Chapter 4

Continuous time and dynamic optimization

Among the themes of the previous chapter was the effect of including a bequest motive due to parental altruism in the two-period OLG model. Through this extension we came close to the basic representative agent model, the Ramsey model (in a discrete time version). The next chapter presents a pure Ramsey model in its standard, continuous time form. In the present chapter some of the building blocks and analytical tools involved are introduced.

Especially two tools are needed before we can cope with the Ramsey model in continuous time, namely the basic concepts of continuous time analysis in general and dynamic optimization in continuous time in particular. There are many problems where continuous time is preferable to discrete time analysis (period analysis). One reason is that continuous time opens up for application of the general mathematical apparatus of differential equations, which is more powerful than the corresponding apparatus in discrete time analysis, difference equations. Another reason is that the mathematical discipline optimal control theory is more developed and potent in its continuous time version. In addition, continuous time analysis is often more straightforward because many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A). So the aim of this chapter is to equip the reader with the basic concepts and methods of continuous time analysis − with an emphasis on dynamic optimization.

As a vehicle for illustrating the principles we consider a household's consumption/saving decision. How does the household assess the choice between consumption today and consumption in the future? We start from a discrete time framework along the lines of the previous chapters and then carry out the transition to continuous time. In contrast to the previous chapters, we shall allow lifetime to consist of an arbitrary number of periods. Thus, the period length considered may be much shorter. This opens up for capturing additional aspects of economic behaviour.

First, we shall specify the market environment in which the optimizing household operates.

4.1 Market conditions

In the Diamond OLG model the loan market was not active and wealth effects through changes in the interest rate were absent. It is otherwise in a setup where agents live for many periods. This motivates a closer look at the asset markets and notions related to intertemporal choices.

We assume perfect competition on all markets, so the household takes all prices as given. Ignoring uncertainty, the various assets (real capital, stocks, loans etc.) that the household invest in gives the same rate of return in equilibrium. Suppose the household can at any date take a loan or issue loans to others at the going interest rate, r_t. That is, one faces the same interest rate whether borrowing or lending (there are no transaction costs). And there are no borrowing restrictions other than the requirement on the part of borrowers to maintain their intertemporal budget constraint. A loan market satisfying these conditions is called a *perfect loan market*. The implications of such a market are:

1. various payment streams can be subject to comparison; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another if it has the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment at the end of period t. As in the previous chapters period t runs from date t to date $t+1$ for $t = 0, 1, ..., T-1$; and r_t is defined as the interest rate on a loan from date t to date $t+1$, i.e., from the beginning of period t to the end of period t. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+r_0} + \frac{x_1}{(1+r_0)(1+r_1)} + \cdots + \frac{x_{T-1}}{(1+r_0)(1+r_1)\cdots(1+r_{T-1})}. \quad (4.1)$$

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$, but wishes to buy today a durable consumption good of value PV_0, she can borrow this amount and use the income stream $\{x_t\}_{t=0}^{T-1}$ to repay it over the periods $t = 1, 2, ..., T$. In general, when Jones wishes to have a different time profile on the income stream than the original one, she can obtain this through appropriate transactions in the loan market leaving her with any stream of income with the same present value.

The good which is traded on the loan market will here be referred to as *bonds*. The borrower issues bonds and the lender buys them. Until further notice, we will consider all loans as short-term, i.e., as one-period bonds. For every unit of account you lend in the form of a one-period bond at the end of period $t-1$, you will get $1 + r_t$ units of account in return at the end of period t. If you are a borrower of one unit of account at the end of period $t-1$, you pay $1 + r_t$ units of account back to the lender at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued and the revenue is spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to a variable-rate demand deposit in a bank.

[1] We use "present value" as synonymous with "present discounted value". Note that our timing convention is that PV_0 denotes the date 0 value of the payment stream, including the discounted value of the payment (or dividend) in period 0.

4.2 The consumption/saving problem

Let the household's labour supply per time unit be inelastic and constant over time. We normalize this labour supply to one unit of labour per time unit. There is only one consumption good so the *composition* of consumption poses no problems. What remains is the question how to distribute the income between consumption and saving.

4.2.1 Discrete time

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. We assume the preferences of the household can be represented by an additive utility function $U(c_0, c_1, c_2, \cdots, c_{T-1}) = u^0(c_0) + u^1(c_1) + ...u^{T-1}(c_{T-1})$. In fact we shall specialize $U(\cdot)$ further and assume a constant *utility discount rate*, ρ:

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \cdots + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \quad (4.2)$$

The function $U(\cdot)$ is called the *intertemporal utility* function whereas the function $u(\cdot)$ is known as the *period utility* function or *elementary utility* function.[2] We assume that for all $c > 0$, we have $u'(c) > 0$ and $u''(c) < 0$. A $\rho > 0$ reflects that if the level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher if it arrives in the earlier period. The literature refers to ρ as the *rate of time preference* or the *rate of impatience.* The number $1 + \rho$ tells how many additional units of utility in the next period that the household insists on to compensate for a decrease of one unit of utility in the current period. The *utility discount factor,* $1/(1 + \rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t (see Box 1).[3] The rate of discount of *utility*, ρ must not

[2] Some authors refer to $u(\cdot)$ as the *subutility* function. In continuous time analysis the corresponding function is known as the *instantaneous utility* function, the *felicity* function or the *utility flow* function.

[3] Multiplying through in (4.2) by $(1+\rho)^{-1}$ would leave the ranking of all possible alternative consumption paths unchanged, and at the same time make the objective function

be mistaken with the rate of discount of *income*, namely the interest rate, r_t, as in (4.1).

Box 1. Admissible transformations of the period utility function

When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function $U(\cdot)$ is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, $U(\cdot)$ will express the same ranking between alternative consumption paths as any increasing transformation of $U(\cdot)$. The period utility function, $u(\cdot)$, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility units" is legitimate. That is, the *size* of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95\left[u(c_1') - u(c_1)\right] \Longleftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality.

Only a *linear* positive transformation of the utility function $u(\cdot)$, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged.[4]

Already the additivity of the intertemporal utility function is a strong assumption. It implies that the trade-off between consumption this period and consumption two periods from now is independent of consumption in the interim. The constant time discount implies further that the marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the

appear in a way similar to (4.1) in the sense that also the first term in the sum becomes discounted. However, for ease of notation the form (4.2) is commonly used.

