Economic Growth. Lecture Note 12

07.04.2010. Christian Groth

Miscellaneous in relation to AK models and their semi-endogenous siblings

Section 1 is a follow-up on simple endogenous growth models and Section 2 is about the distorting effects of a *time-varying* consumption tax.

1 On robustness of simple endogenous growth models

The series of models considered in lecture notes 8 and 9 illustrate the fact that endogenous growth models with exogenous population typically exist in two varieties or cases. One is the fully endogenous growth case where a particular value is imposed on a key parameter. This value is such that there are constant returns (at least asymptotically) to producible inputs in the "growth engine" of the economy.[1] In the "corresponding" semi-endogenous growth case, the key parameter is allowed to take any value in an open interval. The endpoint of this interval appears as the "knife-edge" value assumed in the fully endogenous growth case.

Although the two varieties build on qualitatively the same mathematical model of a certain growth mechanism (say, learning by doing or research and development), the long-run results turn out to be very sensitive to which of the two cases is assumed. In the fully endogenous growth case a positive per-capita growth rate is maintained forever without support of growth in any exogenous factor. In the semi-endogenous growth case, the growth process needs "support" by some growing exogenous factor in order for sustained growth to be possible.[2]

[1]Suppose the aggregate production function is $Y = AK + BK^{\alpha}L^{1-\alpha}$, $A > 0, B > 0, 0 < \alpha < 1$, we have $y \equiv Y/L = Ak + Bk^{\alpha}$, where $k \equiv K/L$. We then get $y/k = A + Bk^{\alpha-1} \rightarrow A$ for $k \rightarrow \infty$, that is, the output-capital ratio converges to a positive constant when the capital-labor ratio goes to infinity. We then say that *asymptotically* there are *CRS wrt. the producible inputs*, here just K. B & S, Chapter 4.5, briefly considers this kind of "asymptotic" AK models where the force of diminishing returns to capital ultimately becomes negligible.

[2]The established terminology is somewhat seductive here. "Fully endogenous" sounds as something going much deeper than "semi-endogenous". But nothing of that sort should be implied.

As Solow (1997, pp. 7-8) emphasizes in connection with learning by investing models (with constant population), the Romer case with $\lambda = 1$ is a very special, indeed an "extreme case, not something intermediate". A value of λ slightly above 1 leads to explosive growth: infinite output in finite time.[3] And a value of λ slightly below 1 leads to growth petering out in the long run.

Whereas the strength of the semi-endogenous growth case is its theoretical and empirical robustness, the strength of the fully endogenous growth case is that it has much simpler dynamics. Then the question arises to what extent a fully endogenous growth model can be seen as a useful approximation to its semi-endogenous growth "counterpart". For example, imagine that we contemplate applying the fully endogenous growth case as a basis for prediction or policy evaluation in a situation where the "true" case is the semi-endogenous growth case. Then we would want to know: Are the impulse-response functions generated by a shock in the fully endogenous growth case an acceptable approximation to those generated by the same shock in the corresponding semi-endogenous growth case for *a sufficiently long time horizon to be of interest*?[4] The answer is "yes" if the critical parameter value is "close" to the knife edge case and "no" otherwise. How close we need it to be depends on circumstances. My tentative impression is that usually it is "closer" than what the empirical evidence warrants.

Even if a single growth-generating mechanism, like learning by doing, does not in itself seem strong enough to generate a reduced-form AK model (the fully endogenous growth case), there might exist complementary factors and mechanisms that in total could generate something close to a reduced-form AK model. The time-series test by Jones (1995) rejects this.[5]

2 A time-varying consumption tax

As we have seen in connection with several AK models, when labor supply is *inelastic*, a time-independent consumption tax acts as a lump-sum tax and is thus non-distortionary. We shall here see that if the consumption tax is *time-dependent*, this no longer holds true.

Consider a Ramsey household with *inelastic* labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time

[3] A demonstration is in the appendix.

[4] Obviously, the ultimate effects of the shock tend to be very different in the two models.

[5] There is an ongoing debate about this and similar empirical issues and we will later return to it.

unit equal to c_t per capita, the household has to pay

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms). Thus, paying $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{1}$$

In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned}
U_0 &= \int_0^{\infty} \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\
\bar{c}_t &\geq 0, \\
\dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \quad a_0 \text{ given}, \\
\lim_{t\to\infty} a_t e^{-\int_0^{\infty}(r_s - n)ds} &\geq 0.
\end{aligned}$$

From now we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta} - 1}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \tag{2}$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \tag{3}$$

and the transversality condition

$$\lim_{t\to\infty} a_t e^{-\int_0^\infty (r_s - n)ds} = 0. \tag{4}$$

We take logs in (2) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating wrt. time gives

$$(\theta - 1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (1), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right] = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming profit maximizing firms and perfect competition in the economy, in equilibrium we have

$$r = \frac{\partial Y}{\partial K} - \delta.$$

But the real rate of return faced by the consumer is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} = \frac{\partial Y}{\partial K} - \delta - \frac{\dot{\tau}}{1+\tau} \lesseqgtr \frac{\partial Y}{\partial K} - \delta \text{ for } \dot{\tau} \gtreqless 0,$$

respectively. If for example the consumption tax is increasing, then the real rate of return faced by the consumer is smaller than the real interest rate because saving implies postponing consumption, and future consumption is more expensive due to the higher consumption tax.

