Economic Growth Lecture Note 5.

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Speed of adjustment and other issues

In this note we discuss an aspect of the transitional dynamics in the Solow model, namely the speed of adjustment toward the steady state. In § 1.2.13, B & S analyze how fast (or rather how slow) are the transitional dynamics in the Solow model. To put it another way: according to the Solow model, how fast does the economy approach its steady state? The answer turns out to be: not very fast - to say the least. This is a rather general conclusion: growth adjustment processes are quite time consuming.

To obtain a conceptual framework for studying such a question, we search for a formula for the *speed of adjustment* of $\tilde{k} - \tilde{k}^*$ and $\tilde{y} - \tilde{y}^*$ in the Solow model. We shall use a more general approach than B & S (pp. 56-59), who concentrate on the Cobb-Douglas case. We use the term speed of adjustment instead of "speed of convergence" (the term used in B & S, pp. 56-59, 111-118 and 167) in order not to confuse the topic with the question of across-country convergence (which is what "σ convergence" is about). Our issue here is "within-country convergence", that is, how fast do variables such as $\tilde{k}$, k, and y approach their steady state paths in a closed economy.

In the last section of this lecture note we touch upon the question: do poor countries necessarily tend to approach their steady state *from below*?[1]

1 The Solow model

Let the aggregate production function be

$$Y = F(K, TL), \tag{1}$$

where F is a neoclassical production function with CRS, Y is output, K is capital input and TL is effective labor input in that T and L are the technology level and labor force, respectively, $T = T(0)e^{xt}$, $T(0) > 0$ given, $L = L(0)e^{nt}$, $L(0) > 0$ given. It is natural to

[1] Sections marked with * are only cursory reading.

assume the parameter x non-negative. We will allow the parameter n to have any sign. In view of CRS, we can write output per unit of effective labor as

$$\tilde{y} \equiv \frac{y}{T} \equiv \frac{Y}{TL} = F(\frac{K}{TL}, 1) \equiv f(\tilde{k}), \quad f' > 0, f'' < 0,$$

where $\tilde{k} \equiv k/T \equiv K/(TL)$ is the (effective) capital intensity. Considering the Solow model, there is a given constant saving rate s, $0 < s < 1$. Thus, in a closed economy, $C = (1-s)Y$ so that

$$\dot{K} = Y - C - \delta K = sY - \delta K, \qquad K(0) > 0 \text{ given},$$

where δ is a constant capital depreciation rate, $\delta \geq 0$. Log-differentiating $\tilde{k} \equiv K/(TL)$ wrt. t, we get the fundamental differential equation for the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + x + n)\tilde{k} \equiv sf(\tilde{k}) - m\tilde{k}. \tag{2}$$

Assume

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) > \frac{m}{s} > \lim_{\tilde{k} \to \infty} f'(\tilde{k}). \tag{A1}$$

Then there exists a unique non-trivial steady state, $\tilde{k}^* > 0$, that is, a unique positive solution to the equation

$$sf(\tilde{k}^*) = m\tilde{k}^*; \tag{3}$$

note that (A1) ensures $m > 0$. Furthermore,

$$\dot{\tilde{k}} \gtreqless 0 \text{ for } \tilde{k} \lesseqgtr \tilde{k}^*. \tag{4}$$

Thus, the steady state, $\tilde{k}^*$, is globally asymptotically stable. Fig. 1 illustrate the dynamics as seen from (2). Fig. 2 illustrates the dynamics as seen from

$$\dot{\tilde{k}} = s\left[f(\tilde{k}) - \frac{\delta + x + n}{s}\tilde{k}\right].$$

In Fig. 3 in Section 4 yet another illustration is exhibited.

