Economic Growth Exercise solutions

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Suggested solution to Problem V.3[1]

For convenience we repeat the equations of the model:

$$\begin{aligned} Y_t &= K_t^\alpha (A_t L_{Yt})^{1-\alpha}, \quad 0 < \alpha < 1, \\ \dot{K}_t &= Y_t - c_t L_t - \delta K_t, \quad \delta \geq 0, & (*) \\ \dot{A}_t &= \mu A_t^\varphi L_{At}, \quad \mu > 0, \varphi < 1, & (**) \\ L_{Yt} + L_{At} &= L_t, \\ L_t &= L_0 e^{nt}, \ n > 0, \ \text{constant}. \end{aligned}$$

a) Dividing through by A_t in (**??**) gives

$$\frac{\dot{A}}{A} \equiv g_A = \mu A^{\varphi - 1} L_A. \tag{1}$$

Presupposing $g_A > 0$, log-differentiating wrt. t gives

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1) g_A + g_{L_A}. \tag{2}$$

Constancy of g_A implies $\dot{g}_A = 0$ so that (2) gives

$$g_A = \frac{g_{L_A}}{1 - \varphi}, \tag{3}$$

where g_{L_A} must be constant. We can then rule out that $g_{L_A} > n$, since $L_A \leq L$ by definition. But whether $g_{L_A} = n$ or $0 < g_{L_A} < n$ we cannot tell without further information

b) As suggested by the hint, instead of first finding the real interest rate in equilibrium (as we do in AK-style models), we take a growth-accounting approach. Log-differentiating wrt. t in the aggregate production function gives

$$g_Y = \alpha g_K + (1 - \alpha)(g_A + g_{L_Y}). \tag{4}$$

[1] At several places in this exercise the analytical method is similar to the one applied in LN 9, Section 1.1.

In view of the capital accumulation equation (*), we have under balanced growth $g_Y = g_K$. Then (4) gives

$$\begin{aligned}(1-\alpha)g_Y &= (1-\alpha)(g_A + g_{L_Y}) \text{ or} \\ g_Y &= g_A + g_{L_Y} = \frac{g_{L_A}}{1-\varphi} + g_{L_Y} = \frac{g_{L_A} + (1-\varphi)g_{L_Y}}{1-\varphi},\end{aligned} \tag{5}$$

from (3). Thus, with both g_Y, g_A, and g_{L_A} constant, also g_{L_Y} must be constant. Then, in view of $L_Y \leq L$, $g_{L_Y} \leq n$. We conclude $g_{L_A} = g_{L_Y} = n$, since $g_{L_A} < n$ would lead to the contradiction that $g_{L_Y} > n$.

Thereby, (5) gives $g_Y = n/(1-\varphi) + n$ and so

$$g_y = g_Y - n = \frac{n}{1-\varphi} = g_A. \tag{6}$$

Remark. If we had been asked to completely solve the model (with Ramsey households), including finding the transitional dynamics, the approach would be to first derive the complete system of differential equations like we did in the standard Ramsey model in B & S, Chapter 2. Then one finds that the dynamics are described by a *four*-dimensional dynamic system (in contrast to the standard Ramsey model which has two-dimensional dynamics). Characterizing the solution to that four-dimensional system is possible, but outside the confines of this course.

c) Defining $C \equiv cL$, under balanced growth $g_C = g_Y$ and so

$$g_c = g_C - n = g_Y - n = \frac{n}{1-\varphi} \equiv g_c^*.$$

d) We consider an R&D subsidy which increases $s_A \equiv L_A/L$. Since the model is saddle-point stable, the economy converges to a balanced growth path (BGP) in the long run with growth rate g_y given by (6).