[4]The point is that a linear positive transformation does not affect the *ratios* of marginal period utilities (the marginal rates of substitution across time).

same in the two periods.[5]

It is generally believed that human beings are impatient in the sense that ρ should be positive; indeed, it seems intuitively reasonable that the distant future does not matter much for current private decisions.[6] There is, however, a growing body of evidence suggesting that the discount rate is not constant, but declining with the time distance from now to the event in question (see, e.g., Loewenstein and Thaler, 1989). Since this last point complicates the models considerably, macroeconomics often, as a first approach, ignores it and assume a constant ρ to keep things simple. Here we follow this practice. More specifically, we assume $\rho > 0$, although this will not be important to begin with, where the time horizon is finite.

Suppose the household considered has income from two sources: work and financial wealth (possibly negative). As numeraire (unit of account) for the wage rate and the rate of return etc. in period t we apply units of the consumption good delivered at the end of period t. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint. The present value of the consumption plan $\{c_t\}_{t=0}^{T-1}$ is,

$$\begin{aligned} PV(c_1, ..., c_{T-1}) &\equiv \frac{c_0}{1+r_0} + \frac{c_1}{(1+r_0)(1+r_1)} \\ + \cdots \frac{c_{T-1}}{(1+r_0)(1+r_1)\cdots(1+r_{T-1})} &= \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)}. \end{aligned}$$

This value can not exceed the household's total wealth, including the present value of expected future labour income, which is

$$h_0 = \frac{w_0}{1+r_0} + \frac{w_1}{(1+r_0)(1+r_1)} + \cdots + \frac{w_{T-1}}{(1+r_0)(1+r_1)\cdots(1+r_{T-1})}. \quad (4.3)$$

In accordance with the assumption of perfect competition, the household does not face any problems selling its labour supply in the market at the

[5] The (strong) assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period utilities discounted at a constant rate are dealt with by Koopmans (1960) and Fishburn and Rubinstein (1982) and – in summary form – by Heal (1998).

[6] If uncertainty were included in the model, $(1+\rho)^{-1}$ might be seen as reflecting the probability of surviving to the next period.

going real wage, w_t. Thus, the household's intertemporal budget constraint is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)} \leq a_0 + h_0, \tag{4.4}$$

where a_0 is the household's financial wealth at the beginning of period 0 (the value of the initial stock of bonds). This can be positive as well as negative (in which case the household is in debt). The household's problem is to choose a consumption plan $\{c_t\}_{t=0}^{T-1}$ so as to achieve a maximum of U_0 subject to this budget constraint.

However, for the study of dynamic problems it is in many cases more convenient to use continuous-time analysis.

4.2.2 Continuous time

Our point of departure is the discrete time framework above: the run of time is divided into successive periods of constant length, taken as the time-unit. Let financial wealth at the beginning of period i be denoted a_i, $i = 0, 1, 2, ...$ Then wealth accumulation in discrete time can be written

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Transition to continuous time analysis

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt be equal to one time unit. Then $a(i\Delta t) = a_i$. Consider the forward first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t+\Delta t)$ approach zero, i.e., let $\Delta t \to 0$. Assuming $a(\cdot)$ is a continuous and differentiable function, then $\lim_{\Delta t \to 0} \Delta a(t)/\Delta t$ exists and is denoted the *derivative of* $a(\cdot)$ at t, usually written $da(t)/dt$ or just $\dot{a}(t)$.

That is,

$$\dot{a}(t) = \frac{da(t)}{dt} = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t}.$$

By implication, wealth accumulation in continuous time is written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given,} \tag{4.5}$$

where $s(t)$ is the saving at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit let us choose one year. Going back to discrete time, if wealth grows at the constant rate $g > 0$ per year, then after i periods of length one year (with annual compounding)

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \ldots . \tag{4.6}$$

When compounding is n times a year, corresponding to a period length of $1/n$ year, then after i such periods:

$$a_i = a_0(1+\frac{g}{n})^i. \tag{4.7}$$

With t still denoting time (measured in years) passed since the initial date (here date 0), we have $i = nt$ periods. Substituting into (4.7) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let n (and so m) $\to \infty$. Then, in the limit there is continuous compounding and

$$a(t) = a_0 e^{gt}, \tag{4.8}$$

where e is the "exponential" defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.718281828...$. The formula (4.8) is the analogue in continuous time (with continuous compounding) to the discrete time formula (4.6) with annual compounding. Thus, a geometric growth factor is replaced by an exponential growth factor.

We can also view these two formulas as the solutions to a difference equation and a differential equation, respectively. Thus, (4.6) is the solution to the simple linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (4.8) is the solution to the simple linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. With a time dependent growth rate, $g(t)$, the corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ with solution

$$a(t) = a_0 e^{\int_0^t g(\tau)d\tau}, \tag{4.9}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (4.9) is called the *basic growth formula* in continuous time analysis and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

Notice that the allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{4.10}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (4.10) gives $K(t) = K_0 e^{-\delta t}$ (exponential decay). This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model converted into continuous time), say 30 years, then $\delta > 1$ may fit better, empirically. Suppose, for example, that physical capital has a half-life of 10 years. Then with 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

Stocks and flows An advantage of continuous time analysis is that it forces one to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption and saving). A *stock* variable is a variable measured as just a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ above are flow variables.

One can not add a stock and a flow, because they have *different denomination.* What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so and so many machines of a certain kind or so and so many litres of oil or so and so many units of payment) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units.* Thus, the capital stock K has the denomination "quantity of machines". In contrast, investment I has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". If we change our time unit, say from quarters to years, the value of a flow variable is quadrupled (presupposing annual compounding). A growth rate or interest rate has the denomination "(quantity/time)/quantity" $=$ "time^{-1}".

Thus, in continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are illegitimate. But one can write $K(t+\Delta t) \approx K(t)+I(t)\Delta t$ and $\dot{K}(t) = I(t)$ (if $\delta = 0$). In the same way, if a bath tub contains 50 litres of water and the tap pours $\frac{1}{2}$ litre per second into the tub, a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 90\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t)\Delta t$. Similarly, $s(t)$ (the flow of individual saving) should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 4.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. But, it is really wealth at the beginning of period i and the saved

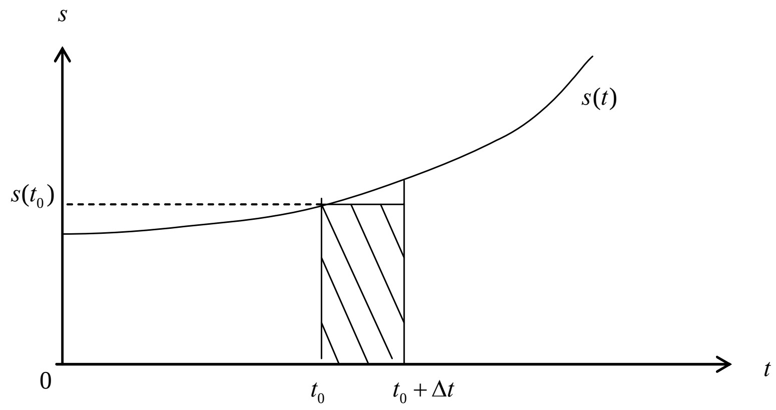

Figure 4.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

amount during period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may be a good reason for putting at least microeconomic analysis in period terms. Yet, it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions and then a formulation in continuous time may be preferable. This also allows variation *within* the usually artificial periods in which the data are chopped up.[7] For example stock markets are more naturally modelled in continuous time because such markets equilibrate almost instantaneously; they respond immediately to new information.