An important variable in the analysis below is the output elasticity wrt. capital:

$$\frac{K}{Y}\frac{\partial Y}{\partial K} = \frac{\tilde{k}}{f(\tilde{k})}f'(\tilde{k}) \equiv \alpha(\tilde{k}). \tag{5}$$

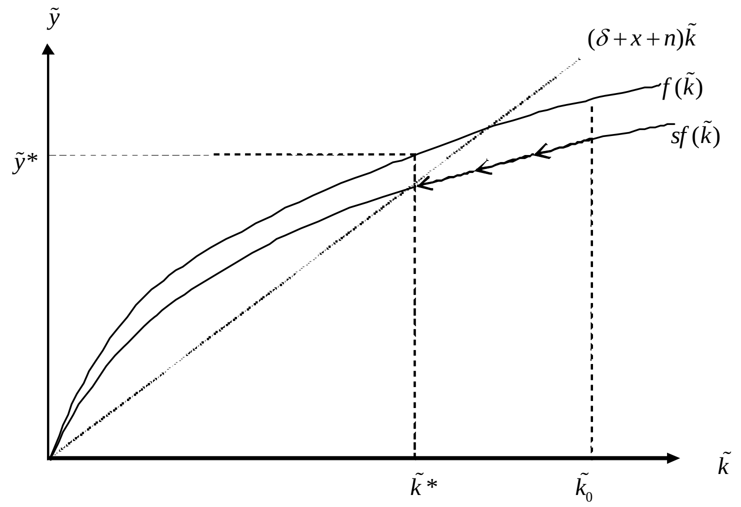

Figure 1:

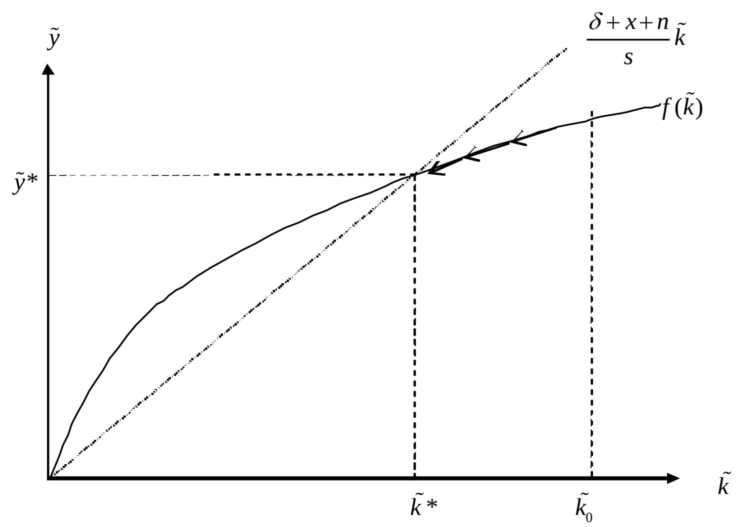

Figure 2:

2 Speed of adjustment close to the steady state

2.1 Adjustment speed for $\tilde{k}$

Let $\varphi(\tilde{k}) \equiv sf(\tilde{k}) - m\tilde{k}$. A first-order Taylor approximation of $\varphi(\tilde{k})$ gives

$$\varphi(\tilde{k}) \approx \varphi(\tilde{k}^*) + \varphi'(\tilde{k}^*)(\tilde{k} - \tilde{k}^*) = 0 + (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*),$$

so that in a neighborhood of the steady state

$$\begin{aligned}
\dot{\tilde{k}} &\approx (sf'(\tilde{k}^*) - m)(\tilde{k} - \tilde{k}^*) \\
&= (\frac{sf'(\tilde{k}^*)}{m} - 1)m(\tilde{k} - \tilde{k}^*) \\
&= -(1 - \frac{\tilde{k}^* f'(\tilde{k}^*)}{f(\tilde{k}^*)})m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (3))} \\
&= -(1 - \alpha(\tilde{k}^*))m(\tilde{k} - \tilde{k}^*) \qquad \text{(from (5))}.
\end{aligned}$$

Dividing through by $(\tilde{k} - \tilde{k}^*)$, we get, since $d\tilde{k}^*/dt = 0$,

$$\frac{d(\tilde{k} - \tilde{k}^*)/dt}{\tilde{k} - \tilde{k}^*} \approx -(1 - \alpha(\tilde{k}^*))m \equiv -\beta(\tilde{k}^*) < 0. \tag{6}$$

Thus, with explicit dating of the time-dependent variable $\tilde{k}$, the (proportionate) rate of *decline* of $\tilde{k}(t) - \tilde{k}^*$ is

$$-\frac{d(\tilde{k}(t) - \tilde{k}^*)/dt}{\tilde{k}(t) - \tilde{k}^*} \approx (1 - \alpha(\tilde{k}^*))(\delta + x + n) \equiv \beta(\tilde{k}^*) > 0, \tag{7}$$

where we have inserted $m \equiv \delta + x + n$. This result tells us how fast the economy approaches its steady state. For example, if $\beta(\tilde{k}^*) = 0.02$ per year (empirically realistic, according to B & S, Ch. 11), then 2 percent of the gap between $\tilde{k}(t)$ and $\tilde{k}^*$ vanishes per year. We call the (proportionate) rate of decline of the distance to steady state the *speed of adjustment*. Hence, $\beta(\tilde{k}^*)$ measures the approximate *speed of adjustment* of $\tilde{k}(t)$ in a neighborhood of the steady state.

