital equipment under different

[21] The link between the ex ante production function f and the technical coefficients A and B was described in Chapter 2.

circumstances at different points in time in the past so that, generally, the equipment would be somewhat heterogeneous and A and B would be index numbers and differ across the firms. To make aggregation simple, however, the analyst may be tempted in a first approach to ignore this complication and assume A and B are the same across the firms.

19.4.2 Capacity utilization and monopolistic-competitive equilibrium

Capacity utilization in a Keynesian equilibrium

There is *full capacity utilization* when $u_i = 1$, which means that each machine is operating "full time" (seven days and nights a week, allowing for surplus time for repairs and maintenance). Capacity is given as $A\bar{k}_i$ per week. Producing efficiently at capacity requires $n_i = A\bar{k}_i/B$. But if demand, y_i^d, is less than capacity, satisfying this demand efficiently requires $n_i = y_i^d/B$ and $u_i = Bn_i/(A\bar{k}_i) < 1$. As long as $u_i < 1$, there is unused capacity, and marginal productivity of labor in firm i is a *constant*, B.

The (pure) profit of firm i is $\Pi_i = P_i y_i - \mathcal{C}(y_i, W, F_i)$, where $\mathcal{C}(y_i) = Wy_i/B + F_i$ is the cost function with F_i denoting the fixed costs deriving from the fixed production factor, $\bar{k}_i$. Average cost is $AC = \mathcal{C}(y_i)/y_i = W/B + F_i/y_i$ and marginal cost is a constant $MC = W/B$ for $y_i < A\bar{k}_i$.

Fig. 19.6 depicts these cost curves together with a downward-sloping demand curve. A monopolistic-competition market structure as described in Section 19.3.1 is assumed but now only a subset of the m firms produce differentiated consumption goods. The other firms produce differentiated capital goods, also under conditions of monopolistic competition and with the same price elasticity of demand.

Firm i presets the price of good i at $P_i = \bar{P}_i$ in the expectation that the level of demand will be as indicated by the downward sloping D curve in Fig. 19.6. The point E_{SR} in the figure represents a standard short-run equilibrium under monopolistic competition with output level such that $MC = MR$. Assuming full symmetry across the different firms, the point E_{SR} would also reflect a Keynesian equilibrium if the actual demand level (position of the demand curve) had turned out to be as expected by the firms when fixing their price. But in the figure it is assumed that the demand level turned out lower. The produced quantity is reduced while the price remains unchanged (because of either menu costs or simply the constant marginal costs combined with constant price elasticity of demand). Firm i ends up with actual production equal to y_i' in Fig. 19.6. The obtained (pure) profit is indicated by the hatched rectangle constructed by the help of the average cost curve AC.

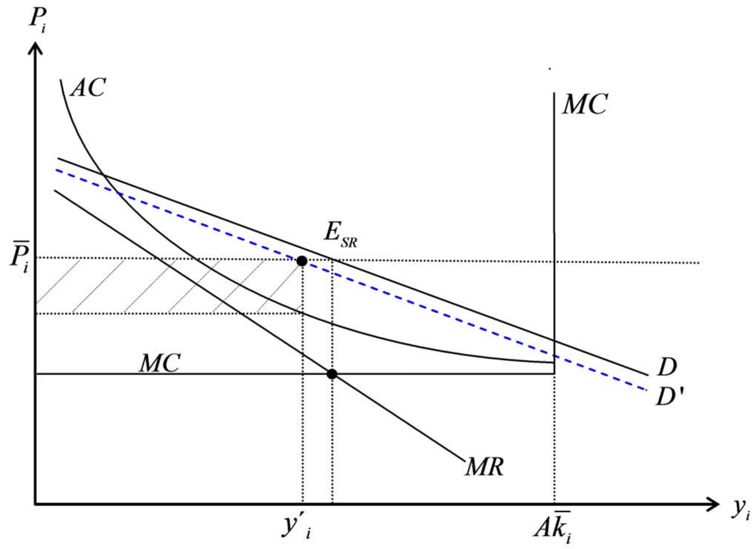

Figure 19.6: Firm i in a Keynesian equilibrium ($A\bar{k}_i$ is production capacity; MC curve first horizontal and then vertical at $y_i = A\bar{k}_i$).

By interpreting y_i' in Fig. 19.6 as actual production we have implicitly assumed that enough labor is available. We let this "labor abundance" be understood throughout this discussion. If the picture in Fig. 19.6 is representative for the economy as a whole, the unemployment in the economy is predominantly *Keynesian.*

If the actual demand level had turned up higher than expected, firm i would be induced to raise production. There is scope for this because price is above marginal costs the whole way up to full capacity utilization. Profits will by this expansion of production become *substantially* higher than expected because the profit on the marginal unit sold is higher than average profit per unit as a result of the AC curve being downward-sloping up to the production level $A\bar{k}_i$.

Anyway, all the way up to $A\bar{k}_i$ we have a situation where the quantity produced is less than the quantity at which average costs are at the minimum. Such a situation is sometimes said to reflect "excess capacity". But "excess" sounds as if the situation reveals a kind of inefficiency, which need not be the case. So we prefer the term "abundant capacity".

A monopolistic-competitive long-run equilibrium

The picture is essentially the same in a "free-entry-and-exit equilibrium" where all pure profit is eliminated; in the theory of industrial organization this is known as a "long-run equilibrium". Suppose that initially some of the firms get positive

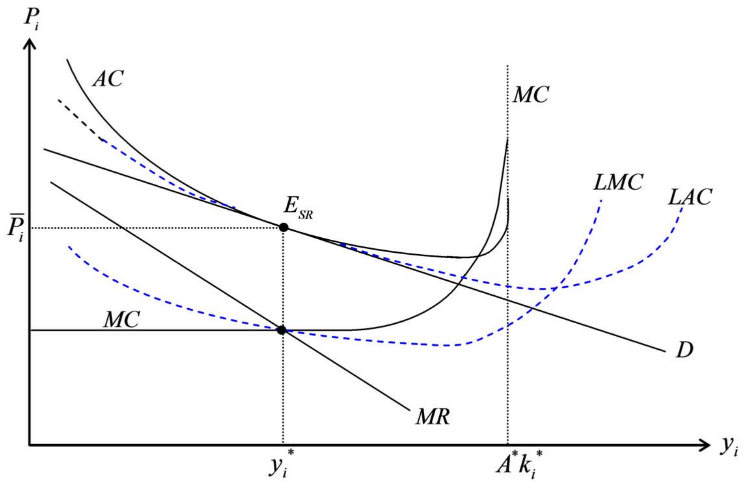

Figure 19.7: Firm i in a monopolistic-competition long-run equilibrium (y_i^* is long-run equilibrium output and $A^*k_i^*$ is long-run production capacity).

pure profits as illustrated in Fig. 19.6. This state of affairs invites entry of new firms. Over time these set up new plants and begin to supply new differentiated goods from a large set of as yet un-utilized *possible* imperfect substitutes for the existing goods. The entry process continues until equilibrium with zero pure profits applies to all product lines. When this state is reached, prices equal both short- and long-run average costs, and each firm operates where the downward-sloping demand curve is *tangent* to both AC curves. By "short-run" we mean a time horizon within which only a subset of the production factors are variable, while "long-run" refers to a time horizon long enough for all production factors to be variable.

From a macroeconomic perspective the important conclusion is that the "transition" from short-run equilibrium to long-run equilibrium in no way tends to lessen the presence of abundant capacity in the firms.

To portray a long-run equilibrium, we only need to let the AC curve for firm i's chosen plant and equipment be tangent to the demand curve at the point E_{LR}. This is what we have done in Fig. 19.7 where also the long-run marginal and average cost curves are visible, denoted LMC and LAC, respectively. The LMC curve is assumed U-shaped. This implies a U-shaped LAC curve. The downward-sloping part of the LMC curve may be due to indivisibilities of plant and equipment. And the upward-sloping part may reflect coordination problems or an implicit production factor which is tacitly held fixed (a special managerial expertise, say).

Independently of the long-run versus short-run perspective, we have in Fig. 19.7 introduced the case where the short-run marginal cost curve, MC, is horizontal only up to certain rate of capacity utilization $\bar{u} < 1$. It then rises gradually and ends up as vertical at full capacity utilization. This is to open up for the possibility that a decision to produce more here and now may imply bringing less efficient standby equipment to use. Also the wear and tear on the machinery may be raised. This amounts to a "rounding off" of the possibly too "sharp" Leontief production function (19.44). Instead, efficient production could here be described by

$$n_i = \begin{cases} y_i/B & \text{if } y_i < \bar{y}_i \equiv \bar{u}A\bar{k}_i, \\ \bar{y}_i/B + (y_i - \bar{y}_i)^{1/\alpha}/B & \text{if } \bar{y}_i \leq y_i \leq A\bar{k}_i, \quad 0 < \alpha < 1. \end{cases}$$

An alternative or additional reason for the MC curve to be upward-sloping at high capacity utilization is wage bonuses for working on the night shift or in the weekend.

Also in this more realistic setup is *abundant capacity* revealed as a sustainable equilibrium phenomenon. In long-run equilibrium each firm produces at a point where:

- price is above marginal costs so as to exactly cover fixed costs;

- the quantity produced is less than the quantity at which average costs are at the minimum (i.e., where the MC curve crosses the AC curve in Fig. 19.7), given the firm's preferred plant and equipment.

The conclusion is that as long as the AC curve does not shift (this would happen if the general wage level changed), firms are more than willing to accommodate an increased demand (outward shift of the demand curve) at an unchanged price or even at a lower price by an increase in production. Similarly, an inward shift of the demand curve will not lead to a temptation to reduce the price, rather the opposite if the menu cost is immaterial. These observations fit well with the huge amount of sales promotion we see. They also fit with the empirical evidence that measured total factor productivity and gross operating profits rise in an economic upturn and fall in a downturn, an issue to which we return in Part VII of this book.

Finer shades*

Oligopoly In the real world some markets are better characterized by strategic interaction between a few big firms than by monopolistic competition. This is a situation where abundant capacity may result not only from falling average costs but also from a strategic incentive. Maintaining abundant capacity will make credible a threat to cut price in response to unwelcome entry by a competitor.

Contestable markets A *contestable market* is a market for a homogeneous good where, because of large economies of scale, the quantity at which average costs are at the minimum exceeds the size of the market so that there is only room for one firm if production costs should be minimized.[22] The mere "threat of entry" induces average cost-pricing by the incumbent. So, with the point E_{LR} interpreted as the long-run equilibrium under these conditions, Fig. 19.7 also portrays this situation. Again "abundant capacity" is displayed.

The role of indivisibilities The downward-sloping part of the LAC curve, reflecting indivisibilities in plant and equipment, is important also from another perspective. Without indivisibilities it may be difficult to see why involuntarily unemployed workers could not just employ and support themselves in the backyard, financing the needed tiny bits of capital by tiny bank loans. This point is developed further in, e.g., Weitzman (1982).

19.4.3 Aggregation over different regimes*

Returning to Figure 19.6, let us assume that the position of the demand curve faced by firm i is shifted to the right so that the production capacity $A\bar{k}_i$ becomes a binding constraint on y_i'. Suppose further that most of the industries are in this situation. If unemployment is still massive, it is no longer mainly Keynesian, but a particular form of *classical* unemployment. Neither aggregate demand nor production costs as such like in Section 19.3.1, but simply the lack of sufficient capital is the binding constraint on employment. This state of affairs amounts to a form of classical unemployment known as *Marxian unemployment* because it was emphasized in the economic writings of Karl Marx.

Even though the phenomenon of insufficient capital is generally regarded as more common in developing countries (and in the pre-industrial period of Western Europe), it also appears from time to time in specific product lines of industrialized countries under structural change. Similarly, the constraint from labor supply may from time to time be binding in other product lines. Macroeconomic analysis should therefore allow for different regimes in the different product lines of the economy.

Let us imagine that firm i (or industry i) not only produces its own differentiated good i but is also distinguished by using a particular type of labor, say "local labor", effectively supplied in the amount n_i^s. According to the rule of the minimum, actual production will then be

$$y_i = \min(y_i^d, y_i^c, y_i^f), \qquad i = 1, \dots, m,$$

[22] Cf. Tirole (1988).

where $y_i^c \equiv A\bar{k}_i$ and $y_i^f \equiv Bn_i^s$. Depending on which is the binding constraint, y_i^d, y_i^c, or y_i^f, firm i is either in the Keynesian regime, the classical regime, or the repressed-inflation regime.

Because of technological change and changes in demand patterns we expect some regime heterogeneity, known as *mismatch*, to evolve in the economy as a whole. For the aggregate level, we define $YD \equiv \sum_i y_i^d$, $YC \equiv \sum_i y_i^c$, and $YF \equiv \sum_i y_i^f$. Then, in general, $Y \equiv \sum_i y_i < \min(YD, YC, YN)$.

As an alternative to an aggregate *min* condition as in Section 19.2, a large multi-country study of Western European unemployment since the 1960s, entitled *Europe's Employment Problem*,[23] introduced a statistical distribution of demands and supplies on micro-markets for goods and labor in each country at a given point in time. The approach can briefly be described as follows. Suppose the micro-market values y^d and y^c are jointly log-normally distributed:

$$\begin{bmatrix} \log y^d \\ \log y^c \end{bmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \xi_d \\ \xi_c \end{pmatrix}, \begin{pmatrix} \sigma_d^2 & \text{cov} \\ \text{cov} & \sigma_c^2 \end{pmatrix}\right].$$

Then $\log(y^d/y^c) \sim \mathcal{N}(\xi_d - \xi_c, \sigma^2)$, where $\sigma^2 = \sigma_d^2 + \sigma_c^2 - 2\text{cov}$. Letting Y^* denote firms' *aggregate desired output* (aggregate output in case labor supply nowhere is the binding constraint), it can be shown[24] that

$$Y^* \approx (YD^{-\rho} + YC^{-\rho})^{-1/\rho}, \qquad \rho > 0, \tag{19.45}$$

that is, Y^* is approximately a constant-returns-to-scale CES function of YD and YC. The inverse of ρ is then an increasing function of the variance of $\log(y^d/y^c)$ and is therefore a measure of the "degree of mismatch" between the demand constraint and the capacity constraint. Indeed, it can be shown that $(YD^{-\rho} + YC^{-\rho})^{-1/\rho} < \min(YD, YC)$ for $\rho \in (0, \infty)$ and that $\lim_{\rho\to\infty}(YD^{-\rho} + YC^{-\rho})^{-1/\rho} = \min(YD, YC)$, saying that as $1/\rho \to 0$, mismatch on the firms' side disappears.

As to mismatch in the labor markets, consider firm i's actual employment $n_i = \min(n_i^d, n_i^s)$, where $n_i^d = \min(y_i^d, y_i^c)$ is effective demand for labor, and n_i^s is the effective supply of labor. Given the Leontief production function (19.44), the aggregate effective demand for labor is

$$N(Y^*) = \frac{Y^*}{B} = \left((\frac{YD}{B})^{-\rho} + (\frac{YC}{B})^{-\rho}\right)^{-1/\rho} \equiv (NY^{-\rho} + NC^{-\rho})^{-1/\rho}. \tag{19.46}$$

In analogy with (19.45) we assume that actual aggregate employment satisfies

$$N \approx (N(Y^*)^{-\rho'} + NS^{-\rho'})^{-1/\rho'}, \qquad \rho' > 0,$$

[23] See Drèze and Bean (1990).

[24] For the math behind this and other claims in this section, see Lambert (1988).

where $NS \equiv \sum n_i^s$ and ρ' measures the "degree of mismatch" between the demand and supply in the labor markets. Substituting (19.46) into this gives

$$N \approx \left((NY^{-\rho} + NC^{-\rho})^{-\rho'/\rho} + NS^{-\rho'}\right)^{-1/\rho'} = \left(NY^{-\rho} + NC^{-\rho} + NS^{-\rho}\right)^{-1/\rho},$$

where the last equality holds *if* $\rho' = \rho$. In that case we also have

$$Y = BN = (YD^{-\rho} + YC^{-\rho} + YS^{-\rho})^{-1/\rho}, \tag{19.47}$$

approximately, where $YS \equiv B \cdot NS$ and where, for convenience, we have replaced "$\approx$" by "$=$", appealing to the law of large numbers.

The parameters ρ and ρ' can be estimated on the basis of business and household survey data (firms' answers to regular survey questions about demand, capacity and labor constraints and households' answers about desired employment). The mentioned *Europe's Employment Problem* study estimated for most of the countries (Denmark included) a falling ρ since middle of the 1960s towards the late 1980s, that is, a rising mismatch. This tends to raise the unemployment rate. To illustrate, imagine the "favorable" case where $NY = NC = NS$ so that without mismatch full-employment equilibrium would prevail. Actual employment will be

$$N = (3NS^{-\rho})^{-1/\rho} = 3^{-1/\rho} NS.$$

The unemployment rate then is

$$u \equiv \frac{NS - N}{NS} = 1 - \frac{N}{NS} = 1 - 3^{-1/\rho} > 0,$$

when $\rho < \infty$, i.e., $1/\rho > 0$. An increased mismatch, $1/\rho$, thus means a higher u.

Another consequence of mismatch is that it reduces the Keynesian spending multiplier. Consider aggregate demand as given by the standard textbook income-expenditure equation

$$YD = C(Y) + \bar{I} + \bar{G} + \bar{X} - IM(Y), \qquad C' > 0,\ IM' > 0, 0 < C' - IM' < 1, \tag{19.48}$$

where $C(Y)$ and $IM(Y)$ are private consumption and imports, respectively, while $\bar{I}$, $\bar{G}$, and $\bar{X}$ are private investment, government purchases, and exports, respectively, all exogenous. To find the multiplier with respect to $\bar{G}$, we substitute (19.48) into (19.47) and differentiate with respect to $\bar{G}$, using the chain rule:

$$\frac{\partial Y}{\partial \bar{G}} = -\frac{1}{\rho}(YD^{-\rho} + YC^{-\rho} + YS^{-\rho})^{-\frac{1}{\rho}-1} \cdot (-\rho) YD^{-\rho-1} \left((C' - IM') \frac{\partial Y}{\partial \bar{G}} + 1\right).$$

By ordering,

$$\frac{\partial Y}{\partial \bar{G}} = \frac{1}{\left(\frac{YD}{Y}\right)^{\rho+1} - C' + M'} < \frac{1}{1 - C' + M'}, \tag{19.49}$$

where the inequality is due to $Y < YD$. In turn, this latter inequality reflects that not all micro-markets are in a Keynesian regime.[25]

The mentioned multi-country study thus concluded that increased mismatch in the preceding years was part of the explanation of the high level of unemployment in Western Europe in the 1980s. Moreover, the increased mismatch was attributed to the collapse of capital investment in the aftermath of the first and second oil price crises 1973 and 1979. As a consequence a higher fraction of industries, $y_i^c - y_i^d$ became negative.

Additional conclusions for the period considered, from the middle of the 1960s to the late 1980s, were:

1. Keynesian unemployment has been the dominant regime.
2. The influence of demand pressure on prices has been negligible; instead demand pressures spill over into increased imports.
3. The degree of capacity utilization has been a significant determinant of investment.
4. The elasticity of prices with respect to wage costs is substantial, ranging from 0.5 in the short run to 1.0 in the long run.
5. Increases in real wages induce capital-labor substitution.
6. The main determinant of output growth in the eighties in Western Europe has been effective demand.

19.5 Concluding remarks

(incomplete)

Let us summarize. This chapter has extended the *income-expenditure model*, known from introductory textbooks, with some microeconomic underpinnings. The framework is based on the idea that for short-run analysis of effects of demand shocks in an industrialized economy it makes sense, as a first approximation, to treat the nominal price level as a predetermined variable. We have built on the

[25] Warning: As $YD/Y > 1$, from (19.49) it may appear that a rise in mismatch, i.e., decrease in ρ, raises the Keynesian multiplier $\partial Y/\partial \bar{G}$. This counter-intuitive impression is false, however. Indeed, a decrease in ρ means an increased YD/Y through a reduced Y, cf. (19.47).

assumption that the wage level is predetermined and that the marginal cost curve is horizontal. When this is combined with constant markups due to a more or less constant price elasticity of demand at firm level, a price level which is independent of output in the short run follows.

An alternative - or supplementary - approach to the explanation of the presumed price insensitivity, building on the menu cost theory, was also described. The idea here is that there are fixed costs (pecuniary or non-pecuniary) associated with changing prices. The main theoretical insight of the menu cost theory is that even *small* menu costs can be enough to prevent firms from changing their price. This is because the opportunity cost of not changing price is only of second order, i.e., "small"; this is a reflection of the envelope theorem. So, nominal prices may be sticky in the short run even if marginal costs, MC, are rising. But owing to imperfect competition (price $> MC$), the effect on aggregate output, employment, and welfare of not changing prices is of first order, i.e., "large".

The described framework allows us to think in terms of general equilibrium, in the sense of a state of rest, in spite of the presence of some non-clearing markets. First and foremost the labor market belongs to the latter category. A key distinction is the one between *effective* supplies and demands and *actual* transactions.

Apart from the border case where all markets clear, three different types of short-run equilibria arise: repressed inflation, classical unemployment, and Keynesian unemployment. The Keynesian view is that the latter type of short-run equilibrium is prevalent in industrialized economies. Repressed inflation seems rare. We may put it this way: wages and prices appear less sticky in situations with upward pressure than in situations with downward pressure. Moreover, as long as there is a positive mark-up, classical unemployment is ruled out, at the theoretical level, if constant short-run marginal costs are assumed. Not all macroeconomists regard such an assumption empirically tenable, especially with a view on peak periods in the business cycle.

Abundant capacity. Micro-markets. Mismatch.

Empirics on price stickiness: Blinder ( ).

Bils, Klenow, and Malin (2012) find evidence in support of Keynesian labor demand

A rigorous general equilibrium model with monopolistic competition, the Blanchard-Kiyotaki model,[26] is set up and analyzed in the next chapter. That model includes a complete description of the households with respect to preferences regarding differentiated consumption goods and supply of different types of labor. Still only a single financial asset is available, base money. Readers eager to attribute to asset markets a more important role may jump directly to

[26] Blanchard and Kiyotaki (1987).

Chapter 21. That chapter presents and analyzes the IS-LM model, based on John R. Hicks' summary of the analytical content of Keynes' main opus, *The General Theory of Employment, Interest and Money* from 1936.[27] This summary became a cornerstone of mainstream short-run macroeconomics after the Second World War.

Throughout this chapter the functioning of labour markets has received scant attention. As alluded to, the Blanchard-Kiyotaki model of the next chapter represents one approach to the integration of labor markets. In Chapter 24 other approaches are discussed.

19.6 Literature notes

(incomplete)

The basic model in Keynes' *General Theory* (1936) relied less on imperfect competition than what became normal in later Keynesian thinking, as articulated for instance by the World's Smallest Macroeconomic Model and the various new-Keynesian contributions to be considered later. In Keynes (1936) only the labor market has imperfect competition, resulting in a predetermined nominal wage level. In the output market, perfect competition with full price flexibility rules. This feature, including Keynes' associated conjecture that real wages would be countercyclical, was criticized on empirical grounds by Dunlop (1938) and Tarshis (1939). In his answer, Keynes (1939) acknowledged the need for reconsideration of this matter.

In the vocabulary of Walrasian economics, the term *equilibrium* is reserved to states where all markets clear unless the price in question has fallen to zero. On this background it may be a surprise that one may talk about Keynesian *equilibrium* with unemployment. But equilibrium is an abstract concept that need not require equality of *Walrasian* demand and supply. Walrasian equilibrium is just *one* type of equilibrium, *one* type of state of rest for an economic system. In this chapter we have introduced *another* kind of state of rest, relevant under other circumstances: *equilibrium with quantity rationing.* By adhering to this terminology, we follow the strand within macroeconomics called "macroeconomics with quantity rationing", to which the French scholar Edmund Malinvaud, cited in the introduction to this chapter, belongs. The first to show existence of general equilibrium in a fully articulated disaggregate setup, but with fixed prices and quantity rationing, was another French economist, Jean-Pascal Benassy (1975). An important precursor is Chapter 14 in Arrow and Hahn (1971).

There is an alternative terminology in which a state of affairs with non-clearing

[27] Hicks,1937.

markets, in the Walrasian sense, is termed a *disequilibrium*. The title, “A general disequilibrium model of income and employment”, of the seminal paper by the American economists Robert Barro and Herchel Grossman (1971) is a case in point as is the vocabulary in the book *On Keynesian Economics and the Economics of Keynes* by Axel Leijonhufvud (1968).[28] Although terminologies differ, contributors to these strands of macroeconomics, including Patinkin (1956), Clower (1965), and Herings (1996), seem to agree that the important aspects are the dynamic processes triggered by non-clearing markets. A thorough account of macroeconomics with quantity rationing is given in Malinvaud (1998b). The theory has been applied to analytical studies of mass unemployment as in, e.g., Malinvaud (1984, 1994) and empirical studies like the large econometric multi-country study entitled *Europe's Employment Problem* (1990), based on the theoretical framework in Sneessens and Drèze (1986) and Lambert (1988).

Already Karl Marx (1867) rejected Say's law by emphasizing the option of hoarding money instead of buying produced goods. A contemporary examination of the role of Walras' law and refutation of Say's law in macroeconomics is contained in Patinkin (2008). Recurring controversy about Say's “Law”, or Say's “fallacy”, as some opponents say, arise during periods of severe recession or depression. During the Great Depression Keynes charged the UK Treasury and contemporary economists for being “‘deeply steeped in the notion that if people do not spend their money in one way they will spend it in another” (Keynes, 1936, p. 20). By a series of citations, DeLong (2009) finds that a similar notion characterizes one side in “the fiscal stimulus controversy” in the US in the aftermath of the financial crisis 2007-08.

Keynes (1937).

Comparison between Keynes (1936) and Keynes (1939). Keynes and Hicks. Kalecki.

In Section 19.2 we assumed that he perceived quantity signals, like the price signals, are deterministic. In Svensson.( ) the theory is extended to include stochastic quantity signals.

Market forms: For estimations of the markup in various U.S. industries, see, e.g., Hall (1988).

Even a dynamic general equilibrium with perfect competition is not a completely lucid thing. In perfect competition all firms are price takers. So who is left to change prices? This is a sign of a logical difficulty within standard competitive theory (as pointed out by Arrow, 1959). Merely assigning price setting to abstract “market forces” is not theoretically satisfactory, and reference to the

[28] A few years after the publication of this paper, Robert Barro lost confidence in the Keynesian stuff and became one of the leading new-Classical macroeconomists. His reasons are given in Barro (1979).

mythical “Walrasian auctioneer” is not convincing.

Reference to inventory dynamics?

The convex price adjustment cost approach, Rotemberg, JME, 52 (4), 1982, 829-852.

Approaches to *menu costs in a dynamic context*:

Caplin and Spulper (1987), briefly summarized in Benassy (2011, p. 317-18).

The inclusion of ongoing inflation, see Blanchard 1990 and Jeanne 1998.

One might conjecture *G*insburg et al., EL, 1991.

Danziger (AER PP, 1999, SJE 2008).

Bursteain and Hellwig (AER, 2008).

Caballero and Engel (2007).

What about costs of changing the production level?

19.7 Appendix

A. The envelope theorem

ENVELOPE THEOREM FOR AN UNCONSTRAINED MAXIMUM Let $y = f(a, x)$ be a continuously differentiable function of two variables, of which one, a, is conceived as a parameter and the other, x, as a control variable. Let $g(a)$ be a value of x at which $\frac{\partial f}{\partial x}(a, x) = 0$, i.e., $\frac{\partial f}{\partial x}(a, g(a)) = 0$. Let $F(a) \equiv f(a, g(a))$. Provided $F(a)$ is differentiable,

$$F'(a) = \frac{\partial f}{\partial a}(a, g(a)),$$

where $\partial f / \partial a$ denotes the partial derivative of $f(a, x)$ w.r.t. the first argument.

Proof $F'(a) = \frac{\partial f}{\partial a}(a, g(a)) + \frac{\partial f}{\partial x}(a, g(a))g'(a) = \frac{\partial f}{\partial a}(a, g(a))$, since $\frac{\partial f}{\partial x}(a, g(a))$ $= 0$ by definition of $g(a)$. $\square$

That is, when calculating the total derivative of a function w.r.t. a parameter and evaluating this derivative at an interior maximum w.r.t. a control variable, the envelope theorem allows us to ignore the terms that arise from the chain rule. This is also the case if we calculate the total derivative at an interior minimum.[29]

B. The opportunity cost of not changing price is of second order

(no text yet available)

[29] For extensions and more rigorous framing of the envelope theorem, see for example Sydsaeter et al. (2006).

19.8 Exercises

Chapter 21

The IS-LM model

After more basic reflections in the previous two chapters about short-run analysis, in this chapter we revisit what became known as the *IS-LM model*. This model is based on John R. Hicks' summary of the analytical core of Keynes' *General Theory of Employment, Interest and Money* (Hicks, 1937). The distinguishing element of the IS-LM model compared with both the World's Smallest Macroeconomic Model of Chapter 19 and the Blanchard-Kiyotaki model of Chapter 20 is that an interest-bearing asset is added. There is then scope for considering money holding as motivated primarily by its liquidity services rather than its role as a store of value.

The version of the IS-LM model presented here is in one respect different from the presentation in many introductory and intermediate textbooks. The tradition has been to see the IS-LM model as just one building block of a more involved aggregate supply-aggregate demand (AD-AS) framework where only the wage level is predetermined while the output price is flexible and adjusts in response to shifts in aggregate demand, triggered by changes in exogenous variables. We interpret the IS-LM model differently, namely as a self-standing short-run model in its own right, based on the assumption that both wages and prices are set in advance and that the wage and price setters operate in imperfectly competitive markets and refrain from frequent price changes.

The model deals with mechanisms supposed to be operative within a "short period". Expectations are extrapolative but their role is diminutive. The interaction between current events and the expected future is kept at a minimum.

We may think of the period length to be a month, a year, or something in between. The focus is on the interaction between the output market and the asset markets. The model conveys the central message of Keynes' theory: the equilibrating forces in the output and money markets are adjustments in the output level and the nominal interest rate. We survey the Keynesian tenets known as *spending multipliers*, the *balanced budget multiplier*, the *paradox of thrift*, and the *liquidity trap*. This will serve as an introduction to the subsequent chapters where we extend the IS-LM model with endogenous forward-looking expectations and consider dynamics.

A by-product of the present chapter is training in *comparative statics* by means of *Cramer's rule* applied to a system of two non-linear equations with two endogenous and many exogenous variables. This provides a first simple approach to insight into how the economy reacts to changes in the "environment", that is, the exogenous variables. The focus is on the mechanisms and mutual dependencies in the system in the short run.

21.1 The building blocks

We consider a closed economy with a private sector, a government, and a central bank. The produce of the economy consists mainly of manufacturing goods and services, supplied under conditions of imperfect competition, imperfect credit markets, and price stickiness of some sort. The "money supply" in the model is usually interpreted as money in the broad sense and thus includes money created by a commercial bank sector in addition to currency in circulation.

The model starts out directly from presumed aggregate behavioral relationships. These are supposed to roughly characterize the economy-wide behavior of heterogeneous populations of firms and households, respectively, with imperfect information. On the one hand this lack of microfundation is of course a limitation of the model. On the other hand, it helps to avoid too many complexities to arise in a first approach, when one leaves the comfortable realm of perfect competition, perfect information, and homogeneous agents.

21.1.1 The output market

Demand

Aggregate output demand is given as

$$\begin{aligned} Y^d &= C(Y^p, Y^e_{+1}, qK, r^e) + I(Y^e_{+1}, K, r^e) + G + \varepsilon_D, && (21.1)\\ C_{Y^p} &> 0, C_{Y^e_{+1}} > 0, I_{Y^e_{+1}} > 0, C_{Y^p} + C_{Y^e_{+1}} + I_{Y^e_{+1}} < 1, && (21.2)\\ C_{(qK)} &> 0, C_{r^e} \leq 0, I_K < 0, I_{r^e} < 0, \end{aligned}$$

where the function $C(\cdot)$ represents private consumption, the function $I(\cdot)$ represents private fixed capital investment, G is public spending on goods and services, and ε_D is a shift parameter summarizing the role of unspecified exogenous variables that suddenly may affect the level of consumption or investment. A rise in the general "state of confidence" may thus be result in a higher level of investment than otherwise and a higher preference for the present relative to the future may result in a higher level of consumption than otherwise. Arguments appearing in the consumption and investment functions include Y^p which is current private disposable income, Y^e_{+1} which is expected output the next period (or period*s*), q and K which are commented on below, and finally r^e which is the *expected* short-term real interest rate. In this first version of the model we assume there are only two assets in the economy, money and a one-period bond with a real interest rate r.

The signs of the partial derivatives of the consumption and investment functions in (21.1) are explained as follows. A general tenet from earlier chapters is that consumption depends positively on household wealth. One component of household wealth is financial wealth, here represented by the market value, $q \cdot K$, of the capital stock K (including the housing stock). Another component is perceived human wealth (the present value of the expected labor earnings stream), which tends to be positively related to both Y^p and Y^e_{+1}. The separate role of disposable income, Y^p, reflects the hypothesis that a substantial fraction of households are credit constrained. The role of the interest rate, r, reflects the hypothesis that the negative substitution and wealth effects on current consumption of a rise in the real interest rate dominate the positive income effect. These hypotheses find support in the empirical literature.

Firms' investment depends positively on Y^e_{+1}. This is because the productive capacity needed next period depends on the expected level of demand next period. In addition, investment in new technologies is more paying when expected sales are high. On the other hand, the more capital firms already have, the less they need to invest, hence $I_K < 0$. Finally, the cost of investing is higher the higher is the real interest rate. These features are consistent with the q-theory of investment when considering an economy where firms' production is demand constrained (cf. Chapter 14).

Disposable income is given by

$$Y^p \equiv Y - \mathbb{T}, \tag{21.3}$$

where Y is aggregate factor income (= GNP) and $\mathbb{T}$ is real net tax revenue in a broad sense, that is, $\mathbb{T}$ equals gross tax revenue minus transfers *and minus* interest service on government debt. We assume a quasi-linear net tax revenue function

$$\mathbb{T} = \tau + T(Y), \qquad 0 \leq T'(Y) < 1,$$

where τ is a constant parameter reflecting "tightness" of discretionary fiscal policy. Fiscal policy is thus described by two variables, G representing government spending on goods and services and τ representing the discretionary element in taxation. A balanced primary budget is the special case $\tau + T(Y) = G$. The endogenous part, $T(Y)$, of the tax revenue is determined by given taxation rules; when $T' > 0$, these rules act as "automatic stabilizers" by softening the effects on disposable income, and thereby on consumption, of changes in output and employment.

With regard to expected output next period, Y_{+1}^e, the model takes a shortcut and assumes Y_{+1}^e is simply an increasing function of current output and nothing else:

$$Y_{+1}^e = \varphi(Y), \qquad 0 < \varphi'(Y) \leq 1. \tag{21.4}$$

We make a couple of simplifications in the specification of aggregate private output demand. First, since we only consider a single period, we treat the amount of installed capital as a given constant, $\bar{K}$, and suppress the explicit reference to $\bar{K}$ in the consumption and investment functions. Second, we ignore the possible influence of q (which may be more problematic). As an implication, we can express aggregate private demand (the sum of C and I) as a function $D(Y, r^e, \tau)$, whereby (21.1) becomes

$$\begin{aligned} Y^d &= D(Y, r^e, \tau) + G + \varepsilon_D, \qquad \text{where} && (21.5)\\ 0 &< D_Y = C_{Y^p}(1 - T'(Y)) + (C_{Y_{+1}^e} + I_{Y_{+1}^e})\varphi'(Y) < 1, && (21.6)\\ D_{r^e} &= C_{r^e} + I_{r^e} < 0, \text{ and } D_\tau = -C_{Y^p} \in (-1, 0). && (21.7) \end{aligned}$$

Behind the scene: production and employment

Prices on goods and services have been set in advance by firms operating in markets with monopolistic competition. Owing to either constant marginal costs or the presence of menu costs, when firms face shifts in demand, they change production rather than price. There is scope for maintaining profitability this way because wages are sticky (due to long-term contracts, say) and the preset prices are normally above marginal costs.

Behind the scene there is an aggregate production function, $Y = F(\bar{K}, N)$, where N is *employment*. The conception is that under "normal circumstances" there is abundant capacity. That is, the given capital stock, $\bar{K}$, is large enough so that output demand can be satisfied, i.e.,

$$F(\bar{K}, N) = Y^d, \tag{21.8}$$

without violating the rule of the minimum as defined in Chapter 19. Assuming $F_N > 0$, we can solve the equation (21.8) for firms' desired employment, N^d, and write $N^d = \mathcal{N}(Y^d, \bar{K})$, where $\mathcal{N}_{Y^d} > 0$ and $\mathcal{N}_K < 0$ under the assumption that $F_K > 0$.

Let $\bar{N}$ denote the size of the *labor force,* i.e., those people holding a job or registered as being available for work. The actual employment, N, must satisfy

$N \leq \bar{N} - \tilde{U}$, where $\tilde{U}$ is *frictional unemployment.* We use this term in a broad sense comprising people inevitably unemployed in connection with change of job and location in a vibrant economy, people unemployed because of mismatch of skills and job opportunities, and people unemployed because their reservation wage is above the market wage. The remainder of the labor force that are unemployed are said to be *involuntarily unemployed* in the sense of being ready and willing to work at the going wage or even a bit lower wage. The IS-LM model deals with the case where firms' desired employment, $\mathcal{N}(Y^d, K)$, can be realized, that is, the case where $\mathcal{N}(Y^d, \bar{K}) \leq \bar{N} - \tilde{U}$.

Terminological remarks With $\hat{U}$ denoting those involuntarily unemployed, total unemployment, U, can be written

$$U = \bar{N} - N = \tilde{U} + \hat{U}.$$

In an alternative decomposition of unemployment one writes

$$U = U^n + U^c,$$

where U^n is the *NAIRU unemployment* level and U^c the remainder unemployment, often called *cyclical* unemployment (a positive number in a recession, a negative number in a boom). So U^n is defined as the level of unemployment prevailing when the unemployment rate, $U/\bar{N}$, equals what is known as the NAIRU, namely that rate of unemployment which generates neither upward nor downward pressure on the inflation rate. The term "NAIRU" (an abbreviation of non-accelerating-inflation-rate-of-unemployment) is in fact a misnomer because the point is not absence of acceleration but merely absence of pressure on the inflation rate in one or the other direction. Nevertheless, we shall stick to this term, because the alternative terms offered in the literature are not better. One is the "natural rate of unemployment"; but there is nothing natural about that unemployment rate – it depends on legal institutions, economic policy, and structural characteristics of the economy. Another – somewhat elusive – name is the "structural rate of unemployment".[1]

In Keynesian theory the *NAIRU unemployment* rate, $U_n/\bar{N}$, is perceived as generally being below $\tilde{U}/\bar{N}$. And business cycle fluctuations in unemployment are perceived as primarily reflecting fluctuations in $\mathcal{N}(Y^d, \bar{K})$ rather than in $\bar{N} - \tilde{U}$. While the size and composition of unemployment generally matter for wage and

[1]Our formulations here implitly presuppose that absence of pressure on the inflation rate can be traced to a *single* rate of unemployment. However, there exist empirics as well as theory implying that under certain conditions there is a *range* of unemployment rates within which no pressure on the inflation rate is generated, neither upward nor downward (see Chapter 24).

price changes, the IS-LM model considers such effects as not materializing until at the earliest the next period. Being concerned about only a single short period, the model is therefore often tacit about production and employment aspects and leave them "behind the scene".

21.1.2 Asset markets

In this first version of the IS-LM model we assume that only two financial assets exist, money and an interest-bearing short-term bond. The latter may be issued by the government as well as private agents/firms. Although not directly visible in the model, it is usually understood that there are commercial banks that accept deposits and provide bank loans to households and firms. Bank deposits are then considered as earning no interest at all.[2] Up to a certain amount bank deposits are nevertheless attractive because for many transactions liquidity is needed. Bank deposits are also a fairly secure store of liquidity, being better protected against theft than cash and being, in modern times, also protected against bank default by government-guaranteed deposit insurance. The interest rate on bank loans allows the banks a revenue over and above the costs associated with banking.

Let M denote the money stock (in the implied broad sense), held by the non-bank public at a given date. That is, in addition to currency in circulation, the bank-created money in the form of liquid deposits in commercial banks is included in M. We may thereby think of M as representing what is in the statistics denoted either M_1 or M_2, cf. Chapter 16. The bank lending rate is assumed equal to the short-term nominal interest rate, i, on government bonds. All interest-bearing assets are considered perfect substitutes from the point of view of the investor and will from now just be called "bonds".

The demand for money is assumed given by

$$M^d = P \cdot (L(Y, i) + \varepsilon_L), \qquad L_Y > 0,\ L_i < 0, \tag{21.9}$$

where P is the output price level (think of the GDP deflator) and ε_L is a shift parameter summarizing the role of unspecified exogenous variables that may affect money demand for any given pair (Y, i). Apart from the shift term, ε_L, real money demand is given by the function $L(Y, i)$, known as the *liquidity preference function.* The first partial derivative of this function is positive reflecting the *transaction motive* for holding money. The output level is an approximate statistic (a "proxy") for the flow of transactions for which money is needed. The negative sign of the second partial derivative reflects that the interest rate, i, is the opportunity cost of holding money instead of interest-bearing assets.

[2]In practice even checkable deposits in banks may earn a small nominal interest, but this is ignored by the model.

The part of non-human wealth not held in the form of money is held in the form of an interest-bearing asset, a one-period bond. We imagine that also firms' capital investment is financed by issuing such bonds. The bond offers a payoff equal to 1 unit of money at the end of the period. Let the market price of the bond at the beginning of the period be v units of money. The implicit nominal interest rate, i, is then determined by the equation $v(1+i) = 1$,[3] i.e.,

$$i = (1 - v)/v. \tag{21.10}$$

There is a definitional link between the nominal interest rate and the expected short-term real interest rate, r^e. In continuous time we would have $r^e = i - \pi^e$ with i as the instantaneous nominal interest rate (with continuous compounding) and π ($\equiv \dot{P}/P$) as the (forward-looking) instantaneous inflation rate, the superscript e indicating expected value. But in discrete time, as we have here, the appropriate way of defining r^e is more involved. The holding of money is motivated by the need, or at least convenience, of ready liquidity to carry out expected as well as unexpected spending in the near future. To perform this role, money must be held in advance, that is, at the beginning of the (short) period in which the purchases are to be made ("cash in advance"). If the price of a good is P euro to be paid at the end of the period and you have to hold this money already from the beginning of the period, you effectively pay $P + iP$ for the good, namely the purchase price, P, plus the opportunity cost, iP. Postponing the purchase one period thus gives savings equal to $P + iP$. The price of the good next period is P_{+1} which, with cash in advance, must be held already from the beginning of that period. So the real gross rate of return obtained by postponing the purchase one period is

$$1 + r = (1+i)P\frac{1}{P_{+1}} = \frac{1+i}{1+\pi_{+1}},$$

where $\pi_{+1} \equiv (P_{+1} - P)/P$ is the inflation rate from the current to the next period. As seen from the current period, P_{+1} and π_{+1} are generally not known. So decisions are based on the expected real interest rate,

$$r^e = \frac{1+i}{1+\pi^e_{+1}} - 1 \approx i - \pi^e_{+1}, \tag{21.11}$$

where the approximation is valid for "small" i and π^e_{+1}.

21.2 Keynesian equilibrium

The model assumes that both the output and the money market clear by adjustment of output and nominal interest rate so that in both markets supply equals

[3] In continuous time with compound interest, $ve^i = 1$ so that $i = -\ln v$.

demand:

$$\begin{aligned} Y &= D(Y, i - \pi^e_{+1}, \tau) + G + \varepsilon_D, \ 0 < D_Y < 1, \ D_{r^e} < 0, \ -1 < D_\tau < 0, \quad \text{(IS)} \\ \frac{M}{P} &= L(Y, i) + \varepsilon_L, \qquad L_Y > 0, \ L_i < 0, \quad \text{(LM)} \end{aligned}$$

where, for simplicity, we have used the approximation in (21.11), and where M is the available money stock at the beginning of the period. In reality, the central bank has direct control only over the monetary base. Yet the traditional understanding of the model is that through this, the central bank has under "normal circumstances" control also over M. With M given by monetary policy, the interpretation of the equations (IS) and (LM) is therefore that output and the nominal interest rate quickly adjust so as to clear the output and money markets.

The equation (IS), known as the *IS equation*, asserts clearing in a *flow* market: so much output *per time unit* matches the effective demand *per time unit* for this output. The name comes from an alternative way of writing it, namely as $I = S$ (investment = saving, where saving $S = Y - C - G - \varepsilon_D$).

In contrast, the equation (LM), known as the *LM equation*, asserts clearing in a *stock* market: so much liquidity demand matches the available money stock, M, at a given point in time. In our discrete time setting we think of asset market openings occurring in a diminutive time interval at the beginning of each period. And we think of changes in the money stock as taking place abruptly from market opening to market opening. Agents' decisions about portfolio composition, consumption, and investment are also thought of as being made at the beginning of each period. Production takes place *during* the period and at the end of the period receipts for work and lending and payment for consumption occur. This interpretation calls for a quite short period length.

At the empirical level we have data for M and i on a daily basis, whereas the period length of data for aggregate output, consumption, and investment, is usually a year or at best a quarter of a year. So, in connection with econometric analyses, instead of linking M and i to a single point in time, one may think of M and i as averages over a year (or a quarter of a year). A possible interpretation would then be that the year still consists of many subperiods with their own asset supplies and demands as well as production and consumption flows. The environment of the system remains unchanged throughout the year, and the system remains in equilibrium with constant stocks and flows.

Having specified the LM equation, should we not also specify a condition for clearing in the market for bonds? Well, we do not have to. The balance sheet constraint of the non-bank private sector guarantees that clearing in the money market implies clearing also in the bond market – and vice versa. To see this, let W denote the nominal financial wealth of the non-bank private sector and let

x denote the number of one-period bonds held on net by the non-bank private sector. Each bond offers a payoff of 1 unit of money at the end of the period and is by the market priced $v = 1/(1+i)$ at the beginning of the period. Then $M + vx \equiv W$. With x^d denoting the on net by the non-bank private sector demanded quantity of bonds, we have $M^d + vx^d = W$. This is an example of a *balance sheet constraint* and implies a "Walras' law for stocks". Subtracting the first from the second of these two equations yields

$$M^d - M + v(x^d - x) = 0. \tag{21.12}$$

Given $v > 0$, it follows that if and only if $M^d = M$, then $x^d = x$. That is, clearing in one of the asset markets implies clearing in the other. Hence it suffices to consider just one of these two markets explicitly. Usually the money market is considered.

The IS and LM equations amount to the traditional IS-LM model in compact form. The exogenous variables are $P, \pi^e_{+1}, \tau, G, \varepsilon_D, \varepsilon_L$, and, in the traditional interpretation, M. Given the values of these variables, a solution, (Y, i), to the equation system consisting of (IS) and (LM) is an example of a *Keynesian equilibrium.* It is an *equilibrium* in the sense that, given the prevailing expectations and preset goods prices, asset markets clear by price adjustment (here adjustment of i) and the traded quantity in the goods market complies with the short-side rule (the rule saying that the short side of the market determines the traded quantity).[4] The model assumes that both the output and the money market clear by adjustment of output and nominal interest rate so that in both markets supply equals demand: It is a *Keynesian* equilibrium because it is aggregate demand in the output market which is the binding constraint on output (and implicitly thereby also on employment).

The current price level, P, is seen as predetermined and maintained through the period. But the price level P_{+1} set for the *next* period will presumably not be independent of current events. So expected inflation, π^e_{+1}, *ought* to be endogenous. It is therefore a deficiency of the model that π^e_{+1} is treated as exogenous. Yet this may give an acceptable approximation as long as the sensitivity of expected inflation to current events is small.

[4] In the introductory paragraph to this section it was briefly said that the IS-LM model assumes that in both the output and the money market "supply equals demand". It is important to be aware what "supply" means in each of the markets. By "supply" in the output market, a flow market where nominal price stickiness rules, is meant the "effective supply", cf. Section 19.2. By "supply" in the money market, a stock market, by "supply" is meant the sum of currency in circulation and the liquid deposits that commercial banks offer to the general public.

21.3 Alternative monetary policy regimes

We shall analyze the functioning of the described economy in three alternative simplistic monetary policy regimes. In the first policy regime the central bank is assumed to maintain the money stock at a certain target level within the period. This is the case of a *money stock rule*. In the second policy regime, trough open-market operations the central bank maintains the interest rate at a certain target level within the period. This is the case of a *fixed interest rate rule* (where "fixed" should be interpreted as "fixed but adjustable"). The third policy regime to be considered is a *countra-cyclical interest rate rule* where both the interest rate and the money stock are endogenous. The static IS-LM model is not suitable for a study of a Taylor-rule regime since that involves dynamics and policy reactions to the rate of inflation.

Some writers interpret the the LM part of the name IS-LM as referring to the specific monetary policy of keeping M constant − the money stock rule. We interpret the name IS-LM as covering a broader framework which can be applied to a range of policies.

21.3.1 Money stock rule

Here the central bank maintains the money stock at a certain target level $M > 0$. We assume that given this M, circumstances are such that the generally nonlinear equation system (IS) - (LM) has a solution (Y, i) and, until further notice, that both Y and i are strictly positive.

The IS-LM diagram

For convenience, we repeat our equation system:

$$Y = D(Y, i - \pi^e_{+1}, \tau) + G + \varepsilon_D, \quad 0 < D_Y < 1, \; D_{r^e} < 0, \; -1 < D_\tau < 0 \tag{IS}$$

$$\frac{M}{P} = L(Y, i) + \varepsilon_L, \qquad L_Y > 0, \; L_i < 0, \tag{LM}$$

The determination of Y and i is conveniently illustrated by an IS-LM diagram, cf. Fig. 21.1. First, consider the equation (IS). We *guess* that this equation defines (determines) i as an implicit function of the other variables in the equation, Y, π^e_{+1}, τ, G, and ε_D:

$$i = i_{IS}(Y, \pi^e_{+1}, \tau, G, \varepsilon_D).$$

The partial derivative of this function w.r.t. Y can be found by taking the

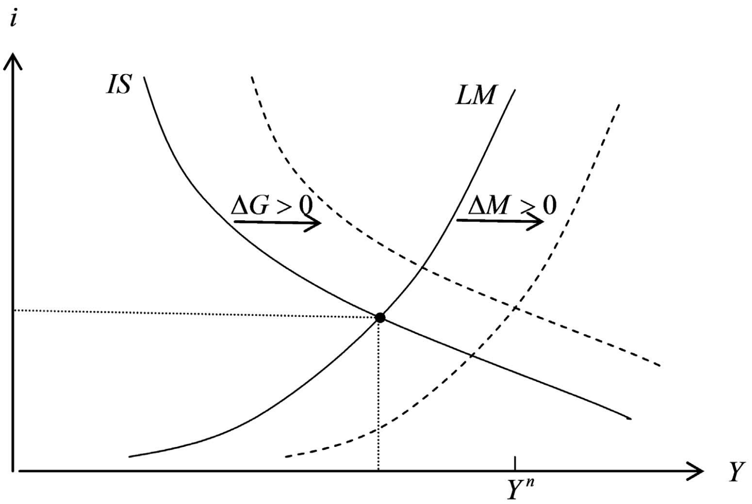

Figure 21.1: The IS-LM cross when M is exogenous; a case with equilibrium output below the NAIRU level, Y^n (π^e_{+1},τ,G,ε_D,M/P, and ε_L given).

differential w.r.t. Y and i on both sides of (IS),[5]

$$dY = D_Y dY + D_{r^e} di,$$

and rearranging:

$$\partial i/\partial Y_{|IS} = \frac{di}{dY} = \frac{1 - D_Y}{D_{r^e}} < 0, \tag{21.13}$$

where the first equality is valid by construction, and where the negative sign follows from the information given (IS). The observation that the denominator, D_{r^e}, in (21.13) is not zero confirms our guess that the equation (IS) defines i as an implicit function of the other variables in the equation.

The solution for the derivative in (21.13) tells that higher aggregate demand in equilibrium requires that the interest rate is lower. In Fig. 21.1, this relationship is illustrated by the downward-sloping *IS curve*, which is the locus of combinations of Y and i that are consistent with clearing in the output market. The slope of this locus is given by (21.13).

Next consider the equation (LM). We *guess* that this equation defines i as an implicit function of the other variables in the equation, Y, M/P, and ε_L :

$$i = i_{LM}(Y, \frac{M}{P}, \varepsilon_L).$$

[5] On the concepts of implicit function and differentials, see Math Tools.

The partial derivative of this function w.r.t. Y can be found by taking the differential w.r.t. Y and i on both sides of (LM),

$$0 = L_Y dY + L_i di,$$

and rearranging:

$$\partial i/\partial Y_{|LM} = \frac{di}{dY} = \frac{-L_Y}{L_i} > 0, \tag{21.14}$$

where the first equality is valid by construction, and the positive sign follows from the information given in (LM). The observation that the denominator in (21.14) is *not* zero confirms our guess that the equation (LM) defines i as an implicit function of the other variables in the equation.

The solution for the derivative in (21.14) tells that for the money market to clear, a higher volume of transactions must go hand in hand with a higher interest rate. In Fig. 21.1, this relationship is illustrated by the upward-sloping *LM curve*, which is the locus of combinations of Y and i that are consistent with clearing in the money market.

A solution (Y, i) to the model is unique (the point of intersection in Fig. 21.1). Hence we can write Y and i as (unspecified) functions of all the exogenous variables:

$$Y = f(\frac{M}{P}, \pi^e_{+1}, \tau, G, \varepsilon_D, \varepsilon_L), \tag{21.15}$$

$$i = g(\frac{M}{P}, \pi^e_{+1}, \tau, G, \varepsilon_D, \varepsilon_L). \tag{21.16}$$

Comparative statics

How do Y and i depend on the exogenous variables? A *qualitative* answer can easily be derived by considering in what direction the IS curve and the LM curve shift in response to a change in an exogenous variable. With minimal training, the directions of these shifts can be directly read off the information given in (IS) and (LM) equations. Alternatively one can use the total differentials (21.17) and (21.18) below also for this purpose.

A *quantitative* answer is based on the standard comparative statics method. Starting afresh with the (IS) - (LM) equation system, we *guess* that the system defines (determines) Y and i as implicit functions, f and g, of the other variables, as in (21.15) and (21.16). The aim is to find formulas for the partial derivatives of these implicit functions, evaluated at an equilibrium point (Y, i), a point satisfying (IS) and (LM). We first calculate the *total* differential on both sides of (IS):

$$dY = D_Y dY + D_{r^e}(di - d\pi^e_{+1}) + D_\tau d\tau + dG + d\varepsilon_D. \tag{21.17}$$

Next we calculate the *total* differential on both sides of (LM):

$$d(\frac{M}{P}) = L_Y dY + L_i di + d\varepsilon_L. \tag{21.18}$$

We interpret these two equations as a *new equation system* with two *new endogenous* variables, the differentials dY and di. The changes, $d\pi^e_{+1}$, dG, $d\tau$, $d\varepsilon_D$, $d(M/P)$, and $d\varepsilon_L$, in the exogenous variables are our *new exogenous* variables. The coefficients, D_Y, D_{r^e}, etc., to these endogenous and exogenous variables in the two equations are derivatives evaluated at the equilibrium point (Y, i). Like the original equation system (IS) - (LM), the new system is simultaneous (not recursive).

The key point is that the new system is *linear*. The further procedure is the following. First rearrange (21.17) and (21.18) so that dY and di appear on the left-hand side and the differentials of the exogenous variables on the right-hand side of each equation:

$$\begin{aligned}(1 - D_Y)dY - D_{r^e}di &= -D_{r^e}d\pi^e_{+1} + D_\tau d\tau + dG + d\varepsilon_D, & (21.19)\\ L_Y dY + L_i di &= d\frac{M}{P} - d\varepsilon_L. & (21.20)\end{aligned}$$

Next, calculate the determinant, Δ, of the coefficient matrix on the left-hand side of the system:

$$\Delta = \begin{vmatrix} 1 - D_Y & -D_{r^e} \\ L_Y & L_i \end{vmatrix} = (1 - D_Y)L_i + D_{r^e}L_Y < 0, \tag{21.21}$$

where the negative sign follows from qualitative information about the functions D and L given in (IS) and (LM), respectively. The observation that the determinant is *not* zero confirms our guess that the (IS) - (LM) system defines Y and i as implicit functions of the other variables.

Now apply Cramer's rule[6] to the linear system (21.19) - (21.20) to determine dY and di:

$$\begin{aligned} dY &= \frac{\begin{vmatrix} -D_{r^e}d\pi^e_{+1} + D_\tau d\tau + dG + d\varepsilon_D & -D_{r^e} \\ d\frac{M}{P} - d\varepsilon_L & L_i \end{vmatrix}}{\Delta} \\ &= \frac{L_i(-D_{r^e}d\pi^e_{+1} + D_\tau d\tau + dG + d\varepsilon_D) + D_{r^e}(d\frac{M}{P} - d\varepsilon_L)}{\Delta}, & (21.22)\end{aligned}$$

[6]See Math Tools.

and

$$\begin{aligned} di &= \frac{\begin{vmatrix} 1-D_Y & -D_{r^e}d\pi^e_{+1} + D_\tau d\tau + dG + d\varepsilon_D \\ L_Y & d\frac{M}{P} - d\varepsilon_L \end{vmatrix}}{\Delta} \\ &= \frac{(1-D_Y)(d\frac{M}{P} - d\varepsilon_L) - L_Y(-D_{r^e}d\pi^e_{+1} + D_\tau d\tau + dG + d\varepsilon_D)}{\Delta} \end{aligned} \tag{21.23}$$

The partial derivatives of f and g, respectively, w.r.t. the exogenous variables can be directly read off these two formulas.

