Economic Growth

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A suggested solution to Problem III.1

For convenience we repeat the basic relations:

$$Y_i = AK_i^{\alpha}(GL_i)^{1-\alpha}, \qquad 0 < \alpha < 1, A > 0, \qquad i = 1, 2, ..., M. \tag{1}$$

$$Y = cL + G + I,$$

$$\dot{K} = I - \delta K,$$

$$G = \bar{g}Y, \tag{*}$$

$$\left[\tau(ra + w) + \tau_{\ell}\right] L = G, \tag{GBC}$$

a) (*) indicates that G is a productive government service, affecting productivity. Since the productivity of every worker depends on the total of G (not the per capita amount, G/L), G is completely nonrival. From (GBC) we see there is no fee for using G.

b) The results to be explained are

$$r = \alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta \equiv \alpha \bar{A} - \delta \equiv \bar{r} \tag{2}$$

and

$$Y = A^{1/\alpha}(\bar{g}L)^{(1-\alpha)/\alpha}kL \equiv \bar{A}K, \tag{3}$$

respectively. The profit maximizing $k_i \equiv K_i/L_i$ of firm i will in equilibrium equal $K/L \equiv k$, given from the supply side. Then we can derive that $Y = Ak^{\alpha}G^{1-\alpha}L$ and from this, together with (*), we find G in terms of predetermined variables and parameters. With this solution for G we finally get (2) and (3).

c) The representative household solves

$$\max_{(c_t)_{t=0}^{\infty}} U_0 = \int_0^{\infty} \frac{c_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$c_t \geq 0,$$

$$\dot{a}_t = (1-\tau)\bar{r}a_t + (1-\tau)w_t - \tau_{\ell} - c_t, \qquad a_0 \quad \text{given}, \tag{4}$$

$$\lim_{t\to\infty} a_t e^{-(1-\tau)\bar{r}t} \geq 0. \tag{NPG}$$

The current-value Hamiltonian is

$$H = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda\left[(1-\tau)(\bar{r}a_t + w_t) - \tau_\ell - c_t\right],$$

where λ can be interpreted as the shadow price of per capita financial wealth along the optimal path. First-order conditions are

$$\partial H/\partial c = c^{-\theta} - \lambda = 0, \text{ i.e., } c^{-\theta} = \lambda, \tag{5}$$

$$\partial H/\partial K = \lambda(1-\tau)\bar{r} = \rho\lambda - \dot{\lambda}, \text{ i.e., } (1-\tau)\bar{r} - \rho = -\dot{\lambda}/\lambda, \tag{6}$$

and the necessary transversality condition (according to the standard formula from the Maximum Principle) is

$$\lim_{t\to\infty} a_t \lambda_t e^{-\rho t} = 0. \tag{TVC}$$

Log-differentiation wrt. t in (5) and inserting into (6) gives the Keynes-Ramsey rule for this model:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}((1-\tau)\bar{r} - \rho) = \frac{1}{\theta}\left[(1-\tau)(\alpha\bar{A} - \delta) - \rho\right] \equiv \gamma, \tag{7}$$

where $\bar{A}$ is given above.

d) From (TVC) combined with (6) follows that $\lambda_t = \lambda_0 e^{-((1-\tau)\bar{r}-\rho)t}$ so that (TVC) can be written

$$\lim_{t\to\infty} a_t e^{-(1-\tau)\bar{r}t} = 0,$$

where we have eliminated the unimportant factor, $\lambda_0 > 0$. *Comment*: if this limiting value were positive, the (NPG) would be "over-satisfied".

e) The model implies a constant real interest rate, $\bar{r}$, and a constant output-capital ratio, $\bar{A}$. Hence, the model belongs to the AK family. From the theory of AK models (in a Ramsey set-up) we know that in equilibrium $\dot{k}/k$ and $\dot{y}/y$ equal $\dot{c}/c$. Thus, from date zero

$$\dot{k}/k = \dot{y}/y = \dot{c}/c = \frac{1}{\theta}((1-\tau)\bar{r} - \rho) \equiv \gamma. \tag{8}$$

There are no transitional dynamics.