Multiplying through by $-(\tilde{k}(t) - \tilde{k}^*)$, the equation (7) takes the standard form of a homogeneous linear differential equation in the gap, $\tilde{k}(t) - \tilde{k}^*$. The solution is

$$\tilde{k}(t) - \tilde{k}^* \approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)t}, \tag{8}$$

which shows how the gap moves and becomes smaller and smaller over time.

Since approximative models are often based on log-linearization, we might ask what the speed of adjustment of $\log \tilde{k}(t) - \log \tilde{k}^*$ is. The answer is: the same! Indeed, by a first-order Taylor approximation of $\log \tilde{k}(t)$ around $\tilde{k}^*$ we have

$$\log \tilde{k}(t) \approx \log \tilde{k}^* + \frac{1}{\tilde{k}^*}(\tilde{k}(t) - \tilde{k}^*), \quad \text{for all } t \geq 0. \tag{9}$$

Dividing by $\tilde{k}^*$ on both sides of (8) gives

$$\begin{aligned} \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*} &\approx \frac{\tilde{k}(0) - \tilde{k}^*}{\tilde{k}^*} e^{-\beta(\tilde{k}^*)t} \qquad \Rightarrow \text{ (by (9))} \\ \log \tilde{k}(t) - \log \tilde{k}^* &\approx (\log \tilde{k}(0) - \log \tilde{k}^*) e^{-\beta(\tilde{k}^*)t} \qquad \Rightarrow \\ -\frac{d(\log \tilde{k}(t) - \log \tilde{k}^*)/dt}{\log \tilde{k}(t) - \log \tilde{k}^*} &\approx \beta(\tilde{k}^*). \end{aligned} \tag{10}$$

We see that (at least in a neighborhood of the steady state) the speed of adjustment of the logarithmic distance to the steady state is the same as the speed of adjustment of the distance of $\tilde{k}$ itself to its steady state.

Comparing with B & S pp. 56-57

By multiplying through by $-(\log \tilde{k}(t) - \log \tilde{k}^*)$ in (10) we get

$$\frac{d\tilde{k}(t)/dt}{\tilde{k}(t)} = \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \approx -\beta(\tilde{k}^*)(\log \tilde{k}(t) - \log \tilde{k}^*) = -\beta(\tilde{k}^*) \log \frac{\tilde{k}(t)}{\tilde{k}^*}.$$

This shows that the B & S result (1.46) holds for a general neoclassical production function.

B & S concentrate on the Cobb-Douglas case:

$$\tilde{y} = A\tilde{k}^\alpha, \qquad A > 0, 0 < \alpha < 1. \tag{11}$$

Then the elasticity of output wrt. capital, $\alpha(\tilde{k})$, equals the constant α for all $\tilde{k} > 0$. In particular $\alpha(\tilde{k}^*) = \alpha$. So, the adjustment speed is

$$(1 - \alpha)(\delta + x + n) \equiv \beta^*, \tag{12}$$

which is independent of $\tilde{k}^*$. This is the same as equation (1.45) in B & S. It is noteworthy that in the Cobb-Douglas case, neither the saving rate, nor the overall level of efficiency of the economy, A, affect the asymptotic speed of adjustment. This is due to two offsetting forces which exactly cancel each other in this case, as explained by B & S, p. 58.

With one year as our time unit, standard parameter values are: $x = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\alpha = 1/3$. Then the adjustment speed for $\tilde{k}$ is $\beta^* = (1-\alpha)(\delta+x+n) = 0.053$ per year.

In the empirical Chapter 11, B & S argue that a lower value of β^*, say 0.02 per year, fits the data better.[2] This requires $\alpha = 0.75$. Such a high value of α ($\approx$ the share of capital income in total income) may seem difficult to defend. But if we reinterpret K in the Solow model to include *human* capital (skills acquired through education and learning by doing), a value of α at that level may not be far out.