Suppose we are interested in the effect on Y and i of a change in the real money supply, M/P. By setting $d\pi^e_{+1} = d\tau = dG = d\varepsilon_D = d\varepsilon_L = 0$ in (21.22) and (21.23) and rearranging, we get

$$\begin{aligned} \frac{\partial Y}{\partial(\frac{M}{P})} &= f_{M/P} = \frac{dY}{d(\frac{M}{P})} = \frac{D_r}{(1-D_Y)L_i + D_{r^e}L_Y} > 0, \\ \frac{\partial i}{\partial(\frac{M}{P})} &= g_{M/P} = \frac{di}{d(\frac{M}{P})} = \frac{1-D_Y}{(1-D_Y)L_i + D_{r^e}L_Y} < 0, \end{aligned}$$

where the signs are due to (21.6), 21.7, and (21.21). Such partial derivatives of the endogenous variables w.r.t. an exogenous variable, evaluated at the equilibrium point, are known as *multipliers*. The approximative short-run effect on Y of a given small increase dM in M is calculated as $dY = (\partial Y/\partial(\frac{M}{P}))dM/P$, where we see the role of the partial derivative w.r.t. M/P as a multiplier on the increase in the exogenous variable, M/P.[7]

The intuitive interpretation of the signs of these multipliers is the following. The central bank increases the money supply through an open market purchase of bonds held by the private sector. In practice it is usually short-term government bonds ("treasury bills") that the central bank buys when it wants to increase the money supply (decrease the short-term interest rate). Immediately after the purchase, the supply of money is higher than before and the supply of bonds available to the public is lower. At the initial interest rate there is now excess supply of money and excess demand for bonds. But the attempt of agents to get rid of their excess cash in exchange for more bonds can not succeed in the aggregate because the supplies of bonds and money are given. Instead, what

[7] Instead of using Cramer's rule, in the present case we could just substitute di, as determined from (21.18), into (21.17) and then find dY from this equation. In the next step, this solution for dY can be inserted into (21.18), which then gives the solution for di. However, if L_i were a function that *could* take the value nil, this procedure might invite a temptation to rule this out by assumption. That would imply an unnecessary reduction of the domain of $f(\cdot)$ and $g(\cdot)$. The only truly necessary assumption is that $\Delta \neq 0$ and that is automatically satisfied in the present problem.

happens is that the price of bonds goes up, that is, the interest rate goes down, cf. (21.10), until the available supplies of money and bonds are willingly held by the agents. Money is therefore *not* neutral.

To find the output multiplier w.r.t. government spending on goods and services, or what is known as the *spending multiplier*, in (21.22) we set $d(M/P) = d\pi^e_{+1} = d\tau = d\varepsilon_D = d\varepsilon_L = 0$ and rearrange to get

$$\frac{\partial Y}{\partial G} = f_G = \frac{dY}{dG} = \frac{L_i}{(1 - D_Y)L_i + D_{r^e} L_Y} = \frac{1}{1 - D_Y + D_{r^e} L_Y / L_i}. \tag{21.24}$$

Under the assumed monetary policy we thus have $0 < \partial Y/\partial G < 1/(1 - D_Y)$. The difference, $1/(1 - D_Y) - \partial Y/\partial G$, is due to the *financial crowding-out effect*, represented by the term $D_{r^e} L_Y / L_i > 0$ in (21.24). Owing to the fixed money stock, the expansionary effect of a rise in G is partly offset by a rise in the interest rate induced by the increased money demand resulting from the "initial rise" in economic activity. If money demand is not sensitive to the interest rate[8] (as the monetarists claimed), the *financial crowding-out* is large and the spending multiplier low in this policy regime.

Another "moderator" comes from the marginal net tax rate, $T'(Y) \in (0, 1)$, which by reducing the private sector's marginal propensity to spend, D_Y in (21.6), acts as an *automatic stabilizer*. When aggregate output (economic activity) rises, disposable income rises less, partly because of higher taxation, partly because of lower aggregate transfers, for example unemployment compensation.[9]

Shifts in the values of the exogenous variables, ε_D and ε_L, may be interpreted as shocks (disturbances) coming from a variety of unspecified events. A positive demand shock, $d\varepsilon_D > 0$, may be due to an upward shift in households' and firms' "confidence". A negative demand shock may come from a "credit crunch" due to a financial crisis. A positive liquidity preference shock may reflect a sudden rise in the perceived risk of default of bond liabilities.

To see how demand shocks and liquidity preference shocks, respectively, affect output under the given monetary policy, in the equation (21.22) we set $d\pi^e_{+1} = d\tau = dG = d\frac{M}{P} = 0$. When in addition we set, first, $d\varepsilon_L = 0$, and next $d\varepsilon_D = 0$, we find the partial derivatives of Y w.r.t. ε_D and ε_L, respectively:

$$\begin{aligned} \frac{\partial Y}{\partial \varepsilon_D} &= f_{\varepsilon_D} = \frac{dY}{d\varepsilon_D} = \frac{L_i}{(1 - D_Y)L_i + D_{r^e} L_Y} = \frac{1}{1 - D_Y + D_{r^e} L_Y / L_i} \\ \frac{\partial Y}{\partial \varepsilon_L} &= f_{\varepsilon_L} = \frac{dY}{d\varepsilon_L} = \frac{-D_{r^e}}{(1 - D_Y)L_i + D_{r^e} L_Y} < 0. \end{aligned}$$

[8] This is the case when $|L_i|$ is low, i.e., the LM curve steep.

[9] Outside our static IS-LM model an additional issue is how current consumers repond to the *increased public debt* in the wake of a not fully tax-financed temporary increase in G. Although this takes us outside the static IS-LM model, we shall briefly comment on it towards the end of this chapter.

As expected, a positive demand shock is expansionary, while a positive liquidity preference shock is contractionary because it raises the interest rate. Note that $\partial Y/\partial\varepsilon_D = \partial Y/\partial G$ (from (21.24)) in view of the way ε_D enters the IS equation.

As now the method should be clear, we present the further results without detailing. From (21.22) and (21.23), respectively, we calculate the output and interest multipliers w.r.t. fiscal tightness to be

$$\begin{aligned}\frac{\partial Y}{\partial \tau} &= \frac{L_i D_\tau}{(1-D_Y)L_i + D_{r^e}L_Y} < 0,\\ \frac{\partial i}{\partial \tau} &= \frac{-L_Y D_\tau}{(1-D_Y)L_i + D_{r^e}L_Y} < 0.\end{aligned}$$

What do (21.22) and (21.23) imply regarding the effect of higher expected inflation on Y, i, and r^e, respectively? We find

$$\begin{aligned}\frac{\partial Y}{\partial \pi^e_{+1}} &= f_{\pi^e_{+1}} = \frac{-L_i D_{r^e}}{(1-D_Y)L_i + D_{r^e}L_Y} > 0,\\ \frac{\partial i}{\partial \pi^e_{+1}} &= g_{\pi^e_{+1}} = \frac{L_Y D_{r^e}}{(1-D_Y)L_i + D_{r^e}L_Y} \in (0,1),\\ \frac{\partial r^e}{\partial \pi^e_{+1}} &= \frac{\partial(i-\pi^e_{+1})}{\partial \pi^e_{+1}} = g_{\pi^e_{+1}} - 1 = \frac{-(1-D_Y)L_i}{(1-D_Y)L_i + D_{r^e}L_Y} \in (-1,0).\end{aligned} \tag{21.25}$$

A higher expected inflation rate thus leads to a less-than-one-to-one increase in the nominal interest rate and thereby a smaller expected real interest rate. Only if money demand were independent of the nominal interest rate ($L_i = 0$), as in the quantity theory of money, would the nominal interest rate rise one–to-one with π^e_{+1} and the expected real interest rate thereby remain unaffected.

Before proceeding, note that there is a reason that we have set up the IS and LM equations in a general nonlinear form. We want the model to allow for the empirical feature that the different multipliers generally depend on the "state of the business cycle". The spending multiplier, for instance, tends to be considerably larger in a slump − with plenty of idle resources − than in a boom. In dynamic extensions of the IS-LM model the length of the time interval associated with the higher G becomes important as does the time profile of the effect on Y. In the present static version of the model it fits intuition best to interpret the rise in G as referring to the "current" period only.

21.3.2 Fixed interest rate rule

Instead of targeting a certain level of the money stock, the central bank now keeps the nominal interest rate at a certain target level $\bar{i} > 0$. The aim may be

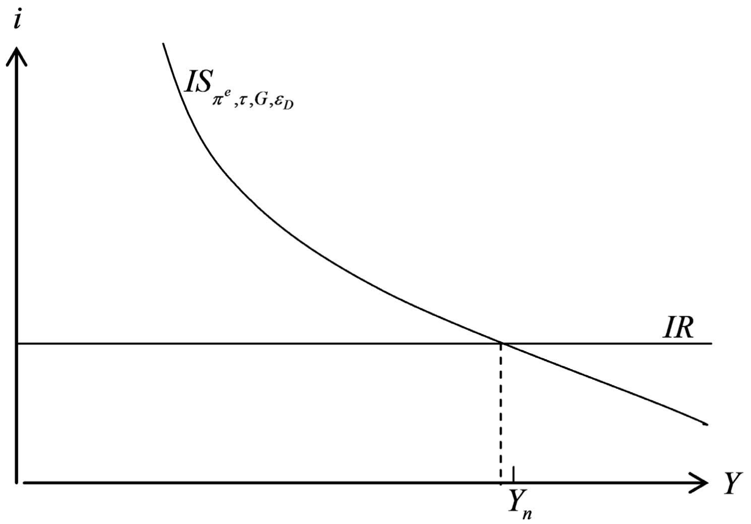

Figure 21.2: A fixed interest rate implying equilibrium output close to the NAIRU level (i,π^e_{+1}, τ, G, and ε_D given).

to have output unaffected by liquidity preference shocks. This monetary policy seems closer to what most central banks nowadays typically do. They announce a target for the nominal interest rate and then, through open-market operations, adjust the monetary base so that the target rate is realized.

In this regime, i is an exogenous constant > 0, while M and Y are endogenous. Instead of the upward-sloping LM curve we get a horizontal line, the IR line in Fig. 21.2 ("IR" for interest rate). The model is now *recursive*. Since M does not enter the equation (IS), Y is given by this equation independently of the equation (LM). Indeed, in view of $D_Y \neq 0$, the equation (IS) defines Y as an implicit function, h, of the other variables in the equation, i.e.,

$$Y = h(r^e, \tau, G, \varepsilon_D) = h(i - \pi^e_{+1}, \tau, G, \varepsilon_D). \tag{21.26}$$

Comparative statics

The partial derivatives of the function h can be directly read off equation (21.17). We find

$$\begin{aligned}
\frac{\partial Y}{\partial i} &= h_{r^e} = \frac{D_{r^e}}{1-D_Y} < 0, \\
\frac{\partial Y}{\partial \pi^e_{+1}} &= -h_{r^e} = -\frac{D_{r^e}}{1-D_Y} > 0, \\
\frac{\partial Y}{\partial \tau} &= h_\tau = \frac{D_\tau}{1-D_Y} < D_\tau < 0, \\
\frac{\partial Y}{\partial G} &= \frac{\partial Y}{\partial \varepsilon_D} = \frac{1}{1-D_Y} > 1, \\
\frac{\partial Y}{\partial \varepsilon_L} &= 0.
\end{aligned} \tag{21.27}$$

The observation that the denominator, $1 - D_Y$, is not zero confirms our guess that the equation (IS) defines Y as an implicit function of the other variables in the equation.

The derivative w.r.t. a liquidity preference shock, ε_L, in the last line of (21.27) reflects the principle that a multiplier w.r.t. an exogenous variable not entering the equation(s) determining the endogenous variable directly or indirectly (see below) is nil. In the present case this means that, with a fixed interest rule, a liquidity preference shock has no effect on equilibrium output. The shock is immediately counteracted by a change in the money stock in the same direction so that the interest rate remains unchanged. Thus, the liquidity preference shock is "cushioned" by this monetary policy.

On the other hand, a shock to output demand has a larger effect on output than in the case where the money stock is kept constant (compare (21.27) to (21.24)). This is because keeping the money stock constant allows a dampening rise in the interest rate to take place. But with a constant interest rate this financial crowding-out effect does not occur.

One is tempted to draw the conclusion (from Poole, 1970):

- a money stock rule is preferable (in the sense of implying less volatility) if most shocks are output demand shocks, while
- a fixed interest rate rule is preferable if most shocks are liquidity preference shocks.

This should be accepted with caution, however, since a static model is not a secure guide for policy rules.

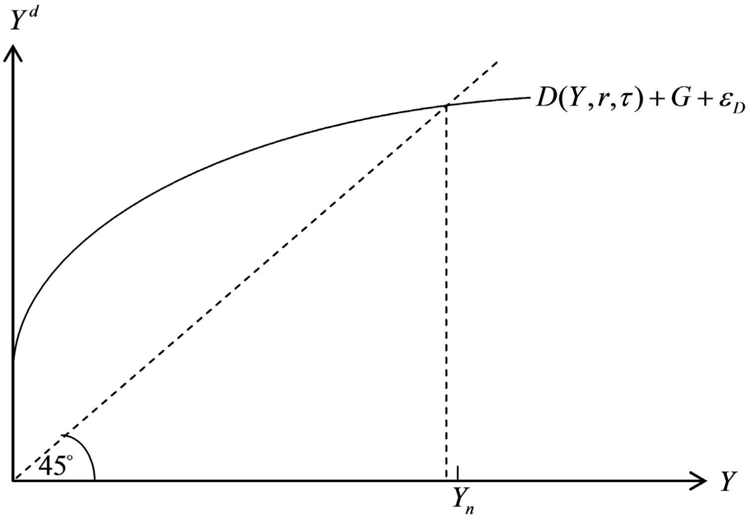

Figure 21.3: Given a fixed interest rate, a "Keynesian cross" diagram is sufficient to display the equilibrium output level (i, π^e_{+1}, τ, G, and ε_D given).

If we are interested also in the required changes in the money stock, we rewrite (LM) as

$$M = P \cdot (L(Y, i) + \varepsilon_L). \tag{LM'}$$

Here, i is exogenous and Y should be seen as already determined from (IS) independently of (LM'), that is, as given by (21.26). In this context we consider (LM') as an equation determining M as an implicit function of the other variables in the equation. To find the partial derivatives of this function, we take the total differential on both sides of (LM'):

$$dM = P(L_Y dY + L_i di + d\varepsilon_L) + (L(Y, i) + \varepsilon_L)dP, \tag{21.28}$$

where dY can be seen as already determined from (21.17) through (21.27), independently of (21.18). For instance, the approximate change in the money stock required for a rise in i of size $di > 0$ to materialize can, by (21.28), be written

$$\Delta M \approx dM = PL_Y dY + PL_i di = PL_Y h_{r^e} di + PL_i di = PL_Y \frac{D_{r^e}}{1 - D_Y} + PL_i di,$$

where the first term after the second equality sign is based on using the chain rule in (LM'). The multiplier of the money stock w.r.t. i is

$$\begin{aligned}
\frac{\partial M}{\partial i}_{|(LM')} &= \frac{\partial M}{\partial Y}_{|(LM')} \cdot \frac{\partial Y}{\partial i}_{|(21.26)} + \frac{\partial M}{\partial i}_{|(LM')} = \frac{dM}{dY}_{|21.28} h_{r^e} + \frac{dM}{di}_{|21.28} \\
&= PL_Y \frac{D_{r^e}}{1 - D_Y} + PL_i < 0,
\end{aligned}$$

where the first term after the last equality sign represents a negative *indirect effect* on the money stock of the rise in the target and the second term a negative *direct effect*. The direct effect indicates the fall in money stock needed to induce a rise in the interest rate of size di for a fixed output level. But also the output level will be affected by the rise in the interest rate since this rise reduces output demand. Through this indirect channel the transactions-motivated demand for money is reduced, and to match this a further fall in the money stock is required. This is the indirect effect.

The multipliers for the money stock w.r.t. the other exogenous variables are found in a similar way from (21.28) and (21.27), again using the chain rule where appropriate. Let us first consider the multiplier w.r.t. the exogenous variables entering (IS) and thereby (21.26). We get :

$$\begin{aligned}
\frac{\partial M}{\partial \pi^e_{+1}} &= PL_Y \frac{\partial Y}{\partial \pi^e_{+1}}_{|(21.26)} = PL_y \cdot (-h_{r^e}) = -PL_Y \frac{D_{r^e}}{1-D_Y} > 0, \\
\frac{\partial M}{\partial \tau} &= PL_Y \frac{\partial Y}{\partial \tau}_{|(21.26)} = PL_y \cdot h_\tau = PL_Y \frac{D_\tau}{1-D_Y} < 0, \\
\frac{\partial M}{\partial G} &= \frac{\partial M}{\partial \varepsilon_D} = PL_Y \frac{\partial Y}{\partial G}_{|(21.26)} = PL_y \cdot h_G = PL_Y \frac{1}{1-D_Y} > 0,
\end{aligned}$$

where the inserted partial derives of h come from (21.27).

We see that higher expected inflation implies that the money stock required to maintain a given interest rate is higher. The reason is that for given i, a higher π^e_{+1} means lower expected real interest rate, hence higher output demand and higher output. Hereby the transactions-motivated demand for money is increased. A higher money stock is thus needed to hinder a rise in the nominal interest rate above target.

Finally, as ε_L and P do not enter (IS) and thereby not (21.26), the multipliers of M w.r.t. these two variables are determined directly by (LM'), keeping Y and i constant. We get

$$\frac{\partial M}{\partial \varepsilon_L} = P > 0, \qquad \frac{\partial M}{\partial P} = L(Y,i) + \varepsilon_L = \frac{M}{P} > 0.$$

The "Keynesian cross" Since for a fixed interest rate there is no financial crowding-out, the production outcome can also be illustrated by a standard 45° "Keynesian cross" diagram as in Fig. 21.3.

The spending multiplier under full tax financing

The spending multiplier in the last line of (21.27) is conditional on the fixed interest rate policy *and* constancy of the "fiscal tightness", τ. Although there will be an automatic rise in net tax revenue via $T'(Y) > 0$, unless increased government spending is fully self-financing (which it will be only if $T'(Y) \geq 1 - D_Y$, as we will see in a moment), the result is $d\mathbb{T} < dG$. This amounts to a larger budget deficit than otherwise and thereby increased public debt and higher taxes in the future. In Section 21.4 below we assess the possible feedback effects of this on the spending multiplier (effects that are ignored by the static IS-LM model).

In the present section we will consider the alternative case, a useful benchmark, where the increase in G is *accompanied* by an adjustment of the fiscal tightness parameter, τ, so as to ensure $d\mathbb{T} = dG$, thereby leaving the budget balance unchanged, it be negative, positive, or nil. The net tax revenue is

$$\mathbb{T} = \tau + T(Y), \qquad 0 \leq T' < 1, \tag{21.29}$$

cf. Section 21.1. We impose the requirement that the primary budget deficit, $G - \mathbb{T}$, remains equal to some constant k in spite of the change in G.This gives the equation

$$G - \tau - T(Y) = k, \tag{*}$$

where both τ and Y are endogenous. We have a second equation where these two variables enter, namely the (IS) equation with i exogenous:

$$Y = D(Y, i - \pi^e_{+1}, \tau) + G, \tag{**}$$

ignoring the shift term ε_D. The equation system (*) - (**) thus determines the pair τ and Y as implicit functions of the remaining variables, all of which are exogenous.

Taking differentials w.r.t. Y, τ, and G on both sides of (*) and (**) gives, after ordering, the linear equation system in $d\tau$ and dY :

$$\begin{aligned} d\tau + T'(Y)dY &= dG \\ -D_\tau d\tau + (1 - D_Y)dY &= dG. \end{aligned}$$

The determinant of the coefficient matrix on the left-hand side of this system is

$$\begin{aligned} \bar{\Delta} &= 1 - D_Y + D_\tau T'(Y) = 1 - D_Y - C_{Y^p}T'(Y) \\ &= 1 - \left[C_{Y^p}(1 - T'(Y)) + (C_{Y^e_{+1}} + I_{Y^e_{+1}})\varphi'(Y)\right] - C_{Y^p}T'(Y) \\ &= 1 - C_{Y^p} - (C_{Y^e_{+1}} + I_{Y^e_{+1}})\varphi'(Y) \in (0, 1), \end{aligned} \tag{21.30}$$

where the second equality sign comes from (21.7) and the third from (21.6). The stated inclusion follows from (21.2) and (21.4). By Cramer's rule

$$\begin{aligned} d\tau &= \frac{(1 - D_Y)dG - T'(Y)dG}{\bar{\Delta}}, \\ dY &= \frac{dG + D_\tau dG}{\bar{\Delta}} = \frac{(1 - C_{Y^p})dG}{\bar{\Delta}}, \end{aligned}$$

where the last equality sign follows from (21.7). Substituting $\bar{\Delta}$ from (21.30), the first line gives

$$\frac{\partial \tau}{\partial G}_{|(*),(**)} = \frac{1 - D_Y - T'(Y)}{1 - C_{Y^p} - (C_{Y^e_{+1}} + I_{Y^e_{+1}})\varphi'(Y)} \gtreqless \text{ for } T'(Y) \lesseqgtr 1 - D_Y, \quad (21.31)$$

The second line gives the derivative of Y w.r.t. G, conditional on full tax financing:

$$\frac{\partial Y}{\partial G}_{|(*),(**)} = \frac{1 - C_{Y^p}}{1 - C_{Y^p} - (C_{Y^e_{+1}} + I_{Y^e_{+1}})\varphi'(Y)} \geq 1. \quad (21.32)$$

Although valid (within the fixed interest rate regime) for any unchanged budget balance, this result is known as the *balanced budget multiplier* in the sense of spending multiplier under a balanced budget. In case $C_{Y^e_{+1}} + I_{Y^e_{+1}} = 0$, the multiplier is exactly 1, which is the original *Haavelmo result* (Haavelmo, 1945).

Let us underline two important results within the IS-LM model:

Result 1: *Even fully tax-financed government spending is expansionary.* Given a constant interest rate, under the unchanged-budget-balance policy (*), $dY \geq dG = d\mathbb{T} > 0$, in view of the spending multiplier being at least 1. Thereby, the change in disposable income is $dY - d\mathbb{T} \geq 0$. Thereby private consumption, C, tends to *rise*, if anything. The rise in G therefore does not crowd out private consumption. It rather crowds it *in*. As Y is raised and monetary policy keeps the interest rate unchanged, according to the model also private investment is "crowded in" rather than "crowded out" (this follows from the assumptions (21.2) and (21.4)).

Result 2: *The timing of (lump-sum) taxes generally matter.* To disentangle the role of timing, we compare the unchanged-budget-balance policy (*) with the case where the rise in G is not accompanied by a change in the fiscal tightness parameter, τ. Only the automatic stabilizer, $T'(Y)dY$, is operative. This will generally result in $dG - T'(Y)dY \neq 0$. If $T'(Y) < 1 - D_Y$ (by many considered the "normal case"[10]), from (21.31) follows that $dG - T'(Y)dY > 0$, that is, the budget balance is allowed to worsen. With fixed interest rate the spending multiplier will be $1/(1 - D_Y)$, cf. (21.27), and exceed that under an unchanged budget balance, given in (21.32).[11] So in the considered case, postponing the taxation needed to provide the ultimate financing of the rise in G makes this rise more expansionary. The timing of taxes matter.

In Section 21.4 we briefly discuss what happens to these two results if we imagine that the household sector consists of a fixed number of utility-maximizing infinitely-lived households.

[10]But in a large downturn it may be otherwise, cf. e.g. DeLong and Summers (2012).

[11]Indeed, $1/(1 - D_Y) \gtreqless (1 - C_{Y^P})/(1 - D_Y - C_{Y^P}T'(Y))$ if $T'(Y) < 1 - D_Y$, respectively.

The paradox of thrift

Another proposition of Keynesian theory is known as the *paradox of thrift.* Consider the following special case of the IS equation:

$$Y = C + I + G = c_0 + c_1(Y - \mathbb{T}) + \tilde{C}(i - \pi^e_{+1}) + c_2 Y + \tilde{I}(i - \pi^e_{+1}) + G, \quad (21.33)$$

where c_0, c_1, and c_2 are given constants satisfying

$$c_0 > 0, 0 < c_1 \leq c_1 + c_2 < 1, \quad (21.34)$$

and $\tilde{C}(\cdot)$ and $\tilde{I}(\cdot)$ are decreasing functions of the expected real interest rate, $i - \pi^e_{+1}$. We have excluded the demand shift parameter ε_D and linearized the income-dependent parts of the consumption and investment functions. We take G, π^e_{+1}, and i as exogenous (fixed interest rate rule).

The paradox of thrift comes out most clear-cut if we ignore the public sector.

No public sector: $G = \mathbb{T} = 0$. In this case equilibrium output is

$$Y = \frac{c_0 + \tilde{C}(i - \pi^e_{+1}) + \tilde{I}(i - \pi^e_{+1})}{1 - c_1 - c_2}.$$

Suppose that all households for some reason decide to save more at any level of income so that c_0 is decreased. What happens to aggregate private saving S^p? We have

$$S^p = Y - C = I = c_2 Y + \tilde{I}(i - \pi^e_{+1}), \quad (21.35)$$

by (21.33) with $G = \mathbb{T} = 0$. Hence,

$$\frac{\partial S^p}{\partial c_0} = c_2 \frac{\partial Y}{\partial c_0} = \frac{c_2}{1 - c_1 - c_2} \geq 0,$$

from (21.27). Considering a reduction of c_0, i.e., $\Delta c_0 < 0$, the resulting change in S^p is thus

$$\Delta S^p = \frac{\partial S^p}{\partial c_0}\Delta c_0 = \frac{c_2}{1 - c_1 - c_2}\Delta c_0 \leq 0.$$

The attempt to save more thus defeats itself. What happens is that income decreases by an amount such that saving is either unchanged or even reduced. More precisely, if the income coefficient in the investment function, c_2, is nil, we get $\Delta S^p = 0$ because aggregate investment remains unchanged and income is reduced exactly as much as consumption, leaving saving unchanged. If $c_2 > 0$, we get $\Delta S^p < 0$ because income is reduced *more* than consumption since also investment is reduced when income is reduced. In this case the attempt to save more is directly counterproductive and leads to *less* aggregate saving.

The background to these results is that when aggregate output and income is demand-determined, the decreased propensity to consume lowers aggregate demand, thereby reducing production and income. The resulting lower income brings aggregate consumption further down through the *Kahn-Keynes multiplier process* (see below). While consumption is reduced, there is nothing in the situation to stimulate aggregate investment (at least not as long as the central bank maintains an unchanged interest rate). Thereby aggregate saving can not rise, since in a closed economy aggregate saving and aggregate investment are in equilibrium just two sides of the same thing as testified by national income accounting, cf. (21.35).

This story is known as the *paradox of thrift*. It is an example of a *fallacy of composition*, a term used by philosophers to denote the error of concluding from what is *locally* valid to what is *globally* valid. Such inference overlooks the possibility that when many agents act at the same time, the conditions framing each agent's actions are affected. As Keynes put it:

> ...although the amount of his own saving is unlikely to have any significant influence on his own income, the reactions of the amount of his consumption on the incomes of others makes it impossible for all individuals simultaneously to save any given sums. Every such attempt to save more by reducing consumption will so affect incomes that the attempt necessarily defeats itself (Keynes 1936, p. 84).

With public sector We return to (21.33) with $G > 0$ and $\mathbb{T} > 0$. The essence of the paradox of thrift remains but it may be partly blurred by the tendency of the government budget deficit to rise when private consumption, and therefore aggregate income, is reduced.

Consider first the case where public dissaving does *not* emerge. This is the case where the government budget is always balanced. Then, net tax revenue is $\mathbb{T} = G$, and private saving is

$$S^p \equiv Y - \mathbb{T} - C = Y - G - C = I = c_2 Y + \tilde{I}(i - \pi^e_{+1}),$$

by (21.33). So in this case the paradox of thrift comes out in the same strong form as above.

Consider instead the more realistic case where alternating budget deficits and surpluses are allowed to arise as a result of the net tax revenue following the rule

$$\mathbb{T} = \tau + \tau_1 Y, \qquad 0 < \tau_1 < 1. \tag{21.36}$$

Equilibrium output now is

$$Y = \frac{c_0 - c_1\tau + \tilde{C}(i - \pi^e_{+1}) + \tilde{I}(i - \pi^e_{+1}) + G}{1 - c_1(1 - \tau_1) - c_2}, \tag{21.37}$$

so that

$$\frac{\partial Y}{\partial c_0} = \frac{1}{1 - c_1(1-\tau_1) - c_2} > 1, \tag{21.38}$$

the inequality being due to (21.34) and (21.36). Private saving is

$$\begin{aligned} S^p &= Y - \mathbb{T} - C = I - (\mathbb{T} - G) = I - S^g = I + G - (\tau + \tau_1 Y) \\ &= c_2 Y + \tilde{I}(i - \pi^e_{+1}) + G - (\tau + \tau_1 Y), \end{aligned}$$

where the second equality comes from (21.33) and the fourth from the taxation rule (21.36). We see that.....(continuation not yet available)

Adjustment: the Kahn-Keynes multiplier process (no text available)

21.3.3 Contra-cyclical interest rate rule

Assuming a fixed interest rate rule may fit the very short run well. If we think of a time interval of a year's length or more, we may imagine a counter-cyclical interest rate rule aiming at dampening fluctuations in aggregate economic activity. Such a policy may take the form

$$i = i_0 + i_1 Y, \qquad i_1 > 0, \tag{21.39}$$

where i_0 and i_1 are policy parameters. The present version of the IS-LM model does not rule out that the parameter i_0 can be negative. But in case $i_0 < 0$, at least i_0 is not so small that even under "normal circumstances", the zero lower bound for i can become operative. The term "contra-cyclical" refers to the attempt to stabilize output by raising i when output goes up and reducing i when output goes down.[12]

If the LM curve in Fig. 21.1 is made linear and its label changed into IRR (for Interest Rate Rule), that figure covers the counter-cyclical interest rate rule (21.39). Instead of a LM curve (which requires a fixed M), we have an upward sloping IRR curve. Both i and M are here endogenous. The fixed interest rate rule from the previous section is a limiting case of this rule, namely the case $i_1 = 0$. By having $i_1 > 0$, the counter-cyclical interest rate rule yields qualitative effects more in line with those of a money stock rule. If $i_1 > \partial i/\partial Y_{|LM}$ from (21.14), the stabilizing response of i to a decrease in Y is *stronger* than under the money stock rule.

[12] The label "contra-cyclical" should not be confused with what is in the terminology of business cycle econometrics named "counter-cyclical" behavior. In that terminology a variable is characterized as "pro-" or "counter-cyclical" depending on whether its correlation with aggregate output is positive or negative, respectively. So (21.39) would in this language exemplify "pro-cyclical" behavior.

Comparative statics

Inserting (21.39) into (IS) gives

$$Y = D(Y, i_0 + i_1 Y - \pi^e_{+1}, \tau) + G + \varepsilon_D.$$

By taking the total differential on both sides we find

$$\begin{aligned}
\frac{\partial Y}{\partial G} &= \frac{\partial Y}{\partial \varepsilon_D} = \frac{1}{1 - D_Y - D_{r^e} i_1} \in (0, \frac{1}{1 - D_Y}), \\
\frac{\partial Y}{\partial i_1} &= \frac{D_{r^e} Y}{1 - D_Y - D_{r^e} i_1} < 0, \\
\frac{\partial Y}{\partial \pi^e_{+1}} &= -\frac{D_{r^e}}{1 - D_Y - D_{r^e} i_1} > 0, \\
\frac{\partial Y}{\partial \varepsilon_L} &= 0.
\end{aligned}$$

We see that all multipliers become become close to 0, if the reaction coefficient i_1 is large enough. In particular, undesired fluctuations due to demand shocks are damped this way.

The corresponding changes in i are given as $\partial i/\partial x = i_1 \partial Y/\partial x$ for $x = G, \varepsilon_D, i_1, \pi^e_{+1}$, and ε_L, respectively. From (21.28) we find the corresponding changes in M as $\partial M/\partial x = P(L_Y + i_1 L_i)\partial Y/\partial x$ for $x = G, \varepsilon_D, i_1$, and π^e_{+1}; finally, from (21.28) we have again $\partial M/\partial \varepsilon_L = P > 0$.

21.4 Further aspects

21.4.1 A liquidity trap

We return to the general IS-LM model,

$$\begin{aligned}
Y &= D(Y, i - \pi^e_{+1}, \tau) + G + \varepsilon_D, && \text{(IS)} \\
\frac{M}{P} &= L(Y, i) + \varepsilon_L, && \text{(LM)}
\end{aligned}$$

where M is again exogenous and Y and i endogenous. Suppose a large adverse demand shock $\varepsilon_D < 0$ takes place. This shock could be due to a bursting housing price bubble making creditors worried and demanding that debtors deleverage. This amounts to decreased consumption and investment and as a consequence, the IS curve may be moved so much leftward in the IS-LM diagram that whatever the money stock, output will end up smaller than the full-employment level, Y^n. Then the economy is in a *liquidity trap*: "conventional" monetary policy is not

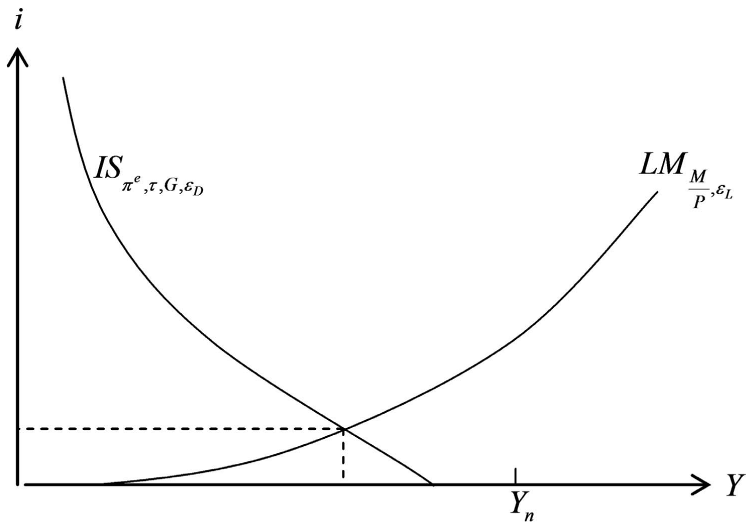

Figure 21.4: A situation where the given IS curve is such that no non-negative nominal interest rate can generate full employment (π^e_{+1}, τ, G, ε_D, M/P, and ε_L given). The value of Y where the IS curve crosses the abscissas axis is dented Y_0.

able to move output back to full employment. By "conventional" monetary policy is meant a policy where the central bank buys bonds in the open market with the aim of reducing the short-term nominal interest rate and thereby stimulate aggregate demand . The situation resembles a "trap" in the sense that when the central bank strives to stimulate aggregate demand by lowering the interest rate through open market operations, it is like attempting to fill a leaking bucket with water. The phenomenon is illustrated in Fig. 21.4.

The crux of the matter is that the nominal interest rate has the lower bound, 0, known as the *zero lower bound*. An increase in M can not bring i below 0. Agents would prefer holding cash at zero interest rather than short-term bonds at negative interest. That is, equilibrium in the asset markets is then consistent with the "=" in the LM equation being replaced by "$\geq$".

Suppose that expected inflation is very low, say nil. Then the (expected) real interest rate can not be brought below zero. The real interest rate *required* for full employment is *negative*, however, given the IS curve in Fig. 21.4. For the given π^e, to solve the demand problem expansionary fiscal policy moving the IS curve rightward is called for. Coordinated fiscal and monetary policy with the aim of raising π^e may also be a way out.

When an economy is at the zero lower bound, the government spending multiplier tends to be relatively large for two reasons. The first reason is the more trivial one that being in a liquidity trap is a *symptom* of a serious deficient aggregate demand problem and low capacity utilization so that there is no hindrance for fast expansion of production. The second reason is that there will be no financial crowding-out effect of a fiscal stimulus as long as the central bank aims at an interest rate as low as possible. (REFER to lit.)

Note that the economy can be in a liquidity trap, as we have defined it, before the zero lower bound on the nominal interest rate has been reached. Fig. 21.4 illustrates such a case. In spite of the current nominal interest rate being above zero, conventional monetary policy is not able to move output back to full employment. Conventional monetary policy can move the LM curve to the right, but the point of intersection with the IS curve can not be moved to the right of Y_0. An alternative – and more common – definition is simply to identify a liquidity trap with a situation in which the short-term nominal interest rate is zero.

Keynes (1936, p. 207) was the first to consider the possibility of a liquidity trap. After the second world war the issue appeared in textbooks, but not in practice, and so it gradually was given less and less attention. Almost at the same time as the textbooks had stopped mentioning it, it turned up in reality, first in Japan from the middle of the 1990, then in several countries, including USA, in the wake of the Great Recession. It became a problem of urgent practical importance and lead to suggestions for non-conventional monetary policies as well as more emphasis on expansionary fiscal policy, aspects to which we return later in this book.

A proviso concerning the exact character of the zero lower bound on the interest rate should be added. The zero bound should only be interpreted as exactly 0.0 if storage, administration, and safety cost are negligible, and – in an open economy – if there is no chance of a sudden appreciation of the currency in which the government debt is denominated. In the wake of the European debt crisis 2010-14, government bonds of some European countries (e.g., Germany, Finland, Switzerland, Denmark, and the Netherlands) were sold at slightly negative yields.

21.4.2 The loanable funds theory of the interest rate

As we have seen, two Keynesian tenets are that involuntary unemployment can be a state of rest and that an increased propensity to save makes things worse. Several of Keynes' contemporaries (for instance NAME, YEAR) objected that the interest rate would adjust so as to bring the demand for new loans by users (primarily home and business investors).in line with an increased supply of new loans by financial savers. This is known as the "loanable funds theory of the interest rate" according to which the interest rate is determined by "the supply and demand for saving". The pre-Keynesian version of this theory does not take into account that aggregate saving depends not only on the interest rate, but also on aggregate income (the same could be said about investment but this is of no help for the pre-Keynesian version).

To clarify the issue, we consider the simple case where $C = C(Y, r^e)$ and $I = I(r^e)$, $0 < C_Y < 1$, $C_{r^e} < 0$, $I_{r^e} < 0$, $r^e = i - \pi^e_{+1}$ and where government spending and taxation are ignored. Let S denote aggregate saving. Then in our closed economy, $S = Y - C = Y - C(Y, r^e) \equiv S(Y, r^e)$, $S_Y = 1 - C_Y > 0$, $S_{r^e} = -C_{r^e} > 0$. Equilibrium in the output market requires $Y = C(Y, r^e) + I(r^e)$. By subtracting $C(Y, r^e)$ on both sides and inserting $r^e = i - \pi^e_{+1}$, we get

$$S(Y, i - \pi^e_{+1}) = I(i - \pi^e_{+1}), \tag{21.40}$$

which may be interpreted as supply of saving being equilibrated with demand for saving. Conditional on a given income level, Y, we could draw an upward-sloping supply curve and a downward-sloping demand curve in the (S, i) plane for given π^e_{+1}. But this would not determine i since the position of the supply curve will depend on the endogenous variable, Y. An extra equation is needed. This is what the money market equilibrium condition, $M/P = L(Y, i)$ delivers, combined with exogeneity of M, P and π^e. In the Keynesian version of the loanable funds theory of the interest rate there are thus two endogenous variables, i and Y, and two equations, (21.40) and $M/P = L(Y, i)$.

If we want to illustrate the solution graphically, we can use the standard IS-LM diagram from Fig. 21.1. This is because the equation (21.40) in the (Y, i) plane is nothing but the standard IS curve. Indeed, by adding consumption on both sides of the equation, we get $Y = C(Y, i - \pi^e_{+1}) + I(i - \pi^e_{+1})$, the standard IS equation. And whether we combine the LM equation with this or with (21.40), the solution for the pair (Y, i) will be the same.

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Unfinished:

Some empirics about spending multipliers and their dependence on the state of the economy.

21.5 Some robustness checks

21.5.1 Presence of an interest rate spread (banks' lending rate $= i + \omega > i$).

(currently no text)

21.5.2 What if households are infinitely-lived?

Here we shall reconsider Result 1 and Result 2 from Section 21.3.2. They were:

Result 1: *Even fully tax-financed government spending is expansionary.*

Result 2: *The timing of (lump-sum) taxes generally matter.*

We ask whether these two results are likely to still hold in some form if we imagine that the household sector consists of a fixed number of utility-maximizing infinitely-lived households. The assumption that involuntary unemployment and abundant capacity are present is maintained.

Concerning Result 1 the answer is *yes* in the sense that the spending multiplier under a balanced budget will remain positive, albeit not necessarily ≥ 1. The reason is that although under a balanced budget the households face a temporary rise in taxes, $d\mathbb{T}$, equal to the temporary rise in spending, dG, they will reduce their *current* consumption by less than $d\mathbb{T}$, if at all. This is because they want to smooth consumption. If they at all have to reduce their total consumption, they will spread this reduction out over all future periods so that the present value of the total reduction is sufficient to cover the rise in taxes. Thereby, $-dC < dG$ so that there is necessarily an "initial" stimulus to aggregate demand equal to $dG - dC > 0$. Owing to unemployment and abundant production capacity, there need not be any crowding out of investment and so aggregate demand, output, and employment will be higher in this "first round" than without the rise in G. This means that current *before*-tax incomes increase and this stimulates private consumption and, therefore, production in the "second round". And so on through the "multiplier process". In the end private consumption in the current period need not at all fall and may even rise. So even with infinitely-lived households, the rise in G is expansionary under the stated circumstances.

Concerning Result 2 the answer depends on whether the credit market is perfect or not. With a perfect credit market current consumption of the infinitely-lived households will *not* depend on the timing of the extra taxes that are needed to finance dG. Whether the tax rise occurs now or later is irrelevant, as long as the present value of the tax rise is the same for the individual household. So the spending multiplier will be the *same* in the two situations. In this case, in spite of the rise in G being expansionary, there is *Ricardian equivalence* in the sense that for a given time path of government expenditures, the time path of (lump-sum) taxes does not matter for aggregate private consumption (whether the taxes are lump-sum or distortionary is in fact not so important in the present context where production and employment are demand-determined rather than supply-determined).

If the credit market is imperfect, however, in a heterogeneous population some of the infinitely-lived households, the less patient, say, may be currently credit constrained. The timing of the extra taxes then *does* matter and Ricardian equivalence is absent. Indeed, the lower current taxes associated with a budget deficit loosens the limit to current consumption of the credit-constrained households. Their consumption demand is thereby stimulated. Aggregate demand and therefore output and employment are thus raised. Through the automatic stabilizers the budget deficit hereby becomes smaller than otherwise. This means that the future extra tax burden becomes lower for everybody, including the households that are not currently credit-constrained. So also *their* current consumption is stimulated, and aggregate demand is raised further. We conclude that in spite of households being infinitely-lived, when credit markets are imperfect, for a given rise in government spending, the spending multiplier is likely to be larger under deficit financing than under balanced budget financing. So even Result 2 seems relatively robust.

21.6 Concluding remarks

The distinguishing feature of the IS-LM model compared with classical and new-classical theory is the treatment of the general price level for goods and services as given in the short run, that is, as a *state variable* of the system, hence very different from an asset price at an auction market. The IS-LM model is not about why it is so (the two previous chapters suggested *some* answers to that question), but about the consequences for how the interaction between goods and asset markets works out. There are two different assets, money and an interest-bearing asset in the form of bonds. Money is held because of its liquidity services while as a store of value money is generally dominated by bonds.

Traditionally, the IS-LM model has been seen as only one building block of a more involved aggregate demand-aggregate supply (AD-AS) framework of many macroeconomic textbooks. In this chapter we have interpreted the IS-LM model another way, namely as an independent short-run model in its own right, based on the approximation that both nominal wages and prices are set in advance by agents operating in imperfectly competitive markets. And these agents are hesitant regarding frequent price changes. Another textbook version of the Keynesian framework with wage- and price-setting agents (for instance Blanchard et al., 2010, Chapter 8) allows adjustment of both the wage and price *within* the period. This implies a risk that the distinction between short-run equilibrium and a sequence of such equilibria is blurred.[13]

[13] If one insists on something related to AD-AS, one could interpret this chapter's model as imposing a horizontal AS curve in the output-price plane. Then we could represent the IS equation as an AD curve in this plane *if* (and only if) monetary policy follows a money stock rule. Under these conditions, however, the original IS-LM diagram is more useful, because it directly describes the two variables, output and interest rate, which adjust in this situation.

Given the pre-set wages and prices, in every short period output is demand-determined. Likewise, but behind the scene, also employment is demand-determined. Not prices on goods and services, but quantities are the equilibrating factors. This is the polar opposite of Walrasian microeconomics and neoclassical long-run theory, cf. Part II-IV of this book, where output and employment are treated as supply-determined − with absolute and relative prices as the equilibrating factors.

A striking implication of this role switch is the *paradox of thrift* which is Keynes' favorite example of a *fallacy of composition.* As Keynes put it:

> ...although the amount of his own saving is unlikely to have any significant influence on his own income, the reactions of the amount of his consumption on the incomes of others makes it impossible for all individuals simultaneously to save any given sums. Every such attempt to save more by reducing consumption will so affect incomes that the attempt necessarily defeats itself (Keynes 1936, p. 84).

Empirically the IS-LM model, in the interpretation given here but extended with an expectation-augmented Phillips curve, does a good job (see Gali, 1992, and Rudebusch and Svensson, 1998). And the surveys in the *Handbook of Macroeconomics* (1999) and *Handbook of Monetary Economics* (201?) support the view that under "normal circumstances", the empirics say that the level of production and employment is significantly sensitive to fiscal and monetary policy.

21.7 Literature notes

The IS-LM model as presented here is essentially based on the attempt by Hicks (1937) to summarize the analytical content of Keynes' *General Theory of Employment, Interest and Money.* Keynes (WHERE?) mainly approved the interpretation. Of course Keynes' book contained many additional ideas and there has subsequently been controversies about "what Keynes really meant" (see, e.g., Leijonhufvud 1968). Yet the IS-LM framework has remained a cornerstone of mainstream short-run macroeconomics. The demand side of the large macroeconometric models which governments, financial institutions, and trade unions apply to forecast macroeconomic evolution in the near future is essentially built on the IS-LM model. At the theoretical level the IS-LM model has been criticized for being *ad hoc,* i.e., not derived from "primitives" (optimizing firms and households, given specified technology, preferences, budget constraints, and market

Still, when it comes to the study of *sequences* of short-run equilibria, a medium-term AD curve in the output-*inflation* plane will arise. Combining this with a Phillips-curve (of some kind) leads to *dynamic* AD-AS analysis, cf. later chapters.

structures combined with an intertemporal perspective with forward-looking expectations) and not ensuring mutual compatibility of agents budget constraints. In recent years, however, more elaborate micro-founded versions of the IS-LM model have been suggested (Goodfriend and King 1997, McCallum and Nelson 1999, Sims 2000, Dubey and Geanakoplos 2003, Walsh 2003, Woodford 2003, Casares and McCallum, 2006). Some of these "modernizations" and consistency checks are considered in Part VII.

To be added:

Barro's and others' critique of the traditional AS-AD interpretation of the IS-LM model.

The case with investment goods industries with monopolistic competition: Kiyotaki, QJE 1987.

Keynes 1937. Comparison between Keynes (1936) and Keynes (1939).

Balanced budget multiplier: Haavelmo (1945).

The natural range of unemployment: McDonald. See also Dixon and Rankin, eds., p. 56.

Keynes and DeLong: Say's law vs. the treasury view.

21.8 Appendix

21.9 Exercises

Chapter 22

IS-LM dynamics with forward-looking expectations

A main weakness of the static IS-LM model as described in the previous chapter is the absence of dynamics and endogenous forward-looking expectations. This motivated Blanchard (1981) to develop a dynamic extension of the IS-LM model. We shall use The key elements are:

- The focus is manifestly on adjustment mechanisms in the "very short run". The model allows for a deviation of aggregate output from aggregate demand – the adjustment of output to demand takes time. In this way the model highlights the interaction between fast-moving asset markets and less-fast-moving goods markets.

- There are three financial assets, money, a short-term bond, and a long-term bond. Accordingly there is a distinction between the short-term interest rate and the long-term interest rate. Thereby changes in the term structure of interest rates, known as the yield curve, can be studied.

- Agents have forward-looking expectations. The expectations are assumed to be rational (model consistent) and thereby endogenous. Since there are no stochastic elements in the model, perfect foresight is effectively assumed.

This richer IS-LM model conveys the central message of Keynesian theory. The equilibrating role in the output market is taken by output changes generated by discrepancies between aggregate demand and production. The distinction between short- and long-term interest rates allows an account of what monetary policy can directly accomplish and what is at least more difficult to accomplish. While the central bank controls the short-term interest rate (as long as it exceeds the zero lower bound), consumption and in particular investment depend on the long-term rate. Finally, at the empirical level the incorporation of the yield curve opens up for a succinct indicator of expectations.

22.1 A dynamic IS-LM model

As in the previous chapter we consider a closed industrialized economy where manufacturing goods and services are supplied in markets with imperfect competition and prices set in advance by firms operating under conditions of abundant capacity. Time is continuous.

Let R_t denote the long-term real interest rate at time t (to be explained below). By replacing the short-term real interest rate in the aggregate demand function from the simple IS-LM model of the previous chapter by the long-term rate, we obtain a better description of aggregate demand:

$$\begin{aligned} Y_t^d &= C(Y_t - (\tau + T(Y_t)), R_t) + I(Y_t, R_t) + G \equiv D(Y_t, R_t, \tau) + G, \text{ where} \\ 0 &< D_Y < 1, D_R < 0, -1 < D_\tau = -C_{Y^p} < 0, \end{aligned}$$

Generally notation is as in the previous chapter although we shift from discrete to continuous time. We should thus interpret the flow variables as *intensities*. Disposable private income per time unit is $Y - \mathbb{T}$ where $\mathbb{T} = \tau + T(Y))$, $0 \leq T'(Y) < 1$, and τ is a constant parameter reflecting "tightness" of discretionary fiscal policy. The symbol G represents government purchases per time unit (spending on goods and services). To avoid too many balls in the air at the same time, we ignore stochastic elements both here and in the money market equation to follow.

The positive dependency of aggregate output demand on current aggregate income, Y, reflects primarily that private consumption depends positively on disposable income. That current disposable income has this role, reflects the empirically supported hypothesis that a substantial fraction of the households are credit-constrained. Perceived human wealth (the present value of the expected stream of after-tax labor income), which in standard consumer theory is a major determinant of consumption, is itself likely to depend positively on current earnings. Similarly, capital investment by demand-constrained firms will depend positively on current economic activity, Y, to the extent that this activity provides internal finance from corporate profits and signals the level of demand in the near future.

The negative dependency of aggregate demand on R reflects first and foremost that capital investment depends negatively on the expected long-term interest rate. Firm's investment in production equipment and structures is normally an endeavour with a lengthy time horizon. Similarly, the households' investment in durable consumption goods (including housing) is based on medium- or long-term considerations. A rise in R induces a negative substitution effect on current consumption and probably also, on average, a negative wealth effect. Increases in household's wealth, whether in the form of human wealth, equity shares and bonds, or housing estate, are triggered by reduction in the long-term interest rate.[1]

Because of the short-run perspective of the model, explicit reference to the available capital stock in the investment function, I, is suppressed.

The continuous-time framework is convenient because we avoid the oddity in period analysis of allowing asset markets to open only at the beginning or end of each period. The continuous-time framework is also convenient by making it easy to operate with different speeds of adjustment for different variables. Regarding the speed of adjustment to changes in demand, we shall operate with a tripartition as envisaged in Table 20.1. Output is understood to consist primarily of goods and services with elastic supply with respect to demand, in contrast to agricultural and mineral products and construction.

Table 20.1. Speed of adjustment of different variables to demand shifts

Variable	*Adjustment speed*
asset prices	high
output	medium
prices on output	low, here assumed nil

The model lets asset prices adjust immediately so that asset markets clear at any instant. The adjustment of output to demand takes time and is gradual. This is modeled as an error-correction:

$$\begin{aligned} \dot{Y}_t &\equiv \frac{dY_t}{dt} = \lambda(Y_t^d - Y_t), && (22.1)\\ &= \lambda(D(Y_t, R_t, \tau) + G - Y_t), && Y_0 > 0 \text{ given,} \end{aligned}$$

where $\lambda > 0$ is a constant adjustment speed. At any point in time the output intensity Y_t is predetermined. During the adjustment process also demand changes (since the output level and fast-moving asset prices are among the determinants of demand). The difference between demand and output is made up of changes in order books and inventories behind the scene. Indeed, the counterpart of $Y_t - Y_t^d$ in national income accounting is unintended inventory investment. In

[1] See, e.g., Case, Quigley, and Shiller (2005, 2011).

a more elaborate version of the model unintended positive or negative inventory investment should result in a feedback on subsequent demand and supply.[2]

The rest of the model consists of the following equations:

$$
\begin{aligned}
M_t &= P_t L(Y_t, i_t), \qquad L_Y > 0,\ L_i < 0. && (22.2)\\
R_t &= \frac{1}{q_t}, && (22.3)\\
r_t^e &\equiv i_t - \pi_t^e, && (22.4)\\
\frac{1+\dot{q}_t^e}{q_t} &= r_t^e, && (22.5)\\
\pi_t &\equiv \frac{\dot{P}_t}{P_t} = \pi. && (22.6)
\end{aligned}
$$

Equation (22.2) is the same equilibrium condition for the money market as in the static IS-LM model. The variable, M_t, is the money stock which we may interpret either as the monetary base or money in the broader sense where private-bank-created money is included. To fix ideas, we choose the former interpretation since no private banking sector is visible in the model.

As financial markets in practice adjust very fast, the model assumes clearing in the asset markets at any instant. The real money demand function, $L(\cdot)$, depends positively on Y (viewed as a proxy for the number of transactions per time unit for which money is needed) and negatively on the short-term nominal interest rate, the opportunity cost of holding money. Like Y_t, the general price level, P_t, is treated as a *state variable*, thereby being historically determined and changing only gradually over time. For a given M_t, money market equilibrium is brought about by immediate adjustment of the short-term nominal interest rate, i_t, so that the available stock of money is willingly held.

In equation (22.3) appears the important "new" variable q_t, which is the real price of a long-term bond, here identified as an inflation-indexed *consol* (a perpetual bond) paying to the owner a stream of payments worth one unit of output per time unit in the indefinite future (no maturity date). The equation tells us that the *long-term* real interest rate at time t is the reciprocal of the real market price of a consol at time t. This is just another way of saying that the long-term rate, R_t, is defined as the internal rate of return on the consol. Indeed,

[2] In the post-war period changes in inventory stocks (inventory investment) account for less than 1% of GDP in the U.S. (Allesandria et al., 2010, Wen, 2011).

That the adjustment of output takes time and is gradual is empirically underpinned by, for instance, Sims (1998) and Estralla and Fuhrer (2002).

the internal rate of return is that number, R_t, which satisfies the equation

$$q_t = \int_t^{\infty} 1 \cdot e^{-R_t(s-t)} ds = \left[\frac{e^{-R_t(s-t)}}{-R_t} \right]_t^{\infty} = \frac{1}{R_t}. \tag{22.7}$$

Thus the long-term interest rate is that discount rate, R_t, which transforms the payment stream on the consol into a present value equal to the real market price of the consol at time t.[3] Inverting (22.7) gives (22.3). An alternative way of presenting the inflation-indexed consol is shown in Appendix A.

Equation (22.4) defines the ex ante *short-term* real interest rate as the short-term nominal interest rate minus the expected inflation rate. Next, equation (22.5) can be interpreted as a no-arbitrage condition saying that the expected real rate of return on the consol (including a possible expected capital gain or capital loss, depending on the sign of $\dot{q}_t^e$) must in equilibrium equal the expected real rate of return, r_t^e, on the short-term bond. We may think of both the short-term bond and the long-term bond as being government bonds. For given expectations ($\dot{q}_t^e$ and r_t^e), the real price of the consol instantaneously adjusts so as to make the available stock of consols willingly held. In general, in view of the higher risk associated with long-term claims, presumably a positive risk premium should be added on the right-hand side of (22.5). We shall ignore uncertainty, however, so that there is no risk premium.[4] Finally, equation (22.6) says that within the relatively short time perspective of the model, the inflation rate is constant at an exogenous level, π. The interpretation is that price changes mainly reflect changes in units costs and that these changes are relatively smooth.

We assume agents' expectations are rational (model consistent). As there is no uncertainty in the model (i.e., no stochastic elements), this assumption amounts to perfect foresight. We thus have $\dot{q}_t^e = \dot{q}_t$ and $\pi_t^e = \pi_t = \pi$. Therefore, equation (22.4) reduces to $r_t^e = i_t - \pi = r_t$ for all t. The wedge between the nominal short-term rate, i_t, which is relevant for the money market equilibrium in (22.2), and the nominal short-term rate that households and firms typically face in a credit market, is absent in the model because uncertainty and default risk are ignored.

[3] Similarly, in discrete time, with coupon payments at the end of each period, we would have

$$q_t = \sum_{s=t+1}^{\infty} \frac{1}{(1+R_t)^{s-t}} = \frac{\frac{1}{1+R_t}}{1 - \frac{1}{1+R_t}} = \frac{1}{R_t}.$$

Consols (though nominal) have been issued by UK governments occationally since 1751 and constitute only a small part of UK government debt. Their form, without a maturity date, make them convenient for dynamic analysis.

[4] If a *constant* risk premium were added, the dynamics of the model will only be slightly modified.

Whichever monetary policy regime to be considered below, the model can be reduced to two coupled first-order differential equations in Y_t and R_t. The first differential equation is (22.1) above. As to the second, note that from (22.3) we have $\dot{R}_t/R_t = -\dot{q}_t/q_t$. Substituting into (22.5), where $\dot{q}_t^e = \dot{q}_t$, and using again (22.3), gives

$$\frac{1}{q_t} + \frac{\dot{q}_t}{q_t} = R_t - \frac{\dot{R}_t}{R_t} = r_t^e = r_t = i_t - \pi, \tag{22.8}$$

in view of (22.4) with $r_t^e = r_t = i_t - \pi$. By reordering,

$$\dot{R}_t = (R_t - (i_t - \pi))R_t, \tag{22.9}$$

where the determination of i_t depends on the monetary regime.