To ensure positive growth we need $(1-\tau)\bar{r} - \rho > 0$, i.e.,

$$(1-\tau)(\alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta) > \rho. \tag{A1}$$

This requires that, given τ, $\bar{g}$ is not too small. On the other hand, to ensure boundedness of the utility integral U_0 we assume

$$\rho > (1-\theta)\gamma. \tag{A2}$$

In case $\theta \geq 1$, (A2) is ensured already by the given assumption that $\rho > 0$. Suppose $0 < \theta < 1$. Then, (A2) requires

$$(1-\tau)(\alpha A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} - \delta) < \frac{\rho}{1-\theta},$$

that is, when $0 < \theta < 1$, there is, in addition to the lower bound on $\bar{g}$ implied by (A1), an upper bound implied by (A2).[1]

f) We get

$$\frac{\partial \gamma}{\partial L} = \frac{(1-\tau)\alpha}{\theta}\frac{\partial \bar{A}}{\partial L} = \frac{(1-\tau)\alpha}{\theta} A^{\frac{1}{\alpha}} \bar{g}^{\frac{1-\alpha}{\alpha}} L^{\frac{1}{\alpha}-2} > 0.$$

There is a scale effect on the growth rate. A combination of two things explains this. First, because of the assumption that the productive public service is a nonrival good, there are economies of scale. Second, the reason that these economies of scale have not just a *level* effect, but an effect on (long-run) growth, is the linearity assumption in (*), namely that the factor multiplied on L_i appears as G and not, for example, as G^{φ} with $0 < \varphi < 1$. This second circumstance is the reason that we end with a reduced-form AK structure and thereby with a fully endogenous growth model in which the scale effect takes the form of a scale effect on growth.

g) In equilibrium in our closed economy $a = k$. Further, $G = \bar{g}Y = \bar{g}\bar{A}K$. We can therefore write the government budget constraint as

$$[\tau(rk + w) + \tau_\ell]\, L = G = \bar{g}\bar{A}K. \tag{9}$$

From firm i's standard first-order condition which equates the firm's marginal product of labor to the labor cost w (not shown above), we find

$$\begin{aligned} w &= \frac{\partial Y_i}{\partial L_i} = (1-\alpha)AK_i^{\alpha}G^{1-\alpha}L_i^{-\alpha} = (1-\alpha)Ak^{\alpha}G^{1-\alpha} \\ &= (1-\alpha)Y/L \equiv (1-\alpha)\bar{A}k. \end{aligned} \tag{10}$$

[1] From the Keynes-Ramsey rule we have $(1-\tau)\bar{r} = \theta\gamma + \rho$, so that the assumption (A2) implies $(1-\tau)\bar{r} > \gamma$, i.e., the after-tax real interest rate is higher than the GDP growth rate (this is a necessary condition for an equilibrium to exist in a representative agent model).

Hence,

$$\bar{r}k + w = (\alpha\bar{A} - \delta)k + (1-\alpha)\bar{A}k = \bar{A}k - \delta k.$$

Given $\tau_\ell = 0$, (9) therefore gives

$$\tau(\bar{A} - \delta)kL = \bar{g}\bar{A}K$$

or

$$\tau = \frac{\bar{g}\bar{A}}{\bar{A} - \delta}, \tag{11}$$

where $\bar{A} - \delta > \alpha\bar{A} - \delta = \bar{r} > 0$, by (A1). We see it *is* possible to fix τ at a constant level such that the government budget is balanced for all $t \geq 0$ in spite of $\tau_\ell = 0$. So $\tau \gtreqless \bar{g}$ for $\delta \gtreqless 0$, respectively. We should also check whether this tax policy is viable. Viability requires

$$\begin{aligned} \frac{\bar{g}\bar{A}}{\bar{A} - \delta} &< 1, \text{ i.e., } \bar{g}\bar{A} < \bar{A} - \delta \text{ or} \\ (1-\bar{g})\bar{A} &\equiv (1-\bar{g})A^{\frac{1}{\alpha}}(\bar{g}L)^{\frac{1-\alpha}{\alpha}} > \delta. \end{aligned} \tag{A3}$$

For a given $\delta > 0$, if we assume A is not "too small" (i.e., the technology of the economy should be sufficiently "productive"), (A3) will hold if $\bar{g}$ is neither "too little" nor too close to 1. Can (A1) still be satisfied? Yes, with $\tau \geq \bar{g}$,

$$(1-\bar{g})\bar{A} \geq (1-\tau)\bar{A} > (1-\tau)(\alpha\bar{A} - \delta) > \rho,$$

by (A1). So there is scope for (A3) to hold if A is large enough. If we impose the empirically realistic assumption that $\theta \geq 1$, we do not have to worry about (A2).