2.2 Adjustment speed for $\tilde{y}$

The variable which we are interested in is usually not so much $\tilde{k}$ in itself, but rather labor productivity, $y \equiv Y/L \equiv \tilde{y}T$. We have

$$\frac{y(t) - y^*(t)}{y^*(t)} \equiv \frac{\tilde{y}(t) - \tilde{y}^*}{\tilde{y}^*},$$

where $y^*(t) \equiv \tilde{y}^* T(t) = f(\tilde{k}^*)T(0)e^{xt}$ is the steady-state path. Thus, the interesting question becomes what the adjustment speed of $\tilde{y}$ is.

Differentiating $\tilde{y} = f(\tilde{k})$ wrt. t gives

$$\dot{\tilde{y}} = f'(\tilde{k})\dot{\tilde{k}} = f'(\tilde{k})\left[s(\tilde{y} - \tilde{y}^*) - m(\tilde{k} - \tilde{k}^*)\right],$$

from (2) and (3). In that $\tilde{y} - \tilde{y}^* = f(\tilde{k}) - f(\tilde{k}^*)$, this gives

$$\begin{aligned}
\frac{d(\tilde{y} - \tilde{y}^*)/dt}{\tilde{y} - \tilde{y}^*} &= f'(\tilde{k})\left(s - m\frac{\tilde{k} - \tilde{k}^*}{f(\tilde{k}) - f(\tilde{k}^*)}\right) = f'(\tilde{k})\left(\frac{s}{m} - \frac{\tilde{k} - \tilde{k}^*}{f(\tilde{k}) - f(\tilde{k}^*)}\right)m \\
&= f'(\tilde{k})\left(\frac{\tilde{k}^*}{f(\tilde{k}^*)} - \frac{\tilde{k} - \tilde{k}^*}{f(\tilde{k}) - f(\tilde{k}^*)}\right)m \qquad \text{(from (3))} \\
&= \frac{f'(\tilde{k})}{\frac{f(\tilde{k}) - f(\tilde{k}^*)}{\tilde{k} - \tilde{k}^*}}\left(\frac{\tilde{k}^*}{f(\tilde{k}^*)}\frac{f(\tilde{k}) - f(\tilde{k}^*)}{\tilde{k} - \tilde{k}^*} - 1\right)m.
\end{aligned}$$

[2] There is not general agreement about this. Some authors, for example Islam (1995), using a panel data approach, find a speed of adjustment which is more than the double of this. McQuinne and Whelan (2007) get similar results. (Yet, this still signifies a quite slow adjustment.)

So, the speed of adjustment for y is

$$-\frac{d(\tilde{y}-\tilde{y}^*)/dt}{\tilde{y}-\tilde{y}^*} = \frac{f'(\tilde{k})}{\frac{f(\tilde{k})-f(\tilde{k}^*)}{\tilde{k}-\tilde{k}^*}}\left(1-\frac{\tilde{k}^*}{f(\tilde{k}^*)}\frac{f(\tilde{k})-f(\tilde{k}^*)}{\tilde{k}-\tilde{k}^*}\right)m \qquad (13)$$

$$\rightarrow \left(1-\frac{\tilde{k}^*}{f(\tilde{k}^*)}f'(\tilde{k}^*)\right)m \text{ for } \tilde{k}\rightarrow\tilde{k}^*,$$

since

$$\lim_{\tilde{k}\rightarrow\tilde{k}^*} f'(\tilde{k}) = f'(\tilde{k}^*) = \lim_{\tilde{k}\rightarrow\tilde{k}^*}\frac{f(\tilde{k})-f(\tilde{k}^*)}{\tilde{k}-\tilde{k}^*}.$$

Comparing with (7), we see that, at least in a small neighborhood of the steady state, the speed of adjustment for $\tilde{y}$, in view of the definition in (5), is the same as the speed of adjustment for $\tilde{k}$, namely $\beta(\tilde{k}^*)$.