Before considering alternative policy regimes, we shall emphasize an equation which is useful for the economic interpretation of the ensuing dynamics. Assuming absence of asset price bubbles (see below), the no-arbitrage formula (22.5) is equivalent to a statement saying that the market value of the consol equals the *fundamental value* of the consol. By fundamental value is meant the present value of the future dividends from the consol, using the (expected) future short-term interest rates as discount rates:

$$\begin{aligned} q_t &= \int_t^\infty 1 \cdot e^{-\int_t^s r_\tau d\tau} ds, \qquad \text{so that} \qquad (22.10)\\ R_t &= \frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds} = \int_t^\infty w_{t,s} r_s ds,\\ \text{where } w_{t,s} &\equiv \frac{e^{-\int_t^s r_\tau d\tau}}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds} \quad \text{and} \quad \int_t^\infty w_{t,s} ds = 1. \qquad (22.11) \end{aligned}$$

The fundamental value is the same as the solution to the differential equation for q given in (22.8), presupposing that there are no asset price bubbles (see Appendix B). The formula (22.11) follows by integration (see Appendix C). This formula shows that the long-term rate, R_t, is a weighted average of the expected future short-term rates, r_s, with weights proportional to the discount factor $e^{-\int_t^s r_\tau d\tau}$. The higher are the expected future short-term rates the lower is q_t and the higher is R_t.

If r_τ is expected to be a constant, r, then (22.10) simplifies to

$$R_t = \frac{1}{\int_t^\infty e^{-r(s-t)} ds} = \frac{1}{1/r} = r.$$

And if for example r_τ is expected to be increasing, we get

$$R_t = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds} > \frac{1}{1/r_t} = r_t.$$

22.2 Monetary policy regimes

We shall consider three alternative monetary policy regimes. The two first are *regime* m (money stock rule), where the real money supply is the policy tool, and *regime* i (fixed interest rate rule), where the short-term nominal interest rate is the policy tool. This second regime is by far the simplest one and is closer to what present-day monetary policy is about. Nevertheless regime m is also of interest, both because it has some historical appeal and because it yields impressive dynamics. In addition, regime m has partial affinity with what happens under a contra-cyclical interest rate rule. Our third monetary policy regime is in fact an example of such a rule, and we name it *regime* i'.

The assumption of perfect foresight means that the agents' expectations coincide with the prediction of our deterministic model. Once-for-all shocks may occur, but only so rarely that agents ignore the possibility that a new surprise may occur later. When a shock occurs, it fits intuition best to interpret the time derivative of a variable as a *right-hand* derivative, e.g., $\dot{Y}_t \equiv \lim_{\Delta t \to 0^+}(Y(t+\Delta t) - Y(t))/\Delta t$. This is also the way $\dot{q}_t$ and $\dot{P}_t$ should be interpreted if a shock at time t results in a kink on the otherwise smooth time path of q and P, respectively. In this interpretation $\dot{P}_t/P_t$ and $\dot{q}_t/q_t$ stand for *forward-looking* growth rates of the nominal price of goods and the real price of the consol, respectively.

Throughout the analysis the following variables are exogenous: the inflation rate, π, and the fiscal policy variables, τ and G. Depending on the monetary policy regime, an additional variable relating to the money market may be exogenous. The initial values, P_0 and Y_0, are historically given since in this short-run model it takes time not only for the price level but also the output level to change.

22.2.1 Policy regime m: Money stock rule

Here we assume that the central bank is capable of controlling the money stock. More specifically, the central bank finds the going inflation rate tolerable and pursues a monetary policy of maintaining the real money stock, M_t/P_t, at a constant level $m > 0$, by letting the nominal money supply follow the path:

$$M_t = P_0 e^{\pi t} m = M_0 e^{\pi t}.$$

A natural interpretation is that a part of the government budget deficit is financed by seigniorage: $\dot{M}_t/P_t = (M_t/P_t)\dot{P}_t/P_t = m\pi$.

Equation (22.2) then reads $L(Y_t, i_t) = m$. This equation defines i_t as an implicit function of Y_t and m, i.e.,

$$i_t = i(Y_t, m), \quad \text{with } i_Y = -L_Y/L_i > 0, \quad i_m = 1/L_i < 0. \tag{22.12}$$

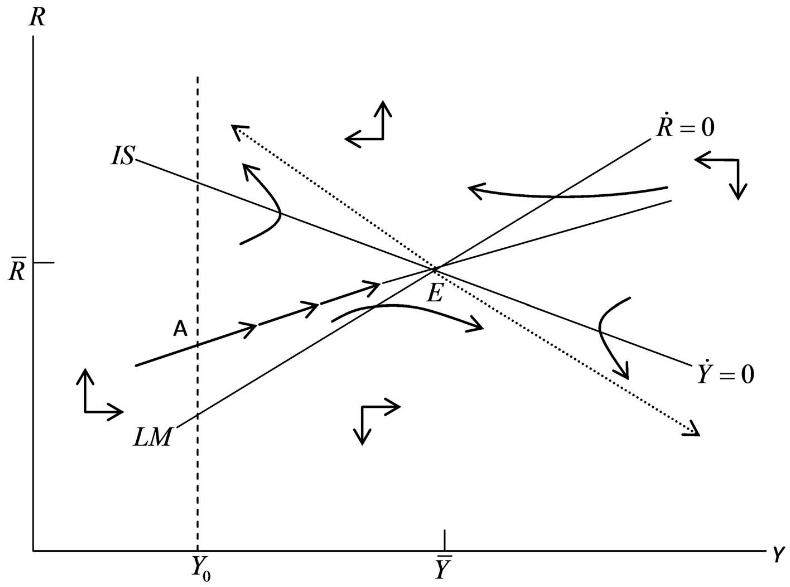

Figure 22.1: Phase diagram when m is the policy instrument.

Inserting this function into (22.9), we have

$$\dot{R}_t = [R_t - i(Y_t, m) + \pi] \, R_t. \tag{22.13}$$

This differential equation together with (22.1) constitutes a dynamic system in the two endogenous variables, Y_t and R_t. For convenience, we repeat (22.1) here:

$$\dot{Y}_t = \lambda(D(Y_t, R_t, \tau) + G - Y_t), \quad \lambda > 0, 0 < D_Y < 1, D_R < 0, D_\tau \in (-1, 0) \tag{22.14}$$

Phase diagram

As long as $R > 0$, (22.13) implies

$$\dot{R} \gtreqless 0 \qquad \text{for} \qquad R \gtreqless i(Y, m) - \pi, \qquad \text{respectively.} \tag{22.15}$$

We have $\frac{\partial R}{\partial Y}|_{\dot{R}=0} = i_Y = -L_Y/L_i > 0$, that is, for real money demand to equal a given real money stock, a higher volume of transactions must go hand in hand with a higher nominal short-term interest rate which in turn, for given inflation, requires a higher real interest rate. The $\dot{R} = 0$ locus is illustrated as the upward sloping curve, LM, in Fig. 22.1.

From (22.14) we have

$$\dot{Y} \gtreqless 0 \qquad \text{for} \qquad D\,(Y, R, \tau) + G \gtreqless Y, \qquad \text{respectively.} \tag{22.16}$$

We have $\frac{\partial R}{\partial Y}|_{\dot{Y}=0} = (1 - D_Y)/D_R < 0$, that is, higher aggregate demand in equilibrium requires a lower interest rate. The $\dot{Y} = 0$ locus is illustrated as the downward sloping curve, IS, in Fig. 22.1. In addition, the figure shows the direction of movement in the different regions, as described by (22.15) and (22.16).

The $\dot{R} = 0$ and $\dot{Y} = 0$ loci intersect at the point E with coordinates $(\bar{Y}, \bar{R})$. For this point to be a steady state obtainable by the economic system, it is required that $\bar{R} > 0$, since $R = 1/q$, and q is the real price of an inflation-indexed consol. Now, $\bar{R} = i(\bar{Y}, m) - \pi$. So, we assume that, given $\bar{Y}$ and π, m is small enough to make $i(\bar{Y}, m) - \pi > 0$, i.e., $\pi < i(\bar{Y}, m)$. If $i(\bar{Y}, m)$ is close to the lower bound, nil,[5] this requires that inflation is essentially negative, which amounts to deflation. To maintain m constant with $\pi < 0$ requires $\dot{M}_t/M_t < 0$ in this quasi- or short-run steady state. We use the the qualifier "short run" because presumably the economy will be subject to further dynamic feedbacks in the system (through a Phillips curve, changed capital stock due to investment, technological change etc.). Owing to the short time horizon, such feedbacks are ignored by the model. The short-run equilibrium may also be called a "short-run equilibrium".

In order to distinguish the short-run steady states from a "genuine" long-run steady state of an economy, we mark the steady-state values by a bar rather than an asterisk. We see that the steady state point, E, with coordinates $(\bar{Y}, \bar{R})$, is a saddle point.[6] So exactly two solution paths – one from each side – converge towards E. These two saddle paths, which together make up the stable arm, are shown in the figure (the slope of the stable arm must be positive, according to the arrows). Also the unstable arm is displayed in the figure (the negatively sloped stippled line which attracts the diverging paths).

The initial value of output, Y_0, is in this model predetermined, i.e., determined by Y's previous history; relative to the short time horizon of the model, output adjustment takes time. Hence, at time $t = 0$, the economy must be somewhere on the vertical line $Y = Y_0$. The question is then whether there can be rational asset price bubbles. An *asset price bubble,* also called a speculative bubble, is present if the market value of an asset for some notable stretch of time differs from its *fundamental value* (the present value of the expected future dividends from the asset, as defined in (22.10)). A *rational* asset price bubble is an asset price bubble that is consistent with the no-arbitrage condition (22.5) under rational expectations.

[5] The nominal interest rate can not go below 0 because agents prefer holding cash at zero interest (or slightly below to cover trivial safe-keeping costs associated with cash holding) rather than short-term bonds at negative interest.

[6] The determinant of the Jacobian matrix for the right-hand sides of the two differential equations, evaluated at the steady-state point, is $\lambda\left[\bar{R}(D_Y - 1) + \bar{R}i_Y D_R\right] < 0$. Hence, the two associated eigenvalues are of opposite sign. This is the precise general mathematical criterion for the steady state to be a saddle point.

Because consols have no terminal date and might be of a unique historical kind available in limited amount, a rational asset price bubble, driven by self-fulfilling expectations, not be ruled out within the model as it stands. In Fig. 22.1 any of the diverging paths with R ultimately falling, and therefore the asset price q ultimately rising, could in principle reflect such a bubble. A *negative* rational bubble can be ruled out, however. Essentially, this is because negative bubbles presuppose that the market price of the consol initially drops below the present value of future dividends (the right-hand side of (22.10)). But in such a situation everyone with rational expectations would want to buy the consol and enjoy the dividends. The resulting excess demand would immediately drive the asset price back to the fundamental value.

In view of its simplistic nature, the model does not provide an appropriate framework for bubble analysis. Here we will simply *assume* that the market participants never have bubbly expectations. An easy way to justify that assumption is to interpret the consols as just a convenient approximation to bonds with long but finite time to maturity (as most bonds in the real world). Now, when market participants never expect a bubbly asset price evolution, bubbles will not arise, hence the implosive paths of R in Fig. 22.1 can not materialize. The explosive paths of R in Fig. 22.1 have already been ruled out, as they would reflect negative bubbles.

We are left with the saddle path, the path AE in the figure, as the unique solution to the model. As the figure is drawn, $Y_0 < \bar{Y}$. The long-term interest rate will then be relatively low so that demand exceeds production and gradually pulls production upward. Hereby demand is stimulated, but less than one-to-one so, both because the marginal propensity to spend is less than one and because also the interest rate rises. Ultimately, say within a year, the economy settles down at the short-run steady state of the model − the short-run equilibrium.

Impulse-response dynamics

Let us consider the effects of level shifts in G and m, respectively. Suppose that the economy has been in its steady state until time t_0. In the steady state we have $r = i = \bar{R}$. Then either fiscal or monetary policy changes. The question is what the effects on r, R, and Y are. The answer depends very much on whether we consider an *unanticipated* change in the policy variable in question (G or m) at time t_0 or an *anticipated* change. As to an anticipated change, we can imagine that the government or the central bank at time t_0 credibly announces a shift to take place at time $t_1 > t_0$. From this derives the term "announcement effect", synonymous with "anticipation effect".

To prepare the ground, consider first the question: how are the IS and LM curves affected by shifts in G and m, respectively? We have, from (22.16), $\frac{\partial R}{\partial G}|_{\dot{Y}=0}$

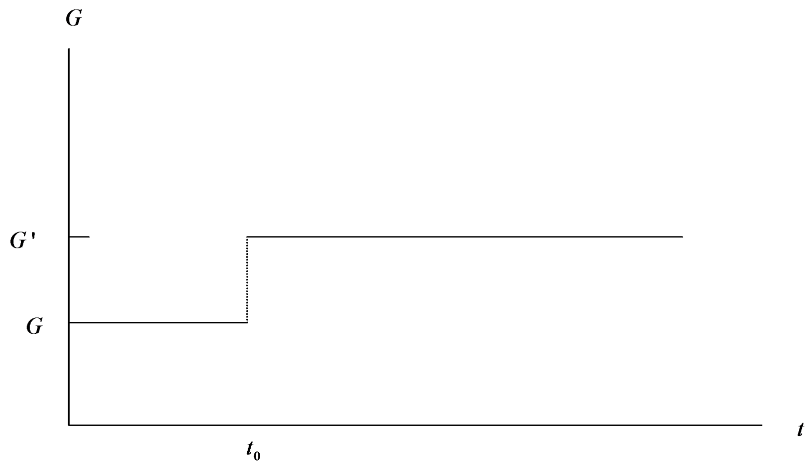

Figure 22.2: Unanticipated upward shift in G (regime m).

$= -1/D_R > 0$, that is, a shift to a higher G moves the $\dot{Y} = 0$ locus (the IS curve) upwards. But the $\dot{R} = 0$ locus is not affected by a shift in G. On the other hand, the $\dot{Y} = 0$ locus is not affected by a shift in m. But the $\dot{R} = 0$ locus (the LM curve) depends on m and moves downwards, if m is increased, since $\frac{\partial R}{\partial m}|_{\dot{R}=0}$ $= i_m < 0$, from (22.15).

We now consider a series of policy changes, some of which are unanticipated, whereas others are anticipated.

(a) The effect of an unanticipated upward shift in G. Suppose the government is unsatisfied with the level of economic activity and at time $t_0 > 0$ decides (unexpectedly) an increase in G. And suppose people rightly expect this higher G to be maintained for a long time.

The upward shift in G is shown in Fig. 22.2.[7] When G shifts, the long-term interest rate jumps up to R_A, cf. Fig. 22.3, reflecting that the market value of the consol jumps down. The explanation is as follows. The higher G implies higher output demand, by (21.1). So an expectation of increasing Y arises (see (22.14)) and therefore also an expectation of increasing i and r, in view of (22.12). The implication is, by (22.10), a lower q_{t_0} and a higher R_{t_0}, as illustrated in Fig. 22.4. After t_0, output Y and the short-term rate r gradually increase toward their new steady state values, $\bar{Y}'$ and $\bar{r}'$, respectively, as shown by Fig. 22.4. As time proceeds and the economy gets closer to the expected high future values of r, these higher values gradually become dominating in the determination of R in

[7] Since m and τ are kept unchanged, the higher G may have to be partly debt financed and thus be associated with a higher amount of outstanding government bonds. Whether this is problematic is not our concern here.

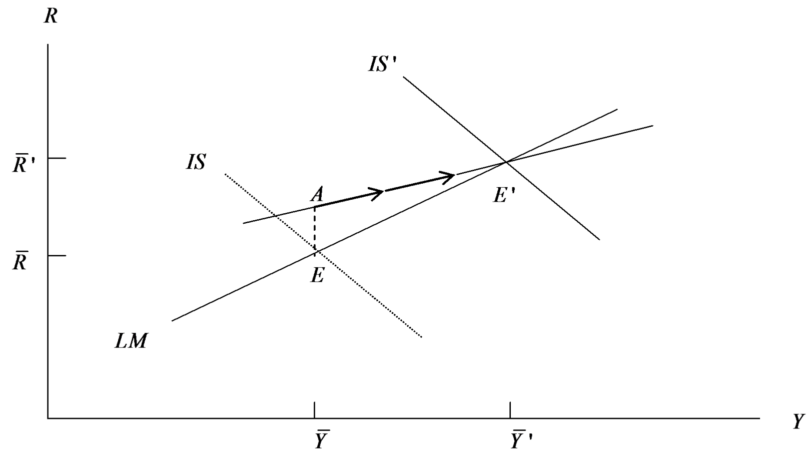

Figure 22.3: Phase portrait of an unanticipated upward shift in G.

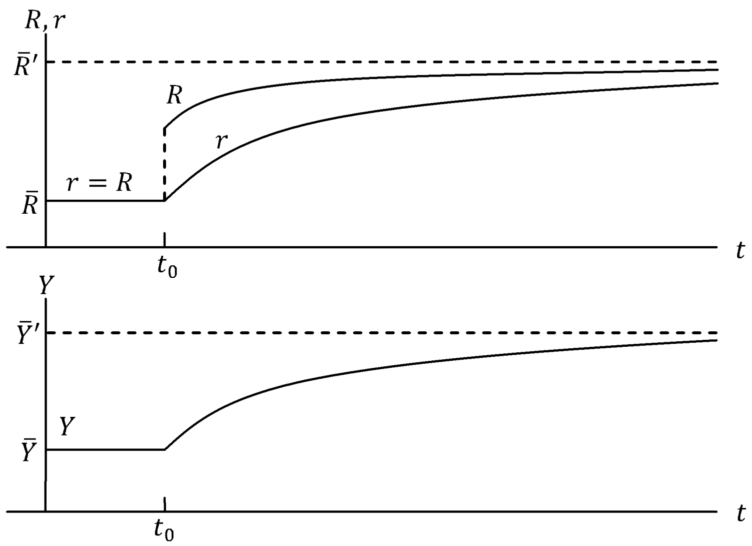

Figure 22.4: Time profiles of interest rates and output to an unanticipated shift in G (regime m).

(22.10). Hence, after t_0 also R gradually increases toward its new steady state value, the same as that for r.

By dampening output demand, the higher R implies a financial crowding-out effect on production.[8] After t_0, during the transition to the new steady state, we have $R > r$ because R "anticipates" all the future increases in r and incorporates them, cf. (22.10). Note also that (22.8) implies

$$R = r + \dot{R}/R \gtreqless r \qquad \text{for} \qquad \dot{R} \gtreqless 0, \qquad \text{respectively.}$$

For example, $\dot{R} > 0$ reflects that $\dot{q} < 0$, that is, a capital loss is expected. To compensate for this, the *level* of R (which always equals $1/q$) must be higher than r such that the no-arbitrage condition (22.5) is still satisfied.

Formulas for the steady-state effects of the change in G can be found by using the comparative statics method of Chapter 21 on the two steady-state equations

$$\bar{Y} = D(\bar{Y}, \bar{R}, \tau) + G \quad \text{and} \quad m = L(\bar{Y}, \bar{R})$$

with the two endogenous variables $\bar{Y}$ and $\bar{R}$ (Cramer's rule). Given the preparatory work already done, a more simple method is to substitute $\bar{R} = i(\bar{Y}, m) - \pi$ into the first-mentioned steady-state equation to get $\bar{Y} = D(\bar{Y}, i(\bar{Y}, m), \tau) + G$. Taking the differential on both sides gives $d\bar{Y} = D_Y d\bar{Y} + D_R i_Y d\bar{Y} + dG$, from which follows, by (22.12),

$$\frac{\partial \bar{Y}}{\partial G} = \frac{1}{1 - D_Y + D_R L_Y / L_i} > 0.$$

From $L(\bar{Y}, \bar{R}) = m$ we get $0 = L_Y d\bar{Y} + L_i d\bar{R} = L_Y(\partial \bar{Y}/\partial G)dG + L_i d\bar{R} = 0$ so that

$$\frac{\partial \bar{R}}{\partial G} = -\frac{L_Y/L_i}{1 - D_Y + D_R L_Y / L_i} > 0.$$

Since our steady-state equations corresponds exactly to the IS and LM equations for the static IS-LM model of Chapter 21, the output and interest rate multipliers w.r.t. G are the same.

As alluded to earlier, one should think about the steady state as only a quasi-steady state. That is, the role of the point $(\bar{Y}, \bar{R})$ is to act as an "attractor" in the short-run dynamics after a policy shift although the point itself would in a larger model be moving slowly due to medium-term dynamics coming from a Phillips curve and/or an increased capital stock. With appropriate parameter values in the model, its adjustment time will be "short". As a rough guess about the order

[8] The crowding out is only partial, because Y still increases.

of magnitude, eliminating 95% of the initial distance to the steady state point might take about a year, say.

Our treatment of the shift in G as permanent should not be interpreted literally. It is only meant to indicate that the fiscal stimulus is durable enough to really matter. A really permanent increase in G in this economy without economic growth might endanger fiscal sustainability, if the automatic budget reaction is not sufficient to, after a while, fully finance the increase in G.

(b) The effect of an anticipated upward shift in G. We assume that the private sector at time t_0 becomes aware that G will shift to a higher level at time t_1, cf. the upper panel of Fig. 22.4. The implied expectation that the short-term interest rate will in the future rise towards a higher level, $\bar{r}'$, immediately triggers an upward jump in the long-term rate, R. To what level? In order to find out, note that the market participants understand that from time t_1, the economy will move along the new saddle path corresponding to the new steady state, E', in Fig. 22.5. The market price, q, of the consol cannot have an *expected* discontinuity at time t_1, since such a jump would imply an infinite expected capital loss (or capital gain) per time unit immediately before $t = t_1$ by holding long-term bonds. Anticipating for example a capital loss, the market participants would want to sell long-term bonds in advance. The implied excess supply would generate an adjustment of q downwards until no longer a jump is expected to occur at time t_1. If instead a capital gain is anticipated, an excess demand would arise. This would generate in advance an upward adjustment of q, thus defeating the expected capital gain. This is the general principle that arbitrage prevents an expected jump in an asset price.

In the time interval (t_0, t_1) the dynamics are determined by the "old" phase diagram, based on the no-arbitrage condition which rules up to time t_1. In this time interval the economy must follow that path (AB in Fig. 22.5) for which, starting from a point on the vertical line $Y = \bar{Y}$, it takes precisely $t_1 - t_0$ units of time to reach the new saddle path. At time t_0, therefore, R jumps to exactly the level R_A in Fig. 22.5.[9] This upward jump has a contractionary effect on output demand. So output starts falling as shown by figures 20.5 and 20.6. This is because the potentially counteracting force, the increase in G, has not yet taken place. Not until time t_1, when G shifts to G', does output begin to rise. In the "long run" both Y, R, and r are higher than in the old steady state.

There are two interesting features. First, in regime m a credible announcement

[9] Note that R_A is unique. Indeed, imagine that the jump, $R_A - \bar{R}$, was smaller than in Fig. 22.5. Then, not only would there be a longer way along the road to the new saddle path, but the system would also start from a position closer to the "old" steady-state point, E. This implies an initially lower adjustment speed.

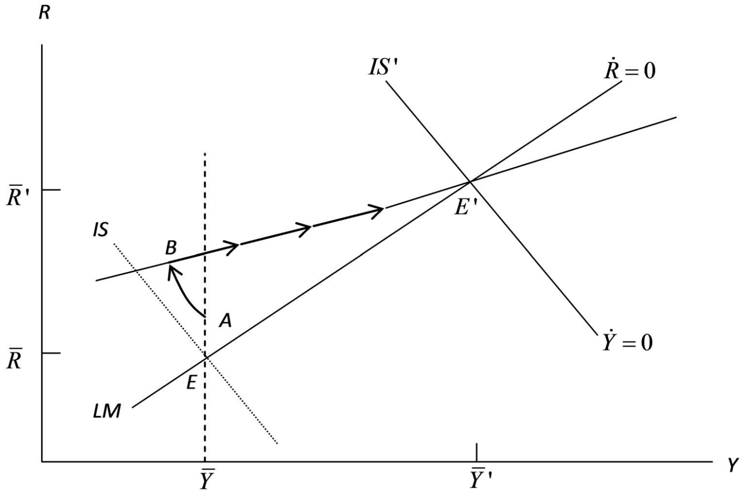

Figure 22.5: Phase portrait of an anticipated upward shift in G (regime m).

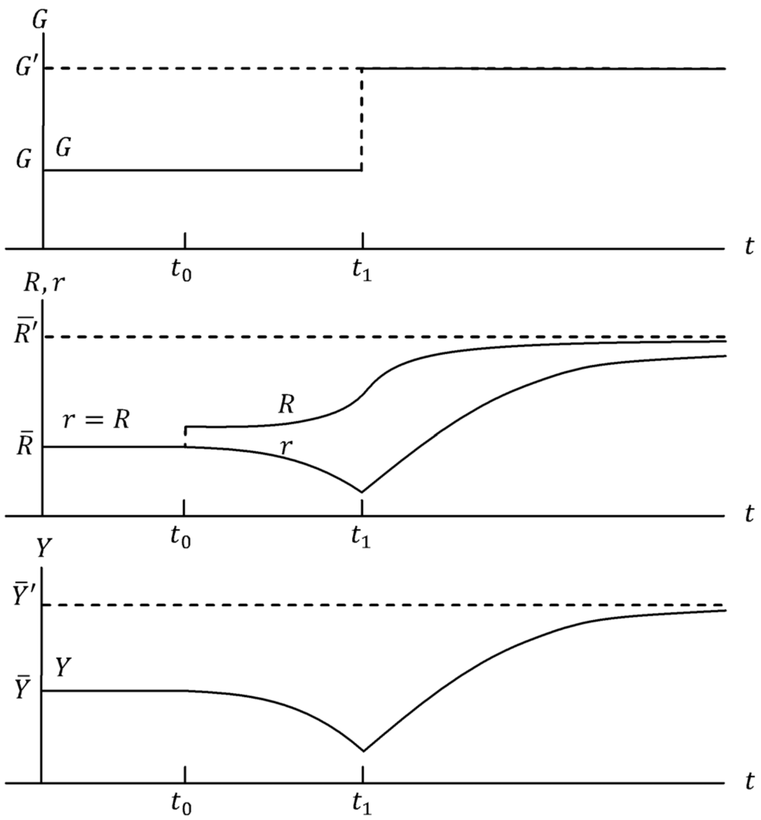

Figure 22.6: An anticipated upward shift in G and time profiles of interest rates and output (regime m).

of future expansive fiscal policy can have a temporary contractionary effect when the announcement occurs. This is due to financial crowding out. There is a way of dampening the problem, namely by letting the central bank announce prolonged open market operations to maintain i low for several years after time t_1, cf. policy regime i below. The second feature relates to the *term structure of interest rates*, also called the *yield curve.* The relationship between the internal rate of return on financial assets and their time to maturity is called the term structure of interest rates. Fig. 22.6 shows that the term structure "twists" in the time interval (t_0, t_1). The long-term rate R rises, because the time where a higher Y (and thereby a higher r) is expected to show up, is getting nearer. But at the same time the short-term rate r is falling because of the falling transaction need for money implied by the initially falling Y, triggered by the rise in the long-term interest rate.[10]

The theory of the term structure

What we have just seen is the *expectations theory of the term structure* in action. Empirically, the term structure of interest rates tends to be upward-sloping, but certainly not always and it may suddenly shift. The theory of the term structure of interest rates generally focuses on two explanatory factors. One is *uncertainty* and this factor tends to imply a positive slope because the greater uncertainty generally associated with long-term bonds generates a risk premium, known as the *term premium*, on these. The present model has nothing to say about this factor since the model ignores uncertainty.

Our model *has* something to say about the other factor, namely *expectations.* Indeed, the model quite well exemplifies what is called the *expectations theory of the term structure.* In its simplest form this theory ignores uncertainty and treats various maturities as perfect substitutes. The theory says that if the short-term interest rate is expected to rise in the future, the long-term rate today will tend to be higher than the short-term rate today. This is because, absent uncertainty, the long-term rate is a weighted average of the expected future short-term rates, as seen from (22.11). Similarly, if the short-term interest rate is expected to fall in the future, the long-term rate today will, everything else equal, tend to be lower than the short-term rate today. Thus, rather than explaining the statistical

[10] A conceivable objection to the model in this context is that it does not fully take into account that consumption and investment are likely to depend positively on expected *future* aggregate income, so that the hypothetical temporary decrease in demand and output never materializes. On the other hand, the model has in fact been seen as an explanation that president Ronald Reagan's announced tax cut in the USA 1981-83 (combined with the strict monetary policy aiming at disinflation) were followed by several years in recession. The concomitant tight monetary policy is an alternative or supplementary explanation of these events.

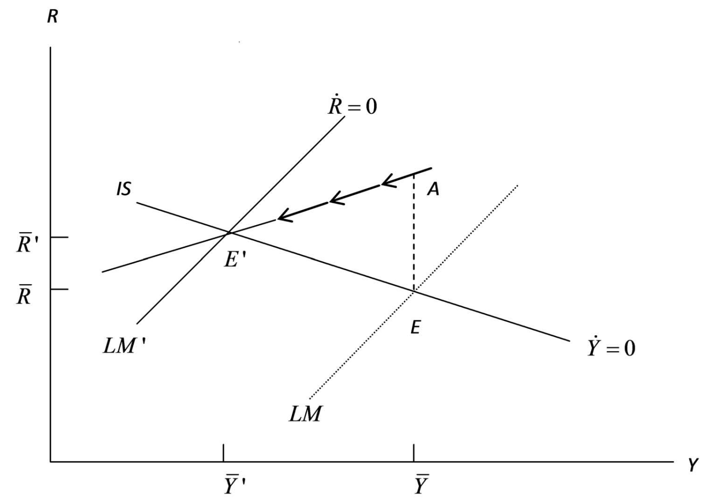

Figure 22.7: Phase portrait of an unanticipated downward shift in m (regime m).

tendency for the slope of the term structure to be positive, changes in expectations are important in explaining *changes* in the term structure. In practice, most bonds are denominated in money. Central to the theory is therefore the link between expected future inflation and the expected future short-term nominal interest rate. This aspect is not captured by the present short-run model, which ignores changes in the inflation rate.

(c) The effect of an unanticipated downward shift in m. The shift in m, brought about by sales of short-term bonds in the open market, is shown in the upper panel of Fig. 22.8. The shift triggers, at time t_0, an upward jump in the long-term rate R to the level of the new saddle path (point A in Fig. 22.7). The explanation is that the fall in money supply implies an upward jump in the short-term rate i, hence also R, at time t_0, cf. (22.10). As indicated by Fig. 22.8, the short-term rate will be expected to *remain* higher than before the decline in m. The rise in R triggers a fall in output demand and so output gradually adjusts downward as depicted in Fig. 22.8. The resulting decline in the transactions-motivated demand for money leads to the gradual fall in the short-term rate towards the new steady state level. This fall is "anticipated" by the long-term rate, which therefore, at every point in time after t_1, is lower than the short-term rate.

It is interesting that when the new policy is introduced, both R and r "overshoot" in their adjustment to the new steady-state levels. This happens, because,

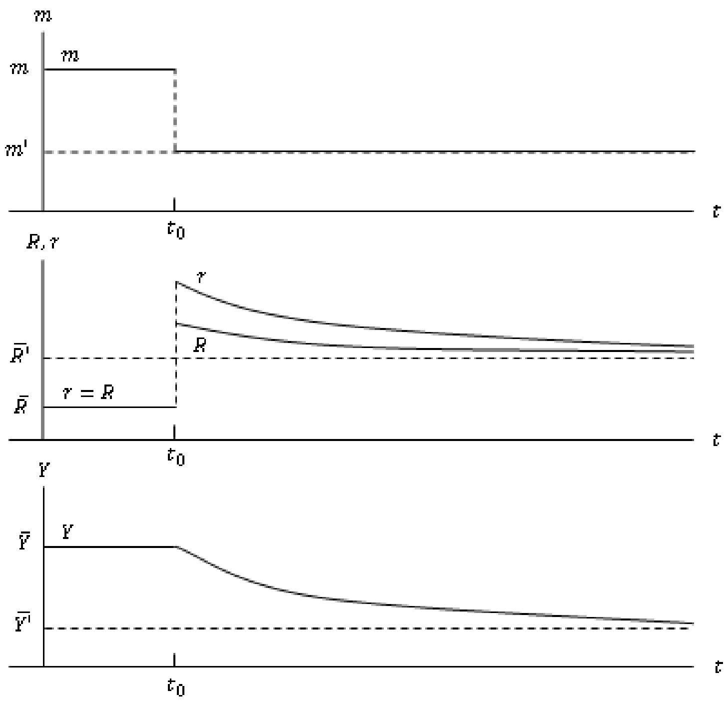

Figure 22.8: An unanticipated downward shift in m and time profiles of interest rates and output (regime m).

after t_0, both R and r have to be decreasing, parallel with the decreasing Y which implies lower money demand. Another noteworthy feature is that the yield curve is negatively sloped for some time after t_0.

Not surprisingly, there is not money neutrality. This is due, of course, to the price level being unaffected in the short run.

To find expressions for the steady-state effects of the change in m, we first take the differential on both sides of $D(\bar{Y}, i(\bar{Y}, m), \tau)+G = \bar{Y}$ to get $(1-D_Y-D_R i_Y)d\bar{Y} = D_R i_m dm$. By (22.12), this gives

$$\frac{\partial \bar{Y}}{\partial m} = \frac{D_R/L_i}{1 - D_Y + D_R L_Y/L_i} > 0.$$

Hence, $d\bar{Y} = (\partial \bar{Y}/\partial m)dm < 0$ for $dm < 0$. From $m = L(\bar{Y}, \bar{R})$ we get $dm = L_Y d\bar{Y} + L_i d\bar{R} = L_Y(\partial \bar{Y}/\partial m)dm + L_i d\bar{R}$ so that

$$\frac{\partial \bar{R}}{\partial m} = \frac{(1 - D_Y)/L_i}{1 - D_Y + D_R L_Y/L_i} < 0.$$

Hence, $d\bar{R} = (\partial \bar{R}/\partial m)dm > 0$ for $dm < 0$. These multipliers are the same as those for the static IS-LM model of Chapter 21.

(d) The effect of an anticipated downward shift in m. The shift in m is announced at time t_0 to take place at time t_1, cf. Fig. 22.9. At the time t_0 of "announcement", R jumps to R_A and then gradually increases until time t_1. This is due to the expectation that the short-term rate will in the longer run be higher than in the old steady state. The higher R implies a lower output demand and so output gradually adjusts downward. Then also the short-term rate moves downward until time t_1. In the time interval (t_0, t_1) the dynamics are determined by the old phase diagram and the economy follows that path (AB in Fig. 22.9) for which, starting from a point on the vertical line $Y = \bar{Y}$, it takes precisely $t_1 - t_0$ units of time to reach the new saddle path. Since in the time interval (t_0, t_1), R increases, while r decreases, we again witness a "twist" in the term structure of interest rates, cf. Fig. 22.10.

Owing to the principle that arbitrage prevents an expected jump in an asset price, exactly at the time t_1 of implementation of the tight monetary policy, the economy reaches the new saddle path generated by the lower money supply (cf. the point B in Fig. 22.9). The fall in m triggers a jump upward in the short-term rate r. This is foreseen by everybody, but it implies no capital loss because the bond is short-term. Output Y continues falling towards its new low steady state level, cf. Fig. 22.10. The transactions-motivated demand for money decreases and therefore r gradually decreases towards the new steady-state level which is

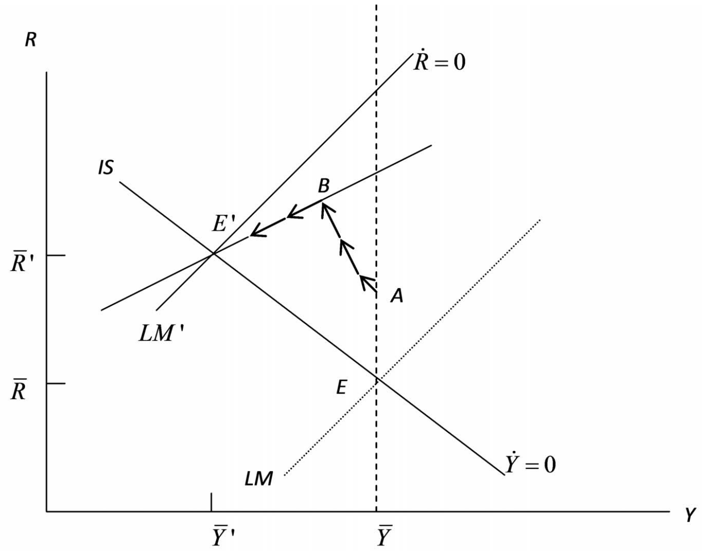

Figure 22.9: Phase portrait of an anticipated downward shift in m (regime m).

above the old because m is smaller than before. The long-term rate in advance "discounts" this gradual fall in r and is therefore, after t_1, always lower than r, implying a negatively sloped yield curve. Nevertheless, over time the long-term rate approaches the short-term rate in this model where there is no risk premium.

22.2.2 Policy regime i: Fixed short-term interest rate

Here we shall analyze a monetary policy regime where the short-term interest rate is the instrument. The model now takes i, the nominal interest rate on short-term government bonds, as an exogenous but adjustable constant. Thus, i is a policy instrument, together with the fiscal instruments, G and τ. Then the real money stock, m, has to be endogenous, which reflects that the central bank through open market operations adjusts the monetary base so that the actual short-term rate equals the one desired (and usually explicitly announced) by the central bank. Common names for this rate are the "target rate", the "policy rate", or "the official interest rate". In the real world, where there usually is a commercial banking sector, the central bank's target rate is often the so-called *interbank rate*. This is the interest rate charged on short-term (typically day-to-day) loans from one bank to another in the private banking sector, cf. Chapter 16. In the Euro area the ECB accordingly announces a certain target for the EONIA (euro overnight index average) and in the US the central bank announces a target for the Federal Funds Rate. Fig. 22.11 shows the evolution of the announced target for the Federal Funds Rate 1978-2013, stating dates of important economic and

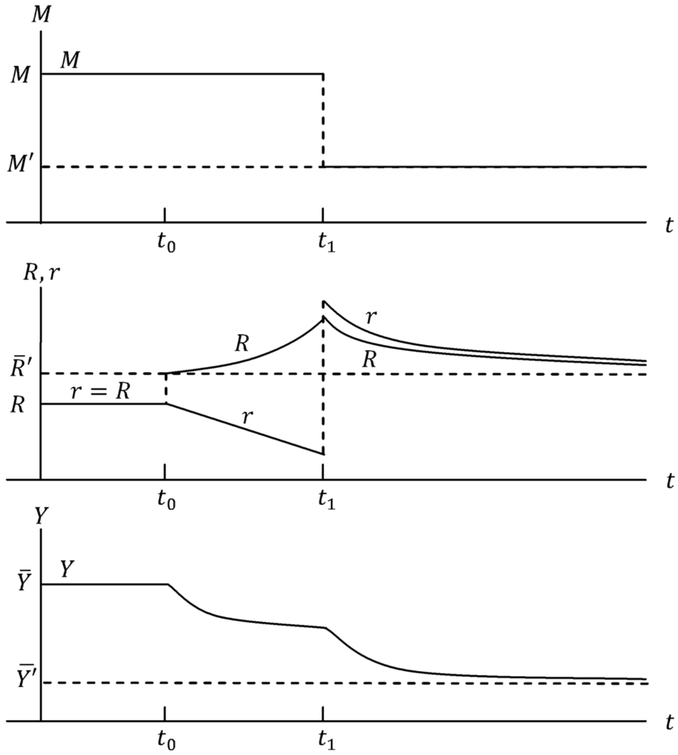

Figure 22.10: An anticipated downward shift in m and time profiles of interest rates and output (regime m).

political events over the period.

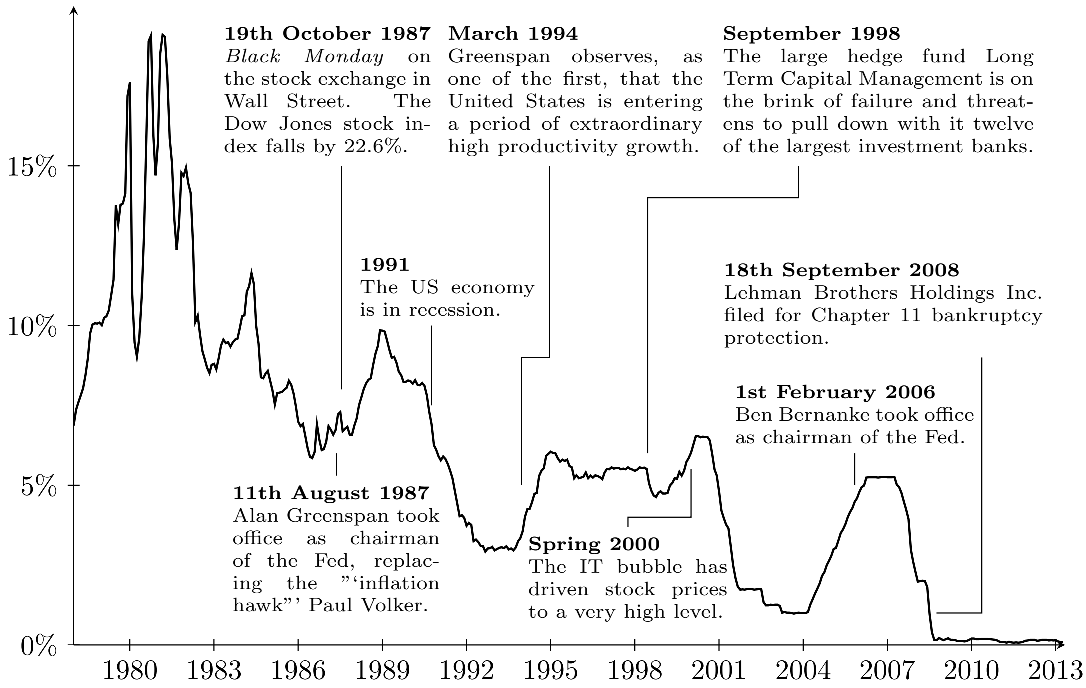

Figure 22.11: The evolution of the US federal funds rate 1978-2013. Source: Federal Reserve Bank of St. Louis.

The dynamic system

With $i > 0$ exogenous and m_t endogenous, the dynamic system consists of (22.9) and (22.1), which we repeat here for convenience:

$$\begin{aligned} \dot{R}_t &= (R_t - i + \pi)R_t, && (22.17)\\ \dot{Y}_t &= \lambda(D(Y_t, R_t, \tau) + G - Y_t), \qquad \text{where} && (22.18)\\ 0 &< D_Y < 1, D_R < 0, -1 < D_\tau < 0. \end{aligned}$$

Because Y_t does not appear in (22.17), the system (22.17) - (22.18) is simpler. The system determines the movement of R_t and Y_t. In the next step the required movement of M_t is determined by $M_t = P_t L(Y_t, i) = P_0 e^{\pi t} L(Y_t, i)$, from (22.2). In practice, an unchanged i will not be maintained forever but is likely to be adjusted according to the circumstances. Using a similar method as before we construct the phase diagram, cf. Fig. 22.12. The $\dot{R} = 0$ locus is now horizontal. The steady state is again a saddle point and is saddle-point stable. Notice, that here the saddle path *coincides* with the $\dot{R} = 0$ locus.

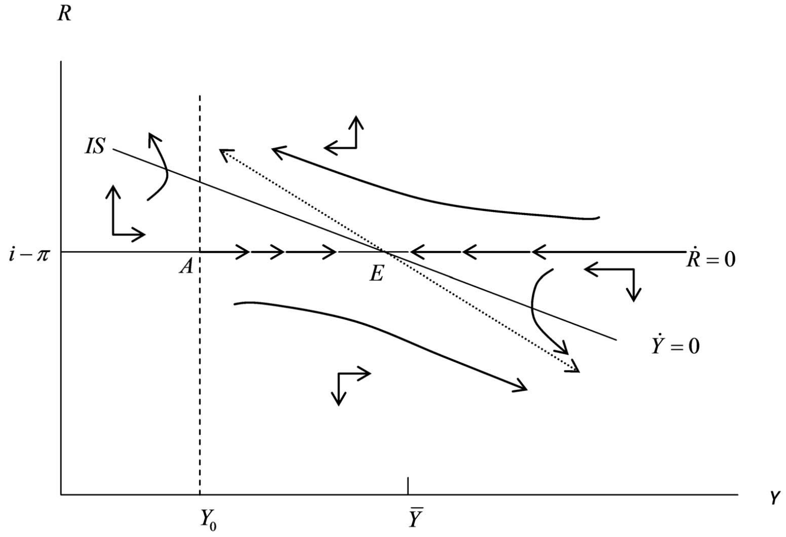

Figure 22.12: Phase diagram when i is the policy instrument.

Dynamic responses to policy changes when the short-term interest rate is the instrument

Let us again consider effects of permanent level shifts in exogenous variables, here G and i. Suppose that the economy has been in its steady state until time t_0. In the steady state we have $R = r = i - \pi$. Then either fiscal policy or monetary policy shifts. We consider the following three shifts in exogenous variables:

(a) An unanticipated decrease of G. See figures 22.13 and 22.14.

(b) An unanticipated decrease of i. See figures 22.15 and 22.16 in Appendix D.

(c) An anticipated decrease of i. See figures 22.17 and 22.18 in Appendix D.

As to the anticipated shift in i, we imagine that the central bank at time t_0 credibly announces the shift in i to take place at time $t_1 > t_0$.

The figures illustrate the responses. The diagrams should, by now, be self-explanatory. The only thing to add is that the reader is free to introduce another interpretation of, say, the exogenous variable G. For example, G could be interpreted as measuring consumers' and investors' "degree of optimism". The shift (a) could then be seen as reflecting the change in the "state of confidence" associated with the worldwide recession in 2001 or in 2008. The shift (b) could be interpreted as the immediate reaction of the Fed in the USA. As the public becomes aware of the general recessionary situation, further decreases of the federal

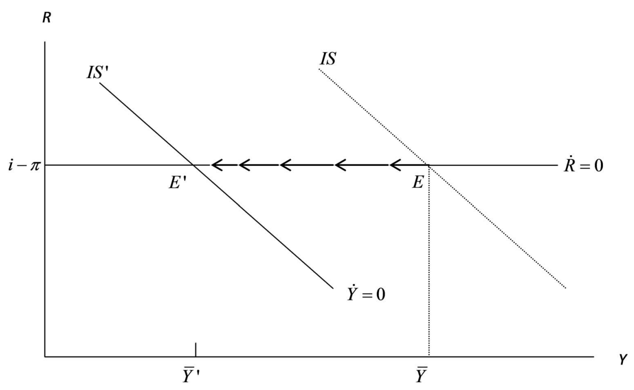

Figure 22.13: Phase portrait of an unanticipated downward shift in G (regime i).

funds rate, i, are expected and tends also to be executed. This is what point (c) is about.

22.2.3 Policy regime i': A contra-cyclical interest rate rule

Suppose the central bank conducts stabilization policy by using the interest rate rule

$$\begin{aligned} i_t &= \alpha_0 + \alpha_1(Y_t - Y^n) + \alpha_2(\pi_t - \hat{\pi}) = \alpha_0 + \alpha_1 Y_t - \alpha_1 Y^n + \alpha_2(\pi^e - \hat{\pi}) \\ &\equiv \alpha_0' + \alpha_1 Y_t, \end{aligned} \tag{22.19}$$

where α_0' is implicitly defined in (22.19), and Y^n is that level of output at which unemployment is at the NAIRU level, $\hat{\pi}$ is the desired inflation rate, and the α's are (in this model) constant policy parameters, $\alpha_1 > 0$. With a longer time horizon than in the present model, also the inflation rate would be treated as endogenous. A policy rule like (22.19) is known as a *Taylor rule.* The American economist John Taylor found the rule (with $\alpha_1 = 0.5$ and $\alpha_2 = 1.5$) to be a good description of actual U.S. monetary policy over a decade and at the same time a recommendable policy (Taylor, 1993).[11] Bernanke and Gertler (1959) present similar empirical evidence for Japan. Such a rule is called *contra-cyclical* because it dampens fluctuations in aggregate demand. This seems a better name than

[11] When inflation, π_t, and expected inflation, π_t^e, are endogenous, and one of the aims of monetary policy is to have a hold over the inflation rate, it is important to let $\alpha_2 > 1$ so that $r_t \equiv i_t - \pi_t$ goes up when π_t goes up.

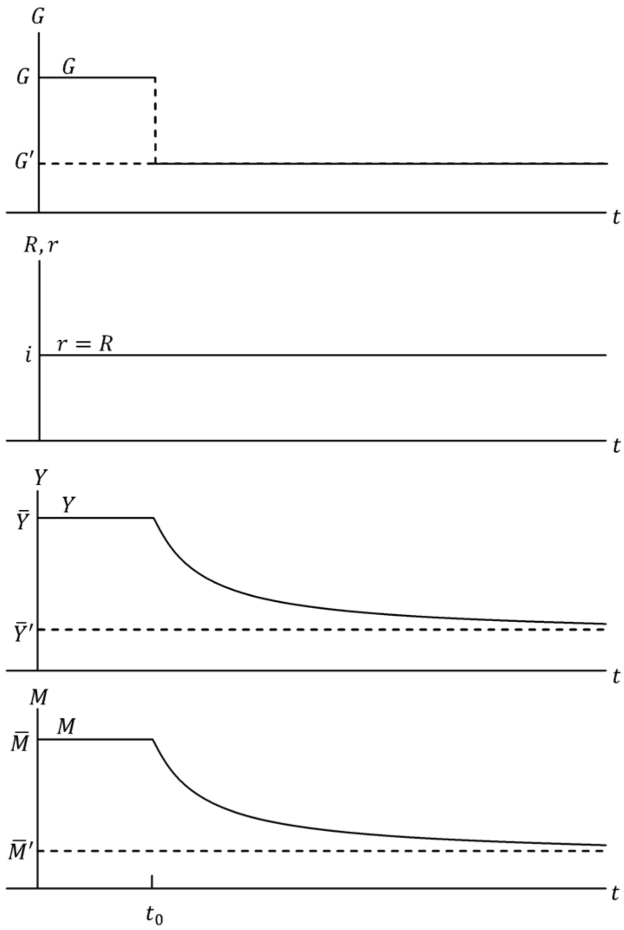

Figure 22.14: An unanticipated downward shift in G and time profiles of interest rates, output, and money supply (regime i; $\pi = 0$).

the sometimes used "counter-cyclical", which may lead to confusion because it generally refers to variables that are *negatively* correlated with aggregate output.

Given the policy rule (22.19), let us consider the dynamic system

$$\begin{aligned} \dot{Y}_t &= \lambda(D(Y_t, R_t, \tau) + G - Y_t), \qquad 0 < D_Y < 1, D_R < 0, -1 < D_\tau < 0, \\ \dot{R}_t &= [R_t - (\alpha_0' + \alpha_1 Y_t) + \pi] \, R_t. \end{aligned}$$

These two differential equations determine the time path of (Y_t, R_t) by a phase diagram similar to that in Fig. 22.1. Responses to unanticipated and anticipated changes in G are qualitatively the same as in regime m, where the money stock was the instrument. Qualitatively, the only difference is that the money stock is no longer an exogenous constant, but has to adjust according to

$$M_t = P_0 e^{\pi t} L(Y_t, \alpha_0' + \alpha_1 Y_t),$$

in order to let the contra-cyclical interest rule work. In Exercise 20.x the reader is asked to show that if $\alpha_1 > -L_Y/L_i$ and inflation is exogenous, this monetary policy regime is more stabilizing w.r.t. output than regime m.

22.3 Discussion

The previous chapter revisited the conventional *static* IS-LM model. Some micro-foundations for this were considered in chapters 19 and 20. In this chapter we have presented a dynamic version of the IS-LM model with endogenous forward-looking expectations. The model deals with the benchmark case of perfect foresight.

The framework captures the empirical tenet that output and employment in the short run tend to be demand-determined − with produced quantities and asset prices as the equilibrating factors, while the path of goods prices respond only little, or not at all, to changes in aggregate demand.

A limitation of simple IS-LM models, whether static or dynamic, is that they are silent about the intertemporal aspects of public and private budget constraints. In addition, aggregate behavior of the agents is postulated and not based on a weighted summation over the actions of different optimizing agent types. Yet the consumption and investment functions *can* to some extent be defended on a microeconomic basis.[12]

The forward-looking expectations in the model capture wealth effects through changes in the long-term interest rate, R. It would be an improvement if also the effect of expected future output demand on current consumption and investment were modeled. Even though we have in earlier chapters seen that Ricardian

[12] See Literature notes to Chapter 21.

equivalence is not plausible, this does not mean that expected future taxes should be ignored.

The simple process assumed for the adjustment of output to changes in demand is of course ad hoc. Nevertheless, it can be seen as a rough approximation to the theory of intended and unintended inventory investment (Wang and Wen, 2009, Wen 2011).

It is a simplification that changes in production and employment have no wage and price effects at all. At least in a medium-run perspective there should be wage and price responses within the model, i.e., some version of a Phillips curve. Then the issue arises under what conditions the dynamic interactions in the system after a disturbance tend to pull Y back to its NAIRU level or further away from it. This issue is taken up in Chapter 24 and later chapters.

The next chapter extends the present short-run framework to a small open economy.

22.4 Literature notes

Our presentation of Blanchard's dynamic IS-LM model builds on the version in Blanchard and Fischer (1989). In the original Blanchard (1981) paper, however, the key forward-looking variable is Tobin's q rather than the long-term interest rate, R. But since the (real) long-term interest rate can, in this context, be considered as inversely related to Tobin's q, there is essentially no difference. Wealth effects come true whether the source is interpreted as changes in Tobin's q or the long-term interest rate.

Treating the inflation rate as a state variable, changing only gradually, is empirically supported by, for instance, Sims (1998) and Estralla and Fuhrer (2002).

Extending the dynamic IS-LM model by some kind of a Phillips curve makes the model substantially more complicated. Blanchard (1981, last section) did in fact take a first step towards such an extension, ending up with a system of three coupled differential equations.

22.5 Appendix

A. An inflation-indexed consol

An alternative way of presenting the inflation-indexed consol is the following. The coupon per time unit at time s in the future amounts to P_s units of account, i.e., the price level at time s. This price level is related to the current price level,

P_t, via the evolution of inflation in the time interval (t, s),

$$P_s = P_t e^{\int_t^s \pi_\tau d\tau}.$$

Starting from a given nominal market value, Q_t, of the consol at time t, we thus have

$$\begin{aligned} Q_t &\equiv P_t q_t = \int_t^\infty P_s e^{-\int_t^s i_\tau d\tau} ds = P_t \int_t^\infty e^{\int_t^s \pi_\tau d\tau} e^{-\int_t^s i_\tau d\tau} ds \\ &= P_t \int_t^\infty e^{-\int_t^s (i_\tau - \pi_s) d\tau} ds = P_t \int_t^\infty e^{-\int_t^s r_\tau d\tau} ds, \end{aligned}$$

by $r_\tau \equiv i_\tau - \pi_s$. Dividing through by P_t gives (22.10).

B. Solving the no-arbitrage equation for q_t in the absence of asset price bubbles

In Section 22.1 we claimed that in the absence of asset price bubbles, the differential equation implied by the no-arbitrage equation (22.8) has the solution

$$q_t = \int_t^\infty 1 \cdot e^{-\int_t^s r_\tau d\tau} ds. \tag{22.20}$$

To prove this, we write the no-arbitrage equation on the standard form for a linear differential equation

$$\dot{q}_t - r_t q_t = -1.$$

The general solution to this is

$$q_t = q_{t_0} e^{\int_{t_0}^t r_\tau d\tau} - e^{\int_{t_0}^t r_\tau d\tau} \int_{t_0}^t e^{-\int_{t_0}^s r_\tau d\tau} ds.$$

Multiplying through by $e^{-\int_{t_0}^t r_\tau d\tau}$ gives

$$q_t e^{-\int_{t_0}^t r_\tau d\tau} = q_{t_0} - \int_{t_0}^t e^{-\int_{t_0}^s r_\tau d\tau} ds.$$

Rearranging and letting $t \to \infty$, we get

$$q_{t_0} = \int_{t_0}^\infty e^{-\int_{t_0}^s r_\tau d\tau} ds + \lim_{t\to\infty} q_t e^{-\int_{t_0}^t r_\tau d\tau}. \tag{22.21}$$

The first term on the right-hand side is the fundamental value of the consol, i.e., the present value of the future dividends on the asset. The second term on the right-hand side thus amounts to the difference between the market price, q_{t_0}, of the consol and its fundamental value. By definition, this difference represents a bubble. In the absence of bubbles, the difference is nil, and the market price, q_{t_0}, coincides with the fundamental value. So (22.20) holds (in (22.21) replace t by T and t_0 by t), as was to be shown.

C. Proof of (22.11)

CLAIM Let $q_t = \lim_{T\to\infty} \int_t^T e^{-\int_t^s r_\tau d\tau} ds < \infty$. Then:

$$\text{(i)} \quad \int_t^\infty e^{-\int_t^s r_\tau d\tau} r_s ds = 1; \qquad \text{and}$$

$$\text{(ii)} \quad \frac{1}{q_t} = \int_t^\infty w_{t,s} r_s ds, \quad \text{where } w_{t,s} \equiv \frac{e^{-\int_t^s r_\tau d\tau}}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds}.$$

Proof. The function $F(s) = e^{-\int_t^s r_\tau d\tau}$ has the derivative

$$F'(s) = -e^{-\int_t^s r_\tau d\tau} r_s.$$

Hence

$$\int_t^\infty e^{-\int_t^s r_\tau d\tau} r_s ds = -\int_t^\infty F'(s) ds = -\left. F(s)\right|_t^\infty = -\left. e^{-\int_t^s r_\tau d\tau}\right|_t^\infty = -(0-1) = 1.$$

This proves (i). We have

$$\frac{1}{q_t} = \frac{1}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds} = \frac{\int_t^\infty e^{-\int_t^s r_\tau d\tau} r_s ds}{\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds} = \int_t^\infty w_{t,s} r_s ds,$$

where the first equality follows from the definition of q_t, the second from (i), and the third by moving the constant $1/(\int_t^\infty e^{-\int_t^s r_\tau d\tau} ds)$ inside the integral and then apply the definition of $w_{t,s}$. This proves (ii). $\square$

D. More examples of dynamics in policy regime i

The figures 22.15 and 22.16 illustrate responses to an *unanticipated* lowering of the short-term interest rate, and figures 22.17 and 22.18 illustrate the responses to an *anticipated* lowering. Throughout it is assumed that $\pi = 0$.