h) The aggregate production function is $Y = Ak^\alpha G^{1-\alpha}L = AK^\alpha G^{1-\alpha}L^{1-\alpha}$ so that

$$\frac{\partial Y}{\partial G} = (1-\alpha)AK^\alpha G^{-\alpha}L^{1-\alpha} = (1-\alpha)\frac{Y}{G}.$$

The net gain by increasing G by one unit is approximately

$$\frac{\partial(Y-G)}{\partial G} = (1-\alpha)\frac{Y}{G} - 1 \gtreqless 0 \text{ for } \frac{G}{Y} \lesseqgtr 1-\alpha.$$

Hence, $\bar{g} = G/Y = 1-\alpha$ is required for static efficiency.

i) We shall suggest an appropriate tax scheme, given that direct lump-sum taxation is out of the question.

1. Labor is inelastic in the model. So a tax on labor income will be non-distortionary. An income tax, however, is a tax not only on labor income but also on capital income and is therefore distortionary in this model where there are no congestion problems. Indeed, the real interest rate in equilibrium is
$$\begin{aligned} r &= \frac{\partial Y_i}{\partial K_i} - \delta = \alpha A K_i^{\alpha-1}(GL_i)^{1-\alpha} = \alpha A k_i^{\alpha-1}G^{1-\alpha} - \delta = \alpha A k^{\alpha-1}G^{1-\alpha} - \delta \\ &= \alpha A K^{\alpha-1}(GL)^{1-\alpha} - \delta = \frac{\partial Y}{\partial K} - \delta, \end{aligned}$$
so that whatever the size of G, the interest rate reflects correctly the intertemporal rate of transformation implied by the technology. But with a tax, τ, on capital income (interest income) the consumer faces the rate of transformation $(1-\tau)r < \partial Y/\partial K - \delta$. The difference ("wedge") between these two rates of transformation measures the distortion. The private rate of return on saving will be smaller than the social rate of return.

2. Is a pure labor income tax τ_w (alone) a viable alternative? To check this, consider the required tax revenue
$$\tau_w w L = G = \bar{g} Y = (1-\alpha)Y. \tag{12}$$
By (10), the real wage in equilibrium is
$$w = (1-\alpha)Ak^{\alpha}G^{1-\alpha} = (1-\alpha)AK^{\alpha}G^{1-\alpha}L^{-\alpha}$$
so that
$$wL = (1-\alpha)AK^{\alpha}G^{1-\alpha}L^{1-\alpha} = (1-\alpha)Y.$$
Comparing with (12), see that the required tax rate on labor income equals 100%! The reason is that the cost of the efficient G equals aggregate labor income (G and L_i have the same exponent in (1)). So a labor income tax, τ_w, alone is not viable.

3. On the other hand, the representative household has to pay the required tax revenue, $(1-\alpha)Y$, for G one way or the other. And it is able to do so and still maintain positive consumption and saving, because it has capital income in addition to the labor income. A non-distortionary way for the government to obtain the needed tax revenue is through a consumption tax, τ_c, satisfying
$$\begin{aligned} \tau_c c L &= G = (1-\alpha)Y, \text{ i.e.,} \\ \tau_c &= (1-\alpha)\frac{Y}{cL}. \end{aligned} \tag{13}$$

Note that this consumption tax rate is *constant* in view of the AK structure of the model (implying that $\dot{c}/c = \dot{y}/y$). Hence, this tax is non-distorting – in the present model. The assumption that ensures this is that labor supply is inelastic and thus not distorted by the consumption tax.

Although not necessary, as the exercise problem is stated, let us calculate the optimal τ_c. By (13) and (3), with $\bar{g} = 1 - \alpha$,

$$\begin{aligned} \tau_c &= (1-\alpha)\frac{A^{1/\alpha}((1-\alpha)L)^{(1-\alpha)/\alpha}kL}{cL} \\ &= \frac{\left[(1-\alpha)A\right]^{1/\alpha} L^{(1-\alpha)/\alpha}}{c/k} = \frac{\left[(1-\alpha)A\right]^{1/\alpha} L^{(1-\alpha)/\alpha}}{\alpha A^{1/\alpha}((1-\alpha)L)^{(1-\alpha)/\alpha} - \delta - \gamma_{SP}}, \end{aligned}$$

where we have inserted the efficient c/k as given from equation (29) in Lecture Note 10, p. 10, and where γ_{SP} is given in equation (26) in the same lecture note.

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