We might also be interested in the adjustment speed of the logarithmic distance of $\tilde{y}$ to the steady state. Using the definition of $\beta(\tilde{k}^*)$ in (6), by (13), $\tilde{y}(t)-\tilde{y}^* \approx (\tilde{y}(0)-\tilde{y}^*)e^{-\beta(\tilde{k}^*)t}$, so that

$$\frac{\tilde{y}(t)-\tilde{y}^*}{\tilde{y}^*} \approx \frac{\tilde{y}(0)-\tilde{y}^*}{\tilde{y}^*}e^{-\beta(\tilde{k}^*)t},$$

or

$$\log\tilde{y}(t)-\log\tilde{y}^* \approx (\log\tilde{y}(0)-\log\tilde{y}^*)e^{-\beta(\tilde{k}^*)t}.$$

This shows that also the logarithmic distance of $\tilde{y}$ to the steady state declines (in a small neighborhood of the steady state) at the rate $\beta(\tilde{k}^*)$.

3 Adjustment in the large*

But the formulas for the adjustment speeds up to now may be valid only in a small neighborhood of the steady state. What if the economy initially is far away from its steady state? In this context, it is convenient to consider the adjustment of the *growth rate* of $\tilde{k}$.

3.1 Adjustment speed for the growth rate of $\tilde{k}$

We may use the fact that the growth rate $g_{\tilde{k}} \equiv (d\tilde{k}/dt)/\tilde{k}$ during the transitional dynamics changes gradually from some non-zero value to zero (asymptotically) in the steady state. The speed of adjustment of this growth rate is, in analogy with (7),

$$\sigma(\tilde{k}) \equiv -\frac{d(g_{\tilde{k}}-0)/dt}{g_{\tilde{k}}-0} = -\frac{\dot{g}_{\tilde{k}}}{g_{\tilde{k}}}, \qquad (14)$$

that is, $\sigma(\tilde{k})$ is the proportionate rate of decline of g_k during the adjustment process.[3] From (2) we have

$$g_{\tilde{k}} = s\frac{f(\tilde{k})}{\tilde{k}} - m, \tag{15}$$

so that

$$\dot{g}_{\tilde{k}} = s\frac{\tilde{k}f'(\tilde{k})\dot{\tilde{k}} - f(\tilde{k})\dot{\tilde{k}}}{\tilde{k}^2} = s\left(\frac{\tilde{k}}{f(\tilde{k})}f'(\tilde{k}) - 1\right)\frac{f(\tilde{k})}{\tilde{k}}g_{\tilde{k}},$$

or

$$\frac{\dot{g}_{\tilde{k}}}{g_{\tilde{k}}} = (\alpha(\tilde{k}) - 1)s\frac{f(\tilde{k})}{\tilde{k}}, \tag{16}$$

using the definition in (5). Thus, by (14), the speed of adjustment of g_k is

$$\sigma(\tilde{k}) = (1 - \alpha(\tilde{k}))s\frac{f(\tilde{k})}{\tilde{k}}. \tag{17}$$

This result is exact whatever the distance to the steady state and holds for any neoclassical $f(\cdot)$.

Comparing (17) with (7), we have, in view of (3) and $m \equiv \delta + x + n$, that close to the steady state, the adjustment speed of the growth rate of $\tilde{k}$, $\sigma(\tilde{k})$, is approximately the same as the adjustment speed of $\tilde{k}$ itself, $\beta(\tilde{k}^*)$.[4]

The Cobb-Douglas case

In the Cobb-Douglas case (11) we have $\alpha(\tilde{k}) = \alpha$ and $\beta(\tilde{k}^*) = \beta^*$, from (12). For this case, comparing (17) with (7), we get

$$\sigma(\tilde{k}) \gtreqless \beta^* \quad \text{for} \quad s\frac{f(\tilde{k})}{\tilde{k}} \gtreqless \delta + x + n, \text{ respectively,}$$
$$\text{that is, for } \tilde{k} \lesseqgtr \tilde{k}^*, \text{ respectively,}$$

in view of (4). Hence, if the Cobb-Douglas economy is below its steady state ($\tilde{k} < \tilde{k}^*$), then the adjustment speed $\sigma(\tilde{k})$ exceeds β^* and is declining over time. On the other hand, if the economy is above its steady state ($\tilde{k} > \tilde{k}^*$), then the adjustment speed $\sigma(\tilde{k})$ is lower than β^* and is rising over time.[5] This difference is due to the concavity of the production function. The result goes through approximately, even with the general production function, $f(\tilde{k})$, if $\alpha(\tilde{k})$ is not too elastic with respect to $\tilde{k}$.