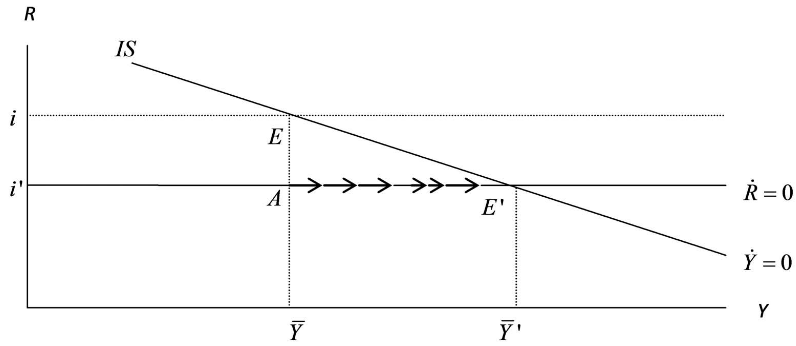

Figure 22.15: Phase portrait of an unanticipated downward shift in i (regime i, $\pi = 0$).

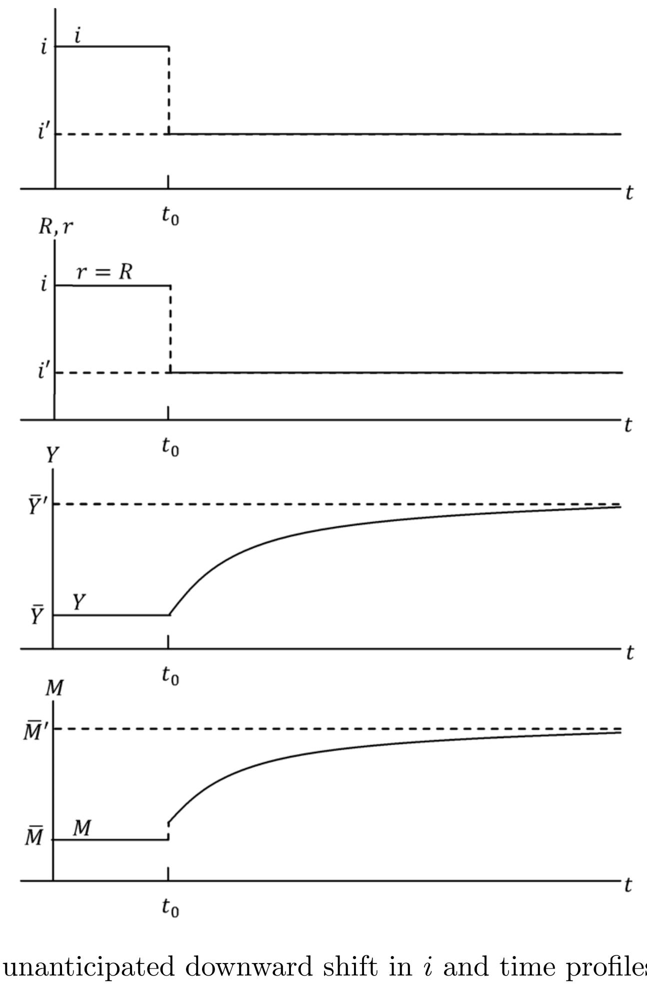

Figure 22.16: An unanticipated downward shift in i and time profiles of the long-term rate, output, and money supply (regime i, $\pi = 0$).

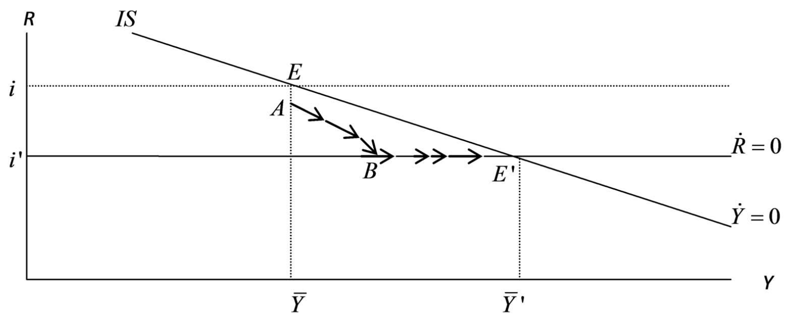

Figure 22.17: Phase portrait of an anticipated downward shift in i (regime i, $\pi = 0$).

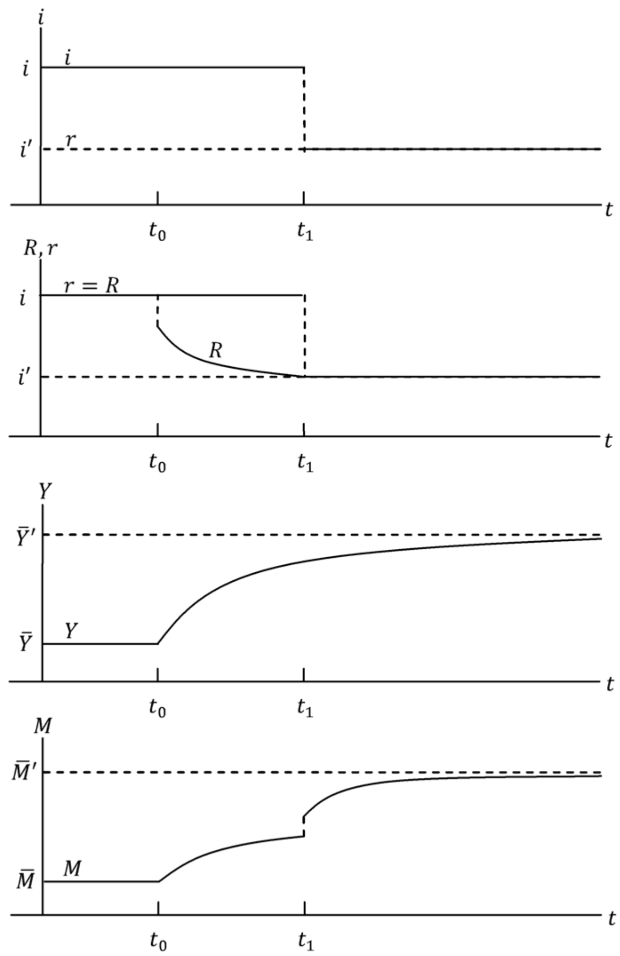

Figure 22.18: An unanticipated downward shift in i and time profiles of the long-term rate, output, and money supply (regime i, , $\pi = 0$).

Chapter 23

The open economy and alternative exchange rate regimes

In this chapter we consider simple open-economy members of the IS-LM family. Section 23.1 revisits the standard static version, the Mundell-Fleming model, in its fixed-exchange-rate as well as floating-exchange-rate adaptations. The Mundell-Fleming model is well-known from elementary macroeconomics and our presentation of the model is merely a prelude to the next sections which address dynamic extensions. In Section 23.2 we show how the *dynamic* closed-economy IS-LM model with rational expectations from the previous chapter can be easily modified to cover the case of a small open economy with fixed exchange rates. In Section 23.3 we go into detail about the more challenging topic of floating exchange rates. In particular we address the issue of exchange rate *overshooting*, first studied by Dornbusch (1976).

Both the original Mundell-Fleming model and the dynamic extensions considered here are *ad hoc* in the sense that the microeconomic setting is not articulated in any precise way. Yet the models are useful and have been influential as a means of structuring thinking about an open economy in the short run.

The models focus on short-run mechanisms in a small open economy. There is a "domestic currency" and a "foreign currency" and these currencies are traded in the foreign exchange market. In this market, nowadays, the volume of trade is gigantic.

The following assumptions are shared by the models in (most of) this chapter:

1. Free mobility across borders of financial capital (i.e., no barriers or restrictions on currency trade).

2. Domestic and foreign bonds are perfect substitutes and hence command the same expected rate of return.

3. Free mobility across borders of goods and services (i.e., no barriers or restrictions on trade in goods and services)

4. No mobility across borders of labor.

5. Domestic and foreign goods are imperfect substitutes.

6. Nominal prices are sluggish and follow an exogenous constant inflation path.

The two first assumptions together make up the case of *perfect mobility of financial capital.* As the two last assumptions indicate, we consider an economy with imperfectly competitive firms. In an asynchronous way, the firms adjust their prices when their unit costs change. The aggregate inflation rate is considered sticky.

We use the same notation as in the previous chapter, with the following clarifications and additions: Y is domestic output (GDP), P the domestic price level, P^* the foreign price level, i the domestic short-term nominal interest rate, i^* the foreign short-term nominal interest rate, and X the nominal exchange rate. Suppose UK is the "home country". Then the exchange rate X indicates the price in terms of British£ (GBP) for one US$ (USD), say. Be aware that a currency trading convention is to announce an exchange rate as, for example, "USD/GBP is X", in words: "USD through GBP is X". The intended meaning is that the exchange rate is X GBP per USD. In ordinary language (as well as in mathematics) a slash, however, means "per". Thus, writing "USD/GBP is X" ought to mean that the exchange rate is X USD per GBP, which is exactly the opposite. Whenever in this text we use a slash, it has this standard mathematical meaning. To counter any risk of confusion, when indicating an exchange rate, we therefore avoid using a slash altogether. Instead we use the unmistakable "per".

When reporting that "the exchange rate is X", our point of view is that of an *importer* in the home country. That is, if UK is the "home country", saying that "the exchange rate is X" shall mean that X GBP must be paid per \$ worth of imports. And saying that "the real exchange rate is x" shall mean that x domestic goods must be paid per imported good. This convention is customary in continental Europe. Note however, that it is the opposite of the British convention which reports the home country's nominal and real "exchange rate" as $1/X$ and $1/x$, respectively.

When considering *terms of trade,* $1/x$, our point of view is that of an *exporter* in the home country. The terms of trade tell us how many foreign goods we get per exported good. In accordance with this, Table 21.1 gives a list of key open economy variables.

Table 21.1	**Open economy glossary**	
Term	*Symbol*	*Meaning*
Nominal exchange rate	X	The price of foreign currency in terms of domestic currency.
Real exchange rate	$x \equiv \frac{XP^*}{P}$	The price of foreign goods in terms of domestic goods (can be interpreted as an indicator of competitiveness).
Terms of trade	$1/x$	In this simple model terms of trade is just the inverse of the real exchange rate (generally, it refers to the price of export goods in terms of import goods).
Purchasing power parity		The nominal exchange rate which makes the cost of a basket of goods and services equal in two countries, i.e., makes $x = 1$.
Uncovered interest parity		The hypothesis that domestic and foreign bonds have the same expected rate of return, expressed in terms of the same currency.
Exports	E	
Imports	IM	
Net exports (in domestic output units)	N	$= E - xIM$.
Net foreign assets	A^f	
Net factor income from abroad (in domestic output units)	$rA^f + w^f L^f$	The present model has $L^f = 0$.
Current account surplus	CAS	$= N + rA^f + w^f L^f$. In this model $L^f = 0$.
Official reserve assets	ORA	
Private net foreign assets	A_p^f	$= A^f - ORA$.
Increase per time unit in some variable z	Δz	
Financial account surplus	FAS	$= -\Delta A_p^f - \Delta ORA = -CAS =$ current account deficit.
Net inflow of foreign exchange	$= CAS$	$= -FAS = -$ (net outflow of foreign exchange)

We simplify by talking of an exchange rate as if the "foreign country" constitutes the rest of the world and the exchange rate is thereby just a bilateral entity. A more precise treatment would center on the *effective exchange rate,* which is a trade-weighted index of the exchange rate vis-a-vis a collection of major trade partners.

23.1 The Mundell-Fleming model

Whether the Mundell-Fleming model is adapted to a fixed or floating exchange rate regime, there is a common set of elements.

23.1.1 The basic elements

Compared with the static closed-economy IS-LM model, the Mundell-Fleming model contains two new elements:

- An extra output demand component, namely a net export function $N(Y, x)$, where $x \equiv XP^*/P$ is the real exchange rate. As a higher income implies more imports, we assume that $N_Y < 0$. And as a higher real exchange rate implies better competitiveness, we assume that $N_x > 0$.[1]
- The uncovered interest parity condition (for short UIP). This says that domestic and foreign financial assets pay the same expected rate of return (measured in the same currency).

Apart from the addition of these open-economy elements, notation is as in the previous chapter. Output demand is given as

$$\begin{aligned} Y^d &= C(Y^p, r^e) + I(Y, r^e) + N(Y, x) + G + \varepsilon_D, \text{ where} \qquad (23.1) \\ 0 &< C_{Y^p} + N_Y < C_{Y^p} \leq C_{Y^p} + I_Y < 1, C_{r^e} + I_{r^e} \leq I_{r^e} < 0, N_x > 0, \end{aligned}$$

and ε_D is a demand shift parameter. This parameter could for instance reflect the level of economic activity in the world economy. Disposable income, Y^p, is

$$Y^p \equiv Y - \mathbb{T}, \qquad (23.2)$$

where $\mathbb{T}$ is real net tax revenue (gross tax revenue minus transfers). We assume a quasi-linear tax revenue function

$$\mathbb{T} = \tau + T(Y), \qquad 0 \leq T' < 1, \qquad (23.3)$$

[1] By assuming $N_x > 0$, it is presupposed that the Marshall-Lerner condition is satisfied, see Appendix A. Throughout the chapter we ignore that it may take one or two years for a rise in x to materialize as a rise in net exports. This is because the price of imports is immediately increased while the quantity of imports and exports only adjust with a time lag (the pattern known as the *J-curve effect*).

where τ is a constant representing the "tightness" of fiscal policy. This parameter, together with the level of public spending, G, describes fiscal policy.

Inserting (23.2) and (23.3) into (23.1), we can write aggregate demand as

$$\begin{aligned} Y^d &= D(Y, r^e, x, \tau) + G + \varepsilon_D, \quad \text{where} \qquad (23.4) \\ 0 &< D_Y = C_{Y^p}(1 - T') + I_Y + N_Y < 1, D_{r^e} = C_{r^e} + I_{r^e} < 0, \\ D_x &> 0, D_\tau = C_{Y^p} \cdot (-1) \in (-1, 0). \end{aligned}$$

The demand for money (domestic currency and checkable deposits in commercial banks) in the home country is, as in the closed economy model,

$$M^d = P \cdot (L(Y, i) + \varepsilon_L), \qquad L_Y > 0, \; L_i < 0, \qquad (23.5)$$

where i is the short-term nominal interest rate on the *domestic bond* which is denominated in the *domestic* currency. The symbol ε_L represents a shift parameter which may reflect a shock to liquidity preferences or the payment technology and thereby the money multiplier.

There is a link between r^e and i, namely $r^e = i - \pi^e$, where π^e denotes the expected value of π which is the domestic forward-looking inflation rate. Recall that with continuous interest compounding, the equation $r^e = i - \pi^e$ is an identity. In a discrete time framework the equation is a convenient approximation. Assuming clearing in the output market as well as the money market, we now have:

$$\begin{aligned} Y &= D(Y, i - \pi^e, X\frac{P^*}{P}, \tau) + G + \varepsilon_D, \qquad \text{(IS)} \\ \frac{M}{P} &= L(Y, i) + \varepsilon_L. \qquad \text{(LM)} \end{aligned}$$

In addition to the domestic short-term bond there is a short-term bond denominated in *foreign* currency, henceforth the *foreign bond.* The nominal interest rate on the foreign bond is denoted i^* and is exogenous, $i^* > 0$. The term "bonds" may be interpreted in a broad sense, including large firms' interest-bearing bank deposits.

The no-arbitrage condition between the domestic and the foreign bond is assumed given by the *uncovered interest parity* condition,

$$i = i^* + \frac{\dot{X}^e}{X}, \qquad \text{(UIP)}$$

where $\dot{X}^e$ denotes the expected increase per time unit in the exchange rate in the immediate future.[2] Imposing (UIP) amounts to assuming that arbitrage quickly

[2] As a preparation for the dynamic extensions to be considered below, we have presented the UIP condition in its continuous time version with continuous interest compounding. In discrete

brings the interest rate on the domestic bond in line with the expected rate of return on investing in the foreign bond, expressed in the domestic currency. This expected rate of return equals the foreign interest rate plus the expected rate of depreciation of the domestic currency.

By invoking the UIP condition the model assumes that asymmetric risk and liquidity aspects can be ignored in a first approximation. So domestic and foreign bonds are considered perfect substitutes. Hence only the expected rate of return matters. If we imagine that in the very short run there is, for example, a ">" in (UIP) instead of "=", then arbitrage sets in. A massive inflow of financial capital will occur (investors dispose of foreign assets and purchase domestic assets), until "=" in (UIP) is re-established. The adjustment will take the form of a lowering of i in case of a fixed exchange rate system. In case of a floating exchange rate system, the adjustment will take the form of an adjustment in X (generally both its level and subsequent rate of change). Only when (UIP) is satisfied, is the system at rest. The primary actors in the foreign exchange market are commercial banks, mutual funds, asset-management companies, insurance companies, exporting and importing corporations, and central banks.

The model assumes that (UIP) holds continuously (arbitrage in international asset markets is very fast). Thus, if for example the domestic interest rate is below the foreign interest rate, it must be that the domestic currency is expected to appreciate vis-à-vis the foreign currency, that is, $\dot{X}^e < 0$. The adjective "uncovered" refers to the fact that the return on the right-hand side of (UIP) is not guaranteed, but only an expectation. On the *covered interest parity*, see Appendix B.

The original Mundell-Fleming model is a static model describing just one short period with the price levels P and P^* set in advance. The model consists of the equations (IS), (LM), and (UIP) with the following partitioning of the variables:

- Exogenous: $P, P^*, G, \tau, \pi^e, i^*, \dot{X}^e, \varepsilon_D, \varepsilon_L$, and either X or M.
- Endogenous: Y, i, and either M or X, depending on the exchange rate regime.

We now consider the polar cases of fixed and floating exchange rates, respectively.

time with X_t denoting the exchange rate at the beginning of period t, the UIP condition reads: $1 + i_t = \frac{1}{X_t}(1 + i_t^*)X_{t+1}^e$, which can be written $1 + i_t = (1 + i_t^*)(1 + (X_{t+1}^e - X_t)/X_t) \approx 1 + i_t^* + (X_{t+1}^e - X_t)/X_t$, when i_t^* and $(X_{t+1}^e - X_t)/X_t$ are "small".

23.1.2 Fixed exchange rate

A fixed exchange rate regime amounts to a promise by the central bank to sell and buy unlimited amounts of foreign currency at an announced exchange rate. So X becomes an exogenous constant in the model.[3] The system requires that the central bank keeps foreign exchange reserves to be able to buy the domestic currency on foreign exchange markets when needed to maintain its value.

We assume that the announced exchange rate is at a sustainable level vis-a-vis the foreign currency so that credibility problems can be ignored. So we let $\dot{X}^e = 0$. Then (UIP) reduces to $i = i^*$, and output is determined by (IS), given $i = i^*$. Finally, through movements of financial capital the nominal money supply (hence also the real money supply) adjusts endogenously to the level required by (LM), given $i = i^*$ and the value of Y already determined in (IS). The system thus has a recursive structure.

There is no possibility of an independent monetary policy as long as there are no restrictions on movements of financial capital. The intuition is the following. Suppose the central bank naively attempts to stimulate output by buying domestic bonds, thereby raising the money supply. There will be an incipient fall in i. This induces portfolio holders to convert domestic currency into foreign currency to buy foreign bonds and enjoy their higher interest rate. This tends to raise X_t, however. Assuming the central bank abides by its commitment to a fixed exchange rate, the bank will have to immediately counteract this tendency to depreciation by *buying domestic assets* (domestic currency and bonds) for foreign currency in an amount sufficient to bring the domestic money supply down to its original level needed to restore both the exchange rate and the interest rate at their original values. That is, as soon as the central bank attempts expansionary monetary policy, it has to reverse it.

The model is qualitatively the same as the static IS-LM model for the closed economy with the nominal interest rate fixed by the central bank. The output and money multipliers w.r.t. government spending are, from (IS) and (LM) respectively,

$$\begin{aligned}
\frac{\partial Y}{\partial G} &= \frac{1}{1-D_Y} = \frac{1}{1-C_{Y^p}(1-T')-I_Y-N_Y} > 1, \\
\frac{\partial M}{\partial G} &= PL_Y \frac{\partial Y}{\partial G} = PL_Y \frac{1}{1-C_{Y^p}(1-T')-I_Y-N_Y} > 0.
\end{aligned}$$

The output multiplier can be seen as a measure of how much a unit increase in G raises aggregate demand and thereby stimulates production and income; indeed,

[3] In practice there will be a small margin of allowed fluctuation around the par value. The Danish krone (DKK) is fixed at 746.038 DKK per 100 euro +/- 2.25 percent.

$\Delta Y \approx \partial Y/\partial G \cdot \Delta G = \partial Y/\partial G$ for $\Delta G = 1$. Thereby the transaction-motivated demand for money is increased. This generates an incipient tendency for both the short-term interest rate to rise and the exchange rate to appreciate as portfolio holders worldwide buy the currency of the SOE to invest in its bonds and enjoy their high rate of return. The central bank is committed to a fixed exchange rate, however, and has to prevent the pressure for a higher interest rate and currency appreciation by *buying foreign assets* (foreign currency and bonds) for domestic currency. When the money supply has increased enough to nullify the incipient tendency for a higher domestic interest rate, the equilibrium with unchanged exchange rate is restored. The needed increase in the money supply for a unit increase in G is given by $\Delta M \approx \partial M/\partial G \cdot \Delta G = \partial M/\partial G$ for $\Delta G = 1$. One may say that it is the accommodating money supply that allows the full unfolding of the output multiplier w.r.t. government spending. Nevertheless, owing to the *import leakage* ($N_Y < 0$), both the output and money multiplier are lower than the corresponding multipliers in the closed economy where the central bank maintains the interest rate at a certain target level. The system ends up with higher Y, the same i and X, and lower net exports because of higher imports.

The output multipliers w.r.t. a demand shock, an interest rate shock, and a liquidity preference shock, respectively, are

$$\begin{aligned}
\frac{\partial Y}{\partial \varepsilon_D} &= \frac{\partial Y}{\partial G} = \frac{1}{1 - C_{Y^p}(1-T') - I_Y - N_Y} > 1, \\
\frac{\partial Y}{\partial i^*} &= \frac{D_{r^e}}{1 - D_Y} = \frac{C_{r^e} + I_{r^e}}{1 - C_{Y^p}(1-T') - I_Y - N_Y} < 0, \\
\frac{\partial Y}{\partial \varepsilon_L} &= 0.
\end{aligned}$$

This last result reflects that after the liquidity preference shock, the money market equilibrium is restored by full adjustment of the money supply ($dM = Pd\varepsilon_L$) at unchanged interest rate.

23.1.3 Floating exchange rate

In a floating exchange rate regime, also called a flexible exchange rate regime, the exchange rate is allowed to respond endogenously to the market forces, supply and demand, in the foreign exchange market. The model treats the money stock, M, as exogenous. More precisely, the money supply is targeted by the central bank and the model assumes this is done successfully. The exchange rate adjusts so that the available supplies of money and domestic bonds are willingly held.

In the present static portrait of the floating exchange rate regime, $\dot{X}^e$ is treated as exogenous. Following Mundell (1963), we imagine that the system has settled down in a steady state with $\dot{X}^e = 0$. The model is again recursive. First (UIP) yields $i = i^*$. Then output is determined by (LM), given $i = i^*$. And finally the required exchange rate is determined by (IS) for a given level of P^*/P. So the story behind the equilibrium described by the model is that the exchange rate has adjusted to a level such that aggregate demand and output is at the point where, given the real money supply, the transactions-motivated demand for money establishes clearing in asset markets for a nominal interest rate equal to the foreign nominal interest rate.

There is no possibility of *fiscal policy* affecting output as long as there are no restrictions on movements of financial capital. The interpretation is the following. Consider an expansive fiscal policy, $dG > 0$ or $d\tau < 0$. The incipient output stimulation increases the transaction demand for money and thereby the interest rate. The rise in the interest rate is immediately counteracted, however, by inflow of foreign exchange induced by the high interest rate. This inflow means higher demand for the domestic currency, which thereby appreciates, thus lowering competitiveness and net exports. The appreciation continues until competitiveness has decreased enough[4] to bring the interest rate back to its initial level. This state of affairs is obtained when the exchange rate has reached a level at which the fall in net exports matches the rise in G or fall in τ, thereby bringing aggregate demand and output back to their initial levels. In effect, the system ends up with unchanged output and interest rate, a lower exchange rate, X, and lower net exports.[5]

On the other hand, *monetary policy* is effective. An increase in the money supply (through an open-market operation) generates an incipient fall in the interest rate. This triggers a counteracting outflow of financial capital, whereby the domestic currency depreciates, i.e., X rises. The depreciation continues until the real exchange rate has increased enough to induce a rise in net exports and output large enough for the transaction demand for money to match the larger money supply and leave the interest rate at its original level. The system ends up with higher Y, the same i, higher X, and higher net exports.[6]

From (LM) and (IS), respectively, we find the output and exchange rate mul-

[4] Recall that, as we have defined the exchange rate, "up is down and down is up" or, perhaps with a little more transparency, "currency up is exchange rate down".

[5] This canonical result relies heavily on the idealized assumption that foreign and domestic bonds are perfect substitutes and move without restraint of any kind.

[6] The implicit assumption that a higher X does not affect the price level P is of course problematic. If intermediate goods are an important part of imports, then a higher exchange rate would imply higher unit costs of production. And since prices tend to move with costs, this would imply a higher price level. If imports consist primarily of final goods, however, it is easier to accept the logic of the model.

tipliers w.r.t. the money supply:

$$\frac{\partial Y}{\partial M} = \frac{1}{PL_Y} > 0, \tag{23.6}$$

$$\frac{\partial X}{\partial M} = \frac{1 - C_{Y^p}(1 - T') - I_Y - N_Y}{D_x P^* L_Y} > 0. \tag{23.7}$$

Finally, the output multipliers w.r.t. a demand shock and a liquidity preference shock, respectively, are

$$\frac{\partial Y}{\partial \varepsilon_D} = \frac{\partial Y}{\partial G} = 0,$$

$$\frac{\partial Y}{\partial \varepsilon_L} = -\frac{1}{L_Y} < 0.$$

Like increased public spending, a positive output demand shock does not affect output. It is neutralized by appreciation of the domestic currency. A positive liquidity preference shock reduces output. The mechanism is that the increase in money demand triggers an incipient rise in the domestic interest rate. The concomitant appreciation of the currency, resulting from the induced inflow of financial capital, reduces net exports.

23.1.4 Perspectives

The message of this simple model is that for a small open economy, a fixed exchange rate regime is better at stabilizing output if money demand shocks dominate, and a floating exchange rate system is better if most shocks are output demand shocks. In any case, there is an asymmetry. Under a fixed exchange rate system, two macroeconomic short-run policy instruments are given up: exchange rate policy and monetary policy. Under a floating exchange rate system and perfect capital mobility, only one macroeconomic short-run policy instrument is given up: fiscal policy. Yet, the historical experience seems to be that an international system of floating exchange rates ends up with higher volatility in both real and nominal exchange rates than one with fixed exchange rates (Mussa, 1990; Obstfeld and Rogoff, 1996; Basu and Taylor, 1999).

The result that fiscal policy is impotent in the floating exchange rate regime does not necessarily go through in more general settings. For example, in practice, domestic and foreign financial claims may often *not* be perfect substitutes. Indeed, the UIP hypothesis tends to be empirically rejected at short forecast horizons, while it does somewhat better at horizons longer than a year (see the literature notes at the end of the chapter). And as already hinted at, treating the price level as exogenous when import prices change is not satisfactory.

Anyway, even the static Mundell-Fleming model provides a basic insight: the *impossible trinity*. A society might want a system with the following *three* characteristics:

- free mobility of financial capital (to improve resource allocation);
- independent monetary policy (to allow a stabilizing role for the central bank);
- fixed exchange rate (to avoid exchange rate volatility).

But it can have only two of them. A fixed exchange rate system is incompatible with the second characteristic. And a flexible exchange rate system contradicts the third.

In the next sections we extend the model with dynamics and rational expectations. We first consider the fixed exchange rate regime, next the flexible exchange rate regime.

23.2 Dynamics under a fixed exchange rate

We ignore the shift parameters ε_D and ε_L. On the other hand we introduce an additional asset, a long-term bond that is indexed w.r.t. domestic inflation. In the fixed exchange rate regime this extension is easy to manage. In addition we assume rational expectations. The formal structure of the model then becomes exactly the same as that of the dynamic IS-LM model for a closed economy with short- and long-term bonds, studied in the previous chapter.

With R_t denoting the real long-term interest rate at time t (defined as the internal real rate of return on an inflation indexed consol), aggregate demand is

$$\begin{aligned} Y_t^d &= C(Y_t^p, R_t) + I(Y_t, R_t) + N(Y_t, x) + G \equiv D(Y_t, R_t, x, \tau) + G, \\ & \quad \text{where} \\ 0 &< D_Y = C_{Y^p}(1 - T') + I_Y + N_Y < 1, D_R = C_R + I_R < 0, D_x > 0, \\ -1 &< D_\tau = -C_{Y^p} < 0, \end{aligned}$$

where x is the real exchange rate, XP_t^*/P_t, with X representing the given and constant nominal exchange rate and the price ratio P_t^*/P_t assumed constant. The latter assumption is equivalent to assuming the domestic inflation rate to equal the foreign inflation rate for all t. Moreover, this common inflation rate is assumed equal to a constant, π.

To highlight the dynamics between fast-moving asset markets and slower-moving goods markets, the model replaces (IS) by the error-correction specification

$$\dot{Y}_t \equiv \frac{dY_t}{dt} = \lambda(Y_t^d - Y_t) = \lambda(D(Y_t, R_t, x, \tau) + G - Y_t), \qquad Y_0 > 0 \text{ given}, \quad (23.8)$$

where $\lambda > 0$ is the constant adjustment speed. Because changing the level of production is time consuming, Y_0 is historically given.

We assume the fixed exchange rate policy is sustainable. That is, the level of X is such that no threatening cumulative current account deficits in the future are glimpsed. In view of rational expectations we then have $\dot{X}_t^e = 0$ for all $t \geq 0$. In effect the uncovered interest parity condition reduces to

$$i_t = i^*, \quad (23.9)$$

where the exogenous foreign interest rate, i^*, is for simplicity assumed constant.

The remaining elements of the model are well-known from Chapter 22:

$$\begin{aligned}
\frac{M_t}{P_t} &= L(Y_t, i^*), \qquad L_Y > 0,\ L_i < 0. && (23.10)\\
R_t &= \frac{1}{q_t}, && (23.11)\\
\frac{1 + \dot{q}_t^e}{q_t} &= r_t^e, && (23.12)\\
r_t^e &\equiv i^* - \pi_t^e, \qquad \pi_t \equiv \frac{\dot{P}_t}{P_t}, && (23.13)\\
P_t &= P_0 e^{\pi t}, && (23.14)
\end{aligned}$$

where q is the real price of the long-term bond (the consol), the superscript e denotes expected value. If a shock occurs, it fits intuition best to interpret the time derivatives in (23.8), (23.12), and (23.13) as right-hand derivatives, e.g., $\dot{Y}_t \equiv \lim_{\Delta t \to 0^+}(Y(t + \Delta t) - Y(t))/\Delta t$. The variables τ, G, i^*, x, P_0, and π are exogenous constants. The first five of these are positive, and we assume $\pi < i^*$.

As there is no uncertainty in this model (no stochastic elements), the assumption of rational expectations amounts to perfect foresight. We thus have $\dot{q}_t^e = \dot{q}_t$ and $\pi_t^e = \pi$ for all t. Therefore, the equations (23.13) and (23.9) imply $r_t^e = r_t = i^* - \pi > 0$ for all t. Combining this with (23.11) and (23.12), we end up with

$$\dot{R}_t = (R_t - i^* + \pi)R_t. \quad (23.15)$$

Assuming no speculative bubbles, the no-arbitrage condition (23.12) is equivalent to a saying that the consol has market value equal to its *fundamental value*:

$$\begin{aligned} q_t &= \int_t^{\infty} 1 \cdot e^{-\int_t^s r_\tau d\tau} ds, \qquad \text{so that} \qquad (23.16) \\ R_t &= \frac{1}{q_t} = \frac{1}{\int_t^{\infty} e^{-\int_t^s r_\tau d\tau} ds} = \int_t^{\infty} \frac{e^{-\int_t^s r_\tau d\tau}}{\int_t^{\infty} e^{-\int_t^s r_\tau d\tau} ds} r_s ds. \end{aligned}$$

In other words: the long-term rate, R_t, is an average of the (expected) future short-term rates, r_τ, with weights proportional to the discount factor $e^{-\int_t^s r_\tau d\tau}$, cf. the appendix of Chapter 22..

The evolution of the economy over time is described by the two differential equations, (23.8) and (23.15), in the endogenous variables, Y_t and R_t. Since the exchange rate X is an exogenous constant, the UIP condition is upheld by movements of financial capital providing the needed continuous adjustment of the endogenous money supply so as to satisfy $M_t = P_t L(Y_t, i^*) = P_0 e^{\pi t} L(Y_t, i^*)$, in view of (23.10) and (23.14). It is presupposed that the central bank keeps foreign exchange reserves to be able to buy the domestic currency on foreign exchange markets when needed to maintain its value.

The dynamics is essentially the same as that of a closed economy with a target short-term interest rate fixed by the central bank. In a phase diagram the $\dot{R} = 0$ locus is horizontal and coincides with the saddle path. In the absence of speculative bubbles and expected future changes in i^*, we thus get $R_t = r_t = i^* - \pi > 0$ for all $t \geq 0$. The only difference compared with the closed economy is that the short-term interest rate is not a policy variable any more, but an exogenous variable given from the world financial market.

As an example of an adjustment process, consider a fiscal tightening (increase in τ or decrease in G). This will immediately decrease output demand. Thereby output gradually falls to a new lower equilibrium level. The fall in output implies lower money demand because the amount of money-mediated transactions becomes lower. The lower money demand generates an incipient tendency for the short-term interest rate to fall and the domestic currency to depreciate. This tendency is immediately counteracted, however. To take advantage of a higher foreign interest rate, portfolio holders worldwide convert home currency into foreign currency at the given exchange rate in order to buy foreign bonds. Owing to its commitment to a fixed exchange rate, the central bank now intervenes by selling foreign currency and domestic bonds. As soon as i is restored at its original value, i^*, the downward pressure on the value of the domestic currency is nullified. By assumption, the price level stays on the time path (23.14), whereby r and R remain essentially unaffected and equal to the constant $i^* - \pi$ during the output contraction. The figures 20.12 and 20.13 of the previous chapter illustrate.

Another kind of demand shock is a shift in the exports demand due to, say, a reduced economic growth in the world economy.

23.3 Dynamics under a floating exchange rate: overshooting

The exogeneity – and in fact absence – of expected exchange rate changes in the static Mundell-Fleming model of a floating exchange rate regime is unsatisfactory. By a dynamic approach we can open up for an endogenous and time-varying $\dot{X}^e$.

The floating exchange rate regime requires one more differential equation compared to the fixed exchange rate system. To avoid the complexities of a three-dimensional dynamic system, we therefore simplify along another dimension by dropping the distinction between short-term and long-term bonds. Hence, output demand is again described as in (23.1) and depends negatively on the expected short-term real interest rate, r^e. We ignore the disturbance term ε_D.

Apart from the exchange rate now being endogenous, money supply exogenous, and long-term bonds absent, the model is similar to that of the previous section. At the same time the model is close to a famous contribution by the German-American economist Rudiger Dornbusch (1942-2002), who introduced forward-looking rational expectations into a floating exchange rate model (Dornbusch, 1976). Dornbusch thereby showed that exchange rate "overshooting" could arise. This was seen as a possible explanation of the rise in both nominal and real exchange rate volatility during the 1970s after the demise of the Bretton-Woods system. In his original article, Dornbusch wanted to focus on the dynamics between fast moving asset prices and sluggishly changing goods prices. He assumed output to be essentially unchanged in the process. In the influential Blanchard and Fischer (1989) textbook this was modified by letting output adjust gradually to spending, while goods prices were in the short run simply unaffected by demand shifts. This seems a more apt approximation, since the empirics tell us that in response to demand shifts, output moves faster than goods prices. We follow this approach and name it the *Blanchard-Fischer version* of Dornbusch's overshooting model.

23.3.1 The model

This modified Dornbusch model has three building blocks. The first building block is the output error-correction process,

$$\begin{aligned} \dot{Y}_t &= \lambda(Y_t^d - Y_t), \qquad \text{where} \qquad (23.17)\\ Y_t^d &= D(Y_t, r_t^e, x_t, \tau) + G, \end{aligned}$$

where $r_t^e \equiv i_t^e - \pi_t^e$, and $x_t \equiv X_t P_t^*/P_t$ is the real exchange rate. As in (23.4), the partial derivatives of the demand function D satisfy $0 < D_Y < 1$, $D_{r^e} < 0$, $D_x > 0$, and $-1 < D_\tau < 0$.

The second building block comes from the money market equilibrium condition, combined with a monetary policy maintaining a constant real money supply, $m > 0$. This requires that the money supply follows the path

$$M_t = mP_t = mP_0 e^{\pi t} = M_0 e^{\pi t},$$

where π is the actual inflation rate, assumed constant. The money market equilibrium condition now reduces to $m = L(Y_t, i_t)$, which defines i_t as an implicit function,

$$i_t = i(Y_t, m), \quad \text{with } \; i_Y = -L_Y/L_i > 0, \; i_m = 1/L_i < 0. \tag{23.18}$$

We assume that m is small enough so that under "normal circumstances", the interest rate, i_t, takes a value above its lower bound, nil.

The third building block is the foreign exchange market. This market is in equilibrium when the uncovered interest parity condition holds. This is the condition

$$i_t = i^* + \frac{\dot{X}_t^e}{X_t}. \tag{23.19}$$

Assuming perfect foresight, this takes the form $i_t = i^* + \dot{X}_t/X_t$, which implies

$$\dot{X}_t = (i_t - i^*)X_t. \tag{23.20}$$

Admittedly, this relationship may be questioned. As mentioned, the empirical support for the combined hypothesis of UIP and rational expectations is, at least for short forecast horizons, weak. Yet, to avoid a complicated model, we shall proceed as if (23.20) holds for all t.

In a steady state of the system, the nominal exchange rate will be a constant, X. For a steady state to be possible, we need that also the real exchange rate, XP_t^*/P_t, is constant, x. We therefore assume that the foreign inflation rate equals the domestic inflation rate, π. As an implication we have $P_t^*/P_t = P_0^*/P_0$ for all $t \geq 0$.

Because of perfect foresight, $\pi_t^e = \pi$ and $r_t^e = r_t = i_t - \pi$ for all $t \geq 0$. In view of these conditions, together with (23.18), output demand can be written

$$Y_t^d = D(Y_t, i(Y_t, m) - \pi, X_t \frac{P_0^*}{P_0}, \tau) + G. \tag{23.21}$$

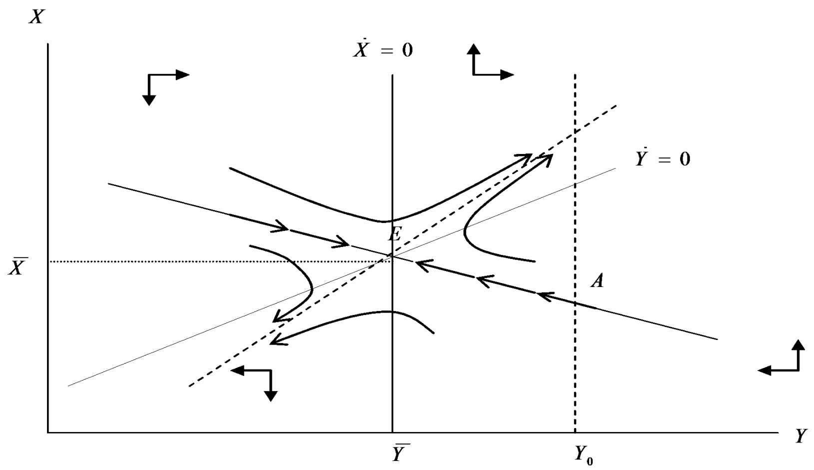

Figure 23.1: Phase diagram in case of a a floating exchange rate.

Inserting this into (23.17) and then (23.18) into (23.20), we end up with the dynamic system

$$\dot{Y}_t = \lambda \left[D(Y_t, i(Y_t, m) - \pi, X_t \frac{P_0^*}{P_0}, \tau) + G - Y_t \right], \; Y_0 > 0 \text{ given}, \quad (23.22)$$

$$\dot{X}_t = [i(Y_t, m) - i^*] X_t. \quad (23.23)$$

This system has Y_t and X_t as endogenous variables, whereas the remaining variables are exogenous and constant: $m, P_0^*/P_0, G$, τ and i^*, all positive.

The phase diagram is shown in Fig. 23.1. By (23.22), the $\dot{Y} = 0$ locus is given by the equation $Y = D(Y, i(Y, m) - \pi, XP_0^*/P_0, \tau) + G$. Taking the differential on both sides w.r.t. Y, X, and m (for later use) gives

$$\begin{aligned} dY &= (D_Y + D_{r^e} i_Y)dY + D_{r^e} i_m dm + D_x \frac{P_0^*}{P_0} dX \Rightarrow \\ (1 - D_Y - D_{r^e} i_Y)dY &= D_{r^e} i_m dm + D_x \frac{P_0^*}{P_0} dX. \quad (23.24) \end{aligned}$$

Setting $dm = 0$, we find

$$\frac{dX}{dY}\Big|_{\dot{Y}=0} = \frac{1 - D_Y - D_{r^e} i_Y}{D_x P_0^*/P_0} > 0. \quad (23.25)$$

It follows that the $\dot{Y} = 0$ locus (the "IS curve") is upward-sloping as shown in Fig. 23.1.

Equation (23.23) implies that $\dot{X} = 0$ for $i(Y, m) = i^*$. The value of Y satisfying this equation is unique (because $i_Y \neq 0$) and is called $\bar{Y}$. That is, $\dot{X} = 0$ for

$Y = \bar{Y}$, which says that the $\dot{X} = 0$ locus (the "LM curve") is vertical. Fig. 23.1 also indicates the direction of movement in the different regions, as determined by (23.22) and (23.23). The arrows show that the steady state is a saddle point. This implies that exactly two solution paths − one from each side − converge towards E.

Since the adjustment of output takes time, Y is a predetermined variable. Thus, at time $t = 0$, the economy must be somewhere on the vertical line $Y = Y_0$. If speculative exchange rate bubbles are assumed away, the explosive or implosive paths of X in Fig. 23.1 cannot arise. Hence, we are left with the segment AE of the saddle path in the figure as the unique solution to the model for $t \geq 0$. Following this path the economy gradually approaches the steady state E. If $Y_0 > \bar{Y}$ (as in Fig. 23.1), output is decreasing and the exchange rate increasing during the adjustment process. If instead $Y_0 < \bar{Y}$, the opposite movements occur.

The steady state can be seen as a "short-run equilibrium" of the economy. Further dynamic interactions will tend to arise in the "medium run", for instance through a Phillips curve and through investment resulting in build up of fixed capital. These ramifications are ignored by the model.

How the steady state and the $\dot{X} = 0$ and $\dot{Y} = 0$ loci depend on m

In steady state we have

$$\bar{Y} = D(\bar{Y}, i^* - \pi, \bar{X}\frac{P_0^*}{P_0}, \tau) + G, \tag{23.26}$$

and

$$m = L(\bar{Y}, i^*). \tag{23.27}$$

First, (23.27) determines $\bar{Y}$ as an implicit function of m and i^* independently of (23.26). To see how $\bar{Y}$ is affected by a change in m, we take the differential on both sides of (23.27) to get $dm = L_Y d\bar{Y} + L_i di^*$. With $di^* = 0$, this gives

$$\frac{\partial \bar{Y}}{\partial m} = \frac{1}{L_Y} > 0. \tag{23.28}$$

Given $\bar{Y}$, we have $\bar{X}$ determined by (23.26). Then, to see how X is affected by a change in m, we take the differential on both sides of (23.26) to get $d\bar{Y} = D_Y d\bar{Y} + D_x P_0^*/P_0 d\bar{X}$. Combining this with (23.28), we end up with

$$\frac{\partial \bar{X}}{\partial m} = \frac{\partial \bar{X}}{\partial \bar{Y}}\frac{\partial \bar{Y}}{\partial m} = \frac{1 - D_Y}{D_x P_0^*/P_0}\frac{1}{L_Y} > 0. \tag{23.29}$$

The intuitive explanation of this sign is linked to that of (23.28). For the higher money supply to be demanded with unchanged interest rate, equilibrium in the money market requires a higher level of transactions, that is, a higher level of economic activity. In steady state this must be balanced by a sufficiently higher output demand. And since the marginal propensity to spend is less than one ($D_Y < 1$), higher net exports are needed; otherwise the rise in output demand is smaller than the rise in output. Thus, higher competitiveness and therefore depreciation of the domestic currency is required which means a higher X.

Taking into account that $dm = dM/P$, we see that the steady state multipliers of Y and X w.r.t. m are the same as the corresponding multipliers in the static model, given in (23.6) and (23.7).

It follows from (23.28) that an increase in m will shift the $\dot{X} = 0$ line to the right, cf. Fig. 23.2. Again, for a higher money supply to be matched by higher money demand at an unchanged interest rate, a higher level of economic activity is needed.

As to the effect of higher m on the $\dot{Y} = 0$ locus, consider Y as fixed at Y_0, i.e., $dY = 0$. Then (23.24) gives

$$\frac{\partial X}{\partial m}_{|\dot{Y}=0,Y=Y_0} = -\frac{D_{r^e} i_m}{D_x P_0^*/P_0} = -\frac{D_{r^e}/L_i}{D_x P_0^*/P_0} < 0.$$

Hence, an increase in m shifts the $\dot{Y} = 0$ locus downward. The intuition is that a rise in m induces a fall in the interest rate; then for output demand to remain unchanged, we need an appreciation, i.e., a fall in X.

Another way of understanding the shift of the $\dot{Y} = 0$ locus is to consider X as fixed at X_0, i.e., $dX = 0$. Then (23.24) yields

$$\frac{\partial Y}{\partial m}_{|\dot{Y}=0,X=X_0} = \frac{D_{r^e} i_m}{1 - D_Y - D_{r^e} i_Y} = \frac{D_{r^e}/L_i}{1 - D_Y + D_{r^e} L_Y/L_i} > 0. \tag{23.30}$$

Hence, we can also say that an increase in m shifts the $\dot{Y} = 0$ locus rightward, cf. Fig. 23.2. The intuition is that, given X, the fall in i induced by higher m increases output demand and therefore also the output level that matches output demand.

The conclusion is that a higher m shifts both the $\dot{X} = 0$ locus and the $\dot{Y} = 0$ locus rightward. Then it might seem ambiguous in what direction $\bar{X}$ moves. But we already know from (23.29) that $\bar{X}$ will unambiguously increase. The explanation is that, first, a higher output level is needed for money market equilibrium to obtain. Second, for the higher output level to be demanded, we need a depreciation of the domestic currency, i.e., a higher X. Fig. 23.2 illustrates.

23.3.2 Unanticipated rise in the real money supply

Now, we are ready to study the dynamic effects of an unanticipated upward shift in the real money supply, m. Suppose the economy has been in its steady

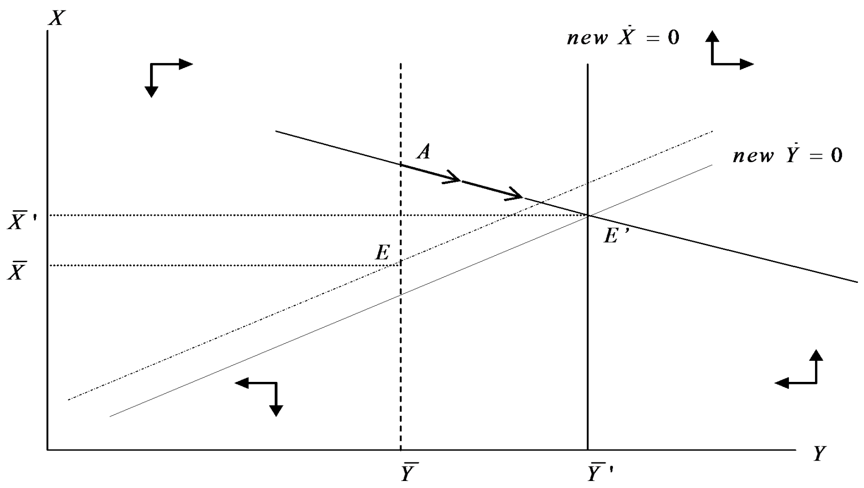

Figure 23.2: Phase portrait of an unanticipated rise in m.

state until time t_0. Then unexpectedly a discrete open-market purchase by the central bank of domestic bonds takes place. This instantly increases the monetary base which through the money multiplier leads to a larger money stock and a smaller stock of bonds held by the private sector. At the same time the nominal interest rate jumps down because output, and thereby the transactions volume, is predetermined in this "very short run" (it takes time to change output). The lower interest rate prompts arbitrage. With the aim of acquiring foreign interest-bearing assets, domestic currency will buy foreign currency until the exchange rate has jumped up to a level from which it is expected to *appreciate* at a rate such that interest parity is reestablished. Very fast, a new "very-short-run" equilibrium is formed where the given supplies of money and domestic bonds are again willingly held by the agents. The essence of the matter is that for a while we have $i_t < i_t^*$ due to the increase in the money supply. To make domestic bonds as attractive as foreign bonds, an expectation appreciation is needed. In turn this requires an *initial depreciation* in excess of the ultimate depreciation implied by the transition from the old to the new steady state.

Fig. 23.2 illustrates this exchange rate *overshooting.* We say that a variable *overshoots* if its initial reaction to a shock is greater than its longer-run response. Fig. 23.3 shows the time profiles of the exchange rate and the other key variables (D is output demand).

To really understand what is going on, let us examine the mechanics of overshooting more closely:

1. What will be the new steady state expected by the market participants? As we have just seen, when m increases, both the $\dot{X} = 0$ locus and the $\dot{Y} = 0$

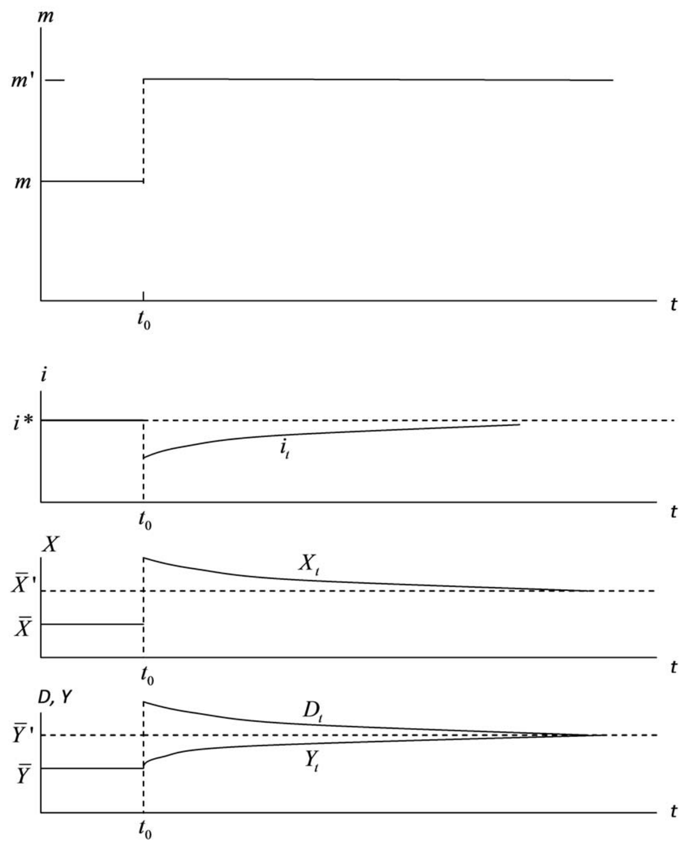

Figure 23.3: Time profiles of r, X, Y, and output demand D in response to an unanticipated rise in m (shown in the upper panel).

locus move to the right. The $\dot{X} = 0$ locus moves to the right, because the higher m tends to decrease i so that an offsetting increase in Y is needed for i to still match i^*, cf. (23.28). And the $\dot{Y} = 0$ locus moves to the right, because output demand depends negatively on the interest rate so that, through this channel, it depends positively on money supply, cf. (23.30). Nevertheless, we can be sure that the new steady-state point is associated with a *higher* exchange rate, as was explained in connection with (23.29) above.

2. Why must the initial depreciation be *larger* than that required in the new steady state? Two things are important here. First, both the outflow of financial capital, prompted by the fall in the interest rate, and the contemporaneous depreciation occur *instantly.* Imagine for a moment they occurred gradually over time. Then there would be expected depreciation of the domestic currency, implying that foreign bonds became even more attractive relative to domestic bonds. This would reinforce the outflow of financial capital and speed up the rise in X, that is, enlarge the drop in the value of the domestic currency. Thus, the financial capital outflow and the depreciation occur very fast, which mathematically corresponds to an upward *jump* in X.

The second issue is: how large will the jump be? The answer is: large enough for the concomitant *expected gradual appreciation* rate to be at the level needed to make domestic bonds not less attractive than foreign bonds *at the same time* as convergence to the new steady state is ensured. This happens where the vertical line $Y = \bar{Y}$ crosses the new saddle path, i.e., at the point A in Fig. 23.2. Ruling out bubbles, agents realize that any incipient jump to a point above or below A offers arbitrage opportunities. Exploiting these, the system is almost instantly brought back to A.

3. Why will output gradually rise and the domestic currency gradually appreciate after the initial depreciation jump? For $t > t_0$ the economy moves along the new saddle path: Y gradually responds to the high output demand generated by the low interest rate $r_t = i_t$ and the high competitiveness, X_t. As output rises, money demand increases and i_t gradually returns to "normal", see Fig. 23.3. Moreover, X_t gradually adjusts downwards and converges towards $\bar{X}'$ as the interest differential, $i_t - i^*$, declines. (Remember that the analysis presupposes that after time t_0 the market participants rightly expect no further changes in the money supply.)

The exchange rate as a forward-looking variable

It helps the interpretation of the dynamics if we recognize that the exchange rate is an asset price, hence forward-looking. Under perfect foresight the uncovered interest parity implies that the exchange rate satisfies the differential equation

(23.23), except at points of discontinuity of X_t. For convenience we repeat the differential equation here:

$$\dot{X}_t = (i_t - i^*)X_t. \tag{23.31}$$

For fixed $t > t_0$ we can write the solution of this linear differential equation as

$$X_\tau = X_t e^{\int_t^\tau (i_s - i^*)ds} \equiv X_t e^{(\bar{i}_{t,\tau} - i^*)(\tau - t)}, \qquad \text{for } \tau > t,$$

where $\bar{i}_{t,\tau}$ is the mean of the interest rates between time t and time τ, i.e., $\bar{i}_{t,\tau} \equiv \int_t^\tau i_s ds/(\tau - t)$. Being a forward-looking variable, X_t is not predetermined. It is therefore more natural to write the solution on the forward-looking form

$$X_t = X_\tau e^{-\int_t^\tau (i_s - i^*)ds} \equiv X_\tau e^{-(\bar{i}_{t,\tau} - i^*)(\tau - t)}, \qquad \text{for } \tau > t, \tag{23.32}$$

where X_τ and i_s should be interpreted as the *expected* future values as seen from time t. Thus, under the UIP hypothesis the exchange rate today equals the expected future exchange rate discounted by the mean interest differential $\bar{i}_{t,\tau} - i^*$ expected to be in force in the meantime.[7] As a consequence, new information implying anticipation of, for instance, a higher X in the future (compared with the reference path) will, for given expectations concerning the mean interest differential, show up immediately as depreciation of the domestic currency today.

From our explanation of the mechanics of overshooting, the reader might think that financial capital movements that prompt an exchange rate adjustment require a lot of exchange transactions to occur. However, what is needed for expected asset returns to be equalized is in principle just that the traders, in possession of the needed currency, in response to new information adjust their bid and ask prices to the new level at which supply and demand are equilibrated. In this way, what we see need not be much more than an international re-evaluation of domestic and foreign bonds. In highly integrated asset markets a new equilibrium may be found very fast. These circumstances notwithstanding, the volume of foreign exchange trading per day has in recent years grown to enormous magnitudes.

Returning to our specific case of a monetary expansion, in (23.32) let $\tau \to \infty$ to get

$$X_t = \lim_{\tau \to \infty} X_\tau e^{-\int_t^\tau (i_s - i^*)ds} = \bar{X}' e^{-\int_t^\infty (i_s - i^*)ds} > \bar{X}', \tag{23.33}$$

where the new steady-state value of the exchange rate after the rise in m is denoted $\bar{X}'$ as in Fig. 23.3. The inequality in (23.33) is due to $i_s < i^*$ during the adjustment process. As time proceeds, the shortfall of the domestic vis-à-vis

[7] The solution formula (23.32) presupposes absence of any jump in X between time t and time τ. Or more to the point: arbitrage prevents any such *expected* jump. Ex post, the formula (23.32) is valid only if no jumps in X actually occurred in the time interval considered.

the foreign interest rate is reduced and the exchange rate converges to its steady-state value from above. When for instance 90% of the initial distance from the steady state has been recovered, we say that the system has essentially reached its steady state. The adjustment process so far may not involve more time than a couple of years, say. Several factors that may matter for further adjustments are ignored by this model. Hence, we avoid to call $\bar{X}'$ a "long-run" value.

In (23.32) and (23.33) we consider the value of the foreign currency in terms of the domestic currency. Similar expressions of course hold for the value of the domestic currency in terms of the foreign currency. Thus, inverting (23.32) gives

$$X_t^{-1} = X_\tau^{-1} e^{-\int_t^\tau (i^* - i_s) ds} \equiv X_\tau^{-1} e^{-(i^* - \bar{i}_{t,\tau})(\tau - t)}.$$

That is, the value of the domestic currency today equals its expected future value discounted by the mean interest differential $i^* - \bar{i}_{t,\tau}$ expected to be in force in the meantime. Inverting (23.33) yields

$$X_t^{-1} = \bar{X}'^{-1} e^{-\int_t^\infty (i^* - i_s) ds} < \bar{X}'^{-1}.$$

As time proceeds, the excess of the foreign over the domestic interest rate decreases and the value of the domestic currency converges to its steady-state value from below.