[7] Allowing for such variations may be necessary to avoid the *artificial* oscillations which sometimes arise in a discrete time model due to a "too" large period length (see Math Tools).

In his discussion of this modelling issue, Allen (1967) concluded that from the point of view of the economic contents, the choice between discrete time or continuous time analysis may be a matter of taste. But from the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable.[8]

Discounting in continuous time When calculating present values in continuous time, as a rule we use continuous compounding. Let $r(t)$ denote the (short-term) real interest rate with continuous compounding (in this context some authors call $r(t)$ the interest intensity). First, assume $r(t)$ is a constant, r. Then the present value of a given consumption plan, $(c(t))_{t=0}^{T}$, as seen from time 0, is

$$PV_0 = \int_0^T c(t)\ e^{-rt}dt. \tag{4.11}$$

Instead of the geometric discount factor from discrete time analysis, $1/(1+r)^t$, we have now an exponential discount factor, $1/(e^{rt})$. This is because when discounting, we reverse an accumulation formula like (4.8) and go from the compounded or terminal value, $a(t)$, to the present value, a_0 (in (4.8) replace g by r).

If the interest rate varies over time, then (4.11) is replaced by

$$PV_0 = \int_0^T c(t)\ e^{-\int_0^t r(\tau)d\tau}dt.$$

The discount factor is now $e^{-\int_0^t r(\tau)d\tau}$, the inverse of the accumulation factor in (4.9).[9]

The household's intertemporal budget constraint

As regards the intertemporal budget constraint, (4.4), we are now ready to state its analogue in continuous time. The intertemporal budget constraint

[8] At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier if one is not familiar with stochastic calculus.

[9] Sometimes the discount factor with time-dependent interest rate is written in a different way, see Appendix B.

of the household is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{4.12}$$

where a_0 is the historically given value of the stock of bonds at time 0 as above, while h_0 is human wealth, i.e.,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{4.13}$$

The analogue in continuous time to the intertemporal utility function, (4.2), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{4.14}$$

In this context, we shall call the utility flow function, $u(\cdot)$, the *instantaneous utility function*. The household's problem is now to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (4.12).

Infinite time horizon

In for example the Ramsey model of the next chapter the idea is used that household's may have an *infinite* time horizon. The interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links and the household may be considered a family dynasty with a time horizon far beyond the life time of the current members of the family. As a mathematical idealization one can then use an infinite planning horizon. Introducing a positive discount rate, less weight is attached to circumstances further away in the future and it may be ensured that achievable discounted utility is bounded. Yet, an infinite time horizon is of course fiction. If for no other reason, then because the sun, as we all know, will eventually (in some billion years) burn out and, consequently, life on earth will become extinct. Nonetheless, an infinite time horizon can be a useful mathematical approximation. The solution for "T large" will in many cases most of the time be close to the solution for "$T = \infty$" (cf. the turnpike proposition in Chapter 3). An infinite time horizon can also be a natural notion when in any given period there is a positive probability that there will also be a next period. If this probability is low, it can be reflected in a high discount rate.

With an infinite time horizon and $\rho > 0$, the household's or dynasty's problem becomes one of choosing a plan $(c(t))_{t=0}^{\infty}$, which maximizes

$$U_0 = \int_0^{\infty} u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{4.15}$$

$$\int_0^{\infty} c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by replacing T in (4.13) with ∞. Working with infinite horizons the analyst should be aware that there may exist technically feasible paths along which the integrals in (4.13), (4.15) and (IBC) go to ∞ for $T \rightarrow \infty$, in which case maximization does not make sense. However, the assumptions that we are going to make when working with the Ramsey market economy, will turn out to guarantee that the integrals converge as $T \rightarrow \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high to ensure that the long-run growth rate of the economy becomes less than the long-run interest rate.

The budget constraint in flow terms

The mathematical method which is particularly apt for solving intertemporal decision problems in continuous time is *optimal control theory*. To apply the method, we have to convert the household's budget constraint into flow terms. By mere accounting, in every short period $(t, t + \Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t) + \dot{a}(t))\Delta t = (r(t)a(t) + w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is saving and thus the same as the increase per time unit in financial wealth. If we divide through by Δt and isolate saving on the left of the equation, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{4.16}$$

This book-keeping equation just tells us by how much and in which direction the stock of bonds is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only constraint, one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until the requirement of *solvency* is added to (4.16) that we get a budget *constraint*. When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (4.12). When $T = \infty$, the relevant solvency requirement is a so-called No-Ponzi-Game condition:

$$\lim_{t \to \infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0, \tag{NPG}$$

i.e., the present value of *debts* ($\equiv -a(t)$) infinitely far out in the future, is not permitted to be positive.[10] This is because of the following equivalency:

PROPOSITION 1 *(equivalence of flow budget constraint and intertemporal budget constraint)* Let $T = \infty$ and assume the integral (4.13), which defines h_0, remains finite for $T \to \infty$. Given the accounting relation (4.16), then:

(i) the solvency requirement, (NPG), is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household (or family dynasty) can remain in debt. This would also be an unnatural requirement as the time horizon is infinite. The condition does imply, however, that there is an upper bound for the speed whereby debts can increase in the long term. In the long term, debts cannot grow at a rate greater than (or just equal to) the interest rate.

To understand the implication of this, let us look at the case where the interest rate is a constant, $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[11]

[10] About Ponzi, see below.

[11] Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have

Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run are able to cover at least part of their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run, that is, debts do not explode.