1. No, a higher s_A will not affect g_y in the long run, since (6) shows that g_y only depends on n and φ, not on s_A. A higher s_A will temporarily increase the growth rate of A and tends to temporarily increase also the growth rate of y. But the fact that $\varphi < 1$ (diminishing returns to knowledge in the growth engine) makes it impossible to maintain the higher growth rate in A forever. This is like in a Solow model where an increase in the saving rate raises the growth rate only temporarily due to the falling marginal productivity of capital.

2. We have

$$y \equiv \frac{Y}{L} = \frac{Y}{L_Y}\frac{L_Y}{L} = \frac{Y}{L_Y}(1-s_A) = \tilde{k}^{\alpha}A(1-s_A), \tag{7}$$

where $\tilde{k} \equiv K/(AL_Y)$. We consider s_A as fixed by policy. Under balanced growth one can infer stocks from flows. Indeed, from (1) and (3) follows

$$\mu A^{\varphi-1}L_A = \frac{n}{1-\varphi},$$

implying

$$A_t = \left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} L_{At}^{\frac{1}{1-\varphi}} = \left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} (s_A L_t)^{\frac{1}{1-\varphi}}.$$

Substituting into (7) gives

$$y_t = (\tilde{k}^*)^{\alpha}\left(\frac{n}{\mu(1-\varphi)}\right)^{\frac{1}{\varphi-1}} (s_A L_0 e^{nt})^{\frac{1}{1-\varphi}}(1-s_A) \tag{8}$$

in balanced growth where $\tilde{k}$ takes some constant value, say $\tilde{k}^*$. If $\tilde{k}^*$ is independent of s_A, (8) unambiguously shows that the path for y_t depends on s_A and thus the answer is: yes, policy has long-run level effects.

We now show that $\tilde{k}^*$ is indeed independent of s_A. From the aggregate production function we have

$$\frac{Y}{K} = K^{\alpha-1}(AL_Y)^{1-\alpha} = \tilde{k}^{\alpha-1} = (\tilde{k}^*)^{\alpha-1}$$

along the BGP. With r denoting the real interest rate, using the household's Keynes-Ramsey rule we have, along the BGP,

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(\alpha^2\frac{Y}{K} - \delta - \rho\right) = \frac{1}{\theta}\left(\alpha^2(\tilde{k}^*)^{\alpha-1} - \delta - \rho\right) = g_c^* = \frac{n}{1-\varphi}. \tag{9}$$

This equation determines $\tilde{k}^*$ independently of s_A as was to be shown.

Remark. Note that the effect on levels is of ambiguous sign. Defining

$$z \equiv s_A^{\frac{1}{1-\varphi}}(1-s_A),$$

we see that

$$\begin{aligned}
\frac{\partial z}{\partial s_A} &= (1-s_A)\frac{1}{1-\varphi}s_A^{\frac{1}{1-\varphi}-1} - s_A^{\frac{1}{1-\varphi}} \\
&= \frac{s_A^{\frac{1}{1-\varphi}-1}}{1-\varphi}\left[1-s_A-(1-\varphi)s_A\right] \\
&= \frac{s_A^{\frac{1}{1-\varphi}-1}}{1-\varphi}\left[1-(2-\varphi)s_A\right] \gtreqless 0 \text{ for } s_A \lesseqgtr \frac{1}{2-\varphi}.
\end{aligned}$$

Thus, if s_A is not "too high", an increase in s_A will have a positive level effect on y via the productivity-enhancing effect of more knowledge creation. But if s_A is already quite high, L_Y will be low, which implies that $\partial Y/\partial L_Y$ is large. This large marginal product constitutes the opportunity cost of increasing s_A and dominates the benefit of a higher s_A, when $s_A > 1/(2-\varphi)$.

e) That s_A under balanced growth is independent of L, follows from the formulas in Jones, 1995, p. 769. By (9) we see that $\tilde{k}^*$ is independent of L. Hence, (8) clearly implies

$$\frac{\partial y_t}{\partial L_0} > 0.$$

So the answer is: yes, there is a scale effect on levels in the model.

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