23.3.3 Anticipated rise in the money supply

As an alternative scenario suppose that the economy has been in steady state until time t_0 when agents suddenly become aware that an increase in the money supply is going to take place at some future time. To be specific let us imagine that the central bank at time t_0 credibly announces that there will be discrete upward shift in M through an open-market operation at time $t_1 > t_0$, while M for a long time after t_1 will grow at the same rate, π, as before. According to the model this credible announcement immediately causes a *jump* in the exchange rate X in the same direction as the longer-run change, that is, a jump to some level like X_B in Fig. 23.4. This is due to the agents' anticipation that after time t_1, the economy will be on the new saddle path. Or, in more economic terms, the agents know that (a) from time t_1, the expansionary monetary policy will cause the interest rate to be lower than i^*, and (b) the exchange rate must therefore at time t_1 have reached a level from which it can have an expected and actual rate of appreciation such that interest parity is maintained in spite of $i < i^*$ after time t_1.

Under these circumstances, at the old exchange rate, $\bar{X}$, there would be excess supply of domestic bonds and excess demand for foreign bonds immediately after

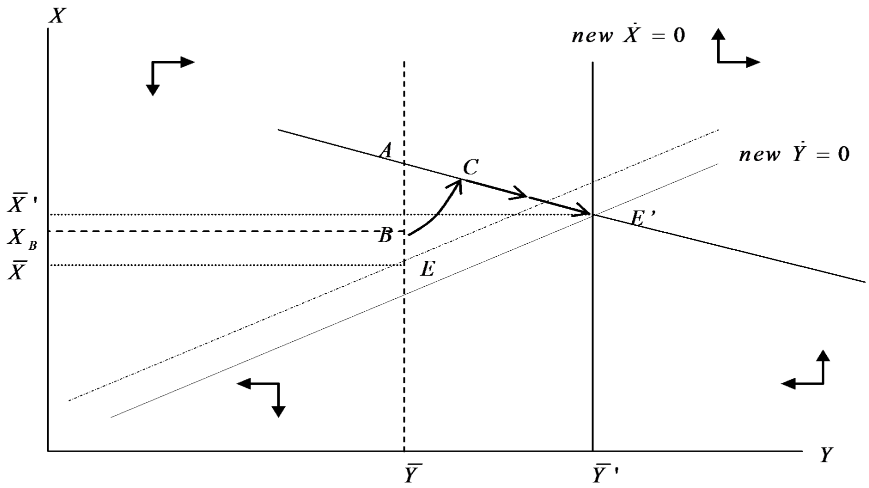

Figure 23.4: Phase portrait of an anticipated rise in m.

time t_0 and this is what instantly triggers the jump to X_B. After this initial jump the exchange rate, X, will adjust continuously. Currency is an asset, hence *anticipated* discrete jumps in the exchange rate are ruled out by arbitrage. In particular, at time t_1 there can be no jump, because no new information has arrived.

In the time interval (t_0, t_1) the movement of (Y, X) is governed by the "old" dynamics. That is, for $t_0 < t < t_1$ the economy must follow a trajectory consistent with the "old" dynamics, reflecting the operation of the no-arbitrage condition,

$$i(Y_t, m) = i^* + \frac{\dot{X}_t^e}{X_t}.$$

which rules as long as the announced policy change is not yet implemented. Under perfect foresight, the market mechanism "selects" that trajectory (BC in Fig. 23.4) along which it takes exactly $t_1 - t_0$ time units to reach the new saddle path. It is in fact this requirement that determines the *size* of the jump in X immediately after time t_0.[8]

The higher competitiveness caused by the instantaneous depreciation implies higher output demand, so that output begins a gradual upward adjustment already before monetary policy has been eased. Along with the rising Y, transaction demand for money rises gradually and so do the interest rate (recall m has not

[8] The level X_B can be shown to be unique and this is also what intuition tells us. Imagine that the jump, $X_B - \bar{X}$, was smaller than in Fig. 21.4. Then, not only would there be a longer way along the road to the new saddle path, but the system would also start from a position closer to the steady state point E, which implies an initially lower adjustment speed.

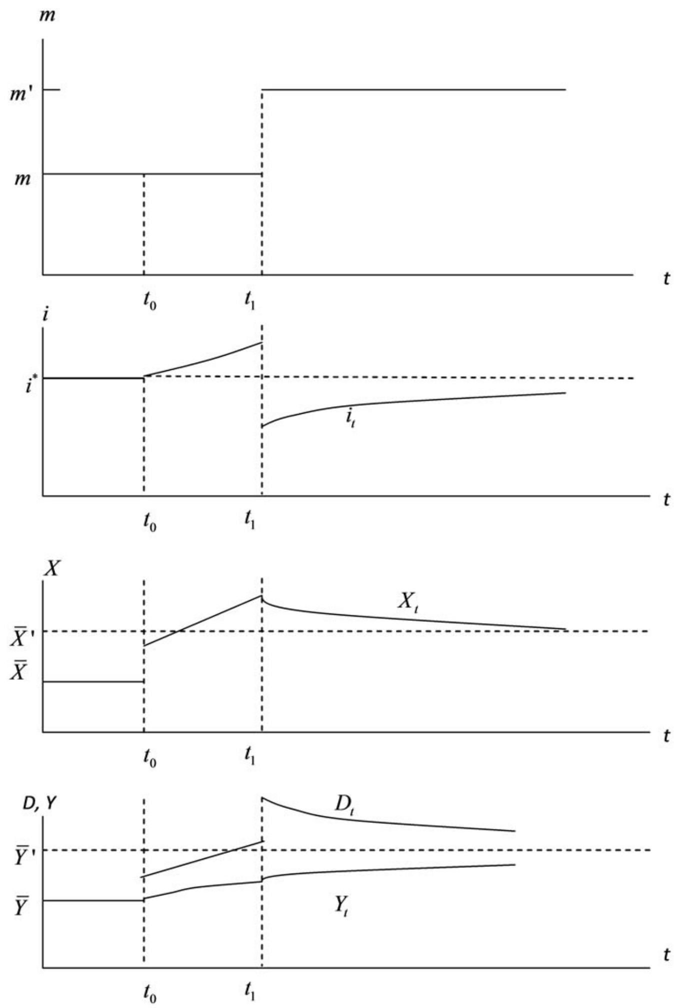

Figure 23.5: Time profiles of r, X, Y, and output demand D in response to an anticipated rise in m (shown in the upper panel).

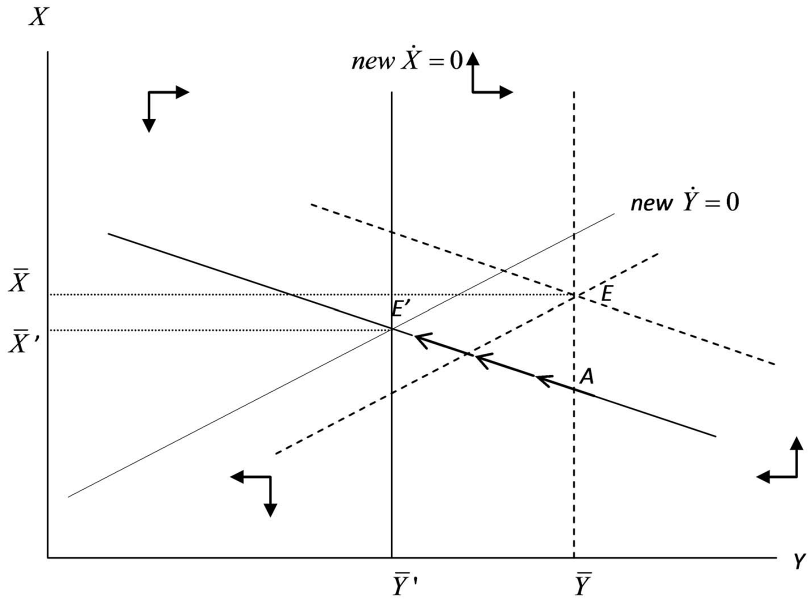

Figure 23.6: Phase portrait of an unanticipated fall in m.

changed yet) and the exchange rate. That is, in the time interval (t_0, t_1) we have both $i_t > i^*$ and $\dot{X}_t > 0$ so as to maintain interest parity. The process continues until the new monetary policy is implemented at time t_1. Exactly at this time the economy's trajectory, governed by the old dynamic regime, crosses the new saddle path, cf. the point C in Fig. 23.4. The actual rise in $m = M/P$ at time t_1 then triggers the anticipated discrete fall in the interest rate to a level $i_{t_1} < i^*$.[9] For $t > t_1$ the economy features gradual appreciation ($\dot{X} < 0$) during the adjustment along the new saddle path. Although output demand is therefore now falling, it is still high enough to pull output further up until the new steady state E' is reached.

The time profiles of Y, X, and r $(= i)$ are shown in Fig. 23.5. The tangent to the X_t curve at $t = t_0$ is horizontal. Hence, infinitely close to t_0 the size of $\dot{X}$ is vanishing. This is dictated by the "old dynamics" ruling in the time interval (t_0, t_1), which entail that the trajectory through the point B in Fig.21.4 is horizontal at B. And this is in accordance with interest parity since it takes time for Y to rise above $\bar{Y}$, hence for i to rise above i^*. Note also that if the length of the time interval (t_0, t_1) were small enough, then X might already immediately after time t_0 be above its new steady-state level, $\bar{X}'$. However, Fig. 23.4 and Fig.

[9] Since right before t_1 we have $i > i^*$, one might wonder whether the fall in the interest rate is necessarily large enough to ensure $i < i^*$ right after t_1. The fall is, indeed, large enough because the dynamics in the time interval (t_0, t_1) ensures $Y_{t_1} < \bar{Y}'$, from which follows $i(Y_{t_1}, m')$ $< i(\bar{Y}', m') = i^*$, where the inequality is due to $i_Y > 0$.

23.5 depict the opposite case, where the time interval (t_0, t_1) is somewhat larger.

23.3.4 Monetary policy tightening

Considering a *downward* shifts in the money supply path, the above processes are reversed.

Unanticipated monetary policy tightening

Suppose the system is in steady state until time t_0. Then, unexpectedly, a discrete open market sale by the central bank of domestic bonds takes place. The new steady state will have a *lower* exchange rate. Indeed, given the lower money supply and unchanged interest rate, equilibrium in the money market requires a lower level of transactions, that is, a lower level of economic activity. In steady state this must be balanced by a sufficiently lower output demand. And since the marginal propensity to spend is less than one ($D_Y < 1$), lower net exports are needed; otherwise the fall in output demand is smaller than the fall in output. Thus, lower competitiveness and therefore appreciation of the domestic currency is required which means a lower X.

In the short run, the nominal interest rate jumps up, prompting an inflow of financial capital, which in turn prompts a jump down in the exchange rate, as shown in Fig. 23.6. This appreciation must be large enough to generate the expected rate of depreciation required for domestic bonds to be no more attractive than foreign bonds in spite of $i_t > i^*$. Very fast a new "very-short-run" equilibrium is formed where the supplies of money and bonds are willingly held by the agents. The fact that there must be expected depreciation is the reason that an initial appreciation in excess of that implied by the new steady-state equilibrium is required. Again the exchange rate "overshoots", this time by taking a greater downward jump than corresponding to the new steady-state level.

For $t > t_0$ the economy moves along the new saddle path: X gradually rises and Y gradually falls in response to the low output demand generated by the high interest rate $r_t = i_t - \pi$ and the low competitiveness, X_t. In the process, money demand decreases and i_t gradually returns to "normal", see Fig. 23.7. Moreover, X_t gradually adjusts upwards and converges towards $\bar{X}'$ as the interest differential, $i_t - i^*$, gradually vanishes.

Anticipated monetary policy tightening

It may happen that the public in advance have a feeling that a monetary policy shift is on the way, due to foreseeable overheating problems, say. To be more

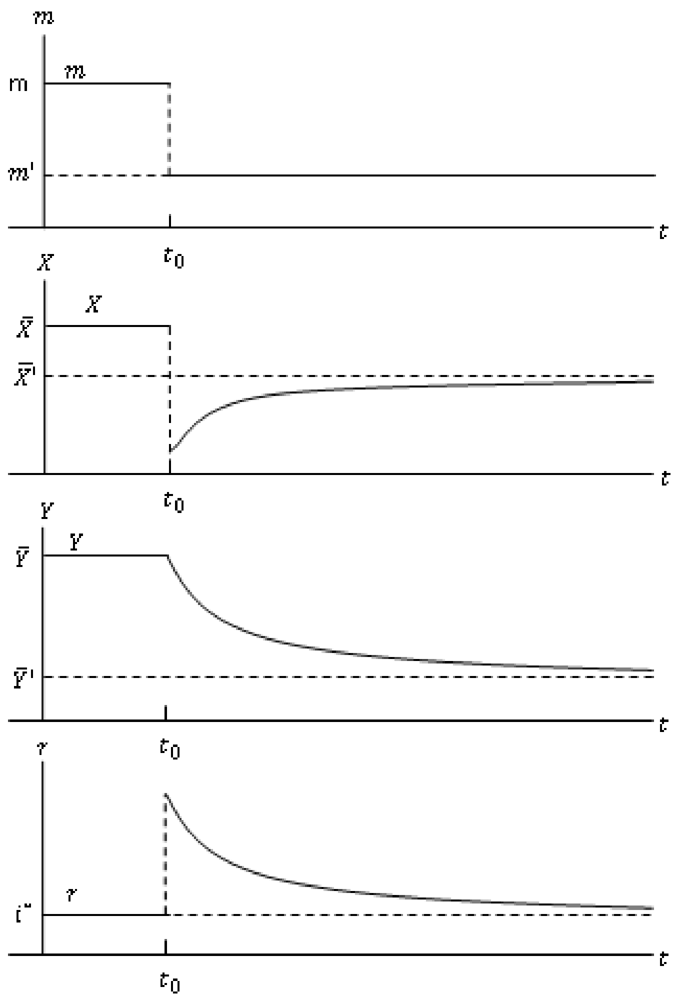

Figure 23.7: Time profiles of X, r, and Y in response to an unanticipated fall in m (shown in the upper panel).

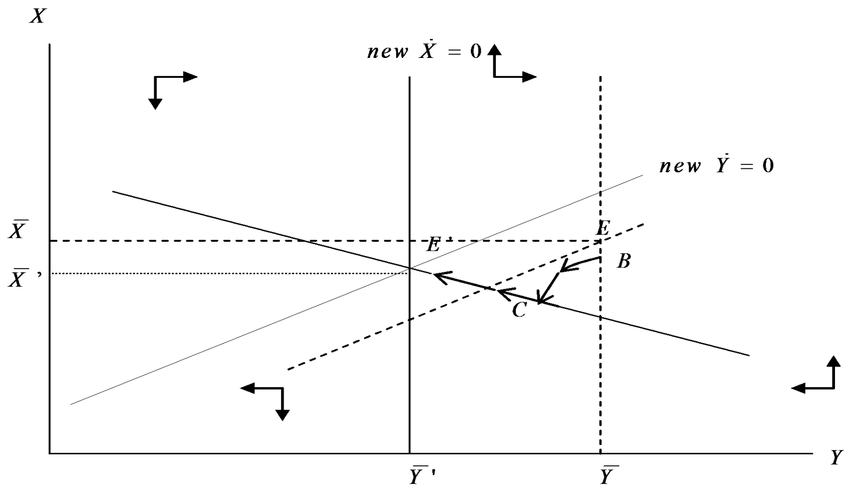

Figure 23.8: Phase portrait of an anticipated fall in m.

specific, suppose that at time t_0 a tightening of monetary policy is credibly announced by the central bank to be implemented at time $t_1 > t_0$ in the form of a reduction in real money supply to the level $m' < m$.

Fig. 23.8 illustrates what happens from time t_0. As soon as the future tight monetary policy becomes anticipated, there is an immediate effect on X in the same direction as the "longer-run" effect, i.e., X drops to some point B as in Fig. 23.8. Indeed, agents anticipate that from time t_1 the tight monetary policy will cause the interest rate to be higher than i^*, thereby engendering gradual depreciation (rise in X) along the new saddle path. Arbitrage prevents any anticipated discrete jump in the exchange rate after time t_0.

In the time interval (t_0, t_1) the economy must follow a trajectory consistent with the "old" dynamics. The market mechanism "selects" that trajectory (BC in Fig. 23.8) along which it takes exactly $t_1 - t_0$ time units to reach the new saddle path. The lower competitiveness caused by the instantaneous appreciation implies lower output demand, so that output begins a gradual downward adjustment already before monetary policy has been tightened.

The time profiles of Y, X, and $r = i - \pi$ are shown in Fig. 23.9. If the length of the time interval (t_0, t_1) is small enough, X may already immediately after time t_0 be below its new steady-state level. However, Fig. 23.8 and Fig. 23.9 depict the opposite case, where the time interval (t_0, t_1) is somewhat larger.

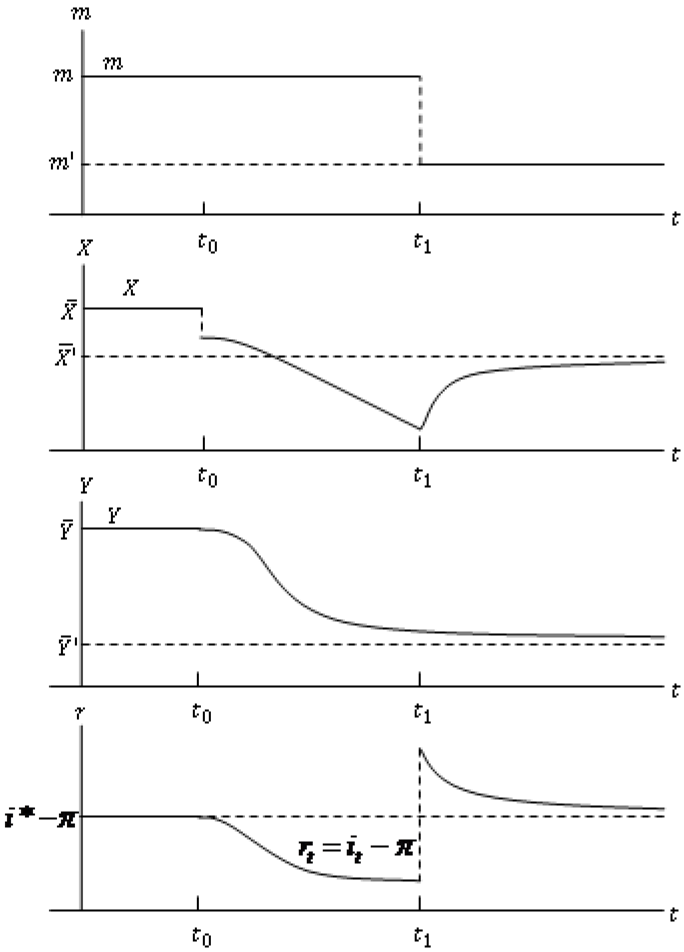

Figure 23.9: Time profiles of X, Y, and r in response to an anticipated fall in m (shown in the upper panel).

23.4 Concluding remarks

The dynamic model of a floating exchange rate regime shows that with fast moving asset markets and nominal rigidities, large volatility in exchange rates can occur. And large volatility of both nominal and real exchange rates under a floating exchange rate regime is in fact what the data show. Nevertheless, there *are* empirical problems with the model. One of them is that it *exaggerates* the exchange rate fluctuations (see Obstfeld and Rogoff, 1996, p. 621 ff.). Moreover, several empirical studies reject the UIP condition or at least they reject the combined hypothesis of UIP and rational expectations (Lewis 1995).

In the model versions considered here the price level moves along a fixed path. An extended model should incorporate that the domestic price level depends on import prices and that persistent changes in aggregate production and employment are likely to activate a Phillips curve of some sort. As they stand, the models imply that monetary shocks have *permanent* real effects, contrary to what the data in general indicates.

Incorporating a medium-run equilibrium level of the real exchange rate, anchored by some kind of expected purchasing power parity, opens up for interesting issues. With x^* denoting a medium-run equilibrium real exchange rate, $\bar{X}'$ in (23.33) would equal $x^*(P/P^*)^e$ where $(P/P^*)^e$ is the expected medium-run value of the relative price level P/P^*. Then, suppose the interest rate differential suddenly rises because the foreign country (the U.S., say) is hit by an economic recession. According to (23.33), this immediately triggers an appreciation of the home currency (China, say). Alternatively, imagine that the expected long-run value of P/P^* goes down due to a strong productivity development in the domestic economy. According to (23.33), also this triggers an appreciation of the home currency (China, say).

23.5 Literature notes

(incomplete)

The origin of the Mundell-Fleming model goes back to Robert Mundell (1963, 1964) and Marcus Fleming (1962). The over-shooting hypothesis goes back to Dornbusch (1976). What we named the *Blanchard-Fischer version* of Dornbusch's overshooting model was presented in the Blanchard and Fischer (1989) textbook. At two points our adaptation departs from Blanchard and Fischer. First, they take the exogenous inflation rate, π, to be zero. Second, they ignore the negative dependence of output demand on the interest rate. Though recognizing this dependency makes the analysis slightly more cumbersome, it is worth the trouble as it underlines the robustness of the results.

An extensive treatment of open economy macroeconomics is contained in the textbook Obstfeld, M., and K. Rogoff, 1996, *Foundations of International Macroeconomics*, The MIT Press, London. In their Chapter 8 the authors discuss the empirical difficulty that the UIP condition is rejected at short prediction horizons although it does somewhat better at horizons longer than one year. Other textbook treatments of this empirical issue include:

Feenstra and Taylor (2012).

Krugman, Obstfeld, and Melitz (2012).

Wickens, M., 2008, Macroeconomic Theory. A Dynamic General Equilibrium Approach, Princeton University Press, Oxford, Ch. 11.4.

Advanced approaches:

Isard, P., 2008, "Uncovered interest parity". In: The new Palgrave Dictionary of Economics. Second edition. Online: http://www.econ.ku.dk/English/libraries/links/

Lewis, K. K., 1995, Puzzles in international financial markets. In: Handbook of International Economics, vol. III, Elsevier, Amsterdam.

The volume of foreign exchange trading per day has in recent years increased to enormous magnitudes. This fact indicates that differences in information and expectations are prevalent. Recent contributions in macroeconomic theory and empirics are considering how to incorporate heterogeneity, imperfect knowledge, and agent's uncertainty about what is the right model of the economy. See e.g. Ellison ( ).

23.6 Appendix

A. The Marshall-Lerner condition

By assuming that net exports depends positively on the real exchange rate ($N_x > 0$), the Mundell-Fleming model presupposes that the Marshall-Lerner condition is satisfied. This is the condition that the weighted sum of the absolute elasticities of exports and imports w.r.t. the real exchange rate is large enough to offset the decrease in the terms of trade implied by a higher real exchange rate. If in the initial situation, net exports are zero, then the sum of the two absolute elasticities should be above 1. The econometric evidence is that the condition is satisfied for industrialized countries if we allow for an adjustment period of one to two years (Artus and Knight, 1984, Table 4, cf. Krugman, Obstfeld, and Melitz, 2012, p. 492). It may be argued that there should be a countervailing effect of the real exchange rate, x, in the consumption function since the purchasing power of domestic income is eroded by an increase in x. The Mundell-Fleming model assumes that the effect of this on aggregate demand is dominated by the role of

$N_x > 0$.

B. The covered interest parity

When buying foreign bonds one can avoid the uncertainty concerning the future exchange rate by entering a reverse *forward exchange* deal with someone else. Today an investor in foreign bonds thus contracts with her bank to sell in thirty days' time a certain amount of foreign currency for domestic currency at a prespecified rate. This rate is called the thirty-day *forward exchange rate*. It is generally different from the spot exchange rate, X. But empirically the two move closely together.

The *covered interest parity* condition, CIP, is the associated no-arbitrage condition. In discrete time it reads:

$$1 + i_t = \frac{1}{X_t}(1 + i_t^*)X_{t+1}^F, \tag{CIP}$$

where X_{t+1}^F is the one-period forward exchange rate. If there is no default risk and no fear that meanwhile regulations will be imposed which restrain the movement of foreign funds, arbitrage will immediately make CIP hold. Indeed, an agent can borrow one unit of the domestic currency, buy $1/X_t$ units of the foreign currency, then for this amount buy foreign one-period bonds paying a return of $1 + i_t^*$ per bond after one period, and finally lock in the future payout in the domestic currency by selling the return forward at the rate X_{t+1}^F. As the whole undertaking can be conducted at time t, there is no risk.

Let us compare with the (UIP) in discrete time:

$$1 + i_t = \frac{1}{X_t}(1 + i_t^*)X_{t+1}^e, \tag{UIP}$$

We see that the UIP will hold if and only if $X_{t+1}^F = X_{t+1}^e$. But to the extent an asymmetric foreign exchange risk plays a role, a positive risk term should be added to one of the sides in (UIP). Anyway, while X_{t+1}^F is observable via (CIP), X_{t+1}^e is not immediately observable.

23.7 Exercises

The bank-lending channel: The IS-BL model

In this note we consider the IS-BL model by Bernanke and Blinder (1988), "BL" being an abbreviation of "bank loans". The model aims at clarifying the monetary transmission mechanism in an economy where commercial banks offer checkable deposits to households and grant long-term bank loans to ultimate borrowers (households and non-bank firms). It is shown that the transmission of monetary policy takes two routes, the well-known *interest rate channel* and the *bank lending channel*.

Small and medium-sized firms are generally unable to issue bonds and equity shares for the centralized financial capital markets. Hence they are dependent on banks for external finance; indeed, in many countries bank loans are the main source of external finance.[1]

Fig. 1 portrays a stylized financial system. Households' *financial saving* is their income minus spending on durable as well as non-durable goods and services. The financial saving is partly channeled directly through centralized bond and stock markets to the ultimate users (government and other households and firms) and partly channeled indirectly to the ultimate users through *financial intermediaries*, the commercial banks.

1 The IS-BL model

A closed economy is considered. There is a public sector with a government in charge of fiscal policy and a partly independent *central bank*, the latter controlling the monetary base. The model gives no details about the credit market imperfections lying behind the existence of financial intermediaries, the *commercial banks*. We may think of asymmetric information and moral hazard problems between lender and borrower.

One-period bonds are issued by the government and traded in a centralized auction market. The private sector consists of the commercial banks and the "non-bank general

[1] See for instance Mishkin (2007, p. 182) and Kashyap and Stein (1994).

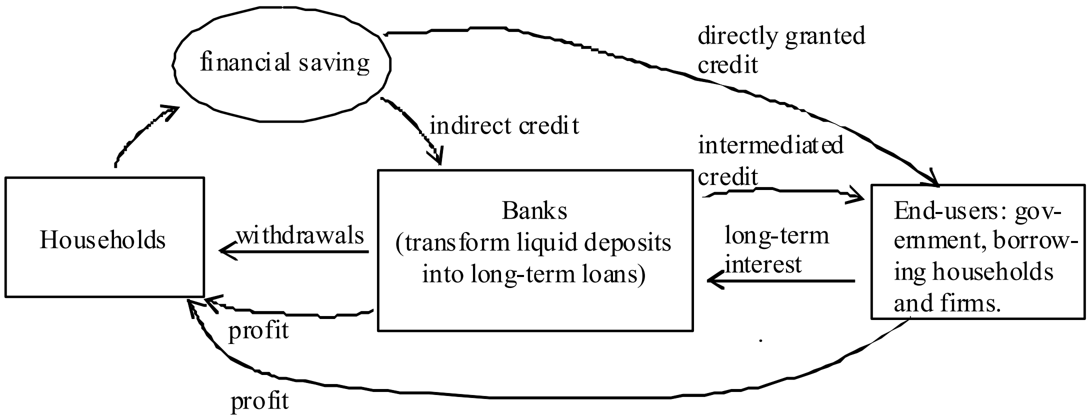

Figure 1: A stylized financial system.

public" (households and non-bank firms). From now "banks" means the commercial banks. The banks act as competitive financial intermediaries which pool a significant part of households' financial savings, placed in *demand deposit accounts*. The pooled deposits are then used to invest in government bonds and to offer long-term loans to households and production firms (the non-bank firms) in a "customer credit market" with limited contract enforcement. The bank loans should be thought of as *variable-rate loans*. The deposits earn no interest (as an approximation for a "low" deposit interest rate). The model ignores the equity market and assumes that firms finance their investment partly by withheld profit, partly by bank loans. The model also ignores so-called *investment banks* that make up a major part of what is known as the "shadow banking system".[2]

The IS-BL model is static in the sense that only one period is considered. Notation is indicated in Table 1.

[2]The investment banks do not offer deposit accounts, and they are subject to significantly less public regulations than "ordinary banks".

Table 1. List of main variable symbols.

$i_B =$ nominal bond interest rate,
$i_L =$ nominal bank lending interest rate (the "lending rate"),
$D =$ deposits of the non-bank private sector (a stock, earns no interest),
$\rho =$ required reserve-deposit ratio, $\rho \in [0, 1)$, exogenous,
$M_0 =$ monetary base,
$mm =$ the money multiplier,
$M_1 \equiv mm \cdot M_0 =$ money supply in the sense of demand deposits in commercial banks (a stock, earns no interest),
$E \equiv M_0 - \rho D =$ excess reserves (a stock, like required reserves earning no interest),
$L^s =$ supply of bank loans ("credit supply", a stock),
$\sigma =$ a shift parameter measuring perceived riskiness of offering bank loans,
$B =$ nominal value of the stock of government bonds held by the private sector,
$B_b =$ nominal value of government bonds held by the banks,
$B_n =$ nominal value of government bonds held by the non-bank public,
$W \equiv M_0 + B =$ aggregate nominal financial wealth of the private sector,
$P =$ price level, exogenous, $P = 1$,
$Y =$ real aggregate output,
$G =$ real government spending on goods and services, a policy parameter,
$\tau =$ "fiscal tightness" (shift parameter).

Contrary to the simple IS-LM model, here L refers to bank *loans,* not liquidity demand. The subscript b stands for commercial *banks*, the subscript n for the *non*-bank private sector, also called the "non-bank public". All deposits are fully liquid in the sense of being demand deposits (i.e., checkable deposits). The superscripts s and d signify *supply* and *demand*, respectively. We *ignore currency.* So the monetary base is identical to the stock of bank reserves. In studies like the present one where there is a distinction between directly granted credit and intermediated credit, it is common to speak of the latter as just "credit" and the "bank lending channel" as just the "credit channel".

The balance sheet[3] of the central bank (CB) is given in Table 2.

Table 2. Balance sheet of the central bank (CB).

Assets	Liabilities
	Currency (for simplicity here $= 0$)
$\bar{B} - B =$ value of gov. bonds held by CB	Deposits held by commercial banks (here $= M_0 = \rho D + E$)
Gold	Net worth
Total $=$	Total

[3] A balance sheet account shows the status (stock of assets and liabilities at a given point in time). An operations account shows the deliveries and uses per period (flows).

The aggregate balance sheet of the commercial banks is shown in Table 3.

Table 3. Merged balance sheet of the commercial banks.

Assets	Liabilities
$M_0 = \rho D + E =$ reserves (deposited in CB)	$D =$ liquid deposits held by the non-bank public
Vault currency (here $= 0$)	Long-term debt (for simplicity here $= 0$)
$L^s =$ loans to the non-bank public	Net worth (for simplicity here $= 0$)
$B_b =$ value of gov. bonds held by commercial banks	
Total $=$	Total

On the assets side, in addition to reserves we have two interest-bearing assets, bank loans and government bonds. In the limiting case where these assets are perfect substitutes, the model becomes a simple IS-LM model.

The fact that the major part of the assets of the banks are long-term, hence comparatively illiquid, while the major part of the liabilities are short-term, gives rise to recurrent historical episodes of bank runs. The term *bank run* refers to situations where many depositors, fearing that their bank will be unable to repay their deposits in full and on time, simultaneously try to withdraw their deposits. This is the reason that since the Great Depression of the 1930s, in developed countries deposits in "ordinary banks" are typically protected by government deposit insurance up to a certain limit. To be enrolled in this kind of insurance, the banks must comply with a set of regulations.

Throughout the analysis, the variables $\rho, \sigma, \bar{B}, G$, and P are exogenous, given the short time horizon of the model. For simplicity, $P = 1$. The expected inflation rate is considered exogenous and, for simplicity, equal to zero. Financial wealth of the private sector, $W \equiv M_0 + B$, is exogenous as well. Until further notice, also the monetary base, M_0, is exogenous in the sense of the CB using M_0 as its policy instrument. The CB can change M_0 through an open market operation whereby $\Delta M_0 = -\Delta B$.

1.1 The supply of bank loans and broad money

From the balance sheet of the commercial banking sector in Table 3 follows the identity

$$M_0 + L^s + B_b \equiv D. \tag{1}$$

We subtract required reserves, ρD, on both sides of (1) to get

$$M_0 - \rho D + L^s + B_b \equiv E + L^s + B_b \equiv (1-\rho)D, \qquad 0 \leq \rho < 1,$$

where E is *excess reserves*. So far this is just accounting, saying that disposable deposits, $(1-\rho)D$, on the liability side make up excess reserves, E, loans, L^s, and government bonds, B_b, on the asset side.

Given $(1-\rho)D$, how are the components of the triple (E, L^s, B_b) determined? Regarding E, we assume that to dispose of sufficient "cash" (be sufficiently liquid), the banks generally have to hold some excess reserves. The desired amount of excess reserves depends negatively on the opportunity cost, the bond interest rate, i_B, forgone. Because of their comparatively high degree of liquidity, government bonds make up a close substitute for reserves. Denoting the desired fraction of disposable deposits held as excess reserves $e(i_B) \in [0,1]$, we have

$$E = e(i_B)(1-\rho)D, \tag{2}$$

where E/D is known as the "excess reserves ratio", and

$$e(i_B) \begin{cases} = 0 \text{ for } i_B \geq \bar{\imath}_B, \\ \in (0,1) \text{ and } e'(i_B) < 0 \text{ for } 0 < i_B < \bar{\imath}_B, \\ \text{is set-valued (i.e., indeterminate) for } i_B = 0. \end{cases} \tag{3}$$

The upper bound, $\bar{\imath}_B$, above which no excess reserves are held, will for our purposes be treated as exogenous and "large" so as to not be binding. There is a zero-lower bound for i_B because agents prefer holding money at zero interest rather than bonds at negative interest. As indicated by (3), at the zero-lower-bound banks are indifferent between holding excess reserves or government bonds.

How is the supply of *bank loans* determined? The fraction, $\ell(i_B, i_L, \sigma) \in [0,1]$, of disposable deposits used as bank loans is assumed to depend positively on the interest rate obtainable on these and negatively on the opportunity cost, the interest rate on bonds. That is,[4]

$$L^s = \ell(i_B, i_L, \sigma)(1-\rho)D, \qquad \ell'_{i_B} < 0, \ell'_{i_L} > 0, \ell'_\sigma < 0 \text{ as long as } \ell(i_B, i_L, \sigma) \in (0,1).$$

Indeed, offering bank loans is less attractive the higher is i_B. And it is more attractive the higher is i_L. A precise no-arbitrage condition from the banks' perspective between what

[4]In the present model, variables may have several subscripts indicating the specific economic interpretation of the variable. To avoid confusion, we shall therefore add a prime, " ′ ", when considering partial derivatives. The partial derivatives of a function $z = f(x, y)$ are thus denoted f'_x and f'_y, respectively.

is held as bank loans and what is held as government bonds is absent. This is due to heterogeneity and unknown risk of default within the stock of bank loans. The negative dependency of the supply of loans on the shift parameter σ reflects that given the lending rate, i_L, less loans will be offered the higher the perceived riskiness.

The remaining part of disposable deposits, $(1-\rho)D-(E+L^s)$, is placed in *government bonds*.

We will concentrate on "normal circumstances" where both $e(i_B)$ and $\ell(i_B, i_L, \sigma)$ belong to the interior of $[0,1]$. That is, we assume "non-crisis circumstances" where $i_B > 0$, and σ is not "too high".

Since the model ignores currency, the monetary base, M_0, equals bank reserves held in the CB. The money supply, M_1^s, available for the non-bank private sector, equals deposits, D. The inverse of the money multiplier, mm, thus equals the reserve-deposit ratio:

$$\frac{1}{mm} = \frac{M_0}{M_1^s} = \frac{M_0}{D} = \frac{\rho D + E}{D} = \rho + e(i_B)(1-\rho) \equiv \frac{1}{mm(i_B)}, \tag{4}$$

where the fourth equality follows from (2). The money multiplier is thereby seen to be a function, $mm(i_B)$, of the interest rate on government bonds. From (4), and the fact that $e'(i_B) < 0$ in (3), follows that $mm'(i_B) > 0$. In words, a higher bond interest rate implies a higher money multiplier because a higher bond interest rate motivates the bank to hold less excess reserves. Moreover, in view of $\rho < 1$, and as long as $0 < e(i_B) < 1$ (which is the case under "normal circumstances"), we have $mm(i_B) > 1$. To summarize, the money supply can be written

$$M_1^s = D = mm(i_B)M_0 > M_0, \qquad mm'(i_B) > 0. \tag{5}$$

1.2 The demand bank loans and M$_1$-money

The balance sheet of the non-bank private sector (households and production firms) is given in Table 4. Aggregate nominal financial wealth of the non-bank private sector is W.

Table 4. The balance sheet of the non-bank private sector.

Assets		Liabilities
D = bank deposits		L^d = bank loans
Currency in circulation (here = 0)		
B_n = value of gov. bonds held by non-bank public		W = net worth
Total	=	Total

The demand function for bank loans (a stock) is assumed given by

$$L^d = L(Y, i_B, i_L), \qquad L'_Y > 0, L'_{i_B} > 0, L'_{i_L} < 0.$$

The demand derives from households' and firms' need for finance to purchase consumption and investment goods, respectively. The positive dependency on Y captures that higher current income and employment stimulates consumption, directly as well as indirectly through raising *expected* future income and thereby the state of confidence. To smooth consumption, some households may need credit. Similarly, from the perspective of production firms, a higher current level of economic activity, Y, may signal higher aggregate demand in the near future ("good times are underway"), thus making investment in increased capacity profitable.

The negative dependency of the demand for bank loans on the lending rate, i_L, reflects that bank loans are less attractive the higher the interest cost. Finally, a liability in the form of a bank loan is likely to be more tolerable the higher is i_B. This is because the borrower may place temporary excess liquidity in government bonds.

The demand for deposits in the banks derives from liquidity being needed for transactions. In addition, deposits offer a convenient book-keeping arrangement and a measure against theft. The stock of checkable deposits willingly held by the non-bank private sector is given by the money demand function

$$M_1^d = M(Y, i_B), \qquad M'_Y > 0, M'_{i_B} < 0.$$

Money demand thus depends partly on aggregate economic activity (due to the transactions motive) and partly on the opportunity cost of holding money, the interest rate on government bonds. As the exogenous price level, P, is set to 1, P is not visible.

In view of the balance sheet constraint

$$M_1^d + B_n = L^d + W, \tag{6}$$

the demand for government bonds coming from the non-bank private sector is given as the residual: $B_n = L^d + W - M_1^d$.

Since the banks are ultimately owned by households, W equals the aggregate nominal financial wealth of the private sector as a whole and consists of the monetary base and the value of outstanding government bonds, B:

$$W \equiv M_0 + B. \tag{7}$$

2 General equilibrium

There are three asset markets and an output market. There is abundant capacity in the output market and the nominal price level is rigid in the short run so that output is demand determined. Behind the scene employment adjusts to the demand for labor.

2.1 Equilibrium in the asset markets

The three asset markets to consider are the money market, the market for bank loans, and the bonds market. Interest rates adjust so as to generate equilibrium in all three asset markets. Since assets are stocks (quantities at a given point in time), we should think of the asset markets as being open at a given moment of time, that is, either at the beginning or the end of the current period. To fix ideas, we choose the beginning of the period.

Considering the two first-mentioned asset markets, we have $M_1^s = M_1^d$, that is,

$$mm(i_B)M_0 = M(Y, i_B), \tag{MM}$$

and $L^s = L^d$, that is,

$$\ell(i_B, i_L, \sigma)(1-\rho)mm(i_B)M_0 = L(Y, i_B, i_L). \tag{BL}$$

Regarding the third market, the bond market, let x denote the *number* of one-period bonds issued by the government at the beginning of the period (recall that the government debt has to be refinanced each period). Let each bond offer a payoff of 1 unit of money at the end of the period. If $v > 0$ denotes the market price of a bond at the beginning of the period, the effective interest rate i_B is given by the equation

$$v = 1/(1 + i_B).$$

The market value of the whole government debt at the beginning of the period is $B \equiv vx$.

Considering the demand side, the banks' demand a quantity of bonds equal to $x_b = B_b/v$ and the non-bank private sector demands $x_n = B_n/v$. Market clearing in the bond market requires

$$x = x_b + x_n.$$

This is equivalent to

$$B \equiv vx = vx_b + vx_n \equiv B_b + B_n.$$

So equilibrium in the bonds market is present if and only if

$$B = B_b + B_n. \tag{8}$$

This equilibrium condition holds automatically, when the two other asset markets are in equilibrium. This is the principle known as "Walras' law for stocks" and is an implication of the *balance sheet constraint* of the private sector. This balance sheet constraint is given by (7) combined with (6):

$$M_1^d - L^d + B_n = W \equiv M_0 + B. \tag{9}$$

Indeed,

$$\begin{aligned} B &\equiv W - M_0 = B_n + M_1^d - L^d - M_0 \qquad \text{(by (9))} \\ &= B_n - L^s + M_1^s - M_0 = B_n - L^s + D - M_0 \qquad \text{(by (MM), (BL), and (5))} \\ &= B_n - L^s + B_b + L^s = B_n + B_b \qquad \text{(by (1))}, \end{aligned}$$

thus confirming that the bond market clears, i.e., (8) holds, if the money market and the market for bank loans clear.

2.2 The output market

The asset market equilibrium conditions should be combined with equilibrium in the market for output.

The expected inflation rate is by assumption nil. So the nominal interest rates, i_B and i_L, on bonds and bank loans, respectively, are at the same time real interest rates. We assume that the sum, $\mathfrak{D}$, of private consumption and investment can be written

$$\mathfrak{D} = C(Y^p, i_B, i_L, W) + I(Y, i_B, i_L), \qquad 0 < C_{Y^p} \leq C_{Y^p} + I_Y < 1,$$

where the partial derivatives of both the consumption and the investment function w.r.t. the two interest rates are negative (see (YY) below). Here $Y^p = Y - (\tau + T(Y))$ is disposable private income, τ being a shift parameter ("fiscal tightness") and $T(Y)$ being the "automatic" net tax revenue, $0 \leq T'(Y) < 1$.

Interpreting $\mathfrak{D}$ as a function, equilibrium in the output market can be expressed on the compact form

$$Y = \mathfrak{D}(Y, W, i_B, i_L, \tau) + G, \qquad 0 < \mathfrak{D}'_Y < 1, \mathfrak{D}'_W > 0, \mathfrak{D}'_{i_B} < 0, \mathfrak{D}'_{i_L} < 0, -1 < \mathfrak{D}'_\tau = -C_{Y^p} < 0. \tag{YY}$$

We now have three equations, (MM), (BL), and (YY), and three endogenous variables, Y, i_B, and i_L.

3 Analysis

Let us derive a graphical representation of the economy in the (Y, i_B) plane in analogy of the standard illustration of IS-LM equilibrium.

3.1 Derivation of the MP curve

For given M_0, equilibrium in the money market provides a *monetary policy curve*. Indeed, the equation (MM) defines i_B as an implicit function of Y and M_0 :

$$i_B = i_{MP}(Y, M_0). \tag{MP}$$

The graph of this function in the (Y, i_B) plane for fixed M_0 is our monetary policy curve, in brief, the *MP curve*. The slope of the MP curve equals the partial derivative of the (MP) function w.r.t. Y and can be found by taking the total differential on both sides of (MM):

$$mm(i_B)dM_0 + M_0 mm'(i_B)di_B = M'_{i_B}di_B + M'_Y dY. \tag{MM'}$$

By setting $dM_0 = 0$ and reordering, we find

$$\frac{\partial i_B}{\partial Y}_{|MP} = \frac{M'_Y}{M_0 mm'(i_B) - M'_{i_B}} > 0, \tag{10}$$

cf. Fig. 2. The MP curve is thus positively sloped. The curve depicts the combinations of i_B and Y that for a given M_0 are consistent with money market equilibrium. The positive slope reflects that, for a given M_0, the higher transactions-motivated demand for money, induced by a higher level of economic activity, results in initial excess supply of bonds, thereby lowering their price and so raising the interest rate, i_B. As i_B is raised, the money multiplier goes up because banks become more eager to turn excess reserves into interest bearing assets.

In passing we note that by setting $dY = 0$ in (MM'), while allowing an increase in the monetary base of size dM_0 and reordering, we find

$$\frac{\partial i_B}{\partial M_0}_{|MP} = \frac{-mm(i_B)}{M_0 mm'(i_B) - M'_{i_B}} < 0. \tag{11}$$

As long as the bond interest rate has not yet changed, the effect of an expansion of the monetary base is an excess supply of money and excess demand for bonds. But thereby the price of bonds goes up which amounts to a fall in the bond interest rate, i_B, thus driving the money multiplier up and money demand down until equilibrium in the money market is obtained, given the hypothetical unchanged level of output. In Fig. 2 below this tells us that a higher M_0 shifts the MP curve downwards and thereby, everything else equal, stimulates output demand. This "channel" for the influence of money supply changes on the economy is called the *(bond) interest rate channel* and represents a mechanism also known from the simple IS-LM model.

We shall now see there is an additional channel, the *bank lending channel.*

3.2 Derivation of the IS curve

The description of equilibrium in the bank loans and output markets is a little more cumbersome. First, consider the bank loan market. The equilibrium condition, (BL), gives the lending rate as an implicit function of Y, i_B, σ, and M_0 :

$$i_L = f(Y, i_B, \sigma, M_0). \tag{12}$$

The partial derivatives can be found by taking the total differential on both sides of (BL):

$$\begin{aligned} & (1-\rho)\left[\ell(i_B, i_L, \sigma)\left(mm(i_B)dM_0 + M_0 mm'(i_B)di_B\right)\right. \\ & \left. + \, mm(i_B)M_0(\ell'_{i_B}di_B + \ell'_{i_L}di_L + \ell'_\sigma d\sigma)\right] \\ = \;& L'_{i_B}di_B + L'_{i_L}di_L + L'_Y dY. \end{aligned} \tag{13}$$

We find the partial derivative of f w.r.t. Y by setting $di_B = d\sigma = dM_0 = 0$ and reordering:

$$f'_Y = \frac{di_L}{dY} = \frac{L'_Y}{(1-\rho)mm(i_B)M_0\ell'_{i_L} - L'_{i_L}} > 0. \tag{14}$$

So, given i_B, σ, and M_0, a higher Y induces a tendency for the lending rate to rise. The reason is that the induced higher transaction demand for money raises the demand for bank loans. On the other hand this higher demand for bank loans is held at bay by this very increase in the lending rate.

We find the partial derivative of f w.r.t. i_B by setting $dY = d\sigma = dM_0 = 0$ in (13) and reordering:

$$f'_{i_B} = \frac{di_L}{di_B} = \frac{L'_{i_B} - (1-\rho)\left[\ell(i_B, i_L, \sigma)M_0 mm'(i_B) + mm(i_B)M_0\ell'_{i_B}\right]}{(1-\rho)mm(i_B)M_0\ell'_{i_L} - L'_{i_L}} > 0. \tag{15}$$

The positivity is imposed by assuming, as Bernanke and Blinder (1988) do, that $mm'(i_B)$ is "not too large". The intuitive explanation that this assumption is needed to get a positive derivative di_L/di_B is as follows. On the one hand, given Y, σ, and M_0, a higher bond interest rate raises the value of the option to place temporary excess liquidity in bonds, thus making a high bank lending rate more tolerable. Moreover, from the banks' perspective a higher bond interest rate makes it attractive to invest more in bonds and offer less bank loans ($\ell'_{i_B} < 0$). This means upward pressure on the lending rate also from the supply side. On the other hand there is a partly offsetting influence coming from the induced rise in the money multiplier along with the rise in the bond interest rate. The imposed assumption is that this influence is only *partly* offsetting.

The partial derivative of f w.r.t. σ is found by setting $dY = di_B = dM_0 = 0$ in (13) and reordering:

$$f'_\sigma = \frac{di_L}{d\sigma} = \frac{-(1-\rho)mm(i_B)M_0\ell'_\sigma}{(1-\rho)mm(i_B)M_0\ell'_{i_L} - L'_{i_L}} > 0. \tag{16}$$

This derivative is positive because a higher perceived riskiness associated with offering bank loans reduces the supply of bank loans. Given the demand for bank loans, the lending interest rate thereby becomes higher.

Finally, the partial derivative of f w.r.t. M_0 is found by setting $dY = di_B = d\sigma = 0$ in (13) and reordering:

$$f'_{M_0} = \frac{di_L}{dM_0} = \frac{-(1-\rho)\ell(i_B, i_L, \sigma)mm(i_B)}{(1-\rho)mm(i_B)M_0\ell'_{i_L} - L'_{i_L}} < 0. \tag{17}$$

An increase in the monetary base through an open-market purchase of bonds thus lowers the bank lending rate. The mechanism is that the inflow of central bank money allows the banks to increase profitable lending and at the same time maintain reserves at the desired level. In turn, the raised supply of bank loans lowers the lending rate − and thereby stimulates aggregate demand and output. This mechanism is called the *bank lending channel*.

Now consider the equilibrium condition (YY) for the output market. Substituting (12) into (YY) gives

$$Y = \mathfrak{D}(Y, i_B, f(Y, i_B, \sigma, M_0), \tau) + G, \tag{IS}$$

where the partial derivatives of $\mathfrak{D}$ are reported in (YY). Instead of the standard IS equation we thus arrive at an IS equation the position of which depends both on the supply of base money, M_0, and the perceived riskiness of offering bank loans. The IS equation defines i_B as an implicit function of Y, σ, M_0, τ, and G :

$$i_B = i_{IS}(Y, \sigma, M_0, G, \tau). \tag{IS'}$$

The graph of this function in the (Y, i_B) plane for fixed σ, M_0, G, and τ, defines our *IS curve.*

To be able to calculate (among other things) the slope of the IS curve, we first take the total differential on both sides of (IS):

$$dY = \mathfrak{D}'_{Y^p}dY + \mathfrak{D}'_{i_B}di_B + \mathfrak{D}'_{i_L}(f'_Y dY + f'_{i_B}di_B + f'_\sigma d\sigma + f'_{M_0}dM_0) + \mathfrak{D}'_\tau d\tau + dG. \tag{18}$$

By setting $d\sigma = dM_0 = d\tau = dG = 0$ and reordering, we get

$$\frac{\partial i_B}{\partial Y}_{|IS} = \frac{1 - \mathfrak{D}'_Y - \mathfrak{D}'_{i_L} f'_Y}{\mathfrak{D}'_{i_B} + \mathfrak{D}'_{i_L} f'_{i_B}} < 0,$$

where f'_Y from (14) can be inserted. This formula gives the slope of the IS curve which is thus negative, cf. Fig. 2.

The interpretation of the IS curve is that it depicts the combinations of i_B and Y that, for given σ, M_0, τ, and G, are consistent with equilibrium in both the market for bank loans and the output market. The IS curve is negatively sloped because a rise in the bond interest rate, i_B, both directly and indirectly, via the associated increase in the lending rate, cf. (15), reduces aggregate demand (via reducing consumption and investment). In contrast to a standard IS curve, the position of this IS curve depends not only on the fiscal policy parameters τ and G, but also on the supply of base money, M_0, and the perceived riskiness, σ, of offering bank loans. By setting $dY = d\sigma = d\tau = dG = 0$ in (18), we find that a higher M_0 shifts the IS curve upwards. By setting $dY = dM_0 = d\tau = dG = 0$, we find that a higher σ shifts it downwards.

3.3 General equilibrium

In general equilibrium both the output market, the money market, and the bank loan market (and thereby also the bond market) clear. The equilibrium is given as the point where the MP and IS curves in Fig. 2 intersect. Since an upward-sloping MP curve and a downward-sloping IS curve can only intersect once, a solution to the model, (Y, i_B), is unique. Assuming existence of a solution, we can thus write Y and i_B as implicit functions of the exogenous variables we are interested in:

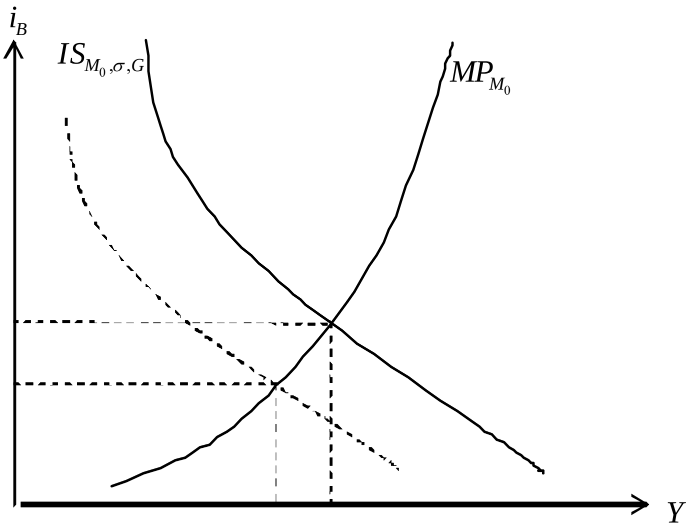

Figure 2: The IS-MP cross (for fixed M_0, σ, and G). A higher σ shifts the IS curve to the stippled position.

$$\begin{aligned} Y &= g(\sigma, M_0, \tau, G), & (19) \\ i_B &= h(\sigma, M_0, \tau, G). & (20) \end{aligned}$$

By construction, the equilibrium is a Keynesian equilibrium. The partial derivatives of the solutions for Y and i_B, respectively, w.r.t. σ, M_0, τ, and G, can be found by using Cramer's rule on the system consisting of (MM) and (IS). If we are only interested in the sign of the effects, "curve shifting" in Fig. 2 is in many cases sufficient.

3.3.1 Crisis

Suppose an economic crisis is under way and that an increased riskiness of supplying bank loans is perceived. The MP curve in Fig. 2 is unaffected, cf. the equation (MP). The IS curve is shifted downwards because a higher σ, for fixed Y and i_B, induces a higher i_L, cf. (16). At a given output level, Y, equilibrium in the output market then requires a lower i_B to compensate for the higher i_L, cf. equation (YY). The conclusion is that the IS curve will now intersect the MP curve South-West from the old equilibrium. So both i_B and Y will be lower in the new equilibrium. These responses of i_B and Y imply a dampening feedback on the "initial" rise in i_L. But they do not eliminate the latter rise. This is because the dampening feedback only exists to the extent that the net effect on

i_L of the higher σ *is* positive. The final outcome is thus an *increased spread:* higher bank lending rate and lower interest rate on government bonds. A concomitant phenomenon is a reduced money multiplier. In summary:

$$\sigma \uparrow \ \Rightarrow i_L \uparrow \ \Rightarrow i_B \downarrow \ \text{(for fixed } Y) \ \Rightarrow \begin{cases} i_L - i_B \uparrow \text{(increased spread)}, \\ \quad mm(i_B) \downarrow \ \text{(reduced money multiplier)} \end{cases}$$

These traits describe well what happened in the U.S. both in the first years of the Great Depression in the 1930s and when the full-scale financial and economic crisis we call the Great Recession broke out in 2008-2009. The comparatively risk-free interest rates, like those on government bonds, fell while risky interest rates, like those on consumer loans (car loans etc.) and corporate bonds, rose.

4 Policy

Monetary policy: Suppose M_0 is increased through an open market operation, implying $\Delta M_0 = -\Delta B > 0$. This affects output through two channels.

The *credit channel*: The banks are now able to offer more bank loans, and so the lending rate decreases, cf. (17). This stimulates demand in view of $\mathfrak{D}'_{i_L} < 0$. Thereby the IS curve is shifted to the right, and output is raised for given i_B.

The *interest rate channel*: For the increased money supply, M_1, to be willingly held by the public, the bond rate, i_B, must fall for given Y, cf. (11). The MP curve thus moves South-East. For unchanged i_B this allows a higher level of output.

The total effect of the monetary expansion on output is thus unambiguously positive, and on the lending rate the effect is unambiguously negative. The total effect on the bond interest rate is ambiguous.

Fiscal policy: Although a rise in G (not accompanied by a change in M_0) will automatically raise tax revenue, this may not be enough to avoid a budget deficit. There will thereby be a larger supply of government bonds next period. Feedback effects from this are ignored in this simple model. The case of a fully financed fiscal expansion can be analyzed by the method used for the simple IS-LM model in Chapter 21.3.

An alternative version of the model would consider i_B as the monetary policy instrument (as long as the zero-lower-bound is not binding) and then let M_0 adjust endogenously.

Allowing corner solutions $-$ desired excess reserves $E = 0$ for instance $-$ the model should be extended to include credit rationing. See Blinder (1987). Stiglitz and Weiss (1981) study the microeconomics of credit rationing from an incomplete-information perspective.

In Exercise Problem X.2 the reader is asked to apply this model for a series of economic questions.

Banks runs *Bank runs* as a self-fulfilling prophesy are studied in Diamond and Dybvig (1983), Diamond (2007), and Gertler and Kiyotaki (2015). In view of government regulation and the institution of deposit insurance since the 1930s, bank runs on ordinary banks are no longer common, but phenomena similar to bank runs, driven by self-fulfilling expectations, may occur vis-a-vis financial intermediaries in the "shadow banking system", like investment banks and mutual funds (think of Northern Rock, London, Sept. 2007, and Lehman Brothers, New York, Sept. 2008).

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IS-MP medium-run dynamics: Phillips curve, Taylor rule, and liquidity trap

This note presents and analyzes a simple Keynesian model of "short-to-medium run" dynamics under the assumption that monetary policy pursues a *Taylor rule*, also known as *inflation targeting*. This is a contra-cyclical monetary policy which uses the short-term nominal interest rate actively to counteract deviations of output and inflation from their structural and desired levels, respectively. To put it differently, through open-market operations the central bank let the money supply respond to business cycle fluctuations in inflation and possibly also output. In some form or another, this is nowadays the prevalent practice of central banks.