The designation of NPG is short-hand for "No-Ponzi-Game condition". The name refers to a guy from Boston, Charles Ponzi, who in the 1920s made a fortune out of an investment scam based on the chain letter principle. The principle behind Ponzi's transactions is to pay off old investors with money from the new investors. The fact that debts grow without bounds is irrelevant for the borrower *if* new lenders can always be found. Then the last period where all remaining debts have to be repaid never occurs. In the real world endeavours to establish this sort of financial eternity machine tend sooner or later to break down, because the flow of lenders dries up. It is exactly such a financial eternity machine the constraint (NPG) precludes. Ponzi was sentenced many years in prison for his transactions; he died poor – and without friends.[12] The so-called pyramid companies are nowadays-examples of attempts to evade the NPG condition.

Generally, a person is defined *solvent* if she is able to meet her financial obligations as they fall due. In the present context we define solvency as being present if *gross* debt does not exceed gross assets. Here, "assets" should be understood in the broadest possible sense, that is, including the present value of the expected future wage income. Considering *net* debt, d_0, the solvency requirement becomes

$$d_0 \leq \int_0^\infty (w(t) - c(t)) e^{-\int_0^t r(\tau)d\tau} dt,$$

where the right-hand side of the inequality is the present value of the expected future primary savings.[13] By use of the definition in (4.13), it can be seen

$d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

[12] A Danish example, though on a smaller scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Nothern Jutland is charged with fraud. In an ad, she offered 200 Dkr to tell you how to make easy money. Some hundred people responded and received the reply: Do like me."

[13] By primary savings is meant the difference between current *earned* income and current

that this requirement is identical to the stock budget constraint (IBC) which consequently expresses solvency. The NPG condition is simply this solvency condition converted into flow terms.

4.3 Solving the household's problem

Our household shall weigh more consumption today vis-a-vis more consumption in the future in order to obtain the largest possible discounted utility. Being formulated in continuous time, the problem is one of choosing a path for the control variable $c(t)$ so as to maximize an integral subject to constraints, partly in the form of a first-order differential equation which determines the time path of a state variable, $a(t)$, and partly in the form of a condition telling us where to begin and possibly a condition telling us where we are allowed to end. Optimal control theory, which was applied to a related discrete time problem in the previous chapter, is a well-suited apparatus for solving this kind of optimization problems. We shall make use of a special case of the continuous time *Maximum Principle* (the basic tool of optimal control theory) in solving the household's optimization problem. We shall consider the case with a finite time horizon as well as the case with an infinite time horizon.[14]

4.3.1 First-order conditions

The household's decision problem is as follows: choose a consumption plan $(c(t))_{t=0}^{T}$ such that a maximum of the criterion function U_0 (where $\rho > 0$) is obtained subject to the constraints given by a dynamic accounting relation

consumption, where earned income means income before interest income (payment) is added (subtracted).

[14]A broader exposition of the Maximum Principle in continuous time is contained in Math Tools, where also the analogy with the discrete time Maximum Principle is briefly exposed. General treatments are available in, e.g., Seierstad and Sydsaeter (1987) and Sydsaeter et al. (2005).

and a solvency requirement. For $T < \infty$ the problem is:

$$\max_{(c(t))_{t=0}^{T}} U_0 = \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{4.17}$$

$$c(t) \geq 0, \qquad \text{(control region)} \tag{4.18}$$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given,} \tag{4.19}$$

$$a(T) \geq 0. \tag{4.20}$$

With infinite time horizon, T is replaced by ∞ and the solvency condition (4.20) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Let I denote the time interval $[0, T]$ if $T < \infty$ and the time interval $[0, \infty)$ if $T = \infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (4.18) and (4.19) for all $t \in I$ as well as the relevant solvency condition, we call $(a(t), c(t))_{t=0}^{T}$ an *admissible path*. If a given admissible path $(a(t), c(t))_{t=0}^{T}$ solves the problem, it is referred to as an *optimal path*. We assume that $w(t)$ and $r(t)$ are positive for all t and, as a minimum requirement for easing finiteness of U_0, the impatience parameter ρ is assumed positive. The solution procedure is as follows:

1. Set up the (current-value) *Hamiltonian*:[15]

   $$H(a, c, \lambda, t) \equiv u(c) + \lambda(ra + w - c),$$

   where λ is an *adjoint variable* (also called a *co-state variable*) associated with the dynamic constraint (4.19), that is, λ is an auxiliary variable which is a function of t (it is analogous to the Lagrange multiplier in static optimization).[16]

2. At every point in time, maximize the Hamiltonian w.r.t. the control variable, in the present case c. If one is looking for an interior solution,[17]

[15] The explicit dating of the time-dependent variables a, c and λ is omitted where it is not needed for clarity.

[16] We apply here the *current-value* Hamiltonian. One may alternatively consider the *present-value* Hamiltonian, that is, $\tilde{H} \equiv He^{-\rho t}$ with the discounted shadow price, μ, substituted for $\lambda e^{-\rho t}$. Then step 3 below is replaced by $\partial\tilde{H}/\partial a = -\dot{\mu}$.

[17] A solution $(a_t, c_t)_{t=0}^{T}$ is an *interior* solution if $c_t > 0$ for all $t \geq 0$.

differentiate H partially with respect to c and set the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0,$$

that is, at every $t \in I$,

$$u'(c) = \lambda. \tag{4.21}$$

3. Differentiate H partially w.r.t. the state variable, in the present case a, and set the result equal to the discount rate (in the integrand of the criterion function) multiplied by λ minus the time derivative of the adjoint variable, λ :

$$\frac{\partial H}{\partial a} = \lambda r = \rho\lambda - \dot{\lambda}.$$

That is, at any point in time, the adjoint variable λ should fulfil the differential equation

$$\dot{\lambda} = (\rho - r)\lambda. \tag{4.22}$$

4. Now, use the *Maximum Principle* which (in this case) says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda(t)$ such that for all $t \in I$, (4.21) and (4.22) hold along the path, and the *transversality condition,*

$$\begin{aligned} e^{-\rho T}\lambda(T)a_T &= 0, \text{ if } T < \infty, \text{ or} \\ \lim_{t\to\infty} e^{-\rho t}\lambda(t)a(t) &= 0, \text{ if } T = \infty, \end{aligned} \tag{TVC}$$

is satisfied.

Thus, in continuous time an optimal path is characterized as a path that maximizes the Hamiltonian associated with the problem. This is because the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, through the implied change in the state variable (here financial wealth); "right" here means in accordance with the opportunities offered by the return on saving vis-a-vis the time preference rate, ρ. The optimality condition (4.21) can be seen as a $MC = MB$ condition: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation. Together with the optimality condition (4.22) this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[18] One of the reasons that for continuous time problems we generally prefer the *current-value* Hamiltonian is that it makes both the calculations and the interpretation slightly simpler. The shadow price, $\lambda(t)$, becomes a current price, along with the other prices in the problem, $w(t)$ and $r(t)$, instead of a discounted price.