[3] Note that $-\sigma(\tilde{k})$ is the growth rate of the growth rate of $\tilde{k}$.

[4] In the Cobb-Douglas case, (17) gives the formula (1.44) in B & S. For further comparison with B & S, see Appendix.

[5] Unfortunately, this last possibility is overlooked by B & S in their statement on p. 57 (in the middle, after (1.45)).

The capital-output ratio in the Cobb-Douglas case

Interestingly, the adjustment speed of the capital-output ratio, $v \equiv K/Y = \tilde{k}^{1-\alpha}$, can in the Cobb-Douglas case be shown to be *independent* of how far away from the steady state the economy is. Indeed, using the closed-form solution for v on p. 44 in B & S, we see that the adjustment speed of v is $(1-\alpha)(\delta+x+n)$ (which is the same as our β^*) independently of the distance of v to its steady-state value, $v^* = sA/(\delta+x+n)$.[6]

3.2 Adjustment speed for the growth rate of $\tilde{y}$

Consider the Cobb-Douglas case. We get, by logarithmic differentiation wrt. t in (11),

$$g_{\tilde{y}} = \alpha g_{\tilde{k}}.$$

It follows that

$$\frac{\dot{g}_{\tilde{y}}}{g_{\tilde{y}}} = \frac{\dot{g}_{\tilde{k}}}{g_{\tilde{k}}} = (\alpha-1)sA\tilde{k}^{\alpha-1} = -\sigma(\tilde{k}),$$

from (16) and (17). Hence, the adjustment speed of the growth rate of $\tilde{y}$ is the same as that of the growth rate of $\tilde{k}$. This is an exact result in the Cobb-Douglas case, but only approximately true in the general case.

4 Intuitive interpretation*

As indicated by the formula (17), the adjustment speed, $\sigma(\tilde{k})$, for $g_{\tilde{k}}$ depends negatively on $\alpha(\tilde{k})$ and positively on the saving-capital ratio, $sf(\tilde{k})/\tilde{k}$. The formula is useful because these determinants are more or less directly observable. We have data for s and can construct data for the output-capital ratio, $f(\tilde{k})/\tilde{k} = Y/K$ (cf. B & S, p. 436 ff.) (and close to steady state, we simply have $sf(\tilde{k})/\tilde{k} \approx \delta+x+n$). Further, $\alpha(\tilde{k})$ can be approximated by the capital income share.

But what is the intuition behind the result that the adjustment speed, $\sigma(\tilde{k})$, depends negatively on $\alpha(\tilde{k})$ and positively on the saving-capital ratio, $sf(\tilde{k})/\tilde{k}$. Or, to put it differently, how comes that high $1-\alpha(\tilde{k})$ and high $sf(\tilde{k})/\tilde{k}$ result in high adjustment speed? To throw light on this, note that:

(i) The product of these two factors (which equals $\sigma(\tilde{k})$) measures a kind of *sensitivity* of the saving-capital ratio, $S/K = sf(\tilde{k})/\tilde{k}$, wrt. $\tilde{k}$.

[6] Here we have extended the result in B & S, p. 44, to the case $x > 0$.

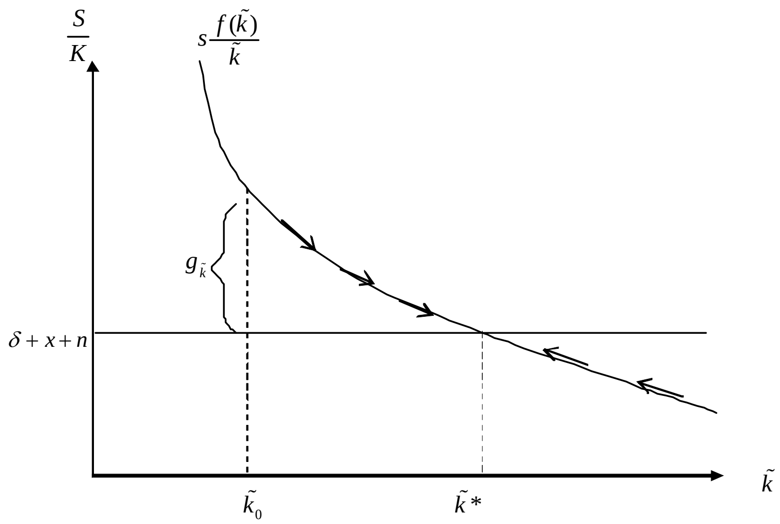

Figure 3:

(ii) The saving-capital ratio is of key importance for how fast $\tilde{k}$ changes.