First a brief overview of two different views concerning monetary policy.

1 Two different views

A Taylor rule contrasts with *monetarist policy*. This term refers to a monetary policy where the central bank tries to maintain a constant but low growth rate in the money supply (be it M1, M2, or M3). In this way, according to monetarists (Milton Friedman and followers), monetary policy would not only restrain inflation but also safeguard stability (automatic and quick return to "full employment").

This view is profoundly questioned by macroeconomists of Keynesian conviction. The argument is that under a constant growth rate in the money supply, an adverse demand shock triggers both a *centripetal* force and a *centrifugal* force. On the one hand, the incipient recession reduces inflation, thereby making real money supply larger than otherwise. The result is a *reduced nominal* interest rate, whereby aggregate demand is *stimulated* − fine! On the other hand, by reducing inflation, also expected inflation is reduced. Through the resulting *higher expected real* interest rate than otherwise, aggregate demand is *dampened* − not so fine! Due to the inherent non-linearity in the money demand function (arising from the Zero Lower Bound), the lower the nominal interest rate has already become, the higher the risk that the centrifugal force dominates the centripetal force. This latter point has received increasing attention after the Zero Lower Bound has become a problem of urgent practical importance, first in Japan in the late 1990s, then in the Western World in the Great Recession 2008-?.

Now to the model.

2 Dynamic IS-MP model

Consider a closed economy. Let time be continuous. Ignore the time lag between output and aggregate demand.

2.1 The private sector

To avoid complicating the model with features of only secondary importance for the issue at hand, we assume that aggregate demand depends on only two endogenous variables, namely current aggregate income and the expected real interest rate faced by borrowing households and firms. At a given point in time we have

$$Y = D(Y, r^e, \eta), \qquad 0 < D_Y < 1, D_{r^e} < 0, D_\eta > 0. \tag{*}$$

Here Y is aggregate output, and r^e is the expected real interest rate faced by the ultimate private borrowers, the superscript e indicating expected value. Finally, η is a shift parameter on which aggregate demand depends positively.

For convenience we will base the analysis on a log-linear approximation to the equation (*). On both sides of (*) we take the total differential, to get $dY = D_Y dY + D_{r^e} dr^e + D_\eta d\eta$. Isolating the dY terms, we have

$$(1 - D_Y)dY = D_{r^e} dr^e + D_\eta d\eta.$$

Dividing through by $(1 - D_Y)Y$ gives

$$\frac{dY}{Y} = d\ln Y = \frac{D_{r^e}}{(1 - D_Y)Y} dr^e + \frac{D_\eta}{(1 - D_Y)Y} d\eta. \tag{**}$$

We assume the coefficients to dr^e and $d\eta$ are constants and name them $-\beta < 0$ and γ, respectively. Integrating on both sides of (**), we then get

$$\ln Y \equiv y = -\beta r^e + \gamma\eta + k,$$

where k is a constant.

Redefining our shift parameter to be $\mu \equiv \gamma\eta + k$ and letting time be explicit, we end up with

$$y_t = \mu - \beta r_t^e, \qquad \mu > 0, \beta > 0. \tag{1}$$

The shift parameter μ is an index of autonomous demand. It varies positively with any exogenous variable having the property that the higher its value, the higher is aggregate demand, everything else equal. So, the "degree of optimism" or "state of confidence" in the economy will affect the size of μ. In our main text we will interpret variations in μ as deriving from this source. government spending on goods and services will affect the size of μ. Alternatively, variation in μ could reflect variation in fiscal policy, an interpretation which we postpone to the concluding section.

Given the "short-to-medium-run" perspective of the model, it should embrace a *Phillips curve* of some sort. For simplicity we assume the simplest specification we can think of:

$$\dot{\pi}_t = \delta(y_t - y^*), \qquad \delta > 0, y^* > 0, \qquad \pi_0 \text{ given}, \tag{2}$$

where π_t in the inflation rate ($\equiv \dot{P}_t/P_t$), $y^* \equiv \ln Y^*$ is the NAIRU level of output, and δ measures the reaction speed.[1] The inflation rate thus speeds up or slows down according to whether output is above or below a certain level, $y^* \equiv \ln Y^*$, respectively. We may interpret this as reflecting a "wage-price spiral". In a boom ($y_t > y^*$) unemployment is low and workers' bargaining position strong. This results in fast nominal wage increases. Via firms' markup pricing fast inflation is induced. As long as the boom continues, faster and faster nominal wage and price increases ensue. In a slump ($y < y^*$) workers' bargaining position is weak and the spiral goes the opposite way.

When $y_t = y^*$, there is no internal pressure on inflation. For simplicity, y^* is assumed to be time independent, that is, the model abstracts from growth in labor force and technology. More to the point is that (2) indicates that the inflation rate is predetermined. So the inflation rate is sticky and can not jump. Inflation changes smoothly over time in response to the *output gap*, $y_t - y^*$.[2] The message of the Phillips curve (2) is that the output gap determines the *change* in inflation rather than the *level* of inflation.

To get further perspective on the Phillips curve (2), we may consider a standard

[1] Although strictly speaking, y is the log of output, we refer to y as "output" when there is no risk of confusion.

A reservation regarding the convenient assumption that δ is constant is made in Section 7.

[2] This seems to be in accordance with the empirics for industrialized economies without hyperinflation, cf. for instance Mankiw (2001).

expectations-augmented Phillips curve in discrete time:

$$\pi_t \equiv (P_{t+1} - P_t)/P_t = \delta(y_t - y^*) + \pi_t^e, \qquad t = 0, 1, 2, \ldots.$$

Now, assume inflation expectations are myopic: $\pi_t^e = \pi_{t-1}$. Then subtract π_{t-1} on both sides. We then get a discrete time analogue to (2).

Returning to our continuous time framework, let $\bar{\mu}$ ($> y^*$) be the value of the autonomous-demand parameter under "normal circumstances". By plugging this value and NAIRU output, y^*, into (1), we find the required value of r^e to be

$$\frac{\bar{\mu} - y^*}{\beta} \equiv \hat{r} > 0. \tag{***}$$

This interest rate level is sometimes called the "natural rate of interest", but we prefer the name *structural rate of interest* since it depends on autonomous demand, $\bar{\mu}$, which in turn depends on for instance fiscal policy. It is the real interest rate required for "full employment" (zero output gap) and stationary inflation under "normal circumstances" and fulfilled expectations.

The nominal interest rate, i_t, on short-term government bonds is, within bounds, controlled by the central bank through open-market operations (see below). We will call i_t the *policy rate*. Given i_t and given the expected inflation rate, π_t^e, the expected real interest rate can be written

$$r_t^e = i_t + \omega - \pi_t^e, \tag{3}$$

where ω is the *spread* (also known as the interest differential). This is the difference between policy rate i_t and the nominal interest rate at which the non-bank public borrows in financial markets (we assume the bank lending rate and the rate on corporate bonds are the same).

In view of government bonds being practically risk-free (usually), while loans to the ultimate borrowers in the private sector are generally risky, the spread will generally be positive, although less than the structural interest rate. We will treat the spread as a quasi-parameter, i.e., as being directly determined by the shift parameter μ (within its relevant range) and nothing else:

$$\omega = \omega(\mu), \qquad \omega'(\mu) < 0, \qquad 0 < \omega(\bar{\mu}) < \hat{r}. \tag{4}$$

So, when the state of confidence shifts, the spread shifts in the opposite direction.[3]

[3] In the IS-BL model of Short Note 3, the spread is "fully endogenous", measured as the difference between the two separate endogenous variables i_L and i_B in that model.

We assume that households, firms, and the central bank have the same expectations. So there is only one π_t^e in the economy. Until further notice, we do not want to be specific about how this expectation is determined, be it rational or adaptive.

2.2 The central bank

The central bank pursues a certain *inflation target*, $\hat{\pi}$. In addition, we simplifying assume that the central bank, owing to its accumulated experience, knows y^*, the structural interest rate, $\hat{r}$, and the "normal" spread $\omega(\bar{\mu})$. Through open-market operations the central bank then establishes its policy rate as the maximum of the "desired level" and nil:

$$\begin{aligned} i_t &= \max\left[0, \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi})\right], \qquad (5) \\ \text{where } \hat{\imath} &\equiv \hat{r} - \omega(\bar{\mu}) + \hat{\pi}, \quad \hat{\pi} > 0,\ \alpha_1 \geq 0,\ \alpha_2 > 1. \end{aligned}$$

This is an example of a *Taylor rule.* As long as the zero lower bound (ZLB) on the nominal interest rate is not binding, the central bank adjusts the policy rate, i_t, depending on the current output gap, $y_t - y^*$, and expected excess inflation, $\pi_t^e - \hat{\pi}$. Thereby the expected real interest rate, r_t^e, in the economy is raised or lowered depending on whether a dampening or stimulation of aggregate demand is called for. The limiting case $\alpha_1 = 0$, such that the policy rate does not at all respond directly to the output gap is included as a special case of the Taylor rule. The imperative $\alpha_2 > 1$ is known as the *Taylor principle.* It ensures that an increase in π_t^e results in a larger increase in i_t so as to raise r_t^e and thereby dampen output demand.

We see that the policy rate is such that when the output and inflation gaps ($y_t - y^*$ and $\pi_t^e - \hat{\pi}$, respectively) are nil and circumstances are "normal", then the expected real interest rate equals the structural rate. Indeed, under these circumstances (3) gives

$$r_t^e = \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) = \hat{r} - \omega(\bar{\mu}) + \hat{\pi} + 0 + 0 + \omega(\bar{\mu}) - \pi_t^e = \hat{r}.$$

If the economy is in recession and the recession is deep enough, the targeted nominal short-term rate implied by the Taylor rule could be negative. Then the zero lower bound indicated by (5) becomes binding, and the actual nominal short-term rate, i_t, stays at nil for some time. Further increases in the money supply can not bring i below 0 because agents prefer holding cash at zero interest rather than short-term government bonds (or demand deposits in banks) at negative interest. Well, strictly speaking, the lower bound is slightly below zero because the alternative to holding bonds or demand deposits is holding cash which gives zero interest but involves costs of storing, insuring, and transporting.

We shall consider the described economy under two alternative scenarios, one where the ZLB is not binding and one where it is binding. The variables α_1, α_2, y^*, $\hat{\pi}$, and μ are independent exogenous variables. There are six endogenous variables in our system: $y_t, r_t^e, i_t, \pi_t^e, \pi_t$, and the quasi-fixed interest spread ω. So far we have only one differential equation, (2) and four static equations, namely (1), (3), (4), and (5); the remaining equations are just preliminaries or define shorthands for combinations of exogenous variables. The lacking element in the model is a specification of how expectations are formed. Below, we shall consider different approaches to this problem.

Empirically there are signs that central banks prefer to "smooth" the time path of the interest rate, letting the policy rate be a weighted average of the rate in the previous period and the current "pure" Taylor-rule value, i_t^T, given from (5). Thereby $i_t = \max\left[0, \rho i_{t-1} + (1-\rho) i_t^T\right]$. In the present exposition we do not integrate this.

3 Short-run equilibrium when the ZLB is not binding

Combining (1) and (3), equilibrium in the output market can be written

$$y_t = \mu - \beta(i_t + \omega(\mu) - \pi_t^e), \tag{IS}$$

which we rewrite as

$$i_t = \frac{\mu - y_t}{\beta} - \omega(\mu) + \pi_t^e. \tag{IS'}$$

Assuming the zero lower bound on the interest rate is not binding, the Taylor rule gives

$$i_t = \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) \equiv \alpha_0 + \alpha_1 y_t + \alpha_2 \pi_t^e, \tag{MP}$$

where MP stands for monetary policy.

3.1 The IS-MP cross

At any given point in time, t, there are historically given expected and actual inflation rates, π_t^e and π_t, respectively. So for fixed t, the combinations of y_t and i_t that are consistent with equilibrium in the output market are given by the equation (IS). In Fig. 1 these combinations are depicted as the downward-sloping IS curve. Although this curve, as well as the MP curve, is here a straight line (due to log-linearization), we shall stick to the standard terminology and speak of both as "curves".

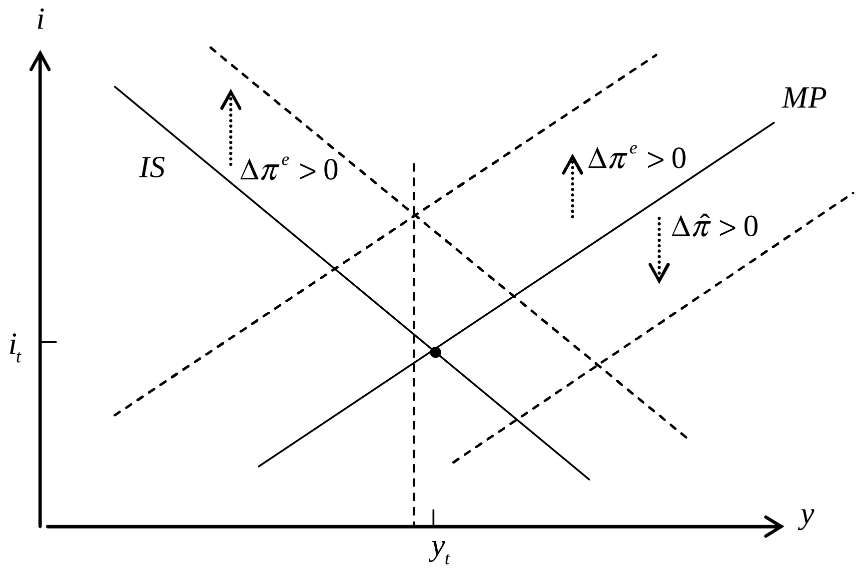

Figure 1: Short-run equilibrium at the IS-MP cross at time t for given π_t^e, $\hat{\pi}$, and μ ($\alpha_1 > 0$).

The upward-sloping MP curve in Fig. 1 represents the combinations of y_t and i_t that are consistent with the Taylor rule (MP). The point of intersection between the IS and MP curves represents the short-run equilibrium, (y_t, i_t), at time t.

Fig. 1 also indicates that for greater expected inflation, both the IS curve and the MP curve move upwards. For a given $\Delta\pi^e$, the MP curve features the largest upward shift in view of $\alpha_2 > 1$ (compare (MP) and (IS')). Hence the new equilibrium value of y_t will be *smaller* than the old. This feature is a first indication of the contra-cyclical role of the Taylor rule. It anticipates what the dynamic analysis below will unfold.

The economic logic behind this result is the following. On the one hand, the higher expected inflation *tends* to reduce the expected real interest rate and thereby stimulate output demand. On the other hand, following the Taylor rule the central bank counteracts this by a rise in the policy rate i_t, indeed a rise *larger* than that of expected inflation. So, in response to the higher expected inflation the central bank effectively *raises* the expected real interest rate. Thereby output demand, hence output, is dampened and the undesired higher inflation averted. If instead the central bank had kept the policy rate unchanged, actual output would have increased and thereby stimulated actual inflation.

In view of the linearity of the model, we get an explicit solution for the short-run equilibrium value of y. Inserting (MP) into (IS) gives

$$y_t = \mu - \beta \left[\hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) + \omega(\mu) - \pi_t^e\right].$$

Isolating y_t, we can thus write

$$y_t = x - \theta \pi_t^e, \tag{AD}$$

where

$$\begin{aligned} x &= \frac{\mu - \beta\left(\hat{\imath} - \alpha_1 y^* + \omega(\mu) - \alpha_2 \hat{\pi}\right)}{1 + \beta\alpha_1} \equiv x(\mu), && (6) \\ \theta &\equiv \frac{\beta(\alpha_2 - 1)}{1 + \beta\alpha_1} > 0. && (7) \end{aligned}$$

The quasi-parameter x shifts when the autonomous demand parameter μ shifts, but is otherwise constant. It measures the level of *aggregate demand in case expected inflation is nil.*

The relationship (AD) tells us that, given the autonomous demand parameter μ, and thereby given the spread, $\omega(\mu)$, aggregate demand and thus output is – through monetary policy – determined by expected inflation. More precisely: the *strong* response (inherent in $\alpha_2 > 1$) of monetary policy to expected inflation determines aggregate demand such that output ends up *depending negatively* on expected inflation. This is first indication that the Taylor rule is a contra-cyclical monetary policy and seems promising for stability.

We call the relationship (AD) the *aggregate demand curve* of the economy. As a preparation for dynamic analysis, we will consider a graphical illustration.

3.2 The AD curve under "normal circumstances"

Fig. 2 depicts in the (y, π^e) plane the AD curve under "normal circumstances", i.e., when confidence is "normal" and thus results in an autonomous demand level equal to $\bar{\mu}$. Then the AD curve reads

$$y_t = x(\bar{\mu}) - \theta \pi_t^e. \tag{$\overline{\text{AD}}$}$$

As long as $\mu = \bar{\mu}$, the AD curve is fixed and the economy must be at some point on this curve (line), depending on the current expected rate of inflation.

As already noted, it is the Taylor rule's $\alpha_2 > 1$ which ensures the *negative* slope of the AD curve. A more "passive" monetary policy, keeping i_t constant or allowing only a modest response to a rise in expected inflation, would make the AD curve positively sloped, cf. (7). This would make stability of the economy precarious, as alluded to in Section 1.

The situation depicted in Fig. 2 is one where at time 0, $\pi^e = \pi_0^e < \hat{\pi}$, cf. the point A in the figure. The corresponding equilibrium output, y_0, is higher than NAIRU output

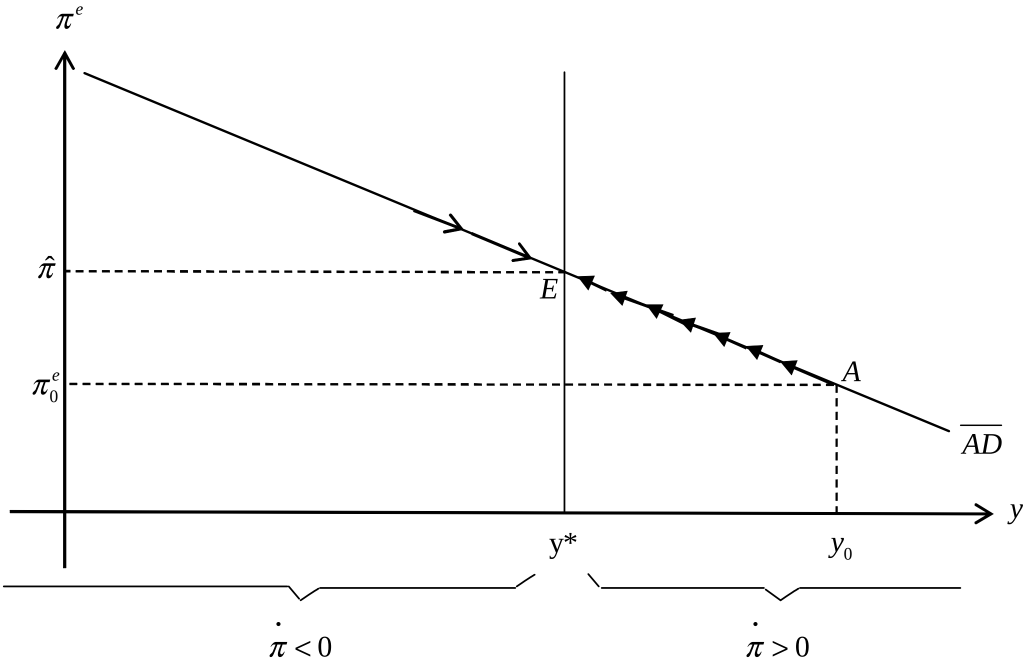

Figure 2: $\overline{\text{AD}}$ is the AD curve under "normal circumstances", i.e., when $\mu = \bar{\mu}$.

as indicated on the horizontal axis. So initially the economy is in a boom. This might seem paradoxical since the initial expected inflation is relatively low. But it is exactly this low expected inflation that invites a slack monetary policy, implying a low expected real interest rate, hence a high level of aggregate demand. Indeed, by (MP) we have

$$r_t^e = i_t + \omega(\bar{\mu}) - \pi_t^e = \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi_t^e - \hat{\pi}) + \omega(\bar{\mu}) - \pi_t^e, \tag{8}$$

where $\partial r_t^e / \partial \pi_t^e = \alpha_2 - 1 > 0$. So, in spite of the conceptual relationship, $r^e \equiv i + \omega(\mu) - \pi^e$, the real interest rate depends, everything else equal, *positively* on the expected inflation as a result of the Taylor rule. The low expected real interest rate needed to get high aggregate demand will be concomitant with the low expected inflation rate via a very low policy rate, i_t.

The arrows in Fig. 2 are explained below.

4 Dynamics when the ZLB is not binding

We shall here characterize the time path of y_t under the assumption that the zero lower bound, ZLB, does not become binding. We start with the easiest case, the benchmark case of rational expectations which here means *perfect foresight* with respect to the inflation rate.

4.1 Dynamics under perfect foresight

Assuming perfect foresight, we have

$$\pi_t^e = \pi_t \quad \text{and} \quad r_t^e = i_t + \omega(\bar{\mu}) - \pi_t \equiv r_t \qquad \text{for all } t.$$

Then the equation ($\overline{\text{AD}}$) reduces to

$$y_t = x(\bar{\mu}) - \theta\pi_t, \tag{9}$$

or, by inverting,

$$\pi_t = \frac{x(\bar{\mu})}{\theta} - \frac{1}{\theta}y_t.$$

Because $\pi_t^e = \pi_t$, we may interpret the (y, π^e) plane in Fig. 2 as an (y, π) plane. The shown AD curve in Fig. 2 is still a valid representation of the economy as long as the state of confidence is unchanged so that $\mu = \bar{\mu}$. Depending on the predetermined initial inflation rate, π_0, the economy must at time 0 be at the corresponding point, (y_0, π_0), on the AD curve.

The initial boom depicted in Fig. 2 is a state of affairs which, in view of the Taylor rule, requires *low* initial (expected and actual) inflation. Via the Phillips curve the boom induces *rising inflation* over time. And because the inflation coefficient, α_2, in the Taylor rule is above 1, rises in the inflation rate prompt even greater rises in the nominal interest rate. The result is a *rising real interest rate.* This gradually dampens aggregate demand and output. Monetary policy is thus in its "tightening mode". We get a leftward adjustment along the $\overline{\text{AD}}$ curve from the initial point A in Fig. 2 towards the steady-state point E. At this point the system is "at rest".

Let us instead imagine that the historically inherited inflation rate is relatively high, i.e., $\pi_0 > \hat{\pi}$. The corresponding equilibrium output, y_0, is then below NAIRU output. Also this may seem a paradoxical situation since the initial inflation is relatively high. A low level of output requires a low level of aggregate demand which in turn requires a high expected real interest rate. This is exactly what the monetary policy in this situation brings about. When the inflation rate is above its steady state level, $\hat{\pi}$, monetary policy chooses a nominal interest rate even more above *its* steady state level, due to the policy parameter α_2 exceeding 1. A high real interest rate and thereby low aggregate demand is the result.

In response to the high inflation, monetary policy has thus brought about a recession. Via the Phillips curve, the recession brings about *falling inflation.* With a policy parameter α_2 less than one, this would result in a rising real interest rate and thus reinforce the recession. But with $\alpha_2 > 1$, it is ensured that the nominal interest rate is lowered *more* than the inflation rate so that a *falling real interest rate* is the result. Monetary policy is here in its "relaxing mode". Aggregate demand and output are gradually stimulated in the rightward process along the AD curve towards the steady state, E.

As a conclusion, if autonomous demand remains at "normal", the economy settles down in steady state at the point E in Fig. 2. So, in contrast to the traditional *static* AS-AD model with an AD curve in the (y, P) plane, in the present model we have an AD curve in the (y, π) plane. A convenient feature of this AD curve is that it does not change its position or slope during the adjustment process. Instead, the dynamic adjustment of the economy takes place in a movement *along* the AD curve (at least if actual and expected inflation coincide).

The dynamics of the economy can also be depicted in the $(\pi, \dot{\pi})$ plane. This is shown in Fig. 6 and 7 in the appendix.

4.2 Dynamics under adaptive expectations

The specification considered of the Taylor rule assumes that monetary policy is forward-looking and responds to anticipated inflation rather than actual inflation. Under perfect foresight this is of course immaterial.

But what can we say in the absence of perfect foresight? First, our Taylor rule will still ensure that equilibrium output at a given point in time is determined uniquely for a given expected inflation rate as in the equation (AD). As long as autonomous demand equals $\bar{\mu}$, Fig. 2 is still applicable.[4]

Second, the ensuing dynamics can be described as follows. At time 0, there is a historically given expected inflation rate, π_0^e. Suppose $\pi_0^e < \hat{\pi}$. Then, through the Taylor rule the corresponding initial aggregate demand is high and y_0 therefore above NAIRU output. Whether or not the actual inflation rate initially differs from the expected, the situation triggers, through the Phillips curve (2), a *rising* actual inflation rate. Expected inflation seems thereby likely to rise as well, in which case the economy represented by the point (y_t, π_t^e) will again move up the AD curve in Fig. 2. As long as $y > y^*$, actual inflation will also rise further although the speed may deviate from that of expected

[4] This would not be true if we had *actual* inflation entering the Taylor rule instead of expected inflation. In that case, aggregate demand would become a function of both expected inflation, via (IS), and actual inflation, via the Taylor rule. We would then need a three-dimensional diagram, which is beyond the scope of this lecture note.

inflation. Intuitively, if no new shocks occur, over time the economy will again settle down in steady state at E in Fig. 2, where both π_t^e and π_t will equal $\hat{\pi}$.

This conjectured stability property definitely holds if we specify expectations to be formed according to the adaptive expectations formula,

$$\dot{\pi}_t^e = \lambda(\pi_t - \pi_t^e), \qquad \lambda > 0. \tag{10}$$

Inserting ($\overline{\text{AD}}$) into the Phillips curve (2), we get

$$\dot{\pi}_t = \delta(\frac{\bar{\mu} - \beta(\alpha_0 + \omega(\bar{\mu}) - \beta(\alpha_2 - 1)\pi_t^e}{1 + \beta\alpha_1} - y^*). \tag{11}$$

Hereby we have a system of two linear differential equations in two endogenous variables, π_t^e and π_t, both of which are predetermined. The steady state is ($\hat{\pi}$, $\hat{\pi}$) and is globally asymptotically stable. That is, for arbitrary initial values, π_0^e and π_0, the solution, (π_t^e, π_t), converges to the steady state for $t \to \infty$.[5]

5 An adverse demand shock

We return to the assumption of rational expectations, here *perfect foresight.*

Suppose that up until time t_1, the economy is in steady state with $\pi = \hat{\pi}$ and the autonomous demand parameter μ has its "normal" value, $\bar{\mu}$, so that aggregate demand in case of zero inflation is $x = x(\bar{\mu})$. Then, unexpectedly, a fall in the general state of confidence occurs so as to shift μ to the level $\mu' < \bar{\mu}$. The background for this adverse demand shock could be a financial crisis in the aftermath of a bursting housing price bubble. The interest spread now rises to $\omega(\mu') > \omega(\bar{\mu})$, which prompts a reduced x, at least for a while.

We first consider the case where the ZLB does not become binding.

5.1 Restoration when the shock is "minor"

Suppose the adverse demand shock is "minor". It shifts the AD curve down to the new position, indicated by AD' in Fig. 4. Immediately after the shock the economy shifts its

[5] This follows by calculating the Jacobian matrix of the right-hand sides of (10) and (11). We get that the trace equals $-\lambda$ and the determinant equals $[\delta\beta(\alpha_2 - 1)\lambda] / (1 + \beta\alpha_1)$. Thereby the trace is negative and the determinant positive (again $\alpha_2 > 1$ is decisive). This is both necessary and sufficient for a two-dimensional linear dynamic system, where both variables are predetermined, to be globally asymptotically stable, cf. Sydsæter et al. (2008, p. 244).

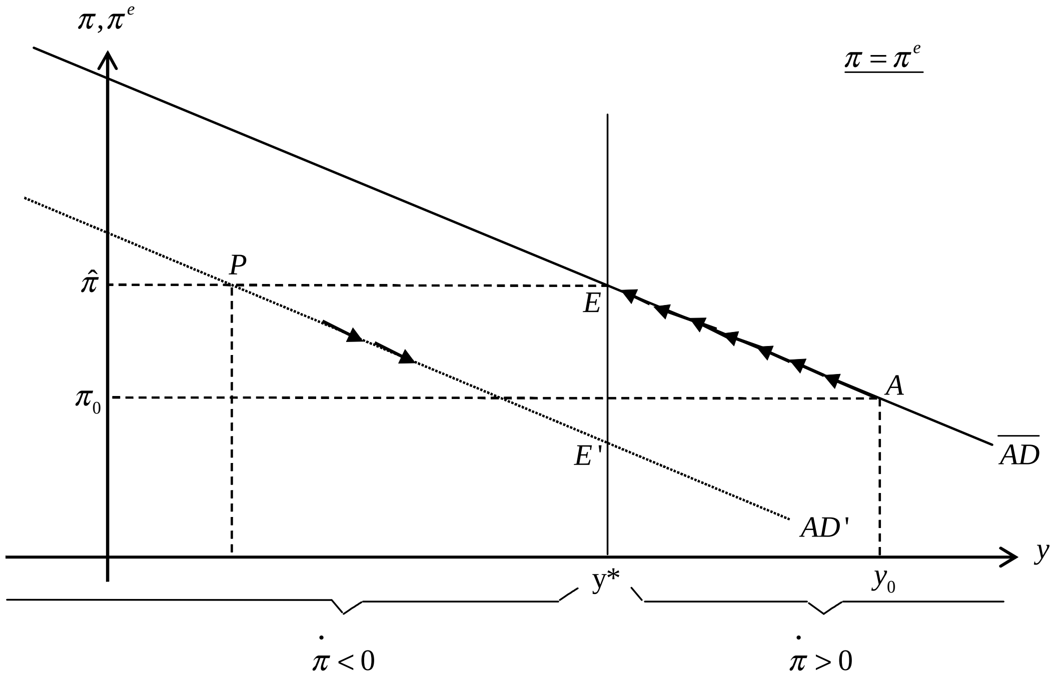

Figure 3: $\overline{\text{AD}}$ is the AD curve under "normal circumstances", i.e., when $\mu = \bar{\mu}$.

position from the point E to the point P in the figure. The implied recession activates the Taylor rule, both via the output gap (if $\alpha_1 > 0$) and, possibly with a delay, via low expected and actual inflation generated by the Phillips curve in response to $y < y^*$. That is, over time the economy travels down the new AD curve, AD', towards a new (quasi-)steady state, E'. So the recession is not lasting. This new steady state has "full" employment, but low inflation and hence low policy rate. This state is conditional on no repair of confidence taking place (hence the qualifier "quasi-").

It may seem more plausible that during the adjustment process, after a while, the experience of a gradual upturn restores confidence. As a crude representation of this, we imagine that a complete restoration of confidence takes place in a discrete jump at time $t_2 > t_1$. So, for $t \geq t_2$, equation ($\overline{\text{AD}}$) with the old $x = x(\bar{\mu})$ is again valid. In Fig. 3 the restoration of confidence shifts the aggregate demand curve back to its original position, $\overline{\text{AD}}$, and the position of the economy to the point A. Instead of settling down at y^*, the economy thus experiences a boom with $y > y^*$. Then inflation begins to rise through the Phillips curve and monetary policy gradually dampens demand and output through the Taylor rule. Over time the economy moves up the AD curve and approaches the old steady-state point E.

The corresponding dynamics in the $(\pi, \dot{\pi})$ plane for $t \geq t_2$ is depicted in Fig. 7 of the appendix.

5.2 Deep recession if the ZLB becomes binding

Suppose again that up until time t_1, the economy is in steady state with $\pi = \hat{\pi}$. Then a *large* adverse demand shock occurs so that the right-hand side of (5) becomes negative. Then the ZLB immediately becomes *binding* and instead of the desired negative interest rate being realized, we have $i_{t_1} = 0$.[6] We maintain the assumption that expected and actual inflation coincides.

According to the equation (IS), immediately after the shock aggregate output is therefore

$$y_{t_1} = \mu' - \beta(0 + \omega(\mu') - \hat{\pi}) < y^*.$$

Owing to the binding ZLB, the nominal interest rate remains at nil for some time. So conventional monetary policy based on adjusting the interest rate does not work $-$ the economy is in a *liquidity trap.* Through the Phillips curve the recession triggers a falling inflation rate. As i_t cannot go negative, the real interest rate *rises,* whereby aggregate demand and output are further reduced, thus sustaining the tendency for the inflation rate to fall. This increases the real interest rate further. A *vicious spiral* is unfolding.

In algebra, for $t \geq t_1$ we have

$$\begin{aligned} y_t &= \mu' - \beta r_t = \mu' - \beta(0 + \omega(\mu') - \pi_t) = \mu' - \beta\omega(\mu') + \beta\pi_t, & (12) \\ \dot{\pi}_t &= \delta(y_t - y^*) = \delta(\mu' - \beta\omega(\mu') + \beta\pi_t - y^*) = \delta(\mu' - \beta\omega(\mu') - y^*) + \delta\beta\pi_t < 0. & (13) \end{aligned}$$

So, for $t \geq t_1$ both output and inflation will be falling and r_t rising. The recession becomes a depression and there will be no recovery unless either *other* monetary policies or fiscal policies are introduced. Alternatively, the crisis may last until (outside the model) the capital stock has been worn down enough $-$ and new innovation possibilities have mounted up enough $-$ to generate a new upturn with rising capital investment and construction activities.

The condition that the lower bound is binding can be represented by a particular area, the *liquidity trap region,* in the (y, π) plane of Fig. 4. In view of (5), the boundary of the liquidity trap region is given by the equation

$$\hat{\imath} + \alpha_1(y - y^*) + \alpha_2(\pi - \hat{\pi}) = 0. \tag{14}$$

[6] When the full-blown financial crisis late in 2008 unfolded, the policy rate in the US and several other countries was quickly reduced to $[0.00 - 0.25)$. In the US this interval remained in force for seven years (Dec. 2008 - Dec. 2015).

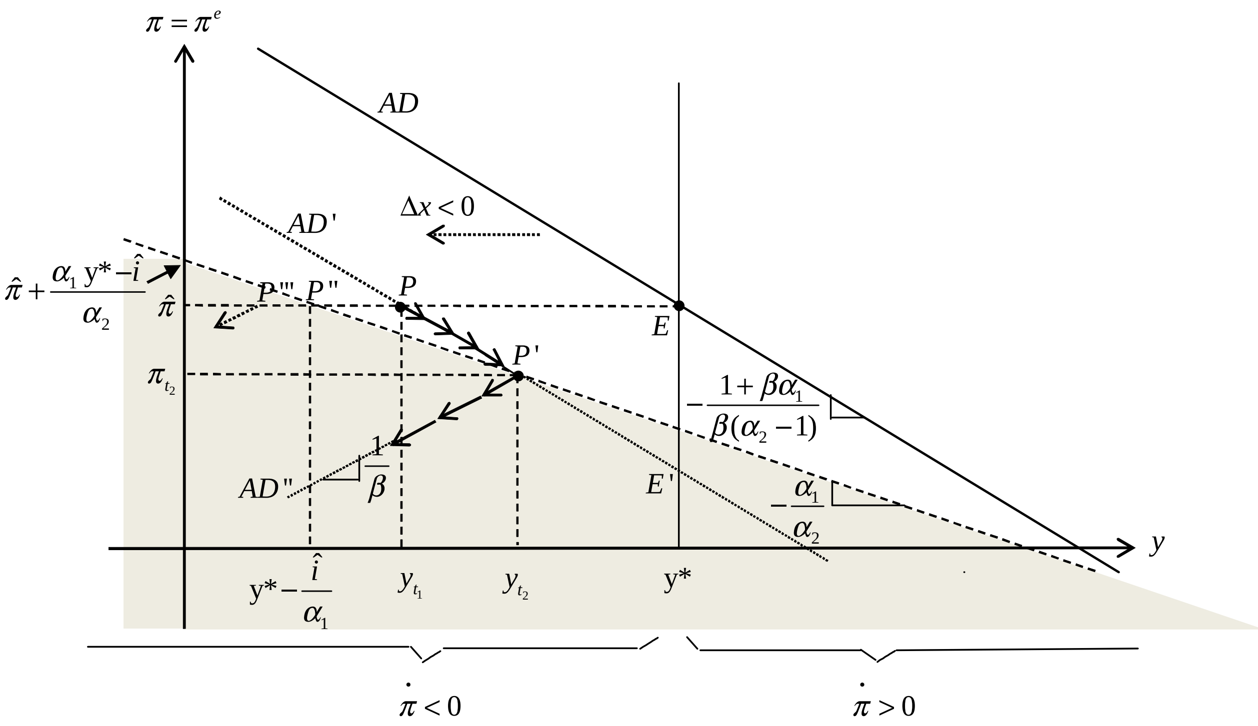

Figure 4: A large demand shock causes the liquidity trap to be operative when the economy hits the point P'. The case $\alpha_1 > 0$.

Rearranging, the boundary of the liquidity trap region thus is

$$\pi = \frac{\alpha_1 y^* + \alpha_2 \hat{\pi} - \hat{\imath}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} y. \tag{15}$$

Comparing the absolute slope of the boundary of the trap region, α_1/α_2. with the slope of the AD and AD' curves, we see that the former is smaller than the latter, as also indicated in Fig. 4.

There are two cases to consider: $\alpha_1 > 0$ and $\alpha_1 = 0$.

5.2.1 The case $\alpha_1 > 0$: Monetary policy responds directly to both gaps

The shaded area in Fig. 4 represents the liquidity trap region for the case $\alpha_1 > 0$, where the boundary of the liquidity trap region is downward sloping. The point P indicates the position of the economy immediately after the adverse demand shock. In the text above we implicitly assumed that P were at P" or to the left of P", say at P"'. In this case the lower bound is immediately operative and forces the economy to move South-West in the diagram as indicated at the point P"'.

As Fig. 4 is drawn, however, P" is to the left of the point P, implying that the lower bound is not immediately operative. Nevertheless, in the process of lowering the nominal interest rate more than inflation falls, monetary policy hits the zero lower bound, at time $t_2 > t_1$, cf. the point P' in Fig. 4. From then on the economy is governed by (12) and (13). The movement is South-West along the *positively* sloped branch, AD", of the total *kinked aggregate demand* curve AD'-AD" in the diagram. Along the positively sloped branch the vicious spiral unfolds with output demand and output falling owing to a rising real interest rate caused by a continuing fall in the inflation rate due the low level of output while there is no longer a falling nominal interest rate.

There is empirical evidence that when the price inflation has become low, it tends to be more and more sticky downwards; similarly with wage inflation (see Hendry and ??, 2013). This *may* end the vicious spiral but does not reverse it. Inflation, π, *may* go negative, which amounts to *deflation,* as we saw under the Great Depression in the 1930s.[7] Also y may go negative. This might seem absurd, but is not, since y is really the *logarithm* of output.

In Fig. 8 of the appendix is depicted what the vicious spiral looks like in the $(\pi, \dot{\pi})$ plane.

A benchmark case* Let us consider the question: How large is the minimum adverse demand disturbance, measured by the change in x, needed to bring the economy immediately into the liquidity trap region? That is, when will the point P in Fig. 4 coincide with point P" on the boundary of the liquidity trap region?

At the point P" we have $\pi^e = \pi = \hat{\pi}$. The associated output level is, by (14), easily found to be

$$y = y^* - \frac{\hat{\imath}}{\alpha_1}.$$

For y_{t_1} to equal this value, we must, in view of (AD), have

$$y_{t_1} = x(\mu') - \theta\hat{\pi} = y^* - \frac{\hat{\imath}}{\alpha_1} = x(\bar{\mu}) - \theta\hat{\pi} - \frac{\hat{\imath}}{\alpha_1},$$

because y^* satisfies (AD) with $\mu = \bar{\mu}$ and $\pi_t^e = \hat{\pi}$. As $\theta\hat{\pi}$ cancels out, we find

$$x(\bar{\mu}) - x(\mu') = \frac{\hat{\imath}}{\alpha_1} \tag{16}$$

[7]At the time of writing (fall 2015), the European central Bank (ECB), facing an inflation rate in the Eurozone down at 0.3 percent on an annual basis, conducts *quantitative easing* (see Section 6 below) in its attempt to stop the vivious spiral and avoid the Eurozone ending up in deflation.

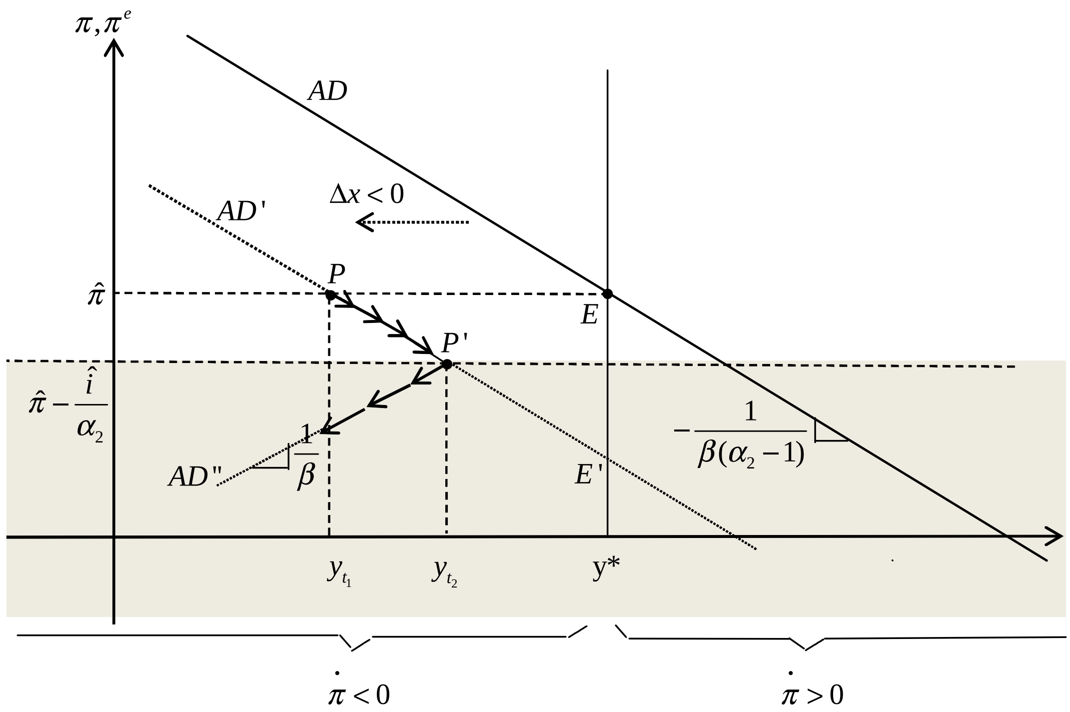

Figure 5: A large demand shock causes the liquidity trap to be operative when the economy hits the point P'. The case $\alpha_1 = 0$.

This is the minimum adverse demand disturbance (drop in x) needed to immediately bring the economy into the liquidity trap region. If the adverse demand disturbance is at least as large as this value, the lower bound becomes binding immediately at time t_1.

We see that the policy coefficient α_1 to the output gap in the the Taylor rule plays a role here. Given the inflation target, $\hat{\imath}$, we have that the larger is α_1, the smaller is the required Δx (considering the Taylor rule formula (5), this is no surprise). Everything else equal, this speaks for choosing a small α_1. Nevertheless, as we shall now see, even $\alpha_1 = 0$ is no guarantee for not ending up in a liquidity trap.

5.2.2 The case $\alpha_1 = 0$: Monetary policy only responds directly to expected inflation

When $\alpha_1 = 0$, the boundary of the liquidity trap region is horizontal. The shaded area in Fig. 5 represents the region in this case. Again the point P indicates the position of the economy immediately after the adverse demand shock at time t_1. By inspection of the figure, since by assumption $\hat{\imath} > 0$, P is necessarily situated above the liquidity trap region. Anyway, falling inflation sets in. In the process of lowering the nominal interest rate even more than inflation falls, monetary policy *may* hit the lower bound during the adjustment. As the figure is drawn, this is what happens at some point in time, t_2, cf. the point P' in Fig. 5. From then on the vicious spiral unfolds and the economy moves South-West along the *positively* sloped branch of the kinked aggregate demand curve AD'-AD" in the diagram.

6 Policy options

We have studied the dynamic interaction between aggregate demand, an "accelerationist" Phillips curve, and monetary policy following a Taylor rule.

Vis-a-vis small demand disturbances of the economy, the Taylor rule works well and tend to stabilize the economy around the "full" employment steady state.

For large adverse demand shocks, the economy may end up in a liquidity trap. In that case an unchanged Taylor rule cannot hinder a vicious circle to arise, leading into prolonged depression.

One policy option is to *raise the inflation target* and thereby inflation expectations. A central bank trying to follow that route may run into credibility problems, however.

Alternative or supplementary policy options are situated outside conventional monetary policy (short-term interest rate policy).

One possibility is *expansionary fiscal policy.* When the economy is in a liquidity trap, fiscal policy multipliers tend to be high. This is so for several reasons. One reason is that there will be no financial crowding out as long as the central bank wants its policy rate to be as low as possible. Another reason is that the economic situation which has triggered the liquidity trap is likely to also be a situation where involuntary unemployment is high.

Another possibility is so-called *quantitative easing* (QE). This can take several forms. The central bank may offer credit to financial intermediaries (banks, mutual funds, mortgage credit companies, insurance firms, etc.) on more gentle conditions than usually. And it may try directly to reduce the spread, ω, by buying long-term government bonds and other assets in the market.

Another form of QE is "helicopter money" as Milton Fridman called it. This is fiscal policy in the form of income transfers to the private sector directly financed by money issue.

7 Discussion

The model, as it stands, makes it appear that the central bank has a strong grip on the economy outside ZLB. Probably stronger than in reality. One circumstance behind this is of course that the model contains no stochastic elements. A second circumstance is that perfect knowledge of the NAIRU and the structural interest rate are strong assumptions, not likely to be fulfilled in practice. Also the assumption that output *immediately* adjusts to the demand changes prompted by changes in the policy rate seems too strong.

The assumed version of the Phillips curve is quite brute. In particular it tends to exaggerate the deflationary pressure in a liquidity trap. The constancy of the reaction speed δ (and perhaps also of y^*) in (2) is not in accordance with the empirical evidence. Hendry and ?? (2013) find that when the inflation rate has become low, it tends to be more and more sticky downwards. According to Stock and Watson (2010), once a slump has lasted 11 quarters at the same rate, no matter how high, unemployment loses its downward pressure on inflation.

Finally two *terminological remarks*:

1. Recognizing the serious limitations of the static AS-AD model in the (Y, P) plane, well-known from many textbooks in the past, several newer textbooks, e.g. Jones (2015), now use the label AS-AD for dynamic models in the (Y, π) plane. To avoid confusion, it may be better to use a label like *dynamic IS-MP model* or *dynamic AD-AS-MP model.*

2. Be aware that in the elder literature "output gap" usually meant $y^* - y$, while in recent literature the opposite meaning, $y - y^*$, has become quite established.

8 Appendix: What the dynamics look like in the $(\pi, \dot{\pi})$ plane

In this appendix we illustrate the dynamics in an alternative way, namely in the $(\pi, \dot{\pi})$ plane rather than the (y, π) plane. We stick to the case of perfect foresight: $\pi_t^e = \pi_t$ for all t.

Dynamics when the lower bound is not binding. After substitution of $(\overline{\text{AD}})$ into the Phillips curve, we have

$$\dot{\pi}_t = \delta(x(\bar{\mu}) - \theta\pi_t - y^*). \tag{17}$$

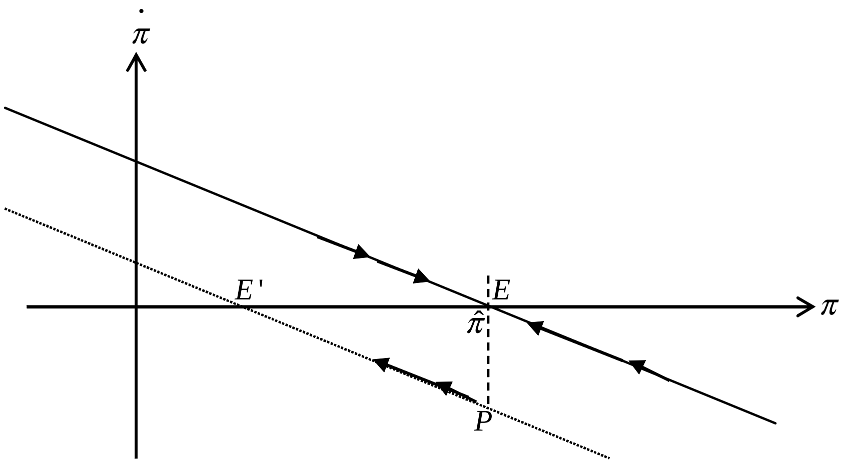

Figure 6:

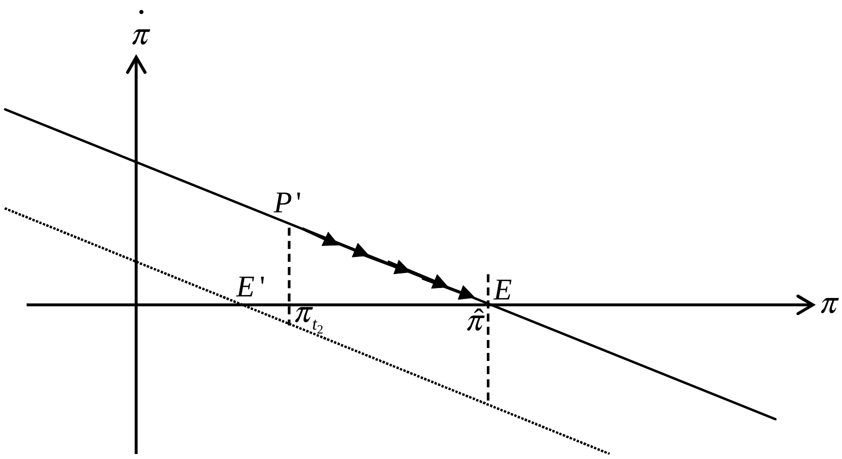

Figure 7:

The graph of this relationship is shown in Fig. 6 as the downward-sloping solid line in the figure. As long as $\mu = \bar{\mu}$, the economy must be at some point on this line. If $\pi < \hat{\pi}$, π will be growing towards $\hat{\pi}$, while if $\pi > \hat{\pi}$, π will be falling towards $\hat{\pi}$. Over time the economy moves along the line representing equation (17) until steady state at the point E is "reached".

Fig. 6 also illustrates the consequence of an adverse demand shock, disturbing an economy initially in steady state. The shock leads to a downward shift of the line representing (17). The position of the economy shifts from the point E to the point P, representing recession. Hereafter, there is a gradual fall in inflation ($\dot{\pi} < 0$) which, by the monetary policy, is accommodated by a faster fall in the nominal interest rate so as to lower the real interest rate.

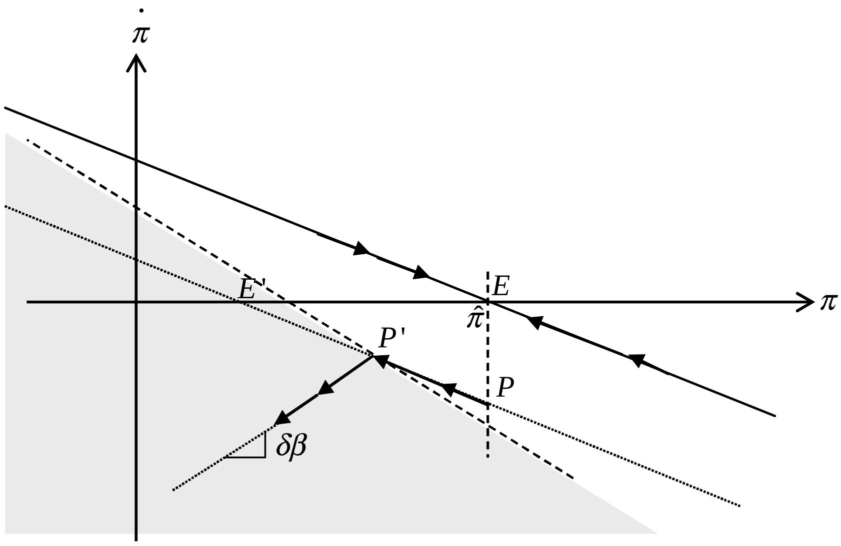

Figure 8:

An ensuing restoration of confidence at time t_2 and the implied dynamics is illustrated in Fig. 7. The favorable restoration of confidence shifts the line representing (17) back to its original position. The resulting boom triggers a gradual rise in inflation. Through the monetary policy the nominal interest rate rises even faster, thereby gradually raising the real interest rate. The boom is thus dampened and the economy is gradually brought back to the original steady state, E.

Dynamics when the lower bound is binding. As we saw in Fig. 4, if the adverse demand shock at time t_1 is large enough, the economy may immediately after the shock be at point P in that figure and then, after some time, enter the liquidity trap region at point P' where the vicious spiral takes over. Fig. 8 shows the corresponding evolution in the $(\pi, \dot{\pi})$ plane.

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On Section 3 of "What explains the 2007-09 drop in employment?"

by Atif Mian and Amir Sufi, Econometrica, Nov. 2014.

Christian Groth

University of Copenhagen

Nov. 2016

From Mian and Sufi's abstract

- Deterioration in household balance sheets, or the *housing net worth channel*, played a significant role in the sharp decline in U.S. employment 2007-09.
- Counties with a larger decline in housing net worth experienced a larger decline in non-tradable employment.
- Result not driven by industry-specific supply side shocks, policy-induced business uncertainty, or credit supply tightening.
- No significant expansion of the tradable sector in counties with the largest decline in housing net worth.
- Little evidence of wage adjustment within or emigration out of the hardest hit counties.

A simple partial-equilibrium model

m counties, $i = 1, 2, \ldots, m$; may differ w.r.t. housing net worth.

Same size $= 1 =$ labor supply in each county.

In each county two sectors:
Sector N produces a *non-tradable good* (only sold at the local market).
- T - *tradable* good (sold economy-wide at one price).

Labor homogeneous, immobile across counties, mobile across sectors within a county.

Production capital not considered.

Houses are treated as non-produced land. Focus is on the channel from a fall in housing net worth to non-tradables production independently of an effect on construction.

Preferences: $U(c^N, c^T) = \alpha \log c^N + (1-\alpha) \log c^T$. Hence,

$$\begin{aligned} P_i^N c_i^N &= \alpha D_i, \\ P^T c_i^T &= (1-\alpha) D_i, \end{aligned}$$

where D_i = nominal consumption demand at i. *Production:*

$$\begin{aligned} y_i^N &= a e_i^N, \\ y_i^T &= b e_i^T. \end{aligned}$$

Walrasian general equilibrium: Households and firms are price takers, markets clear by price adjustment:

$$W_i = P_i^N a = P^T b \equiv W, \quad \text{hence} \quad P_i^N = P^N \ \ \forall i, \ \text{ and } \ P^N / P^T = b/a.$$

$$\begin{aligned} e_i^N + e_i^T &= 1, \qquad \forall i, \\ y_i^N &= c_i^N = \alpha D_i / P_i^N = \alpha D_i / P^N, \qquad \forall i, \\ y_i^T &\neq c_i^T \ \text{ generally, since } D_i\text{'s may differ,} \\ \text{but} \qquad \sum_{i=1}^m y_i^T &= \sum_{i=1}^m c_i^T = \frac{(1-\alpha)\sum_{i=1}^m D_i}{P^T}. \end{aligned}$$

Money neutrality!

Determination of P^T, P^N, and sectoral allocation of employment

$$\sum_{i=1}^{m} y_i^T = \sum_{i=1}^{m} b e_i^T = b\sum_{i=1}^{m}(1-e_i^N) = b\sum_{i=1}^{m}(1-\frac{y_i^N}{a}) = b\sum_{i=1}^{m}(1-\frac{\alpha D_i}{aP^N})$$
$$= b\left(m - \frac{\alpha\sum_{i=1}^{m} D_i}{aP^N}\right) = \frac{(1-\alpha)\sum_{i=1}^{m} D_i}{P^T}.$$

So

$$P^T = \frac{\sum_{i=1}^{m} D_i}{bm}, \qquad P^N = \frac{\sum_{i=1}^{m} D_i}{am}, \qquad W = P^T b = P^N a. \qquad (*)$$

$$e_i^N = \frac{y_i^N}{a} = \frac{\alpha D_i}{aP^N}, \qquad e_i^T = 1 - \frac{\alpha D_i}{aP^N}, \qquad \forall i. \qquad (**)$$

Assume initial symmetry,

$$D_i = D_0, \qquad i = 1, 2, \ldots, m. \qquad \text{Then,}$$
$$P^{*T} = \frac{D_0}{b}, \qquad P^{*N} = \frac{D_0}{a}, \qquad W^* = P^{*T} b = D_0,$$
$$e_i^{*N} = \frac{\alpha D_0}{aP^{*N}} = \alpha, \qquad e_i^{*T} = 1-\alpha, \qquad \forall i.$$

Negative demand shock. Suppose, a negative shock to housing net worth occurs, perhaps due to a bursting housing bubble. Suppose further that this triggers a tightening of borrowing constraints on indebted households.

As a result, to a varying degree across counties, households' nominal demand falls.

Let initial uniform demand, D_0, equal 1, so that

$$D_i = 1 - \delta_i, \ \forall i \qquad \delta_i \in (0,1).$$

Average shock is

$$\frac{\sum_{i=1}^{m} \delta_i}{m} \equiv \bar{\delta}.$$

Non-tradable employment relies heavily on local demand, while tradable employment relies on national or even global demand.