Reordering (4.22) gives

$$\frac{\frac{\partial H}{\partial a} + \dot{\lambda}}{\lambda} = r + \frac{\dot{\lambda}}{\lambda} = \rho.$$

This can be interpreted as a no-arbitrage condition. Along an optimal path the rate of return, measured in utility units, on the marginal unit of saving is equal to the required rate of return, ρ. The household is willing to save the marginal unit of income only if the actual rate of return on saving equals the required rate. The compared rates of return are here in terms of utility. We may also rewrite (4.22) as $r = \rho - \dot{\lambda}/\lambda$. Then we have on the left-hand-side, the actual *real* rate of return on saving. And the right-hand-side can be interpreted as the *required* real rate of return. Indeed, the right-hand-side is the difference between the required rate of return in utility units, ρ, and the growth rate of the shadow price (measured in utility units) of financial wealth, $\dot{\lambda}/\lambda$. Thus, ρ acts as a *nominal* interest rate and $\dot{\lambda}/\lambda$ as an inflation rate, so that the difference is a (required) *real* interest rate.

Substituting (4.21) into the transversality condition for the case $T < \infty$, gives

$$e^{-\rho T} u'(c(T))a(T) = 0. \tag{4.23}$$

This can be read as a standard complementary slackness condition, because we can replace the factor $a(T)$ by $(a(T) - 0)$, since our solvency constraint, $a(T) \geq 0$, can be seen as a general inequality constraint, $a(T) \geq a_T$, where here a_T happens to equal 0. Since $u'(c(T))$ is always positive (saturation is impossible by assumption), an optimal plan must satisfy $a(T) = 0$. The al-

[18] Remember, a *shadow price* (measured in some unit of account) of a good is the number of units of account that the optimizing agent is just willing to offer for one extra unit of the good.

ternative, $a(T) > 0$, would imply that consumption, and thereby U_0, could be increased by a decrease in $a(T)$ without violating the solvency requirement.

As for the infinite horizon case, let $T \to \infty$. Then (NPG) is the solvency requirement and (4.23) is replaced by

$$\lim_{T\to\infty} e^{-\rho T} u'(c(T))a(T) = 0. \tag{4.24}$$

This is the same as (TVC) (replace T by t). Intuitively, a plan that violates this condition indicates scope for improvement and thus can not be optimal. Generally, however, care must be taken when trying to extend a necessary transversality condition from a finite horizon to an infinite horizon, as we just did. But for the present problem, this simple extension *is* valid. Indeed, (TVC) is just a requirement that the NPG condition is not "over-satisfied":

PROPOSITION 2 *(the transversality condition with infinite time horizon)* Let $T = \infty$ and assume the integral (4.13), which defines h_0, remains finite for $T \to \infty$. Provided the adjoint variable, $\lambda(t)$, satisfies the optimality conditions (4.21) and (4.22), we have:

(i) (TVC) holds if and only if (NPG) holds with strict equality; and
(ii) this is the case if and only if (IBC) holds with strict equality.

Proof. See Appendix D.

In view of (i) of this proposition, we can write the transversality condition for $T = \infty$ on the more distinct form

$$\lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} a(t) = 0. \tag{TVC'}$$

4.4 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (4.21) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$ which can be written as $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by applying (4.21) again. Use of (4.22) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{4.25}$$

where

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0, \tag{4.26}$$

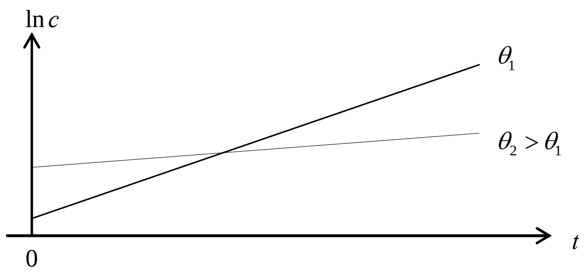

Figure 4.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

that is, $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption c. This elasticity is also referred to as the marginal utility flexibility. It indicates the strength of the consumer's desire to smooth consumption. The inverse of $\theta(c)$ is referred to as the *intertemporal elasticity of substitution* in consumption. It measures the willingness to accept variation in consumption over time when the interest rate makes it attractive, see Appendix D.

The result (4.25) says that an optimal consumption plan is characterized in the following way. The household will completely smooth consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. Indeed, in this case the household would allow for a relatively low level of current consumption with the purpose of enjoying more consumption in the future. This result makes economic sense: the lower the rate of time preference relative to the real interest rate, the more favorable it becomes to defer consumption. Moreover, by (4.25) we see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption, *ceteris paribus*. The reason for this is that a large curvature means that the marginal utility will drop sharply if consumption rises, and will increase sharply if consumption falls. Fig. 4.2 illustrates this in a case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high, has lower slope, but starts from a higher level than when θ is low.

The condition (4.25) which holds for a finite as well as an infinite time horizon is referred to as the *Keynes-Ramsey rule*.[19] The name springs from the English mathematician Frank Ramsey who derived the rule in as early as 1928, while John Maynard Keynes suggested a simple and intuitive way of presenting it. The rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods with the corresponding price ratio. In the present context, the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different points in time. Let us first consider this in discrete time with discrete compounding.

Local optimality in discrete time

Period t is the time interval $[t, t+1)$, $t = 0, 1, ...$. The household's utility function is

$$U_0 = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t},$$

where $T \leq \infty$. Let a given consumption plan $(c_0, c_1, ..., c_t, ...)$ be our "reference path". Imagine an alternative path which equals the reference path except for the periods t and $t+1$. If it is possible to obtain a higher level of total utility compared to the reference path by varying c_t and c_{t+1} subject to an unchanged budget constraint, the reference path cannot be optimal (that is, local optimality is a necessary condition for global optimality). Assume the reference path is optimal. Thus, if the household gives up a marginal unit of consumption in period t and saves this amount, then the increase in consumption next period resulting from the extra interest yield, must be exactly so large as to leave the household indifferent to such a reallocation of consumption over time.