The conclusion is that a time-varying consumption tax is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer and that available from the technology in society. On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

On the other hand, with leisure entering the utility function, even a constant tax on consumption may not be neutral because it reduces the price of leisure *relative to* “genuine”, that is, produced consumption goods. Thereby it creates a disincentive for work.

3 Appendix: Big bang a hair's breadth from the AK

Here we show the statement in Section 1: a hair's breadth from the AK assumption the technology is so productive as to generate infinite output in finite time.

The simple AK model as well as reduced-form AK models end up in an aggregate production function

$$Y = AK.$$

Now we ask the question: what happens if the exponent on K is not exactly 1, but slightly above. For simplicity, let $A = 1$ and consider

$$Y = K^{\alpha}, \qquad \alpha = 1 + \varepsilon, \ \varepsilon \gtrapprox 0.$$

Embed this technology in a Solow-style model with $\delta = n = 0$. We have

$$\dot{K} \equiv \frac{dK}{dt} = sK^{\alpha}, \qquad 0 < s < 1, \qquad K(0) = K_0 > 0 \text{ given.} \tag{5}$$

We see that not only is $\dot{K} > 0$ for all $t \geq 0$, but $\dot{K}$ is increasing over time since K is increasing. So, for sure, $K \to \infty$, but how fast?

To find out, note that (5) is a separable differential equation which implies

$$K^{-\alpha} dK = sdt.$$

By integration,

$$\begin{aligned} \int K^{-\alpha} dK &= \int sdt + \mathcal{C} \ \Rightarrow \\ \frac{K^{-\alpha+1}}{1-\alpha} &= st + \mathcal{C}, \end{aligned} \tag{6}$$

where $\mathcal{C}$ is some constant, determined by the initial condition $K(0) = K_0$. For $t = 0$ (6) gives $\mathcal{C} = K_0^{-\alpha+1}/(1-\alpha)$. Consequently, the solution $K = K(t)$ satisfies

$$\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} = st. \tag{7}$$

As t increases, the left-hand side of this equation follows suit since $K(t)$ increases and $\alpha > 1$. There is a $T < \infty$ such that when $t \to T$ from below, $K(t) \to \infty$. Indeed, by (7) we see that such a T must be the solution to the equation

$$\lim_{K(t)\to\infty} \left(\frac{{K_0}^{1-\alpha}}{\alpha - 1} - \frac{K(t)^{1-\alpha}}{\alpha - 1} \right) = sT.$$

Since

$$\lim_{K(t)\to\infty}\left(\frac{{K_0}^{1-\alpha}}{\alpha-1}-\frac{K(t)^{1-\alpha}}{\alpha-1}\right)=\frac{{K_0}^{1-\alpha}}{\alpha-1},$$

we find

$$T=\frac{1}{s}\frac{{K_0}^{1-\alpha}}{\alpha-1}.$$

To get an idea about the implied order of magnitude, let the time unit be one year and $s = 0.1$, $K_0/Y_0 = K_0^{1-\alpha} = 2$, and $\alpha = 1.05$. Then $T = 400$ years. So the Big Bang ($Y = \infty$) would occur in 400 years from now if $\alpha = 1.05$.

As Solow remarks (Solow 1994), this arrival to the Land of Cockaigne would imply the "end of scarcity", a very optimistic perspective.

In a discrete time setup we get an analogue conclusion. With airframe construction in mind let us imagine that the learning parameter λ is slightly above 1. Then we must accept the implication that it takes only a finite number of labor hours to produce an infinite number of airframes. This is because, given the (direct) labor input required to produce the qth in a sequence of identical airframes is proportional to $q^{-\lambda}$, the total labor input required to produce the first q airframes is proportional to $1/1\ +1/2^{\lambda}\ +1/3^{\lambda}\ +\ ...+1/k^{\lambda}$. Now, the infinite series $\sum_{k=1}^{\infty} 1/q^{\lambda}$ converges if $\lambda > 1$. As a consequence only a finite amount of labor is needed to produce an infinite number of airframes. "This seems to contradict the whole idea of scarcity", Solow observes (Solow 1997, p. 8).

References

Jones, C., 1995. Time series tests of endogenous growth models. *Quarterly Journal of Economics*, 110 (2), 495-525.

Solow, R. M., 1994.

Solow, R. M., 1997.