Observation (ii) is an immediate implication of (15). And (i) can be seen by calculating the fall in S/K triggered by a rise in $\tilde{k}$. We get

$$\begin{aligned}
-\frac{d(S/K)}{d\tilde{k}} &= -\frac{d(S/K)}{d\tilde{k}} = -\frac{d(sf(\tilde{k})/\tilde{k})}{d\tilde{k}} = -s\frac{\tilde{k}f'(\tilde{k}) - f(\tilde{k})}{\tilde{k}^2} \\
&= (1 - \frac{\tilde{k}f'(\tilde{k})}{f(\tilde{k})})\frac{sf(\tilde{k})}{\tilde{k}^2} = (1 - \alpha(\tilde{k}))\frac{sf(\tilde{k})}{\tilde{k}^2}.
\end{aligned}$$

Thus, defining the *sensitivity* of the saving-capital ratio wrt. $\tilde{k}$ as $-d(S/K)/(d\tilde{k}/\tilde{k})$, this sensitivity is

$$(1 - \alpha(\tilde{k}))\frac{sf(\tilde{k})}{\tilde{k}} = \sigma(\tilde{k}),$$

by (7). In this way the speed of change in $\tilde{k}$ is governed by $sf(\tilde{k})/\tilde{k}$, and the offsetting feedback on $sf(\tilde{k})/\tilde{k}$ is stronger, the higher is $\sigma(\tilde{k})$.

The phase diagram in Fig. 3 illustrates some of this. The arrows come from (15) with $m \equiv \delta + x + n$. That a higher $\alpha(\tilde{k})$ results in weaker feedback reflects that $1 - \alpha(\tilde{k})$ is in fact identical to the absolute elasticity of the output-capital ratio wrt. $\tilde{k}$. Indeed,

$$-\frac{\tilde{k}}{f(\tilde{k})/\tilde{k}}\frac{d(f(\tilde{k})/\tilde{k})}{d\tilde{k}} = 1 - \frac{\tilde{k}f'(\tilde{k})}{f(\tilde{k})} \equiv 1 - \alpha(\tilde{k}).$$

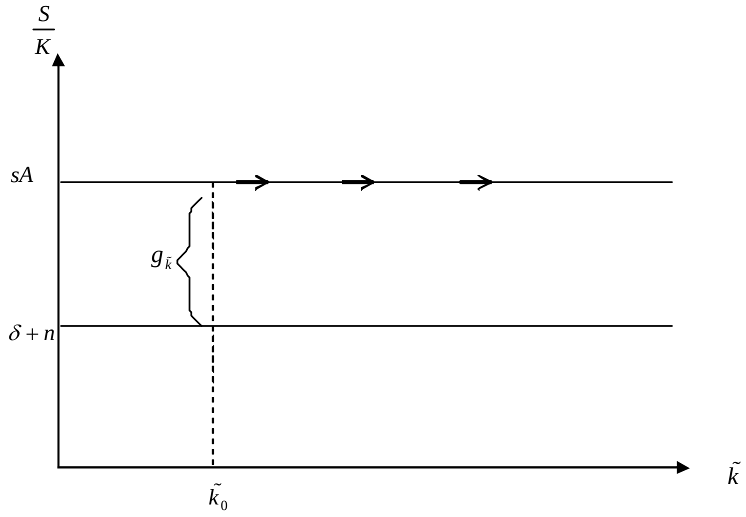

Figure 4:

When $\tilde{k}$ increases, the average product of capital (the output-capital ratio) decreases, but this degree of diminishing returns is smaller, the higher is $\alpha(\tilde{k})$.

It may clarify to go to the extreme. In the Cobb-Douglas case, we can consider the limiting case where $\alpha = 1$ and $x = 0$, that is, the so-called *AK model*, $y = Ak$. Then $\dot{k}/k = sA - (\delta + x + n) > 0$ if $s > (\delta + x + n)/A$. There is *never* "adjustment", because the capital intensity never settles down, cf. Fig. 4.