More precisely: the loan market makes it possible to transform one unit of consumption in period t, to $1 + r_{t+1}$ units of consumption in period $t+1$. Thus, the *marginal rate of transformation*, MRT, to period $t+1$ from period

[19] More generally, such a condition, derived from the first-order conditions of a dynamic optimisation problem, is called an *Euler equation*. The Swiss mathematician L. Euler developed as early as in the 18th century a forerunner of optimal control theory called calculus of variation.

t, is

$$MRT_{t+1,t} \equiv -\frac{dc_{t+1}}{dc_t}\Big|_{\substack{\text{unchanged}\\ \text{budget}}} = 1 + r_{t+1}.$$

The *marginal rate of substitution*, $MRS_{t+1,t}$, of consumption in period $t+1$ for consumption in period t is found by imposing that the total contribution to U_0 from the two periods' consumption must be unchanged for a small reallocation of consumption between the two periods. Hence, the requirement is that $dU_0 = 0$, i.e.,

$$d\left(\frac{u(c_t)}{(1+\rho)^t} + \frac{u(c_{t+1})}{(1+\rho)^{t+1}}\right) = 0,$$

where dx denotes the differential of x. From this we get

$$u'(c_t)dc_t + (1+\rho)^{-1}u'(c_{t+1})dc_{t+1} = 0,$$

that is,

$$MRS_{t+1,t} \equiv -\frac{dc_{t+1}}{dc_t}\,|_{dU_0=0} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}.$$

The reference path can only be optimal if $MRS_{t+1,t} = MRT_{t+1,t}$, i.e., if

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{4.27}$$

Alternatively, this can be expressed in the same way as we did in the previous chapters:

$$u'(c_t) = (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1}).$$

That is, at the margin the cost (in terms of current utility) by lowering consumption in period t by one unit must equal the benefit of having $1+r_{t+1}$ additional units of consumption in the next period. To be comparable with the cost, the extra utility obtained in period $t+1$ must be discounted by the utility discount rate ρ.

Local optimality in continuous time

In continuous time one can, by saving more in the short time interval $(t, t+\Delta t)$, transform $-dc(t)\Delta t$ units of consumption in this time interval to approximately

$$dc(t+\Delta t)\Delta t = -dc(t)\Delta t\; e^{\int_t^{t+\Delta t} r(\tau)d\tau} \tag{4.28}$$

units of consumption in the time interval $(t+\Delta t,\ t+2\Delta t)$. Thus, we have approximately

$$MRT_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}\Big|_{\substack{\text{unchanged}\\ \text{budget}}} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}.$$

Similarly, the marginal rate of substitution, $MRS_{t+\Delta t,t}$, between the two time intervals becomes[20]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{dU_0=0} = \frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))}, \tag{4.29}$$

approximately. The requirement $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ gives

$$\frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}. \tag{4.30}$$

When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (4.27) from discrete time (generally, by a first-order Taylor approximation, $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (4.30), dividing though by Δt, inserting (4.28) and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{4.31}$$

With the definition of $\theta(c)$ in (4.26), this is exactly the same as the Keynes-Ramsey rule (4.25) which, therefore, is merely an expression of the general optimality condition $MRS = MRT$. The household is willing to sacrifice some consumption today for more consumption tomorrow, that is, to go for $\dot{c} > 0$, only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines ($-u''/u'$ large) with increasing consumption.

4.5 The consumption function

The Keynes-Ramsey rule is only a rule which gives the optimal *rate of change* of consumption. It says nothing about the *level* of consumption. In order to

[20] The underlying steps can be found in Appendix E.

determine the level, that is, $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many paths satisfying the Keynes-Ramsey rule, the household will choose the highest one that fulfils the solvency requirement (NPG). Thus, the households acts so that strict equality in (NPG) obtains. As we saw, this is equivalent to satisfying the transversality condition.

Before considering examples, a remark on *sufficient* conditions for optimality is appropriate. The first-order and transversality conditions, (4.21), (4.22) and (TVC), are only *necessary* conditions for a path to be optimal. Hence, up to this point, we have only claimed that if the consumption-saving problem has an interior solution, then it satisfies the Keynes-Ramsey rule and the transversality condition (TVC'). Are these conditions also *sufficient*? The answer is yes in the present case. This follows from Mangasarian's sufficiency theorem (see Math Tools) which applied to the present problem tells us that if the Hamiltonian is *concave* in (a, c) for every t, then the listed necessary conditions, including the transversality condition, are also sufficient. Because the instantaneous utility function $-$ the first term in our Hamiltonian function $-$ is concave and the second term is linear in (a, c), then the Hamiltonian *is* concave in (a, c). Thus our candidate solution *is* a solution.

EXAMPLE 1 *(constant elasticity of marginal utility, infinite time horizon).* We here impose that the elasticity of marginal utility $\theta(c)$, as defined in (4.26), is a constant $\theta > 0$.[21] From Chapter 2 we know that this requirement implies that the utility function can be written:

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{4.32}$$

This "family" of utility functions, the CRRA family, was illustrated in Fig. 2.8 in Chapter 2. The Keynes-Ramsey rule now implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{4.33}$$

cf. (4.9). We know from Proposition 2 that the transversality condition is equivalent to (IBC) being satisfied with strict equality. This result can be

[21] If $\theta \geq 0$, the control region is $c > 0$, since $u(c)$ will in that case not be defined for $c = 0$.

used to determine $c(0)$. Multiply both sides of (4.33) by the discount factor, integrate both sides and finally substitute (IBC) with strict equality.[22] We get

$$c(0)\int_0^\infty e^{\frac{1}{\theta}\int_0^t[(1-\theta)r(\tau)-\rho]d\tau}dt = a_0 + h_0.$$

This can be written

$$\begin{aligned} c(0) &= \beta_0(a_0+h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t[(1-\theta)r(\tau)-\rho]d\tau}dt} \end{aligned} \tag{4.34}$$

is the marginal propensity to consume out of wealth, and

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{4.35}$$

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[23] On the one hand, such an increase makes future consumption cheaper in present value terms. This entails a negative *substitution effect*, which makes it attractive to defer consumption. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth. This entails a positive *income effect* on consumption today. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the income effect, and if $\theta > 1$ (large curvature), the reverse will hold in that a large θ reflects a strong propensity to smooth consumption over time. In the intermediate case $\theta = 1$ (the logarithmic case) we get from (4.34) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{4.36}$$

That is, the marginal propensity to consume is time independent and equal to the rate of time preference. Thus, for a given *total* wealth, a_0+h_0, current consumption is independent of the expected path of the interest rate. That

[22] This method also applies if instead of $T = \infty$ we have $T < \infty$.

[23] By an increase in the interest rate *level* we mean an upward shift in the entire time-profile of the interest rate.

is, in the logarithmic case the *substitution* and *income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents at the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect.[24] Thus, even in the logarithmic case – and *a fortiori* when $\theta < 1$ – the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get $\beta_0 = [(\theta - 1)r + \rho]/\theta$ and $a_0 + h_0 = a_0 + w/r$. When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and h_0, hence the total effect on $c(0)$ is even "more negative". When $\theta > 1$, a higher r gives a higher β_0 and this more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 2, available empirical studies suggest a value of θ somewhat above 1. □

To avoid any misunderstanding, Example 1 should not be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the household's maximization problem with infinite horizon. There is generally no guarantee that integrals converge for $T \to \infty$. But the evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary. It is determined by the requirement of equilibrium. We shall return to this issue in the next chapter.