Case 1: Complete nominal price flexibility.

$$\sum_{i=1}^{m} D_i = \sum_{i=1}^{m}(1-\delta_i) = m - \sum_{i=1}^{m}\delta_i = m - m\bar{\delta} = m(1-\bar{\delta})$$

Prises fall in proportion to fall in nominal demand:

$$\begin{aligned} P^T &= \frac{1-\bar{\delta}}{b} \quad \text{and} \quad P^N = \frac{1-\bar{\delta}}{a}, \\ W &= aP^N = bP^T = 1-\bar{\delta}. \end{aligned}$$

Still $e_i^N + e_i^T = 1$ (full employment $\forall i$), but with local sectoral reallocation:

$$\begin{aligned} e_i^T &= 1 - \frac{\alpha(1-\delta_i)}{1-\bar{\delta}} \gtreqqless e_i^{*T} \ (= 1-\alpha) \quad \text{for} \quad \delta_i \gtreqqless \bar{\delta}, \text{ respectively,} \\ e_i^N &= \frac{y_i^N}{a} = \frac{\alpha D_i}{aP^N} = \frac{\alpha(1-\delta_i)}{1-\bar{\delta}} \lesseqqgtr e_i^{*N} \ (= \alpha) \quad \text{for} \quad \delta_i \gtreqqless \bar{\delta}, \text{ respectively.} \end{aligned}$$

Predictions: Still full emplyment everywhere. In counties faced by a large local shock workers move from N-employment to T-employment. The reverse if shock is small.

Case 2: Complete nominal price rigidity

$$P^T = \frac{D_0}{b} = \frac{1}{b}, \qquad P^N = \frac{D_0}{a} = \frac{1}{a}.$$

From (**):

Tradables:

$$\begin{aligned}
\sum_{i=1}^{m} y_i^T &= \frac{(1-\alpha)\sum_{i=1}^{m} D_i}{P^T} = \frac{(1-\alpha)m\bar{D}_i}{P^T} = \frac{(1-\alpha)m(1-\bar{\delta})}{P^T} \\
e_i^T &= \frac{y_i^T}{b} = \frac{\sum_{i=1}^{m} y_i^T}{mb} = \frac{(1-\alpha)(1-\bar{\delta})}{P^T b} = (1-\alpha)(1-\bar{\delta}) \\
&< e_i^{*T} = 1-\alpha, \ \forall i.
\end{aligned}$$

$$\text{hence T-fall} \equiv e_i^{*T} - e_i^T = 1-\alpha-(1-\alpha)(1-\bar{\delta}) = (1-\alpha)\bar{\delta}.$$

Non-tradables:

$$\begin{aligned}
e_i^N &= \frac{\alpha D_i}{aP^N} = \frac{\alpha(1-\delta_i)}{1} < e_i^{*N} = \alpha, \quad \forall i, \\
\text{hence N-fall} &\equiv e_i^{*N} - e_i^N = \alpha - \alpha(1-\delta_i) = \alpha\delta_i.
\end{aligned}$$

Predictions:

1. Fall in total employment.
2. Fall in local T-employment should have no corr. with local shock δ_i.
3. Fall in local N-employment should have pos. corr. with local shock δ_i.

Data complies.

Likely explanation:

Lower housing net worth$\Rightarrow$ lower wealth

$$\Rightarrow \left\{ \begin{array}{l} \text{consumption} \downarrow \\ \text{value of collateral } \downarrow \Rightarrow \text{credit contraction} \end{array} \right\} \Rightarrow \text{consumption} \downarrow\downarrow$$

$\Rightarrow$ investment $\downarrow\Rightarrow$ consumption $\downarrow\downarrow\downarrow$

and so on in a vicious circle.

—

NOTES

Sizes of the adverse demand shocks are ordered in this way:

$$\delta_i < \delta_{i+1}, \qquad i = 1, 2, \ldots, m-1,$$

but of no use here.

Case 2:

Total employment in county i is

$$e_i = e_i^T + e_i^N = (1-\alpha)(1-\bar{\delta}) + \alpha(1-\delta_i) < 1 = e_i^*.$$

Fall in total employment in county i is

$$1 - e_i = 1 - ((1-\alpha)(1-\bar{\delta}) + \alpha(1-\delta_i)) = (1-\alpha)\bar{\delta} + \alpha\delta_i.$$

Prediction:

Fall in total local employment should have pos. corr. with local shock δ_i.

Data complies (no surprise given the above).

Chapter 29

Business fluctuations

This chapter presents stylized facts and basic concepts relating to business fluctuations. The next chapters go more into depth with specific business cycle theories.

The term *business cycles* refers to the empirical phenomenon of economy-wide fluctuations in output and employment around the trend, observed in industrialized market economies. By "trend" is meant a persistent long-term movement over time. That the fluctuations around trend are often called business "cycles" should not be taken too literally. The sequence of expansions and contractions is not periodic like sinus waves. But the sequence shows many statistical regularities. It is the job of business cycle analysts to characterize and explain these regularities.

29.1 Some business cycle facts

Compared with "white noise fluctuations", business cycle fluctuations are characterized by composite stochastic regularities. In a short list we emphasize the following regularities displayed by time series data:

1. GDP and employment exhibit considerable *fluctuations around their trends.* (Whether the trend is best described as stochastic or deterministic is a recurrent theme in econometric time series analysis.)

2. The expansions and contractions exhibit *persistence* (duration) in that positive deviations from trend are likely to be followed by further deviations of the same sign.

3. The ups and downs tend to be *hump-shaped* rather than saw-tooth shaped (amplification).

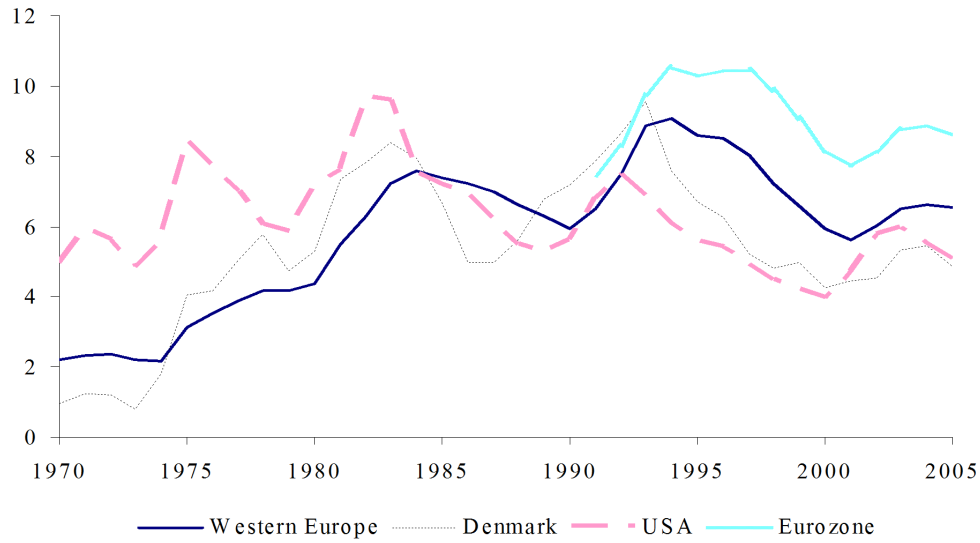

Figure 29.1: The rate of unemployment in Denmark, Western Europe, the Eurozone, and the United States, 1970-2005. Note: Unemployment is measured as the number of unemployed relative to the labor force. Western Europe comprises the EU-15 as well as Norway, Switzerland and Iceland. Germany is included after the reunification in 1991. Source: OECD, Economic Outlook.

4. The fluctuations are recurrent, but *neither periodic nor easily predictable.* The distance from peak to peak may be aboutOne should10 years.

5. The fluctuations exhibit systematic *co-movement* across production sectors, GDP components, and countries. Some facts that have played a central role for the theoretical debate are:

   (a) Employment (aggregate labor hours) is *procyclical*, i.e., varies in the same direction as GDP, and fluctuates almost as much as GDP.

   (b) Aggregate consumption and employment are markedly positively correlated.

   (c) Real wages are weakly procyclical and do not fluctuate much.

   (d) Firms' inventory holdings are procyclical, while the inventory-to-sales ratio is countercyclical.

Some of the regularities identified may only be valid for a subset of countries, depending on the structural characteristics of these. For example Fig. 28.1 shows that unemployment in Europe as well as the US fluctuates considerably. Only in the US, however, has unemployment appeared stationary since the early 1970s.

The next section gives a list of definitions of terms often used by business cycle analysts.

29.2 Key terms from the business cycle vocabulary

Impulse versus *response*. The "impulse" is a disturbance to the economic system coming "from the outside". Is synonymous with a "shock" to an exogenous variable (an unanticipated sudden shift in its value). The "response" refers to the reaction of the economic system, i.e., the effect on endogenous variables.

Propagation and *propagation mechanism*. "Propagation" refers to the spreading of effects of the impulse through the economic system (synonymous with "dissemination", "transmission" or "proliferation"). And "propagation mechanism" is the economic mechanism involved in this spreading.

The propagation mechanism can lead to *amplification, persistence* and *comovement*:

Amplification is present when an α per cent deviation (from normal) of an exogenous variable results in a more than α per cent deviation (from normal) of an endogenous variable. Is more or less synonymous with "magnification", "multiplier effect" or "blow up effect".

Table 1 Glossary concerning shocks and their effects

	Shock type		
Effect on dependent variable	Temporary	Persistent	Permanent
Temporary			
Persistent			
Permanent			

Persistence refers to effects on endogenous variables along another dimension, namely the time dimension. A shock has "persistent" effects to the extent that the effects last long. Is synonymous with *durability* of the effect. Is often measured by the auto correlation coefficient calculated from the time series of the endogenous variable. Sometimes the shock itself is said to be persistent, usually meaning that there is a relatively durable change in an exogenous variable. One should be aware that the distinction between "temporary" and "persistent" may refer to either the effect of the shock or the shock itself. Table 1 gives a reminder, where also the possibility of permanence is included.

Co-movement refers to the presence of significant correlation between two or more de-trended variables (usually in logs).

Finally, *volatility* usually refers to the standard deviation (sometimes variance) of the deviations of a variable from its trend value. Fixed capital investment is much more volatile than GDP whereas consumption is considerable less volatile.

29.3 A quick glance at the Great Recession and its aftermath

Some data on labor market flows in the USA published by the Bureau of Labor Statistics is shown in the figures 28.2 - 28.4. The terminology used is the following: *total separations* equal the sum of *quits* and *layoffs and discharges*, *quits* being separations on the initiative of the worker and *layoffs and discharges* being separations initiated by the firm. Large fluctuations in employment are envisaged. The shaded areas in the figures indicate periods of *recession* as diagnosed by the NBER (National Bureau of Economic Research). The NBER defines an economic recession as: "a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales".[1] It is noteworthy that after the 2008-2009 outbreak of the "Great Recession" the trough level of employment is lower than it was after the dot.com-bubble 2001 recession.

At least two different stories could in principle explain this sharp fall in employment.[2] One is a "Schumpeterian story" about reallocation of labor from old to new industries due to technological change (new industries blossom and old suffer). During such structural changes "above-normal" frictional unemployment due to "mismatch" arises.

The other story is a "Keynesian story" about an overall fall in aggregate demand triggered by a financial crisis. A believer of the Schumpeterian story would expect *total separations*, *hiring*, and *quits* to rise during the recession, as workers move from obsolete industries to blossoming industries. The figures 28.2 and 28.3 indicate the opposite: *total separations*, *hiring,* and *quits* behave procyclically not countercyclically.

A believer of the Keynesian story would expect *layoffs and discharges* to rise and *hiring* to fall during the recession, as firms generally need fewer workers to satisfy the slack demand. In addition, this story predicts that *quits* should fall,

[1] A simpler definition, popular in the press, is that a recession is present if in two consecutive quarters real GDP falls.

[2] Krugman, New York Times, Dec. 11, 2010.

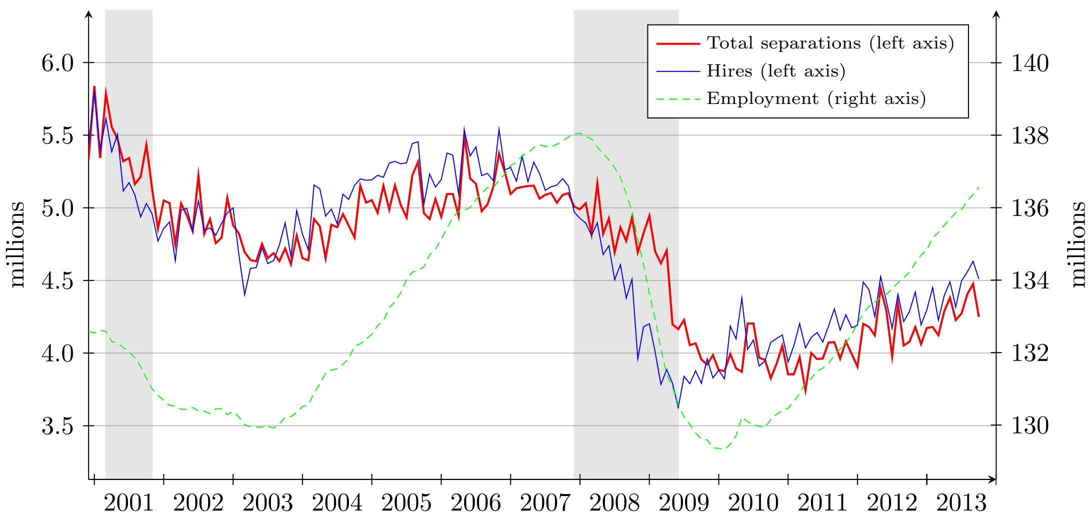

Figure 29.2: Total separations, hires, and employment (seasonally adjusted). USA December 2000 - October 2013. Recessions according to NBER in gray. Source: Bureau of Labor Statistics.

as there is a perception that vacant jobs are scarce. These three predictions are confirmed by the figures. The combination of a rise in *layoffs and discharges* and a fall in *quits* implies that the direction in which *total separations* move is ambiguous according to the Keynesian story. Fig. 28.2 indicates that *total separations* fell during both the dot.com-bubble recession in 2001 and the Great recession 2008-2009; so we can conclude that the fall in *quits* dominated. Moreover, for the whole decade Fig. 28.3 suggests a negative correlation between *quits* and *layoffs and discharges*.

In Fig. 28.4 we see a *Beveridge curve* for the U.S. based on observations over a decade. The variable drawn along the horizontal axis in Fig. 28.4 is the *unemployment rate* in different months since year 2000 (number of unemployed people as a percentage of the labor force). The variable drawn along the vertical axis in the figure is the "job openings rate" in the same months; an alternative name for this variable is the *vacancy rate* (number of vacant jobs as a percentage of the labor force). As expected, the Beveridge curve (so named after the British economist William Henry Beveridge, 1879-1963) is negatively sloped. In a boom, unemployment is low and vacancies plenty because recruitment is difficult, as few workers are searching for a job. In a slump unemployment is high and the vacancy rate low because recruitment is easy, as many workers are searching for a job. In this way, the economy's position on the downward sloping Beveridge curve can be interpreted as reflecting the state of the business cycle. Indeed, Fig. 28.4 shows that from the start of the recent recession in December 2007 until

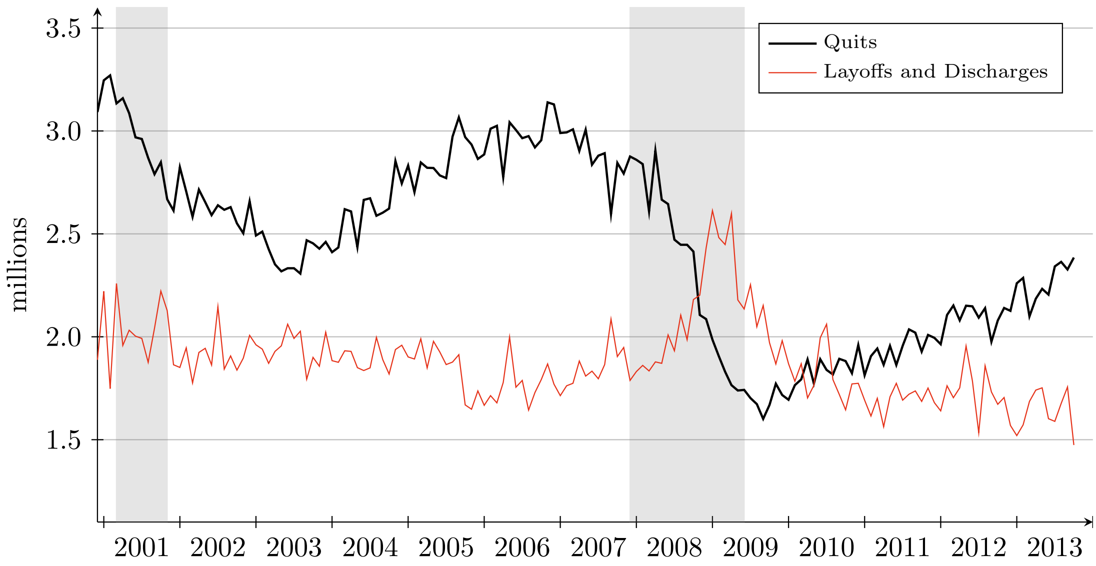

Figure 29.3: Quits and layoffs and discharges (seasonally adjusted). USA December 2000 - October 2013. Recessions according to NBER in gray. Source: Bureau of Labor Statistics.

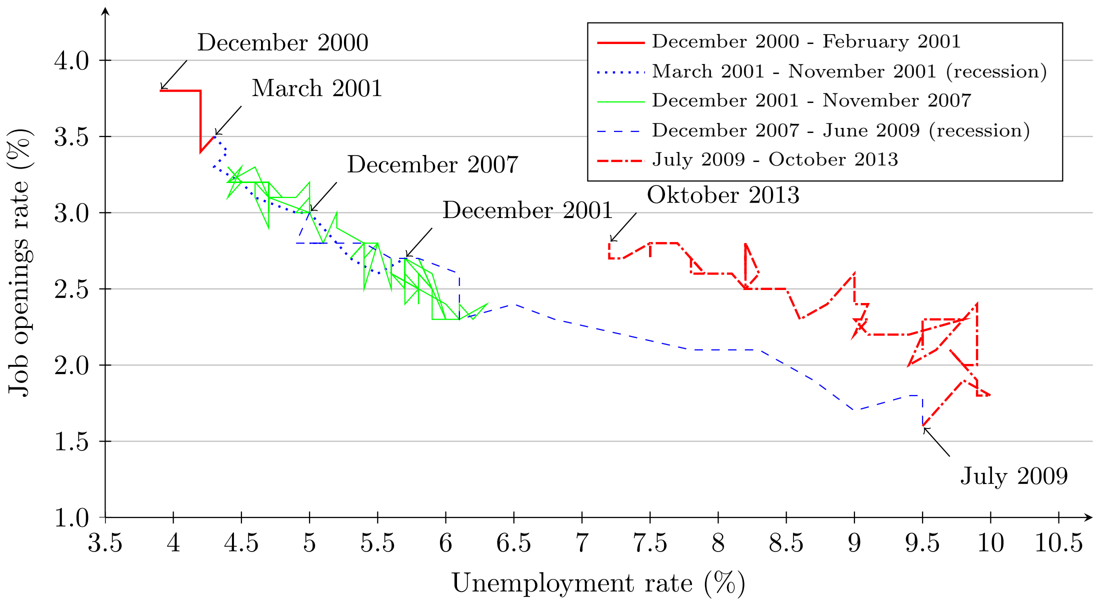

Figure 29.4: The Beveridge curve (seasonally adjusted). USA December 2000 - October 2013. By "job openings rate" is meant vacancy rate. Source: Bureau of Labor Statistics.

October 2009, the economy moved down the curve as the vacancy rate fell and "layoffs and discharges" rose.

An outward shift of the Beveridge curve is a sign of reduced matching efficiency in the labor market. Such a *mismatch* phenomenon can be due to fast technological and structural change. Firms in the new industries have vacant jobs but it is hard to find appropriate workers. Since October 2009, the economy has moved somewhat up and to the left. This is a sign of increased mismatch. On the other hand, as Barlevy (2011) concludes and the figure suggests, increased mismatch can account for only 2 of the 5 percentage point increase in the unemployment rate since December 2007. So in his Nobel laureate lecture, Dale Mortensen (2011) concluded: "The real problem is that demand for goods and services has not recovered because real interest rates have remained too high".

29.4 Conclusion

In the next chapters we consider different theoretical approaches to the explanation of business cycle regularities.

29.5 Literature notes

Articles in Handbook of Macroeconomics (1999) and for example the macroeconomics textbook by Abel and Bernanke (2001) describe in more detail the empirical regularities that characterize business cycle fluctuations, including both the direction and the timing of the cyclical behavior of economic variables.

29.6 Exercises

Chapter 30

The real business cycle theory

Since the middle of the 1970s two quite different approaches to the explanation of business cycle fluctuations have been pursued. We may broadly classify them as either of a new-classical or a Keynesian orientation. The new-classical school attempts to explain output and employment fluctuations as movements in productivity and labor supply. The Keynesian approach attempts to explain them as movements in aggregate demand and the degree of capacity utilization.

Within the new-classical school the monetary mis-perception theory of Lucas (1972, 1975) was the dominating approach in the 1970s. We described this approach in Chapter 27. The theory came under serious empirical attack in the late 1970s.[1] From the early 1980s an alternative approach within new-classical thinking, the Real Business Cycle theory, gradually took over. This theory (RBC theory for short) was initiated by Nobel laureates Finn E. Kydland and Edward C. Prescott (1982) and is the topic of this chapter.[2]

The shared conception of new-classical approaches to business cycle analysis is that economic fluctuations can be explained by adding stochastic disturbances to the neoclassical framework with optimizing agents, rational expectations, and market clearing under perfect competition. Output and employment are seen as supply determined, the only difference compared with the standard neoclassical growth model being that there are fluctuations around the growth trend. These fluctuations are not viewed as deviations from a Walrasian equilibrium, but as a constituent part of a moving stochastic Walrasian equilibrium. In Lucas' monetary mis-perception theory from the 1970s shocks to the money supply were the driving force. When the RBC theory took over, the emphasis shifted to recurrent technology shocks and other supply shocks as a driving force behind economic

[1] For a survey, see Blanchard (1990).

[2] In 2004 Kydland and Prescott were awarded the Nobel prize, primarily for their contributions in two areas: policy implications of time inconsistency and quantitative business cycle research.

fluctuations. In fact, money is typically absent from the RBC models. The empirical positive correlation between money supply and output is attributed to reverse causation. The fluctuations in employment reflect fluctuations in labor supply triggered by real wage movements reflecting shocks to marginal productivity of labor. Government intervention with the purpose of stabilization is seen as likely to be counterproductive. Given the uncertainty due to shocks, the market forces establish a Pareto-optimal moving equilibrium. "Economic fluctuations are optimal responses to uncertainty in the rate of technological change", as Edward Prescott puts it (Prescott 1986).

Below we present a prototype RBC model.

30.1 A simple RBC model

The RBC theory is an extension of the non-monetary Ramsey growth model, usually in discrete time. The key point is that endogenous labor supply and exogenous stochastic recurrent productivity shocks are added. The presentation here is close to King and Rebelo (1999), available in *Handbook of Macroeconomics*, vol. 1B, 1999. As a rule, our notation is the same as that of King and Rebelo, but there will be a few exceptions in order not to diverge too much from our general notational principles. The notation appears in Table 29.1. The most precarious differences vis-a-vis King and Rebelo are that we use ρ in our customary meaning as a utility discount rate and θ for elasticity of marginal utility of consumption.

The firm

There are two categories of economic agents in the model: firms and households; the government sector is ignored. First we describe the firm.

Technology

The representative firm has the production function

$$Y_t = A_t F(K_t, X_t N_t), \tag{30.1}$$

where K_t and N_t are input of capital and labor in period t, while X_t is an exogenous deterministic labor-augmenting technology level, and A_t represents an exogenous random productivity factor. The production function F has constant returns to scale and is neoclassical (i.e., marginal productivity of each factor is positive, but decreasing in the same factor).

Table 29.1. Notation

Variable	*King & Rebelo*	*Here*
Aggregate consumption	C_t	same
Deterministic technology level	X_t	same
Growth corrected consumption	$c_t \equiv C_t/X_t$	same
Growth corrected investment	$i_t \equiv I_t/X_t$	same
Growth corrected output	$y_t \equiv Y_t/X_t$	same
Growth corrected capital	$k_t \equiv K_t/X_t$	same
Aggregate employment (hours)	N_t	same
Aggregate leisure (hours)	$L_t \equiv 1 - N_t$	same
Effective capital intensity	$\frac{k_t}{N_t}$	$\tilde{k}_t \equiv \frac{K_t}{X_t N_t}$
Real wage	$w_t X_t$	w_t
Technology-corrected real wage	w_t	$\tilde{w}_t \equiv w_t/X_t$
Real interest rate from end period t to end period $t+1$	r_t	r_{t+1}
Auto-correlation coefficient in technology process	ρ	ξ
Discount factor w.r.t. utility	b	$\frac{1}{1+\rho}$
Rate of time preference w.r.t. utility	$\frac{1}{b} - 1$	ρ
Elasticity of marginal utility of cons.	σ	θ
Elasticity of marginal utility of leisure	η	same
Elasticity of output w.r.t. labor	α	same
Steady state value of c_t	c	c^*
The natural logarithm	log	same
Log deviation of c_t from steady state value	$\hat{c}_t \equiv \log \frac{c_t}{c}$	$\hat{c}_t \equiv \log \frac{c_t}{c^*}$
Log deviation of N_t from steady state value	$\hat{L}_t \equiv \log \frac{L_t}{L}$	$\hat{N}_t \equiv \log \frac{N_t}{N^*}$

It is assumed that X_t grows deterministically at a constant rate, $\gamma - 1$, i.e.,

$$X_{t+1} = \gamma X_t, \qquad \gamma > 1; \tag{30.2}$$

so γ is a deterministic technology growth factor. The productivity variable A_t is stochastic and assumed to follow the process

$$A_t = A^{*1-\xi}(A_{t-1})^{\xi} e^{\varepsilon_t}.$$

This means that $\log A_t$ is an AR(1) process:

$$\log A_t = (1-\xi)\log A^* + \xi \log A_{t-1} + \varepsilon_t, \qquad 0 \leq \xi < 1. \tag{30.3}$$

The last term, ε_t, represents a productivity shock which is assumed to be *white noise* with variance σ_ε^2. The auto-correlation coefficient ξ measures the degree of *persistence* over time of the effect on $\log A$ of a shock. If $\xi = 0$, the effect is only temporary; if $\xi > 0$, there is some persistence. The unconditional expectation of $\log A_t$ is equal to $\log A^*$ (which is thus the expected value "in the long run"). The shocks, ε_t, may represent accidental events affecting productivity, perhaps technological changes that are not sustainable, including technological mistakes (think of the introduction and later abandonment of asbestos in the construction industry). Negative realizations of the noise term ε_t may represent technological regress. But it need not, since moderate negative values of ε_t are consistent with overall technological progress, though temporarily below the trend represented by the deterministic growth of X_t.

The reason we said "not sustainable" is that sustainability would require $\xi = 1$, which conflicts with (30.3). Yet $\xi = 1$, which turns (30.3) into a random walk with drift, would correspond better to our general conception of technological change as a *cumulative* process. Technical knowledge is cumulative in the sense that a technical invention continues to be known. But in the present version of the RBC model this cumulative part of technological change is represented by the deterministic trend γ in (30.2). Anyway, what the stochastic supply shock A_t really embodies remains somewhat vague. A broad interpretation includes abrupt structural changes, cartelization of markets, closures of industries, shifts in legal and political systems, harvest failures, wartime destruction, natural disasters, and strikes. For an open economy, shifts in terms of trade might be a possible interpretation for example due to oil price shocks.

Factor demand

The representative firm is assumed to maximize its value under perfect competition. Since there are no convex capital installation costs, the problem reduces to that of static maximization of profits each period. And since period t's technological conditions (F, X_t, and the realization of A_t) are assumed known to the firm in period t, the firm does not face any uncertainty. Profit maximization simply implies a standard factor demand (K_t, N_t), satisfying

$$A_t F_1(K_t, X_t N_t) = r_t + \delta, \qquad 0 \leq \delta \leq 1, \tag{30.4}$$

$$A_t F_2(K_t, X_t N_t) X_t = w_t, \tag{30.5}$$

where $r_t + \delta$ is the real cost per unit of the capital service and w_t is the real wage.

The household

There is a given number of households, or rather dynastic families, all alike and with infinite horizon. For simplicity we ignore population growth. Thus we consider a representative household of constant size. The household's saving in period t amounts to buying investment goods that in the next period are rented out to the firms at the rental rate $r_{t+1}+\delta$. Thus the household obtains a net rate of return on financial wealth equal to the interest rate r_{t+1}.

A decision problem under uncertainty

The preferences of the household are described by the expected discounted utility hypothesis. Both consumption, C_t, and leisure, L_t, enter the period utility function. The total time endowment of the household is 1 in all periods:

$$N_t + L_t = 1, \qquad t = 0, 1, 2, \ldots, \tag{30.6}$$

where N_t is labor supply in period t. The fact that N has now been used in two different meanings, in (30.1) as employment and in (30.6) as labor supply, should not cause problems since in the competitive equilibrium of the model the two are quantitatively the same.

The household has rational expectations and solves the problem:

$$\begin{aligned}
\max E_0 U_0 &= E_0[\sum_{t=0}^{\infty} u(C_t, 1-N_t)(1+\rho)^{-t}] \qquad \text{s.t.} && (30.7)\\
C_t &\geq 0, 0 \leq N_t \leq 1, \qquad \text{(control region)} && (30.8)\\
K_{t+1} &= (1+r_t)K_t + w_t N_t - C_t, \qquad K_0 \geq 0 \text{ given}, && (30.9)\\
K_{t+1} &\geq 0 \qquad \text{for } t = 0, 1, 2, , \ldots . && (30.10)
\end{aligned}$$

The period utility function u satisfies $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$ and is concave, which is equivalent to adding the assumption $u_{11}u_{22} - (u_{12})^2 \geq 0$. The decreasing marginal utility assumption reflects, first, a desire of smoothing over time both consumption and leisure; or we could say that there is aversion towards variation over time in these entities. Second, decreasing marginal utility reflects aversion towards variation in consumption and leisure over different "states of nature", i.e., risk aversion. The parameter ρ is the rate of time preference and is assumed positive (a further restriction on ρ will be introduced later).

The symbol E_0 signifies the expected value, conditional on information available in period 0. More generally, E_t is a shorthand for $E(\cdot\,|I_t)$, where I_t denotes information revealed up to and including period t. The only source of uncertainty derives from the stochastic productivity variable A_t. We assume the ex ante uncertainty about A_t is resolved at time t, by which we mean the beginning of period t, the latter being identified with the time interval $[t, t+1)$. Knowledge of the market clearing values of r_t and w_t is included in the conditioning information I_t. There is uncertainty about *future* values of r and w, however. Nonetheless, the household is assumed to know the stochastic processes which these variables follow. Indeed, the household is assumed to know the "true" model of the economy as well as the stochastic process followed by the productivity variable A_t).

First-order conditions and transversality condition

For each t there are three endogenous variables in the household's problem, the control variables C_t and N_t and the state variable K_{t+1}. The decision, as seen from period 0, is to choose a concrete *action* (C_0, N_0) and a series of *contingency plans* $(C(t, K_t), N(t, K_t))$ saying what to do in each of the future periods $t = 1, 2, \ldots,$ as a function of the as yet unknown circumstances, including the financial wealth, K_t, at that time. The decision is made so that expected discounted utility is maximized. The pair of functions $(C(t, K_t), N(t, K_t))$ is named a *contingency plan* because it refers to what level of consumption and labor supply, respectively, will be chosen optimally in the future period t, contingent on the financial wealth at the beginning of period t. In turn this wealth, K_t, depends on the realized path, up to period $t-1$, of the ex ante unknown productivity factor A and the optimally chosen values of C and N. In order to choose the action (C_0, N_0) in a rational way, the household must take into account the whole future, including what the optimal contingent actions in the future will be.

Letting period t be an arbitrary period, i.e., $t \in \{0, 1, 2, ...\}$, we rewrite U_0 in the following way

$$\begin{aligned} U_0 &= \sum_{s=0}^{t-1} u(C_s, 1-N_s)(1+\rho)^{-s} + \sum_{s=t}^{\infty} u(C_s, 1-N_s)(1+\rho)^{-s} \\ &= \sum_{s=0}^{t-1} u(C_s, 1-N_s)(1+\rho)^{-s} + (1+\rho)^{-t} U_t, \end{aligned}$$

where $U_t = \sum_{s=t}^{\infty} u(C_s, 1-N_s)(1+\rho)^{-(s-t)}$. When deciding the "action" (C_0, N_0), the household knows that in every new period, it has to solve the remainder of the problem in a similar way, given the information revealed up to and including that period.

As seen from period t, the objective function can be written

$$\begin{aligned} E_t U_t &= u(C_t, 1-N_t) + (1+\rho)^{-1} E_t \left[u(C_{t+1}, 1-N_{t+1}) \right. \\ &\quad \left. + u(C_{t+2}, 1-N_{t+2})(1+\rho)^{-1} + ... \right]. \end{aligned} \tag{30.11}$$

since there is no uncertainty concerning the current period. To find first-order conditions we will use the *substitution method.* First, from (30.9) we have

$$C_t = (1+r_t)K_t + w_t N_t - K_{t+1}, \qquad \text{and} \tag{30.12}$$

$$C_{t+1} = (1+r_{t+1})K_{t+1} + w_{t+1}N_{t+1} - K_{t+2}. \tag{30.13}$$

Substituting this into (30.11), the decision problem is reduced to an essentially unconstrained maximization problem, namely one of maximizing the function $E_t U_t$ w.r.t. $(N_t, K_{t+1}), (N_{t+1}, K_{t+2}), \ldots$. We first take the partial derivative w.r.t. N_t in (30.11), given (30.12), and set it equal to 0 (thus focusing on interior solutions):

$$\frac{\partial E_t U_t}{\partial N_t} = u_1(C_t, 1 - N_t) w_t + u_2(C_t, 1 - N_t)(-1) = 0,$$

which can be written

$$u_2(C_t, 1 - N_t) = u_1(C_t, 1 - N_t) w_t. \tag{30.14}$$

This first-order condition describes the trade-off between leisure in period t and consumption in the same period. The condition says that in the optimal plan, the opportunity cost (in terms of foregone current utility) associated with decreasing leisure by one unit equals the utility benefit of obtaining an increased labor income and using this increase for extra consumption. In brief, marginal cost = marginal benefit, both measured in current utility.

Secondly, in (30.11) we take the partial derivative w.r.t. K_{t+1}, given (30.12) and (30.13).[3] This gives the first-order condition

$$\frac{\partial E_t U_t}{\partial K_{t+1}} = u_1(C_t, 1 - N_t)(-1) + (1 + \rho)^{-1} E_t[u_1(C_{t+1}, 1 - N_{t+1})(1 + r_{t+1})] = 0,$$

which can be written

$$u_1(C_t, 1 - N_t) = (1 + \rho)^{-1} E_t[u_1(C_{t+1}, 1 - N_{t+1})(1 + r_{t+1})], \tag{30.15}$$

where r_{t+1} is unknown in period t. This first-order condition describes the trade-off between consumption in period t and the uncertain consumption in period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss associated with decreasing consumption by one unit equals the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit. In brief, again marginal cost = marginal benefit in utility terms.

The condition (30.15) is an example of a *stochastic Euler equation*. If there is no uncertainty, the expectation operator E_t can be deleted. Then, apart from

[3] Generally speaking, for a given differentiable function $f(X, \alpha_1, \ldots, \alpha_n)$, where X is a stochastic variable and $\alpha_1, \ldots, \alpha_n$ are parameters, we have

$$\frac{\partial E(f(X, \alpha_1, \ldots, \alpha_n))}{\partial \alpha_i} = E \frac{\partial f(X, \alpha_1, \ldots, \alpha_n)}{\partial \alpha_i}, \quad i = 1, \ldots, n.$$

leisure entering as a second argument, (30.15) is the standard discrete-time analogue to the Keynes-Ramsey rule in continuous time.

For completeness, let us also derive the first-order conditions w.r.t. the future pairs (N_{t+i}, K_{t+i+1}), $i = 1, 2, \ldots$. We get

$$\frac{\partial E_t U_t}{\partial N_{t+i}} = (1+\rho)^{-1} E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) w_{t+i} + u_2(C_{t+i}, 1 - N_{t+i})(-1) \right] = 0,$$

so that

$$E_t \left[u_2(C_{t+i}, 1 - N_{t+i}) \right] = E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) w_{t+i} \right].$$

Similarly,

$$\begin{aligned} \frac{\partial E_t U_t}{\partial K_{t+i+1}} &= E_t \Big[u_1(C_{t+i}, 1 - N_{t+i})(-1) + (1+\rho)^{-1} u_1(C_{t+i+1}, 1 - N_{t+i+1}) \\ \cdot (1 + r_{t+i+1}) &= 0, \end{aligned}$$

so that

$$E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) \right] = (1+\rho)^{-1} E_t \left[u_1(C_{t+i+1}, 1 - N_{t+i+1})(1 + r_{t+i+1}) \right]$$

So, it suffices to say that for the current period, t, the first-order conditions are (30.14) and (30.15), and for the future periods similar first-order conditions hold in *expected* values.

As usual in dynamic optimization problems the first-order conditions say something about optimal *relative* levels of consumption and leisure over time, not about the absolute initial levels of consumption and leisure. The absolute initial levels are determined as the highest possible levels consistent with the requirement that first-order conditions of form (30.14) and (30.15), together with the non-negativity in (30.10), hold for period t and, in terms of expected values as seen from period t, for all future periods. This requirement can be shown to be equivalent to requiring the transversality condition,

$$\lim_{t \to \infty} E_0 \left[K_t u_1(C_{t-1}, 1 - N_{t-1})(1+\rho)^{-(t-1)} \right] = 0,$$

satisfied in addition to the first-order conditions.[4] Finding the resulting consumption function requires specification of the period utility function. But to characterize the equilibrium path, the consumption function is in fact not needed.

[4] In fact, in the budget constraint of the household's optimization problem, we could replace K_t by financial wealth and allow borrowing, so that financial wealth could be negative. Then, instead of the non-negativity constraint (30.10), a No-Ponzi-Game condition in expected value would be relevant. In a representative agent model with infinite horizon, however, this does not change anything, since the non-negativity constraint (30.10) will never be binding.

The remaining elements in the model

It only remains to check market clearing conditions and determine equilibrium factor prices. Implicitly we have already assumed clearing in the factor markets, since we have used the same symbols for capital and employment, respectively, in the firm's problem (the demand side) as in the household's problem (the supply side). The equilibrium factor prices are given by (30.4) and (30.5). We will rewrite these two equations in a more convenient way. In view of constant returns to scale, we have

$$Y_t = A_t F(K_t, X_t N_t) = A_t X_t N_t F(\tilde{k}_t, 1) \equiv A_t X_t N_t f(\tilde{k}_t), \tag{30.16}$$

where $\tilde{k}_t \equiv K_t/(X_t N_t)$ is the effective capital-labor ratio. In terms of the intensive production function f, (30.4) and (30.5) yield

$$\begin{aligned} r_t + \delta &= A_t F_1(K_t, X_t N_t) = A_t f'(\tilde{k}_t), && (30.17)\\ w_t &= A_t F_2(K_t, X_t N_t) X_t = A_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] X_t. && (30.18) \end{aligned}$$

Finally, equilibrium in the output market requires that aggregate output equals aggregate demand, i.e., the sum of aggregate consumption and investment:

$$Y_t = C_t + I_t. \tag{30.19}$$

We now show that this equilibrium condition is automatically implied by previous equations. Indeed, adding δK_t on both sides of the budget constraint (30.9) of the representative household and rearranging, we get

$$\begin{aligned} K_{t+1} - K_t + \delta K_t &= (r_t + \delta) K_t + w_t N_t - C_t && (30.20)\\ &= A_t f'(\tilde{k}_t) K_t + A_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] X_t N_t - C_t \\ &= A_t X_t N_t f(\tilde{k}_t) - C_t = Y_t - C_t \equiv S_t, \end{aligned}$$

where the second equality comes from (30.17) and (30.18) and the fourth from (30.16). Now, in this model aggregate gross saving, S_t, is directly an act of investment so that $I_t = S_t$. From this follows (30.19).

Specification of technology and preferences

To quantify the model we have to specify the production function and the utility function. We abide by the standard assumption in the RBC literature and specify the production function to be Cobb-Douglas:

$$Y_t = A_t K_t^{1-\alpha} (X_t N_t)^{\alpha}, \qquad 0 < \alpha < 1. \tag{30.21}$$

We then get

$$\begin{aligned} f(\tilde{k}_t) &= A_t \tilde{k}_t^{1-\alpha}, &(30.22)\\ r_t + \delta &= (1-\alpha) A_t \tilde{k}_t^{-\alpha}, &(30.23)\\ w_t &= \alpha A_t \tilde{k}_t^{1-\alpha} X_t. &(30.24) \end{aligned}$$

As to the utility function we follow King and Rebelo (1999) and base the analysis on the additively separable CRRA case,

$$u(C_t, 1-N_t) = \frac{C_t^{1-\theta}}{1-\theta} + \omega \frac{(1-N_t)^{1-\eta}}{1-\eta}, \qquad \theta > 0, \eta > 0, \omega > 0. \tag{30.25}$$

Here, θ is the (absolute) elasticity of marginal utility of consumption, equivalently the desire for consumption smoothing, η is the (absolute) elasticity of marginal utility of leisure, equivalently, the desire for leisure smoothing, and ω is the relative weight given to leisure. In case θ or η take on the value 1, the corresponding term in (30.25) should be replaced by $\log C_t$ or $\omega \log(1 - N_t)$, respectively. In fact, most of the time King and Rebelo (1999) take both θ and η to be 1.

With (30.25) applied to (30.14) and (30.15), we get

$$\begin{aligned} \theta(1-N_t)^{-\eta} &= C_t^{-\theta} w_t, \quad \text{and} &(30.26)\\ C_t^{-\theta} &= \frac{1}{1+\rho} E_t \left[C_{t+1}^{-\theta} (1 + r_{t+1}) \right], &(30.27) \end{aligned}$$

respectively.

30.2 A deterministic steady state*

For a while, let us ignore shocks. That is, assume $A_t = A^*$ for all t.

The steady state solution

By a steady state we mean a path along which the growth-corrected variables like $\tilde{k}$ and $\tilde{w} \equiv w/X_t$ stay constant. With $A_t = A^*$ for all t, (30.23) and (30.24) return the steady-state relations between $\tilde{k}$, r, and $\tilde{w}$:

$$\begin{aligned} \tilde{k}^* &= \left[\frac{(1-\alpha)A^*}{r^* + \delta} \right]^{1/\alpha}, &(30.28)\\ \tilde{w}^* &= \alpha A^* \tilde{k}^{*1-\alpha}. &(30.29) \end{aligned}$$

We may write (30.27) as

$$1 + \rho = E_t \left[(\frac{C_{t+1}}{C_t})^{-\theta} (1 + r_{t+1}) \right]. \tag{30.30}$$

In the non-stochastic steady state the expectation operator E_t can be deleted, and r and C/X are independent of t. Hence, $C_{t+1}/C_t = \gamma$, by (30.2), and (30.30) takes the form

$$1 + r^* = (1 + \rho)\gamma^{\theta}. \tag{30.31}$$

In this expression we recognize the *modified golden rule* discussed in chapters 7 and 10.[5] Existence of general equilibrium in our Ramsey framework requires that the long-run real interest rate is larger than the long-run output growth rate, i.e., we need $r^* > \gamma - 1$. This condition is satisfied if and only if

$$1 + \rho > \gamma^{1-\theta}, \tag{30.32}$$

which we assume.[6] If we guess that $\theta = 1$ and $\rho = 0.01$, then with $\gamma = 1.004$ (taken from US national income accounting data 1947-96, using a quarter of a year as our time unit), we find the steady-state rate of return to be $r^* = 0.014$ or 0.056 per annum. Or, the other way round, observing the average return on the Standard & Poor 500 Index over the same period to be 6.5 per annum, given $\theta = 1$ and $\gamma = 1.004$, we estimate ρ to be 0.012.

Using that in steady state N_t is a constant, N^*, we can write (30.20) as

$$\gamma \tilde{k}_{t+1} - (1 - \delta)\tilde{k}_t = A^* \tilde{k}_t^{1-\alpha} - \tilde{c}_t, \tag{30.33}$$

where $\tilde{c}_t \equiv C_t/(X_t N^*)$. Given r^*, (30.28) yields the steady-state capital intensity $\tilde{k}^*$. Then, (30.33) returns

$$\tilde{c}^* \equiv \frac{c^*}{X_t} = A^* \tilde{k}^{*1-\alpha} - (\gamma + \delta - 1)\tilde{k}^*.$$

Consumption dynamics around the steady state in case of no uncertainty

The adjustment process for consumption, absent uncertainty, is given by (30.30) as

$$(\frac{C_{t+1}}{C_t})^{-\theta}(1 + r_{t+1}) = 1 + \rho,$$

or, taking logs,

$$\log \frac{C_{t+1}}{C_t} = \frac{1}{\theta}\left[\log(1 + r_{t+1}) - \log(1 + \rho)\right]. \tag{30.34}$$

[5] King and Rebelo, 1999, p. 947, express this in terms of the growth-adjusted discount factor $\beta \equiv (1 + \rho)^{-1}\gamma^{1-\theta}$, so that $1 + r^* = (1 + \rho)\gamma^{\theta} = \gamma/\beta$.

[6] Since $\gamma > 1$, only if $\theta < 1$ (which does not seem realistic, cf. Chapter 3), is $\rho > 0$ not sufficient for (30.32) to hold.

This is the deterministic Keynes-Ramsey rule in discrete time under separable CRRA utility. For any "small" x we have $\log(1 + x) \approx x$ (from a first-order Taylor approximation of $\log(1 + x)$ around 0). Hence, with $x = C_{t+1}/C_t - 1$, we have $\log(C_{t+1}/C_t) \approx C_{t+1}/C_t - 1$, so that (30.34) implies the approximate relation

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - \rho). \tag{30.35}$$

There is a supplementary way of writing the Keynes-Ramsey rule. Note that (30.31) implies $\log(1 + r^*) = \log(1 + \rho) + \theta \log \gamma$. Using first-order Taylor approximations, this gives $r^* \approx \rho + \theta \log \gamma \approx \rho + \theta g$, where $g \equiv \gamma - 1$ is the trend rate of technological progress. Thus $\rho \approx r^* - \theta g$, and inserting this into (30.35) we get

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - r^*) + g.$$

Then the technology-corrected consumption level, $c_t \equiv C_t/X_t$, moves according to

$$\frac{c_{t+1} - c_t}{c_t} \approx \frac{1}{\theta}(r_{t+1} - r^*),$$

since g is the growth rate of X_t.

30.3 On the approximate solution and numerical simulation*

In the special case $\theta = 1$ (the log utility case), still maintaining the Cobb-Douglas specification of the production function, the model can be solved analytically provided capital is non-durable (i.e., $\delta = 1$). It turns out that in this case the solution has consumption as a constant fraction of output. Further, in this special case labor supply equals a constant and is thus independent of the productivity shocks. Since in actual business cycles, employment fluctuates a lot, this might not seem to be good news for a business cycle model.

But assuming $\delta = 1$ for a period length of one quarter or one year is unreasonable anyway. Given a period length of one year, δ is generally estimated to be less than 0.1. With $\delta < 1$, labor supply *is* affected by the technology shocks, and an exact analytical solution can no longer be found.

One can find an *approximate* solution based on a log-linearization of the model around the steady state. Without dwelling on the more technical details we will make a few observations.

30.3.1 Log-linearization

If x^* is the steady-state value of the variable x_t in the non-stochastic case, then one defines the new variable, the log-deviation of x from x^* :

$$\hat{x}_t \equiv \log(\frac{x_t}{x^*}) = \log x_t - \log x^*. \tag{30.36}$$

That is, $\hat{x}_t$ is the logarithmic deviation of x_t from its steady-state value. But this is approximately the same as x's *proportionate* deviation from its steady-state value. This is because, when x_t is in a neighborhood of its steady-state value, a first-order Taylor approximation of $\log x_t$ around x^* yields

$$\log x_t \approx \log x^* + \frac{1}{x^*}(x_t - x^*),$$

so that

$$\hat{x}_t \approx \frac{x_t - x^*}{x^*}. \tag{30.37}$$

Working with the transformation $\hat{x}_t$ instead of x_t implies the convenience that

$$\begin{aligned}\hat{x}_{t+1} - \hat{x}_t &= \log(\frac{x_{t+1}}{x^*}) - \log(\frac{x_t}{x^*}) = \log x_{t+1} - \log x_t \\ &\approx \frac{x_{t+1} - x_t}{x_t}.\end{aligned}$$

That is, relative changes in x have been replaced by absolute changes in $\hat{x}$.

Some of the equations of interest are exactly log-linear from start. This is true for the equations (30.22), (30.23), and (30.24), as well as for the first-order condition (30.26) for the household. For other equations log-linearization requires approximation. Consider for instance the time constraint $N_t + L_t = 1$. This constraint implies

$$N^* \frac{N_t - N^*}{N^*} + L^* \frac{L_t - L^*}{L^*} = 0$$

or

$$N^* \hat{N}_t + L^* \hat{L}_t \approx 0, \tag{30.38}$$

by the principle in (30.37). From (30.26), taking into account that $1 - N_t = L_t$, we have

$$\begin{aligned}\theta L_t{}^{-\eta} &= C_t^{-\theta} w_t \equiv (c_t X_t)^{-\theta} \tilde{w}_t X_t \\ &= c_t{}^{-\theta} \tilde{w}_t X_t^{1-\theta}.\end{aligned} \tag{30.39}$$

In steady state this takes the form

$$\theta L^{*-\eta} = c^{*-\theta} \tilde{w}^* X_t^{1-\theta}. \tag{30.40}$$

We see that when there is sustained technological progress, $\gamma > 1$, we *need* $\theta = 1$ for a steady state to exist (which explains why in their calibration King and Rebelo assume $\theta = 1$). This quite "narrow" theoretical requirement is an unwelcome feature and is due to the additively separable utility function.

Combining (30.40) with (30.39) gives

$$\left(\frac{L_t}{L^*}\right)^{-\eta} = \left(\frac{c_t}{c^*}\right)^{-\theta}\frac{\tilde{w}_t}{\tilde{w}^*}.$$

Taking logs on both sides we get

$$-\eta \log \frac{L_t}{L^*} = \log \frac{\tilde{w}_t}{\tilde{w}^*} - \theta \log \frac{c_t}{c^*}$$

or

$$-\eta \hat{L}_t = \hat{w}_t - \theta \hat{c}_t.$$

In view of (30.38), this implies

$$\hat{N}_t = -\frac{L^*}{N^*}\hat{L}_t = \frac{1-N^*}{N^*\eta}\hat{w}_t - \frac{1-N^*}{N^*\eta}\theta\hat{c}_t. \tag{30.41}$$

This result tells us that the elasticity of labor supply w.r.t. a temporary change in the real wage depends negatively on η; this is not surprising, since η reflects the desire for leisure smoothing across time. Indeed, calling this elasticity ε, we have

$$\varepsilon = \frac{1-N^*}{N^*\eta}. \tag{30.42}$$

Departing from the steady state, a one per cent increase in the wage ($\hat{w}_t = 0.01$) leads to an ε per cent increase in the labor supply, by (30.41) and (30.42). The number ε measures a kind of compensated wage elasticity of labor supply (in an intertemporal setting), relevant for evaluating the pure substitution effect of a temporary rise in the wage. King and Rebelo (1999) reckon N^* in the US to be 0.2, that is, out of available time one fifth is working time. With $\eta = 1$, we then get $\varepsilon = 4$. This elasticity is much higher than what the micro-econometric evidence suggests, at least for men, namely typically an elasticity below 1 (Pencavel 1986). But with labor supply elasticity as low as 1, the RBC model is far from capable of generating a volatility in employment comparable to what the data show.

For some purposes it is convenient to have the endogenous time-dependent variables appearing separately in the stationary dynamic system. Then, to describe the supply of output in log-linear form, let $y_t \equiv Y_t/X_t \equiv A_t f(\tilde{k}_t)N_t$ and $k_t \equiv K_t/X_t \equiv \tilde{k}_t N_t$. From (30.21),

$$y_t = A_t k_t^{1-\alpha} N_t^{\alpha},$$

and dividing through by the corresponding expression in steady state, we get

$$\frac{y_t}{y^*} = \frac{A_t}{A^*}(\frac{k_t}{k^*})^{1-\alpha}(\frac{N_t}{N^*})^{\alpha}.$$

Taking logs on both sides, we end up with

$$\hat{y}_t = \hat{A}_t + (1-\alpha)\hat{k}_t + \alpha\hat{L}_t. \tag{30.43}$$

For the demand side we can obtain at least an approximate log-linear relation. Indeed, dividing trough by X_t in (30.19) we get

$$c_t + i_t = y_t,$$

where $i_t \equiv I_t/X_t$. Dividing through by y^* and reordering, this can also be written

$$\frac{c^*}{y^*}\frac{c_t - c^*}{c^*} + \frac{i^*}{y^*}\frac{i_t - i^*}{i^*} = \frac{y_t - y^*}{y^*},$$

which, using the hat notation from (30.37), can be written

$$\frac{c^*}{y^*}\hat{c}_t + \frac{i^*}{y^*}\hat{\imath}_t \approx \hat{y}_t. \tag{30.44}$$

to be equated with the right hand side of (30.43).

30.3.2 Numerical simulation

After log-linearization, the model can be reduced to two coupled *linear* stochastic first-order difference equations in k_t and c_t, where k_t is predetermined, and c_t is a jump variable. There are different methods available for solving such an approximate dynamic system analytically.[7] Alternatively, based on a specified set of parameter values one can solve the system by numerical simulation on a computer.

In any case, when it comes to checking the quantitative performance of the model, RBC theorists generally stick to *calibration,* that is, the method based on a choice of parameter values such that the model matches a list of data characteristics. In the present context this means that:

(a) the structural parameters $(\alpha, \delta, \rho, \theta, \eta, \omega, \gamma, N^*)$ are given values that are taken or constructed partly from national income accounting and similar data, partly from micro-econometric studies of households' and firms' behavior;

[7] For details one may consult Campbell (1994, p. 468 ff.), Obstfeld and Rogoff (1996, p. 503 ff.), or Uhlig (1999).

(b) the values of the parameters, ξ and σ_ε, in the stochastic process for the productivity variable A are chosen either on the basis of data for the Solow residual[8] over a long time period, or one or both values are chosen to yield, as closely as possible, a correspondence between the statistical moments (standard deviation, auto-correlation etc.) predicted by the model and those in the data.

The first approach to ξ and σ_ε is followed by, e.g., Prescott (1986). It has been severely criticized by, among others, Mankiw (1989). In the short and medium term, the Solow residual is very sensitive to labor hoarding and variations in the degree of utilization of capital equipment. It can therefore be argued that it is the business cycle fluctuations that explain the fluctuations in the Solow residual, rather than the other way round.[9] The second approach, used by, e.g., Hansen (1985) and Plosser (1989), has the disadvantage that it provides no independent information on the stochastic process for productivity shocks. Yet such information is necessary to assess whether the shocks can be the driving force behind business cycles.

As hitherto we abide to the approach of King and Rebelo (1999) which like Prescott's is based on the Solow residual. The parameters chosen are shown in Table 19.2. Remember that the time unit is a quarter of a year.

Table 29.2. Parameter values

α	δ	ρ	θ	η	ω	γ	N^*	ξ	σ_ε
0.667	0.025	0.0163	1	1	3.48	1.004	0.2	0.979	0.0072

Given these parameter values and initial values of k and A in conformity with the steady state, the simulation is ready to be started. The shock process is activated and the resulting evolution of the endogenous variables generated

[8] Given the Cobb-Douglas production function (30.21), take logs on both sides and rearrange to get

$$\log Y_t - (1-\alpha)\log K_t - \alpha \log L_t = \log A_t + \alpha \log X_t.$$

Based on time series for Y, K , and L, and estimating α by data on the labor income share, the left-hand side can be computed and used to uncover the productivity process $\log A_t + \alpha \log X_t$. In growth accounting the left-hand side makes up the "raw material" for calculating the Solow residual,

$$SR_t \equiv \Delta \log Y_t - ((1-\alpha)\Delta \log K_t + \alpha \Delta \log L_t).$$

Data on the degree of utilization is fragmentary. Hence, correction for variation over time in utilization is difficult.

[9] King and Rebelo (1999, p. 982-993) believe that the problem can be overcome by refinement of the RBC model.

through the propagation mechanism of the model calculated by the computer. From this evolution the analyst next calculates the different relevant statistics: standard deviation (as a measure of volatility), auto-correlation (as a measure of persistence), and cross correlations with different leads and lags (reflecting the co-movements and dynamic interaction of the different variables). These model-generated statistics can then be compared to those calculated on the basis of the empirical observations.

In order to visualize the economic mechanisms involved, *impulse-response functions* are calculated. Shocks before period 0 are ignored and the economy is assumed to be in steady state until this period. Then, a positive once-for-all shock to A occurs so that productivity is increased by, say, 1 % (i.e., given $A_{-1} = A^* = 1$, we put $\varepsilon_0 = 0.01$ in (30.3) with $t = 0$). The resulting path for the endogenous variables is calculated under the assumption that no further shocks occur (i.e., $\varepsilon_t = 0$ for $t = 1, 2, \ldots$). An inpulse-response diagram shows the implied time profiles for the different variables.

Remark. The text should here show some graphs of impulse-response functions. These graphs are not yet available. Instead the reader is referred to the graphs in King and Rebelo (1999), p. 966-970. As expected, the time profiles for output, consumption, employment, real wages, and other variables differ, depending on the size of ξ in (30.3). If we consider a purely temporary productivity shock, the case $\xi = 0$, we get the graphs in King and Rebelo (1999), p. 966. A highly persistent productivity shock, the case $\xi = 0.979$, gives rise to the responses on p. 968. We see that these responses are more drawn out over time. This persistence in the endogenous variables is, however, just inherited from the assumed persistence in the shock. And amplification is limited. In case of a permanent productivity shock, $\xi = 1$, wealth effects on labor supply are strong and tend to offset the substitution effect. □

30.4 The two basic propagation mechanisms

We have added technology shocks to a standard neoclassical growth model. The conclusion is that correlated fluctuations in output, consumption, investment, work hours, output per man-hour, real wages, and the real interest rate are generated. So far so good. Two basic propagation mechanisms drive the fluctuations:

1. *The capital accumulation mechanism.* To understand this mechanism in its pure form, let us abstract from the endogenous labor supply and assume an inelastic labor supply. A positive productivity shock increases marginal productivity of capital and labor. If the shock is not purely temporary, the household feels more wealthy. Both output, consumption and saving go

up, the latter due to the desire for consumption smoothing. The increased capital stock implies higher output also in the next periods. Hence output shows positive persistence. And output, consumption, and investment move together, i.e., there is co-movement.