5 Adjustment time

Let τ_ω be the time that it takes for the fraction $\omega \in (0, 1)$ of the gap between $\tilde{k}$ and $\tilde{k}^*$ to be eliminated, i.e., τ_ω satisfies the equation

$$\frac{\tilde{k}(\tau_\omega) - \tilde{k}^*}{\tilde{k}(0) - \tilde{k}^*} = 1 - \omega.$$

Now, by (8), we have

$$\begin{aligned} \tilde{k}(\tau_\omega) - \tilde{k}^* &\approx (\tilde{k}(0) - \tilde{k}^*)e^{-\beta(\tilde{k}^*)\tau_\omega}, \qquad \text{or} \\ 1 - \omega &\approx e^{-\beta(\tilde{k}^*)\tau_\omega}. \end{aligned}$$

Taking logs on both sides and solving for τ_ω gives

$$\tau_\omega \approx -\frac{\log(1-\omega)}{\beta(\tilde{k}^*)}. \tag{18}$$

This is the *adjustment time* required to eliminate the fraction ω of the distance to steady state, when the adjustment speed is $\beta(\tilde{k}^*)$.

Often we consider the *half-life*, that is, the time that it takes for half the initial gap to be eliminated. To find the half-life, we put $\omega = \frac{1}{2}$ in (18). Again we use one year as our time unit. With the standard parameter values: $x = 0.02$, $n = 0.01$, $\delta = 0.05$, and $\alpha = 1/3$ (we consider the Cobb-Douglas case), we have $\beta^* = (1-\alpha)(\delta + x + n) = 0.053$ per year and thus

$$\tau_{\frac{1}{2}} \approx -\frac{\log\frac{1}{2}}{0.053} \approx \frac{0.69}{0.053} = 13,1 \text{ years.}$$

As mentioned earlier, in Chapter 11 B & S argue that a lower value of β^*, say 0.02 per year, fits the data better. This requires $\alpha = 0.75$. Such a high value of α may be allowed if we reinterpret K in the Solow model as including *human* capital. With $\beta^* = 0.02$, the half-life is

$$\tau_{\frac{1}{2}} \approx -\frac{\log\frac{1}{2}}{0.02} \approx \frac{0.69}{0.02} = 34.7 \text{ years.}$$

The time needed to eliminate three quarters of the distance to steady state, $\tau_{3/4}$, is about 70 years ($= 2 \cdot 35$ years, since $1 - 3/4 = \frac{1}{2} \cdot \frac{1}{2}$).

6 Do poor countries tend to approach their steady state *from below*?

In contrast to the impression you get from B & S, we cannot be sure that an economy (which can be approximately described by, for example, the Solow model or the Ramsey model) will tend to approach its steady state *from below*. This is because the size of $\tilde{k}_t$ relative to $\tilde{k}^*$ for a poor country is ambiguous. Indeed,

$$k_t \equiv \frac{K_t}{T_t L_t},$$

where K_t/L_t is typically low for a poor country, but the technology level, T_t, is also typically relatively low for a poor country. Hence, whether $\tilde{k}_t < \tilde{k}^*$ or $\tilde{k}_t > \tilde{k}^*$ is not obvious at first sight.[7]

[7] To put it differently, several places B & S implicitly presuppose that K in poor countries grows at a rate higher than $x + n$. But the opposite inequality is *a priori* equally plausible.

There is empirical evidence indicating that, "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions" (Cho and Graham, 1996).

7 Appendix

Our result (17) may be a surprise, because B & S define in their (1.42) their β (our $\sigma(\tilde{k})$) in a seemingly different (and, in our opinion, less intuitive) way compared to our definition in (14). But our (14) *implies* (1.42) in B & S. Indeed, elaborating on (14) we can write

$$\sigma(\tilde{k}) = -\frac{d\gamma_{\tilde{k}}/dt}{d\log\tilde{k}/dt} = -\frac{d\gamma_{\tilde{k}}}{d\log\tilde{k}} \qquad \text{(by eliminating } dt\text{)},$$

which is the same as (1.42) in B & S.

8 References

Cho, D., and S. Graham, 1996, The other side of conditional convergence, *Economics Letters 50*, 285-290.

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