EXAMPLE *2* (*constant absolute sensitivity of marginal utility, infinite time horizon*). The requirement is that the sensitivity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α.[25] The utility function can then be written

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{4.37}$$

[24] If $a_0 < 0$, and this debt ($-a_0$) is not a variable-rate loan (as hitherto assumed), but a fixed-rate mortgage loan for example, then a rise in the interest rate lowers the present value of the debt and thereby *counteracts* the negative substitution effect on current consumption.

[25] By the *sensitivity,* sometimes called *semi-elasticity*, of a function $f(x)$ we mean the absolute value of $f'(x)/f(x)$.

The Keynes-Ramsey rule becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find $c(0) = r(a_0+h_0)-(r-\rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

Here there is "constant absolute variability aversion", whereas the degree of *relative* variability aversion is $\theta(c) = \alpha c$ which is thus greater, the larger is c. In the theory of behavior under uncertainty, (4.37) is referred to as the CARA function ("Constant Absolute Risk Aversion"). One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 3 *(finite time horizon, logarithmic utility).* We consider a life-cycle saving problem. A worker enters the labour market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known) and does not wish to pass on bequests. For simplicity, we assume that $r_t = r > 0$ for all $t \in [0, T]$ and $w(t) = w > 0$ for $t \leq t_1 \leq T$, while $w(t) = 0$ for $t > t_1$ (no wage income after retirement, which takes place at time t_1). The decision problem is

$$\begin{aligned} \max_{(c(t))_{t=0}^{T}} U_0 &= \int_0^T (\ln c(t)) e^{-\rho t} dt \quad \text{s.t.} \\ c(t) &> 0, \\ \dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\ a_T &\geq 0. \end{aligned}$$

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r - \rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{4.38}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation whose solution (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt} \left[\frac{w}{r}(1 - e^{-rz}) - \frac{c_0}{\rho}(1 - e^{-\rho t}) \right], \tag{4.39}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (4.39), we get

$$c(0) = (\rho w/r)(1 - e^{-rt_1})/(1 - e^{-\rho T}). \tag{4.40}$$

Substituting this into (4.38) gives the optimal consumption plan.[26]

If $r = \rho$, consumption is constant over time at the level given by (4.40). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself (dissaving). □

The examples illustrate the importance of forwardlooking *expectations*, here the expected evolution of interest rates and wages. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 1) and through their impact on the present value, h_0, of expected future labour income (or of expected future dividends on shares in a more general setup). Yet the examples – and the consumption theory in this chapter in general – should only be seen as a first, rough approximation to consumption/saving behaviour. Real world factors such as uncertainty and credit constraints (absence of perfect credit markets) also affect the behaviour. Including these factors in the analysis tend to make current income an additional and independent determinant of consumption.

4.6 Appendix

A. Growth formulas in continuous time

Let the variables z, x and y be continuous and differentiable functions of time t. Suppose $z(t)$, $x(t)$ and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

[26] For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (4.34) gives when $a(0) = 0$ and $\theta = 1$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time, the analogue formulas are only approximately true and the approximation can be quite bad unless $\dot{x}(t)/x(t)$ and $\dot{y}(t)/y(t)$ are "small".

B. The cumulative mean of interest rates

Sometimes in the literature the discount factor in continuous time analysis is written otherwise than we did in Section 4.2. Let $\bar{r}_{0,t}$ denote the cumulative mean of the (short-term) interest rates from time 0 to time t, i.e.,

$$\bar{r}_{0,t} \equiv \frac{\int_0^t r(\tau)d\tau}{t}. \tag{4.41}$$

Then we can write the present value of the consumption stream $(c(t))_{t=0}^{T}$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$. The discount factor is $e^{-\bar{r}_{0,t}t}$ and this PV formula has a form similar to that obtained with a constant discount rate r, namely $PV = \int_0^T c(t)e^{-rt}dt$.

In discrete time the analogue average interest rate is defined by the requirement $(1 + \bar{r}_{0,t})^t = (1 + r_0)(1 + r_1)\cdots(1 + r_{t-1})$, which gives $\bar{r}_{0,t} = \sqrt[t]{(1 + r_0)(1 + r_1)\cdots(1 + r_{t-1})} - 1$. Hence, $1 + \bar{r}_{0,t}$ is here a geometric average of the gross interest rates for the periods involved. If the period length is short, say a quarter of a year, the interest rates $r_1, r_2, \ldots$, will generally be not far from zero so that the approximation $\ln(1 + r_n) \approx r_n$ is acceptable. Then $\bar{r}_{0,t} \approx \frac{1}{t}(r_0 + r_1 + \ldots + r_{t-1})$, a simple arithmetic average as in (4.41).

C. Notes on Proposition 1 (equivalence between the budget constraint in flow terms and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{4.42}$$

where $a(0) = a_0$ (given), and impose the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Before proceeding, a clarifying remark is apt. The expression in (NPG) is understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths and therefore replace "$\lim_{t\to\infty}$" in (NPG) by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0, \infty)$ is defined as $\lim_{t\to\infty} \inf \{f(s)|\ s \geq t\}$.[27] However, due to strict concavity, infinitely fluctuating paths never turn up in the optimization problems considered in this book. Hence, nothing is lost by using the more narrow definition of solvency in (NPG) and apply the simpler notation "lim" instead of "lim inf", as we also argued (in a slightly different context) in Appendix H of Chapter 3.

On the background of (4.42) Proposition 1 claimed that (NPG) is equivalent to the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

being satisfied, where h_0 is defined as in (4.35) and is assumed to be a finite number. In addition, Proposition 1 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG).

Proof. By isolating $c(t)$ in (4.42) and multiplying through by $e^{-\int_0^t r(\tau)d\tau}$, we get

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

[27] By "inf" is meant *infinum* of the set, that is, the largest number less than or equal to all numbers in the set.

