5 Appendix B: homogeneous functions

In Section 1.1, p. 2, I referred to Euler's theorem on homogeneous functions. In its narrow interpretation Euler's theorem includes only statement (i) below. In its broad interpretation (the one I use) Euler's theorem also includes statement (ii).

Recall that a homogeneous function is defined in the following way:

Assume that the function $f(x) = f(x_1, x_2, \ldots, x_n)$ is defined in a domain D with the property that if $x \in D$ and $\lambda > 0$, then $\lambda x \in D$. Then $f(x)$ is said to be homogeneous of degree h if for all $\lambda > 0$ and all $x \in D$,

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^h f(x_1, x_2, \ldots, x_n).$$

Euler's theorem on homogeneous functions. Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be a function with continuous first-order partial derivatives. Then:

(i) $f(x)$ is homogeneous of degree h if and only if

$$\sum_{i=1}^{n} x_i f_i'(x) = hf(x).$$

(ii) The partial derivatives, $f_i'(x)$, $i = 1, 2, \ldots, n$, are homogeneous of degree $h-1$, i.e., for all $\lambda > 0$ and all $x \in D$,

$$f_i'(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^{h-1} f_i'(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n.$$

For proofs, see for example K. Sydsæter and P. Hammond, *Essential Mathematics for Economic Analysis*, Prentice Hall: London, 2005, p. 432 ff.