2. *Intertemporal substitution in labor supply.* An immediate implication of increased marginal productivity of labor is a higher real wage. To the extent that this increased real wage is only temporary, the household is motivated to supply more labor in the current period and less later. This is the phenomenon of intertemporal substitution in leisure. By the adherents of the RBC theory the observed fluctuations in work hours are seen as reflecting this.

30.5 Limitations

During the last couple of decades there has been an increasing scepticism towards the RBC theory. The central limitation of the theory comes from its insistence upon interpreting fluctuations in employment as reflecting fluctuations in labor supply. The critics maintain that, starting from market clearing based on flexible prices, it is not surprising that it becomes difficult to match the business cycle facts arise.

We may summarize the objections to the theory in the following four points:

a. *Where are the productivity shocks?* As some critics ask: "If productivity shocks are so important, why don't we read about them in the Wall Street Journal or in The Economist?" Indeed, technology shocks occur within particular lines of a multitude of businesses and sum up, at the aggregate level, to an upward *trend* in productivity, relevant for growth theory. It is not easy to see they should be able to drive the business cycle component of the data. Moreover, it seems hard to interpret the absolute economic contractions (decreases in GDP) that sometimes occur in the real world as due to productivity shocks. If the elasticity of output w.r.t. productivity shocks does not exceed one (as it does not seem to, empirically, according to Campbell (1994)), then a *backward* step in technology at the aggregate level is needed. Although genuine technological knowledge as such is inherently increasing, mistakes could be made in choosing technologies. At the disaggregate level, one can sometimes identify technological mistakes, cf. the use of DDT and its subsequent ban in the 1960's due to its damaging effects on health. But it is hard to think of technological drawbacks at the *aggregate* level, capable of explaining the observed economic recessions.

Think of the large and long-lasting contraction of GDP in the US during the Great Depression (27 % reduction between 1929 and 1933 according to Romer (2001), p. 171). Sometimes the adherents of the RBC theory have referred also to other kinds of supply shocks: changes in taxation, changes in environmental legislation etc. (Hansen and Prescott, 1993). But significant changes in taxation and regulation occur rather infrequently.

b. *Lack of internal propagation.* Given the available micro-econometric evidence, the two mechanisms above seem far from capable at generating the *large* fluctuations in output and employment that we observe. Both mechanisms imply little amplification of the shocks. This means that to replicate the stylized business cycle facts, standard RBC models must rely heavily on exogenous shocks dynamics. Indeed, the intertemporal substitution in labor supply as described above is not able to generate much amplification. This is related to the fact that changes in real wages tend to be permanent rather than purely transitory. Permanent wage increases tend to have little or no effect on labor supply (the wealth effect tends to offset the substitution and income effects). Given the very minor temporary movements in the real wage that occur at the empirical level, a high intertemporal elasticity of substitution in labor supply is required to generate large fluctuations in employment as observed in the data. But the empirical evidence suggests that this requirement is not met. Micro-econometric studies of labor supply indicate that this elasticity, at least for men, is quite small (in the range 0 to 1.5, typically below 1).[10] Yet, Prescott (1986) and Plosser (1989) assume it is around 4.

c. *Correlation puzzles.* Sometimes the sign, sometimes the size of correlation coefficients seem persevering wrong (see King and Rebelo, p. 957, 961). As Akerlof (2003, p. 414) points out, there is a conflict between the empirically observed pro-cyclical behavior of workers' quits[11] and the theory's prediction that quits should increase in cyclical *downturns* (since variation in employment is voluntary according to the theory). Considering a dozen of OECD countries, Danthine and Donaldson (1993) find that the required positive correlation between labor productivity and output is visible only in data for the U.S. (and not strong), whereas the correlation is markedly negative for the majority of the other countries.

d. *Disregard of non-neutrality of money.* According to many critics, the RBC

[10] *Handbook of Labor Economics*, vol. 1, 1986, Table 1.22, last column. See also Hall (1999, p. 1148 ff.).

[11] See Chapter 29.

theory conflicts with the empirical evidence of the real effects of monetary policy.

Numerous, and more and more imaginative, attempts at overcoming the criticisms have been made. King and Rebelo (1999, p. 974-993) present some of these. In particular, adherents of the RBC approach have looked for mechanisms that may raise the size of labor supply elasticities at the aggregate level over and above that at the individual level found in micro-econometric studies.

30.6 Technological change as a random walk with drift

Above we have considered technical change as a mean-reverting process with a deterministic trend. This is the approach followed by Prescott (1986) and King and Rebelo (1999). In contrast, Plosser (1989) assumes that technological change is a random walk with drift. The representative firm has the production function

$$Y_t = Z_t F(K_t, N_t),$$

where Z_t is a measure of the level of technology, and the production function F has constant returns to scale. In the numerical simulation Plosser used a Cobb-Douglas specification.

The total factor productivity, Z_t, is an exogenous stochastic variable. In contrast to the process for the logarithm of A_t in the Prescott version above, where we had $\xi < 1$, we now assume that $\xi = 1$ so that the process assumed for $z_t \equiv \log Z_t$ is

$$z_t = \beta + z_{t-1} + \varepsilon_t, \tag{30.45}$$

which is a *random walk*. This corresponds to our general conception of technical knowledge as *cumulative*. If the deterministic term $\beta \neq 0$, the process is called a random walk *with drift*. In the present setting we can interpret β as some underlying deterministic component in the productivity trend, suggesting $\beta > 0$.[12] A stochastic trend component, which can go both ways, is generated by the noise term ε_t. Negative occurrences of this term need not represent *technological regress*, but just a technology development below trend (which will occur when $-\beta \leq \varepsilon_t < 0$). In an open economy, adverse shocks to terms of trade is a candidate interpretation.

[12] The growth rate in total factor productivity is $(Z_t - Z_{t-1})/Z_{t-1}$. From (30.45) we have $E_{t-1}(z_t - z_{t-1}) = \beta$, and $z_t - z_{t-1} = \log Z_t - \log Z_{t-1} \approx (Z_t - Z_{t-1})/Z_{t-1}$ by a 1. order Taylor approximation of $\log Z_t$ about Z_{t-1}. Hence, $E_{t-1}(Z_t - Z_{t-1})/Z_{t-1} \approx \beta$. In Plosser's model all technological change is represented by changes in Z_t, i.e., in (30.2) Plosser has $\gamma \equiv 1$.

Embedded in a Walrasian equilibrium framework the specification (30.45) tends to generate too little fluctuation in employment and output. This is because, when shocks are permanent, large wealth effects offset the intertemporal substitution in labor supply. On top of this comes limitations similar to points a, c, and d in the previous section.

30.7 Concluding remarks

It is advisory to make a distinction between on the one hand *RBC theory* (based on fully flexible prices and market clearing in an environment where productivity shocks are the driving force behind the fluctuations) and on the other hand the broader quantitative modeling framework known as *DSGE models.* A significant amount of research on business cycle fluctuations has left the RBC *theory* and the predominant emphasis on productivity shocks but applies similar *quantitative methods.* This approach is nowadays known as of an attempt at building *Dynamic Stochastic General Equilibrium (DSGE)* modeling. The economic contents of such a model *can* be new-classical (as in the tradition of Kydland and Prescott), emphasizing technology shocks and similar supply side effects. Alternatively it can be new-Keynesian of some variety, based on a combination of imperfect competition with nominal and real price rigidities and with emphasis on monetary policy and demand shocks (see, e.g., Jeanne, 1998, Smets and Wouters, 2003 and 2007, and Danthine and Kurmann, 2004, Gali, 2008). There are many varieties of these new-Keynesian models, some small and analytically oriented, some large and simulation- and forecasting-oriented. We consider an example of the "small" type in Chapter 32.

The aim of medium-run theory is to throw light on business cycle fluctuations and to clarify what kinds of countra-cyclical economic policy, if any, may be functional. This seems to be the area within macroeconomics where there is most disagreement – and has been so for a long time. Some illustrating quotations (TO BE UPDATED):

> Indeed, if the economy did not display the business cycle phenomena, there would be a puzzle. ... costly efforts at stabilization are likely to be counterproductive. Economic fluctuations are optimal responses to uncertainty in the rate of technological change (Prescott 1986).
>
> My view is that real business cycle models of the type urged on us by Prescott have nothing to do with the business cycle phenomena observed in the United States or other capitalist economies. ... The image of a big loose tent flapping in the wind comes to mind (Summers 1986).

30.8 Literature notes

The RBC theory was initiated by Finn E. Kydland and Edward C. Prescott (1982), where a complicated time-to-build aspect was part of the model. A simpler version of the RBC theory was given in Prescott (1986) where also the "economic philosophy" behind the theory was proclaimed. The King and Rebelo (1999) exposition followed here builds on Prescott's 1986 version which has become the prototype RBC model. Plosser's version (Plosser 1989), briefly sketched in Section 30.6, makes up an alternative regarding the modeling of the technology shocks.

In dealing with the intertemporal decision problem of the household we applied the substitution method. More advanced approaches include the discrete time Maximum Principle (see Chapter 8), the Lagrange method (see, e.g., King and Rebelo, 1999), or Dynamic Programming (see, e.g., Ljungqvist and Sargent, 2004).

The empirical approach, *calibration,* is different from econometric estimation and testing in the formal sense. Criteria for what constitutes a good fit are not clear. The calibration method can be seen as a first check whether the model is logically capable of matching main features of the data (say the first and second moments of key variables). Calibration delivers a quantitative example of the working of the model. It does not deliver an econometric test of the validity of the model or of hypotheses based on the model. Whether it provides a useful guide as to what aspects of the model should be revised is debated, see Hoover, 1995, pp. 24-44, Quah, 1995.

30.9 Exercises

Chapter 31

Keynesian perspectives on business cycles

Applying a Vector-Autoregression time series approach with two kinds of shocks interpreted as demand and supply shocks, respectively, Blanchard and Quah (1989) found on the basis of quarterly US data 1950-87 that demand shocks explain more that two thirds of the fluctuations in output and even more of the fluctuations in unemployment. Working with a somewhat larger system and quarterly US data for 1965-1986, Blanchard (1989) summarized the results this way:

(a) Demand shocks explain most of the short-run fluctuations in output.

(b) Positive demand shocks are associated with gradual increases in nominal prices and wages.

(c) Supply shocks dominate the medium and the long run, and positive supply shocks are associated with decreases in nominal prices and wages (relative to trend).

Demand shocks may arise from shifts in the state of confidence, shifts in exports, shifts in government spending, shifts in liquidity preference, a sudden tightening of credit, and similar. Points (a), (b), and (c) lead to a Keynesian interpretation of macroeconomic fluctuations. A prevalent interpretation of point (a) is that nominal and relative price rigidities are present. Then, point (b) supports the view that even though prices in the major sectors of the economy respond only sluggishly, they *do* respond to cost push from changes in the level of economic activity. Finally, point (c) says that durable influences on output come from supply factors, such as the labor force, capital, and technological change.

The Keynesian understanding of economic fluctuations is that they emanate from "large" specific events, often connected to the financial sector. Some of these events trigger *virtuous circles* in the economic system as a whole while others trigger *vicious circles*. In continuation of the emphasis on nominal price stickiness, a crucial element in this understanding is the *refutation of Say's law*. This is the "law" claiming that "supply creates its own demand", cf. Chapter 19. At the microeconomic level, refutation of this doctrine leads to replacement of the Walrasian budget constraint with an *effective budget constraint*, when trade occurs outside Walrasian equilibrium.

31.1 A minimalist Keynesian medium-run model in discrete time

Notation:

$$\begin{aligned}
y &\equiv \ln Y, \\
m &\equiv \ln M, \\
p &\equiv \ln P, \\
\pi_t &\equiv p_t - p_{t-1}, \\
i_t &= \text{ policy rate,} \\
\mu &= \text{ autonomous demand (reflecting perhaps) the state of confidence,} \\
\omega(\mu) &= \text{ interest spread (interest differential),}
\end{aligned}$$

and

x_t and z_t are exogenous stochastic variables.

Output market equilibrium in reduced form:

$$y_t = \alpha y^e_{t+1} - \beta(i_t + \omega(\mu) - \pi^e_{t+1}) + \mu + x_t, \quad \omega(\mu) \geq 0, \ \omega'(\mu) < 0, \ \alpha > 0, \ \beta > 0, \tag{IS}$$

Phillips curve with both forward- and backward-looking elements $+ z_t$, (Ph)

but not clear how it should precisely be specified (weight of forward- versus backward-looking elements, non-linearity, "natural rate" or "natural range"?).

Taylor rule (inflation targeting):

$$i_t = \max\left[0, \hat{\imath} + \alpha_1(y_t - y^*) + \alpha_2(\pi^e_{t+1} - \hat{\pi})\right], \quad \hat{\imath} > 0, \alpha_1 \geq 0, \alpha_2 > 1, \tag{Taylor rule}$$

and, finally, specification of

expectation formation, (exp)

but far from clear what gives a good approximation: rational, adaptive, extrapolative, or "natural"[1], or possibly a mix with shifting weights due to shifting circumstances. To think of a "mix" is appropriate if different categories of agents in the economy form their expectations in very different ways (heterogeneity).

A traditional approach to the Phillips curve There are alternative ways of modelling the details of a Phillips curve. Building on Blanchard and Katz (1999), we here give a broad picture, starting with a wage Phillips curve.

Macroeconometric evidence indicates, in particular for the US after the Second World War, a negative relationship between the rate of change of wages and the unemployment rate:

$$\begin{aligned} w_t - w_{t-1} &= a + (p_{t-1} - p_{t-2}) - bu_t + z_t \\ &= g + (p_{t-1} - p_{t-2}) - b(u_t - u^*) + z_t, \end{aligned} \tag{31.1}$$

where $w = \ln W$, $p = \ln P$, u is the unemployment rate, and a and b are positive constants, whereas z_t is an error term. This is a *wage Phillips curve.* One interpretation is this. As appears in the second line of (31.1), the parameter a can be split into a sum of two terms, g which indicates the long-run growth rate in labor productivity, Y/N, and a term bu^*, where $u^* \equiv (a-g)/b$ (to be interpreted below as the NAIRU rate of unemployment). A straightforward reading of the role of the (lagged) inflation term, $p_{t-1} - p_{t-2}$, in (31.1) is that it represents *expected inflation.* Let p_t^e denote the expected price level in period t as seen from the end of period $t-1$ and let π_t^e denote the expected inflation rate, i.e., $\pi_t^e \equiv (P_t^e - P_{t-1})/P_{t-1} \approx p_t^e - p_{t-1}$. Then, according to the hypothesis of *static expectations of the inflation rate* we have

$$p_t^e - p_{t-1} = p_{t-1} - p_{t-2}. \tag{31.2}$$

In fact, if inflation follows a random walk (which the data does not reject[2]), this hypothesis is consistent with rational expectations.

Substituting (31.2) into (31.1) and ordering gives the expected change of the real wage as a decreasing function of unemployment:

$$w_t - p_t^e - (w_{t-1} - p_{t-1}) = g - b(u_t - u^*) + z_t. \tag{31.3}$$

In this way the empirical Wage Phillips curve, (31.1), is seen as reflecting an *expected-real-wage Phillips curve.* If expectations are not systematically wrong and the trend rate of unemployment is close to u^*, this says that real wages tend

[1] On.the hypothesis of "natural expectations", see Fuster, Laibson, and Mendel (JEP, 2010).

[2] See Hendry (2008).

in the long run to grow at the same rate as labor productivity, Y/N. The data for the US roughly confirms this picture. Consequently, a first interpretation of u^* is that it is that rate of unemployment which is consistent with real wages tending to grow at the same rate as labor productivity.

Whatever the interpretation of (31.1), it can under a certain condition be transformed into a price Phillips curve. Suppose prices are formed by a more or less constant mark-up on marginal cost, $P_t = (1+\mu)W_t/A_t$, where A_t is labor productivity. Then roughly the price inflation rate equals the wage inflation rate minus the productivity growth rate,

$$p_t - p_{t-1} = w_t - w_{t-1} - g.$$

Substituting this into (31.1) gives a standard *backward-looking Phillips curve*

$$p_t - p_{t-1} = p_{t-1} - p_{t-2} - b(u_t - u^*) + z_t. \tag{31.4}$$

Thus, if $u_t < u^*$, inflation increases, and if $u_t > u^*$, inflation decreases. This corresponds to the interpretation of u^* as the NAIRU (for non-accelerating-inflation-rate of unemployment) in the sense of that rate of unemployment which is consistent with a constant inflation rate (other names for u^* are the "natural" or the "structural" rate of unemployment).

As discussed by Blanchard and Katz (1999), the wage Phillips curve (31.1) fits European data less well than US data. And at the theoretical level it is in fact not obvious why a Phillips curve should hold in the first place. According to the theories of the functioning of labor markets (efficiency wages, social norms, search theories, and bargaining) it is the *level* of the expected real wage, rather than the expected change in the real wage, that is negatively related to unemployment. Theory thus predicts a *wage curve:*

$$w_t - p_t^e = \beta v_t + (1-\beta)\alpha_t - bu_t + z_t, \tag{31.5}$$

where β is a constant $\in [0,1]$, v_t is the *reservation wage* (the minimum real wage at which the worker is willing to supply labor), and α_t a measure of labor productivity.

By reasonable hypotheses about how the reservation wage depends on the actual real wage (in the previous period) and on productivity, a *level* formulation as in (31.5) *may* be consistent with a *change* formulation as in (31.1). Blanchard and Katz (1999) find such consistency to be plausible for US labor markets, but not for the typical European labor market with more influential labor unions, more stringent hiring and firing regulations, and perhaps also a greater role of the underground economy. An interesting implication of this theory is that in Europe, the NAIRU should be sensitive to permanent shifts in factors such as the level of energy prices, payroll taxes, or real interest rates, whereas in the US it should not.[3]

31.2 Vicious and virtuous circles

As mentioned, a characteristic feature of the Keynesian approach to business cycle fluctuations is the emphasis on the sometimes *vicious*, sometimes *virtuous circles* that arise, due to production being in the short term demand-determined rather than supply-determined. A vicious circle may for example come about in the following way.

Suppose that during an economic boom a housing price bubble evolves. Sooner or later the bubble bursts, collaterals for bank loans loose value (the *balance sheet channel*), defaults occur, confidence is shaken, credit is *squeezed*, and further defaults occur.[4] The financial crisis spills over to the goods market in the form of an adverse demand disturbance leading to a contraction of production and employment. The fired workers with less income buy fewer consumption goods (in particular fewer durable consumption goods). The process tends to be self-reinforcing in that the fear of being fired increases *precautionary saving*.

Seeing their demand curves continue the inward movement, firms cut production further. The utilization rate of capital equipment falls and so does average and marginal q. The fall in consumption is thus not offset by firms' investment being stimulated, rather the opposite. Firms' access to credit is cut down further as the balance sheets deteriorate. An economic recession or depression may develop if not offset by contra-cyclical monetary and/or fiscal policy.

There are several self-reinforcement mechanisms that bring these "circles" forth, whether they are negative, as above, or positive. Below we list six examples of such mechanisms. We describe them in their negative mode, that is, when they lead to *vicious circles*. They could just as well, however, be described in their positive mode as when they lead to virtuous circles and thereby a boom.

1. *The spending multiplier* (Kahn 1931, Keynes 1936). Recall that a *multiplier* is the ratio of a change in an endogenous variable, here output or employment, to a change in an exogenous variable, for example an autonomous part of private investment or government spending. A decrease in an autonomous demand component leads to a decrease in production and income, which further reduces demand. The government spending multiplier is larger in a depression, especially in a liquidity trap because there will be no financial

[3] Different versions of the so-called *new-Keynesian Phillips curve* are presented in, e.g., Mankiw (2001) and Gali (2015), but not discussed here.

[4] Below we elaborate on the terms in italics.

crowding out and typically a lot of available labor. Households' and firms' *precautionary saving* (see Section 30.4) aggravates the downturn.[5]

2. *Destabilizing price flexibility* (Keynes, Mundell, Tobin). When some nominal price and wage rigidity is present, more flexibility may be destabilizing. Suppose there is an adverse shock to investor's and firms' general long-term confidence which leads to a downturn of investment, aggregate demand, production, and employment. Through the Phillips curve mechanism, inflation and expected inflation also go down. Will high price flexibility be a good or a bad thing? Under a "passive" monetary policy (the k percent rule), comparatively high price flexibility (though less than "full") may turn the incipient recession into a downward wage-price spiral rather than a transitory dip. This is because opposing effects on aggregate demand are in play, giving rise to a *centripetal force* and a *centrifugal force*. On the one hand, the fall in inflation increases real money supply and lowers the nominal rate of interest, thereby stimulating aggregate demand. And in an open economy net exports are stimulated. On the other hand, the fall in *expected* inflation raises the *real* rate of interest,

$$r = i + \omega - \pi^e,$$

for a given short-term nominal rate of interest i (the policy rate) and a given interest differential, $\omega \geq 0$, thereby reducing demand. Depending on the circumstances, this effect may be the strongest and lead to a self-sustaining economic contraction. In particular this may happen, when the nominal rate of interest is already low and therefore near its floor, the "slightly below zero" bound.[6]

3. *The balance sheet channel* (Kiyotaki and Moore, 1997, Bernanke et al., 1999, Eggertsson and Krugman, 2012). Suppose an adverse shock reduces the net worth of credit-constrained borrowers (entrepreneurs and households), whose assets serve as collateral for loans. This depresses aggregate demand

[5] Formally, a *multiplier* is the ratio of a change in an endogenous variable, here output or employment, to a change in an exogenous variable, for example autonomous government spending.

[6] Nominal interest rates cannot fall much below zero, since potential lenders would then prefer holding cash rather than assets paying a negative interest rate.

The scenario described may take the even more pregnant form of a *deflationary spiral* leading to ever-widening economic crisis. The Great Depression in the US in the 1930's is a conspicuous example and the problems in Japan since the early 1990s also have affinity with this. As witnessed by the repercussions of the global financial crisis 2007-09, also under a countracyclical monetary policy, like the Taylor rule, may the lower bound on the short-term nominal interest rate be reached and thereafter such a vicious spiral arise.

in two ways. Because of the reduced wealth and precautionary saving, consumption is decreased. In addition, if expected to persist, the reduced net worth is likely to lead to a credit contraction. In need of liquidity some agents are forced to sell illiquid assets at "fire sale" prices, thereby further reducing the net worth and credit worthiness of debtors. This means less borrowing, faster debt repayment, and thereby less capital investment and consumption.

4. *The bank lending channel* (Bernanke and Blinder, 1988, 1992). If an economic downturn is on the way, banks may perceive that the riskiness of loans has increased. A *credit squeeze* vis-a-vis other banks and the non-bank public may result whereby the spread between the short-term nominal interest rate on, say, government bonds, and the interest rate that the ultimate borrowers must pay is increased. This limits capital investment and spending on durable consumption goods, thus reinforcing the economic downturn.

5. *Coordination failures* and *multiple equilibria.* There are circumstances, e.g., "spillover complementarity", where more than one general equilibrium is possible. Universally held pessimistic expectations lead to prudent actions that sum to a low-level outcome, thus confirming the pessimistic expectations. But if agents held optimistic expectations, they would make confident upbeat decisions. Aggregate demand would boom, thus confirming the expectations that brought it about in the first place (see Heller 1986, Kiyotaki 1988, Xiao, 2004). As expressed by Wren-Lewis (2015):

   "The largest component of aggregate demand is consumption, and consumption depends on expected income, which can depend itself on actual output, and therefore on aggregate demand. The macroeconomy is therefore set up to allow self-fulfilling multiple equilibria".

6. *Hysteresis.* The described demand-side dynamics may interact with the supply side. This occurs when the initial creation of unemployment, through a de-qualification or discouragement effect on the unemployed or through insider-outsider wage-setting behavior, turns a spell of unemployment into long-term unemployment. Such a phenomenon, where unemployment in the longer run depends positively on unemployment in the short run, is called *unemployment hysteresis.*[7] This has implications for the trade-off between short-run benefits of a deficit-financed expansionary fiscal policy

[7] See Blanchard and Summers (1987), Blanchard (1990), and DeLong and Summers (2012). A corresponding *virtuous* hysteresis can arise through the qualification or learning-by-doing effect of being employed. More generally on hysteresis, see Fiorillo (1999).

in a liquidity trap and possible longer-run costs in the form of a higher government debt.

More generally we say that *medium-run hysteresis* is present if the current state of the economy affects the state in the medium run in "the same direction". And *long-run hysteresis* is present if the current state of the economy affects the state in the long run in "the same direction".

One factor contributing to the vicious circles under the headings 1 and 5 is the phenomenon of *precautionary saving* to which we now turn.

31.3 Precautionary saving

We say that *precautionary saving* is present if increased uncertainty, everything else equal, results in increased saving.

In the first years after the crash at the New York stock exchange in 1929 a sharp fall in private consumption and investment occurred. Many economists argue that this should be seen in the light of the fact that the consumption/saving decision is sensitive to increased uncertainty. Similarly, the international financial crisis, triggered by the subprime mortgage crisis in the US in 2007, created a massive worldwide economic recession 2008- (the "Great Recession"). In this downturn again precautionary saving is likely to have played an important role. If people feel more uncertain about what is going to happen, they tend to be more prudent and increase their saving in order to have a "buffer-stock". But this may aggravate the negative spiral of falling aggregate demand and production.

To clarify the issue, we first consider a simple model of a household's consumption/saving decision under uncertainty. Second, we discuss the possible macroeconomic implications and relate the discussion to the different business cycle "schools".

31.3.1 Consumption/saving under alternative forms of uncertainty

Consider a given household facing uncertainty about future labor income and capital income. For simplicity, assume the household supplies one unit of labor inelastically each period. The household can never be sure whether it will be able to sell that amount of labor in the next period. As seen from period 0, the decision problem is:

$$\max E_0 U_0 = E_0[\sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t}] \quad \text{s.t.} \tag{31.6}$$

$$c_t \geq 0, \tag{31.7}$$

$$a_{t+1} = (1+r_t)a_t + w_t n_t - c_t, \qquad a_0 \text{ given}, \tag{31.8}$$

$$a_T \geq 0. \tag{31.9}$$

where $u' > 0$ and $u'' < 0$ (so there is risk aversion). The rate of time preference w.r.t. utility is $\rho > -1$ (usually $\rho > 0$ seems realistic, but here the sign of ρ is not important). We think of "period t" as the time interval $[t, t+1)$. Hence, the last period within the planning horizon T is period $T-1$. Real financial wealth is denoted a_t and w_t (> 0) is the real wage, whereas n_t is the exogenous amount of employment offered to the household by the labor market in period t, $0 \leq n_t \leq 1$.[8] The (net) real rate of return on financial wealth is called r_t (> -1). The symbol E_0 stands for the expectation operator, conditional on the information available in period 0. This information includes knowledge of all relevant variables up to and including period 0. There is uncertainty about future values of r, w, and n, but the household knows the stochastic processes that these variables follow.[9] The risk associated with the uncertainty is assumed to be not insurable.

There are two endogenous variables, the control variable, c_t, and the state variable, a_t. The constraint (31.7) defines the "control region", whereas (31.8) is the dynamic budget identity, and (31.9) is the solvency condition, given the finite planning horizon T. The decision as seen from period 0 is to choose a concrete *action* c_0 and a set of *contingency plans* $c(t, a_t)$ about what to do in the future periods. This decision is made so that expected discounted utility, $E_0 U_0$, is maximized. We call the function $c(t, a_t)$ a contingent plan because it tells what consumption will be in period t, *depending* on the realization of the as yet unknown variables up to period t, including the state variable a_t. To choose c_0 in a rational way, the household must take into account the whole future, including what the optimal conditional actions in the future will be.

Letting period t be an arbitrary period, i.e., $t \in \{0, 1, 2, ..., T-1\}$, we rewrite

[8] More generally, $w_t n_t$ could be replaced by y_t, interpreted as any kind of *exogenous* income, say an uncertain pension.

[9] Or at least the household has beliefs about these processes and calculates subjective conditional probability distributions on this basis.

U_0 in the following way

$$\begin{aligned} U_0 &= \sum_{s=0}^{t-1} u(c_s)(1+\rho)^{-s} + \sum_{s=t}^{T-1} u(c_s)(1+\rho)^{-s} \\ &= \sum_{s=0}^{t-1} u(c_s)(1+\rho)^{-s} + (1+\rho)^{-t}\sum_{s=t}^{T-1} u(c_s)(1+\rho)^{-(s-t)} \\ &\equiv \sum_{s=0}^{t-1} u(c_s)(1+\rho)^{-s} + (1+\rho)^{-t} U_t. \end{aligned}$$

When deciding the "action" c_0, the household knows that in every new period, it has to solve the remainder of the problem in a similar way, given the information revealed up to and including that period. As seen from period t, the objective function is

$$E_t U_t = u(c_t) + (1+\rho)^{-1} E_t[u(c_{t+1}) + u(c_{t+2})(1+\rho)^{-1} + ...] \tag{31.10}$$

To solve the problem as seen from period t we will use the substitution method. First, from (31.8) we have

$$\begin{aligned} c_t &= (1+r_t)a_t + w_t n_t - a_{t+1}, \qquad \text{and} \\ c_{t+1} &= (1+r_{t+1})a_{t+1} + w_{t+1}n_{t+1} - a_{t+2}. \end{aligned} \tag{31.11}$$

Substituting this into (31.10), the problem is reduced to an essentially unconstrained maximization problem, namely one of maximizing the function $E_t U_t$ w.r.t. $a_{t+1}, a_{t+2}, ..., a_T$ (thereby indirectly choosing $c_t, c_{t+1}, .., c_{T-1}$). Hence, we first take the partial derivative w.r.t. a_{t+1} in (31.10) and set it equal to 0:

$$\frac{\partial E_t U_t}{\partial a_{t+1}} = u'(c_t)\cdot(-1) + (1+\rho)^{-1} E_t[u'(c_{t+1})(1+r_{t+1})] = 0.$$

Reordering gives the stochastic Euler equation,

$$u'(c_t) = (1+\rho)^{-1} E_t[u'(c_{t+1})(1+r_{t+1})], \qquad t = 0, 1, 2, ..., T-2. \tag{31.12}$$

This first-order condition describes the trade-off between consumption in period t and period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss by decreasing consumption by one unit is equal to the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit. Considering $\partial E_t U_t/\partial a_{t+i}$ for $i = 2, 3, ..., T-t-2$, we get similar first-order conditions, in expected value, for each i.

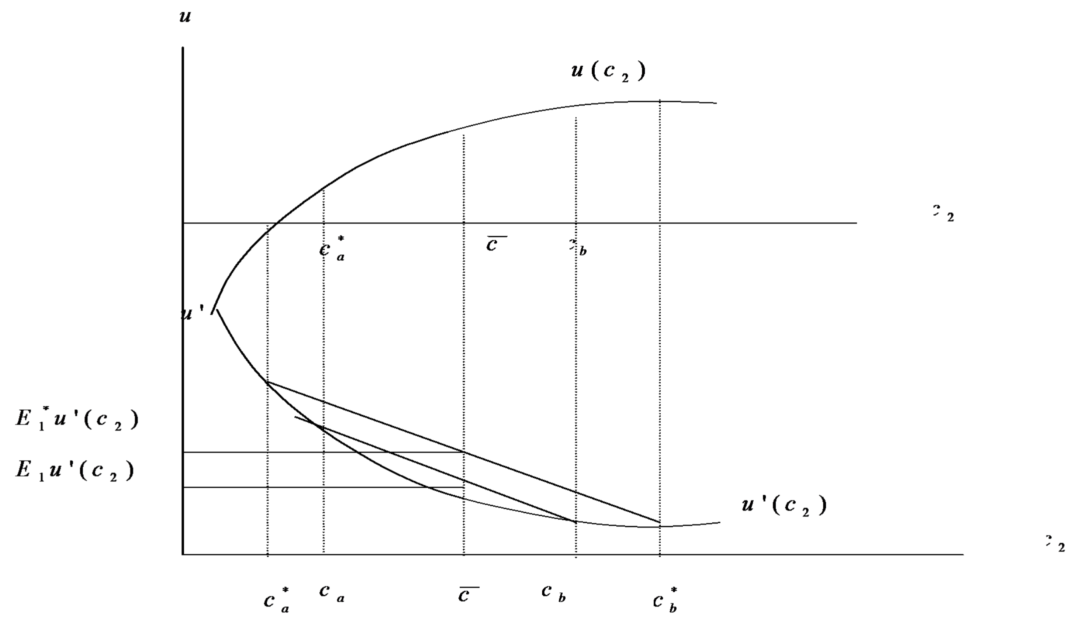

Figure 31.1: Graph of $u(c)$ (upper panel) and graph of $u'(c)$ (lower panel). The case $u'''(c) > 0$.

In the final period, given the solvency condition $a_T \geq 0$, the decision must be to choose $a_T = 0$ (the transversality condition). The alternative, $a_T > 0$, could always be improved upon by increasing c_{T-1} without violating the solvency condition. So, the optimal c_{T-1} satisfies

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1}n_{T-1}. \tag{31.13}$$

First-order conditions only tell us about relative levels of consumption over time, however. The absolute level of consumption is determined by the condition that the current level of consumption, c_t, must be the highest possible consistent with: a) (31.12) for the given t; b) for t replaced by $t+i$, $i = 1, 2, ..., T - t - 2$, (31.12) in expected value as seen from period t, i.e., $E_t u'(c_{t+i}) = (1 + \rho)^{-1} E_t[u'(c_{t+i+1})(1+r_{t+i+1})]$; and c) (31.13) in expected value as seen from period t.

We will first consider the case where there is no uncertainty about the future real interest rates, only about future labor income.

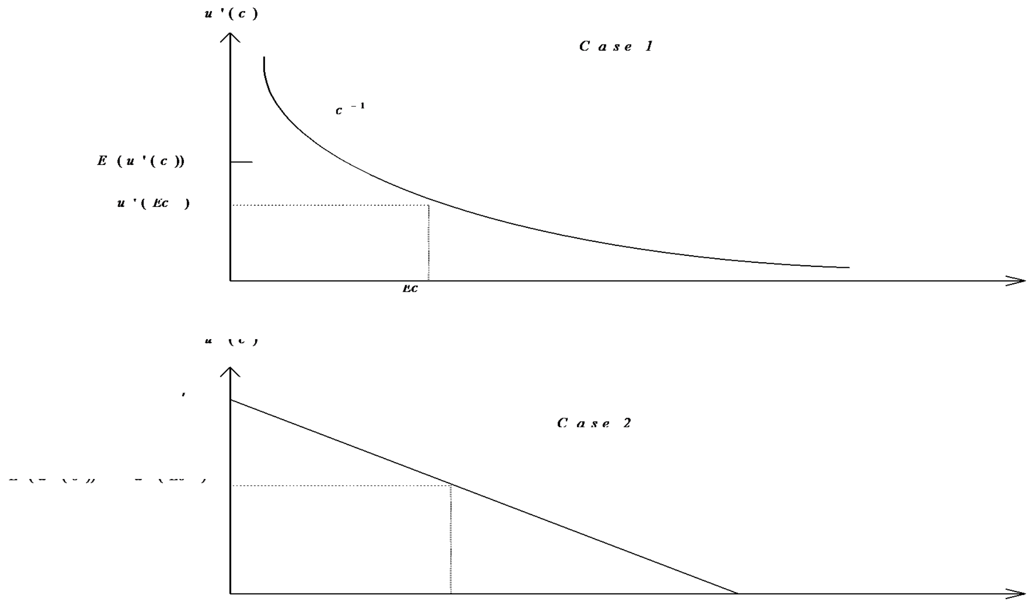

Figure 31.2: Graph of $u'(c)$ when $u(c) = \ln c$ (case 1) and when $u(c) = \eta c - \frac{1}{2}c^2$ (case 2).

Risk-free rate of return

Ruling out uncertainty about the future real interest rates, (31.12) reduces to

$$u'(c_t) = \frac{1 + r_{t+1}}{1+\rho} E_t \left[u'(c_{t+1}) \right], \qquad t = 0, 1, 2, ..., T-2. \tag{31.14}$$

It is natural to assume that higher wealth is associated with lower absolute risk aversion, $-u''/u'$. In that case, it can be shown that marginal utility u' is a strictly convex function of c, that is, $(u')'' > 0$.[10] But this implies that increased uncertainty in the form of a mean-preserving spread will lead to lower consumption "today" (more saving) than would otherwise be the case. This is what precautionary saving is about.

Fig. 30.1 gives an illustration. We can choose any utility function with $(u')'' > 0$. The often used logarithmic utility function is an example since $u(c) = \ln c$ gives $u'(c) = c^{-1}$, $u''(c) = -c^{-2}$ and $u'''(c) = 2c^{-3} > 0$. In the figure it is understood that $T = 3$ (so that the last period is period 2) and that we consider the decision problem as seen from period 1. There is uncertainty about labor income in period 2. It can be because the real wage is unknown or because employment is unknown

[10]See Section 31.4 below.

or both. Suppose, for simplicity, that there are only two possible outcomes for labor income y_t ($\equiv w_t n_t$), say y_a and y_b, each with probability $\frac{1}{2}$. That is, given a_2, there are, in view of (31.13), two possible outcomes for c_2:

$$c_2 = \begin{cases} c_a = (1 + r_2)a_2 + y_a, & \text{with probability} = \frac{1}{2} \\ c_b = (1 + r_2)a_2 + y_b & \text{with probability} = \frac{1}{2}. \end{cases} \tag{31.15}$$

Mean consumption will be $\bar{c} = (1 + r_2)a_2 + \bar{y}$, where $\bar{y} = \frac{1}{2}(y_a + y_b)$.

Suppose c_1 has been chosen optimally. Then, with $t = 1$, (31.14) is satisfied, and a_2 is fixed, by (31.8) with $t = 1$. The lower panel of Fig. 30.1 shows graphically, how $E_1 u'(c_2)$ is determined, given this a_2.

Compare this outcome with a case of *higher uncertainty* in the form of a *mean-preserving spread.* By this is meant that the spread, $|y_b - y_a|$, is larger but the mean, $\bar{y}$, is unchanged. So, if a_2 remains unchanged, now the two possible outcomes for c_2 are c_a^* and c_b^*, while the average equals $\bar{c}$ as before. Fig. 31.1 illustrates. Owing to the strict convexity of marginal utility, the expected marginal utility of consumption is now greater than before, as indicated by $E_1 u'(c_2^*)$ in the figure. In order that (31.14) can still be satisfied, a lower value than before of c_1 must be chosen (since $u'' < 0$), hence, more saving occurs.

Yet, *this* lower value of c_1 is not the final outcome. Indeed, as soon as c_1 tends to be lowered, saving in period 1 tends to be raised. This means a higher a_2 so that the expected value of c_2 is in fact larger than $\bar{c}$ on the figure. This dampens, but does not eliminate, the effect of the mean-preserving spread on $E_1 u'(c_2)$. This expected value ends up somewhere between the old $E_1 u'(c_2)$ and $E_1 u'(c_2^*)$ in the figure. The conclusion is still that the new c_1 has to be lower than the original c_1 in order for the first-order condition (31.14) to be satisfied in the new situation.

If instead the increased uncertainty pertains to period 0, the effect is again to decrease current consumption to provide for a buffer.

What we see here is a manifestation of *precautionary saving:* increased saving as a result of increased uncertainty. In our example there is increased uncertainty about future labor income and as a result lower consumption "today". Consumption is postponed in order to have a buffer-stock. The intuition is that the household wants to be prepared for meeting bad luck, because it wants to avoid the risk of having to end up starving ("save for the rainy day").

Note that the mathematical basis for the phenomenon is the strict convexity of marginal utility, i.e., the assumption that $(u')'' > 0$. This implies $E(u'(c)) > u'(Ec)$ in view of Jensen's inequality. Case 1 in Fig. 30.2 shows the example $u(c) = \ln c$, i.e., $u'(c) = c^{-1}$.

If instead, $(u')'' = 0$, as with a quadratic utility function, then the graph for $u'(c_2)$ is a straight line (cf. case 2 in Fig. 30.2), and then precautionary saving can not occur. Indeed, a quadratic utility function can be written

$$u(c) = \begin{cases} \eta c - \frac{1}{2}c^2 & \text{if } 0 \leq c \leq \eta, \\ \frac{1}{2}\eta^2 & \text{if } c > \eta, \end{cases} \tag{31.16}$$

where $\eta > 0$. We have $u'(c) = \eta - c$ (a negatively sloped line), if $c < \eta$. At $c = \eta$, satiation occurs, and $u'(c) = 0$ for $c > \eta$. If in a given context we want the point of satiation to never be realized in practice, we may assume that η is "large".

The case of quadratic utility is an example of what is known as *certainty equivalence*. We say that certainty equivalence is present, if the decision under uncertainty follows the same rule as under certainty, only with actual values of the conditioning variables replaced by the expected values. Compare a situation where the relevant exogenous variables take on their expected values with probability one (certainty) with a situation where they do that with a probability *less* than one (uncertainty). If the decision is the same in the two situations, certainty equivalence is present. So, when there is certainty equivalence, the decision under uncertainty is independent of the degree of uncertainty, measured, say, by the variance of the relevant conditioning variable(s) for a fixed mean. Quadratic utility implies certainty equivalence. Yet, since (31.16) gives $u'' = -1 < 0$, a household with quadratic utility is risk averse. Hence, for precautionary saving to arise, more than risk aversion is needed.

What is needed for precautionary saving to occur is $u''' > 0$, i.e., "prudence". Just as the degree of (absolute) risk aversion is measured by $-u''/u'$ (i.e., the degree of concavity of the utility function), the degree of (absolute) prudence is measured by $-u'''/u''$ (i.e., the degree of convexity of marginal utility). The degree of risk aversion is important for the size of the *required compensation* for uncertainty, whereas the degree of prudence is important for how the household's *saving behavior* is affected by uncertainty.

Uncertain rate of return

We have just argued that strictly convex marginal utility is a necessary condition for precautionary saving. But, strictly speaking, it is not a sufficient condition. This is so because there may be uncertainty not only about future labor income, but also about the rate of return on saving.

Consider the case where, as seen from period t, r_{t+1} is unknown. Then the relevant first-order condition is (31.12), not (31.14). Now, at least at the theoretical level, the tendency for precautionary saving to arise may be dampened or even turned into its opposite by an offsetting factor. For simplicity, assume first that there is no uncertainty associated with future labor income so that the only uncertainty is about the rate of return, r_{t+1}. In this case it can be shown that there is positive precautionary saving if the *relative* risk aversion, $-cu''/u'$, is larger than 1 ("it is good to have a buffer in case of bad luck") and *negative* precautionary saving (a mean-preserving spread of the ex ante rate of return reduces saving) if the relative risk aversion is less than 1 ("get while the getting is good").

It is generally believed that the empirically relevant assumption from a macroeconomic point of view is that $-cu''/u' > 1$. Thus, increased uncertainty about the rate of return should lead to more saving. The resulting precautionary saving then *adds* to that arising from increased uncertainty about future labor income.

31.3.2 Precautionary saving in a macroeconomic perspective

Simple calculations as well as empirical investigations (for references, see Romer 2001, p. 357) indicate that precautionary saving is not only a theoretical possibility, but can be quantitatively important. A sudden increase in perceived uncertainty seems capable of creating a sizeable fall in consumption expenditure (in particular expenditure on durable consumption goods) and thereby in aggregate demand. According to a study by Christina Romer (1990), this played a major role for the economic downturn in the US after the crash at the stock market in 1929 (see also Blanchard, 2003, p. 471 ff.).

Note that the conception of precautionary saving as an important business cycle force does not fit equally well in all business cycle theories. In new-classical theories (since the 1980s, in practice the RBC theory) a lower propensity to consume is immediately and automatically compensated by higher investment demand and perhaps a larger labor supply and employment in the economy. According to the RBC model from the previous chapter, aggregate demand continues to be sufficient to absorb output at full capacity utilization. Higher uncertainty just leads to a change in the composition of demand, a manifestation of Say's law.

According to many empiricists, this story is contradicted by the data. Less consumption spending seems far form being automatically offset by higher investment spending. The Keynesian interpretation is that output is demand-determined in the short run. An adverse demand shock, triggered by a bursting housing price bubble, say, will, through precautionary saving, lead to a contraction of demand and therefore a downturn of production.

Also firms' behavior may in an economic crisis have aspects of precautionary financial saving. A deep crisis generates a lot of uncertainty: firms are unsure about what has happened and no one knows what actions to choose. The natural thing to do is to pause and wait until the situation becomes clearer. This entails a cutback in the plans for further purchase of investment goods. So on top of households' precautionary saving we have prudent investment behavior by the firms.

31.4 On the distinction between risk aversion and prudence*

We end this chapter with a more general account of basic concepts from the theory of decisions under uncertainty, including the concept of prudence. The aim is to clarify the distinction between the degree of risk aversion and the degree of prudence. We relate this distinction to commonly used utility functions such as the CARA, CRRA, and quadratic utility functions.

31.4.1 Risk aversion and risk premium

Let c be consumption and let E be the expectational operator. Consider a von Neumann-Morgenstern utility index $U = E[u(c)]$ where u is a twice continuously differentiable (sub-) utility function. Assume $u' > 0$. If $u'' < 0$, then the individual in question is said to be *risk averse*. Let $ARA(c)$ be the degree of *Absolute Risk Aversion* at consumption level c, i.e.,

$$ARA(c) \equiv -\frac{u''(c)}{u'(c)}.$$

For a risk-averse individual, this measure is a positive number.[11] As an example, suppose the utility function is

$$\text{CARA:} \qquad u(c) = -\alpha^{-1}e^{-\alpha c},$$

where α is a positive constant. For this function, $ARA(c) = \alpha > 0$, a constant (CARA stands for Constant Absolute Risk Aversion).

The economic significance of the ARA measure is that it is approximately proportional to the (required) *risk premium* (to be defined below). Let ℓ denote the "lottery" that the individual confronts, "lottery" in the sense of a random draw from the given probability distribution for c. For a risk-averse individual $u'' < 0$ (i.e., $u(c)$ is a strictly concave function) and therefore

$$E[u(c)] < u(Ec)$$

[11] The measure $ARA(c)$ is unaffected by an increasing linear transformation of u.

by Jensen's inequality. The *certainty equivalent* for the lottery ℓ is the number c^* satisfying

$$E[u(c)] = u(c^*). \tag{31.17}$$

In words, the certainty equivalent c^* is that *certain* consumption level which the individual is just willing to exchange for the lottery ℓ.

The *risk premium* for the lottery ℓ is defined as the number π satisfying

$$E(c) - \pi = c^*. \tag{31.18}$$

In words, the risk premium is the decrease in expected consumption that the individual is just willing to accept to get rid of the uncertainty and obtain a safe consumption level. Or, since $\pi = E(c) - c^*$, we may look at the matter from the opposite angle and define the risk premium as the increase in expected consumption that the individual requires to just accept an exchange of a safe consumption level c^* for the lottery ℓ.

Let $\bar{c} \equiv Ec$, i.e., $c = \bar{c} + \varepsilon$, where ε is white noise. Now, (31.17) and (31.18) imply

$$E[u(c)] = u[E(c) - \pi] = u(\bar{c} - \pi). \tag{31.19}$$

From this relation we can find an approximate value of π. As to the left-hand-side of (31.19), a *second*-order Taylor approximation of $u(c)$ gives

$$\begin{aligned} u(c) &\approx u(\bar{c}) + u'(\bar{c})\varepsilon + \frac{1}{2}u''(\bar{c})\varepsilon^2 \qquad \Rightarrow \\ E[u(c)] &\approx u(\bar{c}) + 0 + \frac{1}{2}u''(\bar{c})\sigma_\varepsilon^2, \end{aligned} \tag{31.20}$$

where $\sigma_\varepsilon^2 = Var(c) = Var(\varepsilon)$. As to the RHS of (31.19), a first-order Taylor approximation gives

$$u(\bar{c} - \pi) \approx u(\bar{c}) + u'(\bar{c})(-\pi) = u(\bar{c}) - \pi u'(\bar{c}).$$

Inserting this and (31.20) into (31.19) gives

$$\begin{aligned} &u(\bar{c}) + \frac{1}{2}u''(\bar{c})\sigma_\varepsilon^2 \approx u(\bar{c}) - \pi u'(\bar{c}) \;\Rightarrow \\ &\qquad\qquad \pi \approx -\frac{1}{2}\sigma_\varepsilon^2 \frac{u''(\bar{c})}{u'(\bar{c})} = \frac{1}{2}\sigma_\varepsilon^2 ARA(\bar{c}) = \frac{1}{2}\sigma_\varepsilon^2 ARA(Ec). \end{aligned}$$

Hence, ARA evaluated at the consumption level $E(c)$ is approximately proportional to the risk premium.

It seems natural to suppose that as a individual becomes richer - higher $E(c)$ - she cares less and less about the risks she takes. This would say that $\pi'(E(c)) < 0$, i.e., π decreases - hence ARA decreases - as $E(c)$ increases. Therefore, the CARA utility function, defined above, does not seem very realistic. But CARA is just one member of a large family of convenient and more or less realistic utility functions that is called the HARA family.

The HARA family of utility functions is important for at least two reasons.[12] First, if labor income is "diversifiable" (so that the individual can sell shares against future labor income - which is not very realistic, it must be admitted), then it is possible to derive an explicit solution to standard optimum consumption and portfolio problems (as formulated in, e.g., Blanchard and Fischer, 1989, p. 280), if the utility function belongs to the HARA family. Second, the HARA family is the only class of concave utility functions which imply that the consumption function and the portfolio selection function become *linear* in financial wealth. The HARA family as a whole is described mathematically in Appendix B.

Here we shall just meet some prominent members of the family:

$$\text{Quadratic:} \qquad u(c) = \eta c - \frac{1}{2}c^2, \qquad 0 \leq c < \eta,\ \eta \text{ “large”}. \tag{31.21}$$

$$\text{CARA (or the exponential utility function):} \quad u(c) = -\alpha^{-1}e^{-\alpha c},\ \alpha > 0. \tag{31.22}$$

$$\text{CRRA with parameter } \theta > 0: \qquad u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{if } \theta \neq 1, \\ \ln\ c & \text{if } \theta = 1, \end{cases} \tag{31.23}$$

where CRRA is an abbreviation for

31.4.2 The degree of prudence

The *degree of absolute prudence* is defined as the ratio

$$-u'''/u''.$$

As we saw, quadratic utility implies that marginal utility is linear in c (i.e., $u''' = 0$). Hence, in this case the degree of prudence is zero, and the phenomenon of precautionary saving does not arise. But still the quadratic function has $u'' = -1 < 0$, and therefore indicates risk aversion.

The CARA function features the desirable properties of risk aversion ($u'' < 0$) *and* prudence $-u'''/u'' > 0$. On the other hand, the CARA function implies that the required risk premium is constant (independent of wealth), which is probably not a realistic property. The CRRA function, however, has all three desirable properties (so there is no conflict between them).

[12] The name HARA stands for Hyperbolic Absolute Risk Aversion.

CLAIM 1. Given $u' > 0, u'' < 0$, and assuming u to be three times continuously differentiable, non-increasing ARA implies $u''' > 0$.

Proof. From the definition of ARA, we have

$$\begin{aligned}
\frac{dARA}{dc} &= -\frac{u'u''' - (u'')^2}{(u')^2} = (\frac{u''}{u'})^2 - \frac{u'''}{u'} \\
&= \frac{u''}{u'}\left(\frac{u''}{u'} - \frac{u'''}{u''}\right) = ARA(ARA + \frac{u'''}{u''}) \leq 0 \\
&\Rightarrow u''' > 0
\end{aligned} \tag{31.24}$$

since $ARA > 0$, and $u'' < 0$. □

We saw above that when the individual faces a larger future income risk, then, if $u''' > 0$, she has a tendency to consume less in the current period. In other words, precautionary saving tends to occur. The degree of absolute prudence, the ratio $(-u'''/u'')$, can be seen as a measure of the "degree of convexity" of marginal utility $u'(c)$.

The CRRA class of utility functions is characterized by the fact that the measure of *relative risk aversion*

$$RRA \equiv -c\frac{u''}{u'} \equiv c \cdot ARA = \theta$$

is constant (which explains the name CRRA for Constant Relative Risk Aversion). Obviously, this function has the property that ARA ($= \theta/c$) is decreasing in c (as is desirable). Further, $u''' = (\theta + 1)\theta c^{-\theta-1} > 0$ (as expected from Claim 1).

The members of the CRRA class have the (sometimes inconvenient) property that when entering an additively time separable intertemporal utility index, the intertemporal elasticity of substitution becomes equal to $1/\theta$ and hence cannot vary without implying variation in the relative risk aversion measure RRA in the opposite way and in the same proportion. Unsatisfied with this property, ...TO BE CONTINUED

The HARA family is a much richer class, including the four standard cases shown in (31.21) - (31.23) above. By suitable adjustment of the parameters one can get a utility function with decreasing, increasing, or constant absolute or relative risk aversion. As an example, the *general* log utility function

$$u(c) = \ln(\eta + c) \tag{31.25}$$

has decreasing, constant, or increasing RRA as η is negative, zero, or positive, respectively. Indeed, (31.25) has RRA $= c/(\eta + c)$. The case $\eta < 0$ may be interpreted in terms of a subsistence minimum, the subsistence minimum being $|\eta|$.[13]

The case $\eta > 0$ can be interpreted as that c refers to consumption of a luxury good.

31.5 Literature notes

(incomplete)

Paul Krugman's *The Return to Depression Economics* (Krugman 2000) reflects on the need for macroeconomic theory to include depression economics as one of its concerns.

The self-fulfilling prophesy investment theory by Kiyotaki (1988) and the inventory investment theory by Blinder ( ) are examples of business cycle theory emphasizing firms' investment.

Merton (1975).

31.6 Appendix

Jensen's inequality

Jensen's inequality is the proposition that when X is a stochastic variable, and the function f is *convex*, then

$$Ef(X) \geq f(EX)$$

with strict inequality, if f is *strictly* convex (unless X with probability 1 is equal to a constant). It follows that if f is *concave* (i.e., $-f$ is convex), then

$$Ef(X) \leq f(EX)$$

with strict inequality, if f is *strictly* concave (unless X with probability 1 is equal to a constant).

The HARA family of utility functions

Let $c \geq 0$ be consumption, and $u(c)$, $u' > 0$, $u'' < 0$, be a utility function entering a von Neumann-Morgenstern utility index. The measure of *absolute risk tolerance*, ART, is defined as the inverse of the measure of absolute risk aversion, ARA, that is

$$ART(c) \equiv \frac{1}{ARA(c)} \equiv -\frac{u'(c)}{u''(c)} > 0.$$

[13] We have $\lim_{c \to -\eta} \ln(\eta + c) = -\infty$.

A HARA utility function is defined as a utility function $u(c)$ with linear absolute risk tolerance, i.e., the requirement is that

$$ART(c) = \eta + \beta c, \tag{31.26}$$

where η and β are constant parameters[14]. Hence, we get the HARA family of utility functions by solving the second order differential equation

$$\frac{u''}{u'} = -\frac{1}{\eta + \beta c} \tag{31.27}$$

defined on the domain

$$\eta + \beta c > 0. \tag{31.28}$$

Depending on β, the solution is

$$u(c) = \begin{cases} \frac{(\eta+\beta c)^{1-1/\beta}}{\beta-1} + k, & \text{if } \beta \neq 0,\ \beta \neq 1 \\ \ln(\eta + c), & \text{if } \beta = 1, \\ -\eta e^{-c/\eta}, & \text{if } \beta = 0, \end{cases} \tag{31.29}$$

where k is an arbitrary constant (which can be chosen according to what is convenient).

(31.29) is the HARA family of utility functions. This family includes widely used functional forms as special cases: quadratic utility, the CRRA function, the log function, and the CARA function. Each of these, however, are often written in a slightly more convenient way. It is always allowed to add a constant to the function $u(c)$ and multiply by a positive constant (any increasing linear transformation of $u(c)$ will always represent the same von Neumann-Morgenstern preferences).

For example, when $\beta = -1$, $\eta > 0$, and $k = -\eta^2/2$, (31.29) gives

$$\text{the quadratic case:} \qquad u(c) = \eta c - \frac{1}{2}c^2, \qquad 0 \leq c < \eta.$$

When $\beta = 0$, hence $\eta > 0$ by (31.28), (31.29) gives

$$\text{the CARA or the exponential case:} \qquad u(c) = -\alpha^{-1}e^{-\alpha c},\ \alpha \equiv 1/\eta > 0.$$

Letting $\theta \equiv 1/\beta$, where $\beta > 0$, $\beta \neq 1$, $\eta = 0$, and $k = -\beta^{-\theta}/(1-\theta)$, (31.29) gives (multiply through by β^{θ})

$$\text{the CRRA case:} \qquad u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \qquad \theta > 0,\ \theta \neq 1.$$

[14] The HARA definition can be generalized to include cases where η and β are functions of time.

When $\beta = 1$ and $\eta = 0$, (31.29) gives

$$\text{the (standard) logarithmic case:} \qquad u(c) = \ln\, c.$$

As seen by (31.26), the sign of β determines whether risk tolerance is increasing ($\beta > 0$), constant ($\beta = 0$), or decreasing ($\beta < 0$). Increasing risk tolerance - decreasing absolute risk aversion - is considered as the most realistic case. Hence, the CARA utility function (which has $\beta = 0$) should be interpreted as only a theoretical benchmark case which is sometimes mathematically convenient, but probably not realistic. The quadratic utility function is even less plausible (since it has β negative and, in contrast to the other standard functions, it has $u''' \equiv 0$).

Further unfinished notes: HARA $\Rightarrow$ Engel curves are linear $\Rightarrow$ Gorman's aggregation criteria are satisfied (see Basetto and Benhabib, RED 9, 211-23, 2006, and Pollak, Additive utility functions and linear Engel curves, RES, 38 (4), 401-14.

Key terms

mean-preserving spread
degree of risk aversion
risk premium
degree of prudence
precautionary saving
certainty equivalence
vicious circles
virtuous circles

31.7 Exercises