We now integrate from 0 to $T > 0$, which gives $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt$$
$$= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right)$$
$$+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt$$
$$= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),$$

where the second last equality follows by integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogeneous first-order differential equation (in that (4.42) is a special case of this) and then let $T \to \infty$. Since this is a more general approach, we will show how it works and use it for an extended version of Proposition 1 and for the proof of Proposition 2. The extended version of Proposition 1 will for example prove useful in Problem 4.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{4.43}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t g(s)ds} \geq 0 \tag{4.44}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau \leq x_{t_0}. \tag{4.45}$$

Moreover, if and only if (4.44) is satisfied with strict equality, then (4.45) is also satisfied with strict equality.

Proof. The linear differential equation, (4.43), has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds}d\tau. \tag{4.46}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (4.44) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds}d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (4.45). We also see that if and only if (4.44) holds with strict equality, then (4.45) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds}d\tau, \tag{4.47}$$

assumed to be a finite number. Then, given

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{4.48}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds}d\tau \leq a_{t_0} + h_{t_0}, \tag{4.49}$$

where a strict equality on the left of "⇔" implies a strict equality on the right, and vice versa.

Proof. Let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$ in (4.43), (4.44) and (4.45). Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the transversality condition with an infinite time horizon)

In the differential equation (4.43) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$ and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (4.22) in Section 4.3. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 4.3 yields

$$\lambda(t_0) \lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{4.50}$$

From the first-order condition (4.21) in Section 4.3 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (4.50) can be ignored. Thus, (TVC) in Section 4.3 is equivalent to the condition (NPG) in that section being satisfied with strict equality (let $t_0 = 0 = n$). The fact that the latter is equivalent to the (IBC) being satisfied with strict equality is known from the last part of the corollary above. Hence, we have shown Proposition 2. □

E. Intertemporal substitution of consumption in continuous time

We claimed in Section 4.4 that equation (4.29) gives approximately the marginal rate of substitution of consumption in the time interval $(t+\Delta t,\, t+2\Delta t)$ for consumption in $(t,\, t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t etc., even though they are still considered as continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}
\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right) \\
&= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right) \\
&= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].
\end{aligned}$$

Requiring $dU_0 = 0$ is thus approximately the same as requiring $d[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (4.29).

The exact local optimality condition, equation (4.31), can be derived from (4.30) in the following way: take logs on both sides of (4.30) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (4.28) and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{4.51}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (4.51) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

The relationship between the elasticity of marginal utility and the concept of *elasticity of intertemporal substitution* in consumption in continuous time can be made clear in the following way: consider the indifference curve for consumption in the non-overlapping time intervals $(t, t+\Delta t)$ and $(s, s+\Delta t)$. The curve is depicted in Fig. 4.3. The consumption path outside the two time intervals is the same. At a given point $(c_t\Delta t, c_s\Delta t)$ on the curve, the marginal rate of substitution of s-consumption for t-consumption, $MRS_{s,t}$, is given by the absolute value of the slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, $MRS_{s,t}$ is growing along the curve when c_t decreases (and thereby c_s increases). Conversely, we can consider $MRS_{s,t}$ as the independent variable and consider the corresponding point on the indifference curve (and thereby the ratio c_s/c_t) as a function of $MRS_{s,t}$. If we let $MRS_{s,t}$ grow, the corresponding value of the ratio c_s/c_t will also grow (see Fig. 4.3).

The *elasticity of substitution* between consumption in the time interval $(t, t+\Delta t)$ and consumption in the time interval $(s, s+\Delta t)$ is then defined as the elasticity of the consumption ratio, c_s/c_t, with respect to $MRS_{s,t}$, when

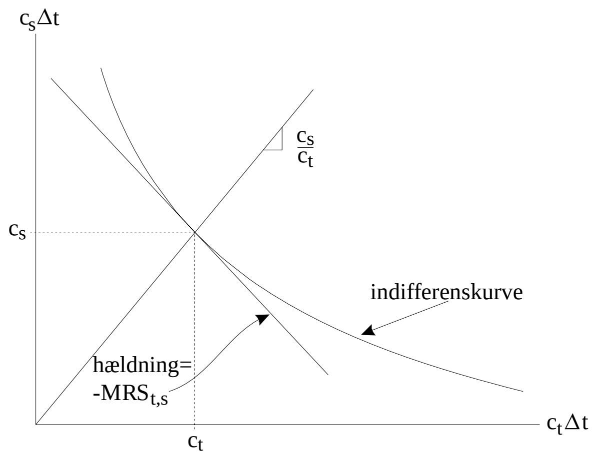

Figure 4.3: Substitutability of s-consumption for t-consumption as $MRS_{s,t}$ changes.

we move along the indifference curve. Let this elasticity be denoted $\sigma(c_t, c_s)$. Thus,

$$\sigma(c_t, c_s) = \frac{MRS_{s,t}}{c_s/c_t}\frac{d(c_s/c_t)}{dMRS_{s,t}} = \frac{\frac{d(c_s/c_t)}{c_s/c_t}}{\frac{dMRS_{s,t}}{MRS_{s,t}}}.$$

At an optimum point, $MRS_{s,t}$ equals the ratio of the discounted prices of good t and good s. Thus, the *elasticity of substitution can be interpreted as the percentage increase in the ratio of the chosen goods, c_s/c_t, by a one percentage increase in the inverse price ratio, holding utility unchanged*; if $s = t + \Delta t$ and the interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (4.27).[28] Inserting $MRS_{s,t}$ from (4.29) we get

$$\begin{aligned}\sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t}\frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}}\\ &= \frac{u'(c_t)/u'(c_s)}{c_s/c_t}\frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))}, \qquad (4.52)\end{aligned}$$

[28] This characterization is equivalent to saying that the elasticity of substitution measures the percentage *decrease* in the ratio of the chosen quantities of goods (when moving along a given indifference curve) induced by a one-percentage *increase* in the *corresponding* price ratio.

by eliminating the factor $e^{-\rho(t-s)}$. Calculating the differentials this gives

$$\sigma(c_t, c_s) = \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$, hence,

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption and is our measure of the willingness to shift consumption over time when the interest rate changes. Thus it reflects the opposite of the desire for consumption smoothing. Indeed, we have $\sigma(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility.

4.7 Problems

Problem 4.1 We look at a household with infinite time horizon and a constant rate of time preference, ρ. Assume the household's labour supply is inelastic but grows at the constant rate $n > 0$. There is always full employment so labour supply and employment are the same. The household's "effective" utility discount rate is $\rho - n > 0$ and the instantaneous utility function is $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, where θ is a positive constant. There is no uncertainty. Show that the household's consumption plan (using standard notation) satisfies

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta} + n)d\tau} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w(t) e^{-\int_0^t (r(\tau)-n)d\tau} dt. \end{aligned}$$

Hint: use the corollary to Claim 1 in Appendix C and the method of Example 1.