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Some basic relationships in growth theory

In this note we present some fundamental propositions about balanced growth. The first and the third proposition are not presented in B & S, and the second proposition (Uzawa's theorem) is presented in a less general way (B & S, pp. 53, 78-80).

1 Balanced growth and constancy of key ratios

First we shall attempt to clarify the terms "balanced growth" and "steady state", often used in growth theory. These terms are not always defined in the same way by different authors. In the literature you will sometimes find, for example, "steady state" defined in a way somewhat different from how B & S define it. Moreover, what are the connections between the terms and how do they relate to the hypothesis of Harrod-neutral technical progress and Kaldor's stylized facts?

1.1 The concepts of steady state and balanced growth

A basic equation from one-sector growth models for a closed economy in continuous time is

$$\dot{K} = S - \delta K \equiv Y - C - \delta K, \tag{1}$$

where K is aggregate capital, S aggregate gross saving (in a closed economy equal to aggregate gross investment), Y aggregate output, C aggregate consumption, and $\delta \geq 0$ is a constant physical capital depreciation rate. We may for example take the Solow model as our point of reference. Then gross saving equals sY, where s is a constant, $0 < s < 1$. Given a neoclassical production function, F, with CRS and Harrod-neutral technical progress, we have $Y = F(K, TL) = TLF(\tilde{k}, 1) \equiv TLf(\tilde{k})$, where L is the labor force, T is the level of technology, $\tilde{k} \equiv K/(TL)$ is the "effective" capital intensity, and $f' > 0$ and $f'' < 0$. Solow assumes $L(t) = L(0)e^{nt}$ and $T(t) = T(0)e^{\gamma t}$, where $n \geq 0$ and

$\gamma \geq 0$ are the constant growth rates of the labor force and technology, respectively.[1] By log-differentiating $\tilde{k}$ wrt. t,[2] the model can be reduced to the *fundamental differential equation* ("law of movement") of the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + \gamma + n)\tilde{k}. \tag{2}$$

Often in the theoretical literature on dynamic models, a *steady state* is defined in the following way:

Definition 1 *A steady state of a model is a stationary solution of the fundamental differential equation(s) of the model.*

That is, in the Solow model, given (2), a (non-trivial) steady state is a $\tilde{k}^* > 0$ such that, if $\tilde{k} = \tilde{k}^*$, then $\dot{\tilde{k}} = 0$. More generally, a *steady state is a stationary state of a dynamic process.*

On pp. 33-34 B & S define a steady state in a slightly different way. They define a steady state as a path $(Y, K, C)_{t=0}^{\infty}$ along which Y, K, and C grow at *constant* rates (not necessarily positive). In the Solow model as well as in many other models for a closed economy steady state in the first meaning implies steady state in the second meaning - and vice versa. In those cases, whether one or the other definition is used is not of great importance.

But there *are* cases where this equivalence does not hold (some open economy models and models with investment-specific learning). Therefore, it seems better to make, right from the beginning, a terminological distinction between the two concepts. Since the first definition of a steady state comes from a broadly used terminology in macroeconomics, also outside growth theory, I will stick to that definition and use the term *balanced growth,* for the second concept. That is, assuming Y, K, and C are positive for all t considered (otherwise growth rates are not defined), we have:

Definition 2 *A balanced growth path is a path $(Y, K, C)_{t=0}^{\infty}$ along which the quantities Y, K, and C grow at constant rates (not necessarily positive).*

[1] Our γ corresponds to what B & S call x.

[2] Or by directly using the fraction rule, see Appendix.

Note that this definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, there may be, under certain conditions, structural change in the sense of a changing sectoral composition of the economy, cf. the Kuznets facts (see Lecture Note 2).[3]

1.2 A general result on balanced growth

We now leave the specific Solow model. Interestingly, given (1), we have always, independently of how saving is determined and how the labor force and technology change, that if there is balanced growth, then the ratios Y/K and C/K are constant. And also the other way round: constancy of the Y/K and C/K ratios is enough to ensure balanced growth.

Since this very useful general result is not explicit in B & S, we shall here state it in a precise way and prove it. Letting g_x denote the growth rate of the (positive) variable x, i.e., $g_x \equiv \dot{x}/x$, we claim:

Proposition 1 *(equivalence of balanced growth and constancy of key ratios). Let* $(Y, K, C)_{t=0}^{\infty}$ *be a path along which* Y, K*,* C*, and* $S \equiv Y - C$ *are positive for all* $t \geq 0$*. Then, given the accumulation equation (1), the following holds:*

(i) if there is balanced growth, then the ratios Y/K*,* C/K*,* C/Y*, and* S/Y *are constant;*

(ii) if Y/K *and* C/K *are constant, then* Y, K, C*, and* S *all grow at the same constant rate, i.e., there is balanced growth.*

Proof (i) Consider a path $(Y, K, C)_{t=0}^{\infty}$ along which Y, K, C and $S \equiv Y - C$ are positive for all $t \geq 0$. a) Assume there is balanced growth. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (1), we have that $S/K = g_K + \delta$ is constant, implying

$$g_S = g_K. \tag{3}$$

Further, since $Y = C + S$,

$$\begin{aligned} g_Y &= \frac{\dot{Y}}{Y} = \frac{\dot{C}}{Y} + \frac{\dot{S}}{Y} = g_C\frac{C}{Y} + g_S\frac{S}{Y} = g_C\frac{C}{Y} + g_K\frac{S}{Y} \qquad \text{(by (3))} \\ &= g_C\frac{C}{Y} + g_K\frac{Y - C}{Y} = \frac{C}{Y}(g_C - g_K) + g_K. \end{aligned} \tag{4}$$

[3] A nice reference on this is Kongsamut et al. (2001).

Now, let us provisionally assume that $g_K \neq g_C$. Then (4) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{5}$$

a constant, so that $g_C = g_Y$. Inserting this latter equation into (5) gives $C/Y = 1$. In view of (1), however, this result contradicts the given condition that $S > 0$. Hence, our provisional assumption is wrong, and we have $g_K = g_C$ without the condition $C/Y = 1$ being implied. By (4), this implies $g_Y = g_K = g_C$, from which follows that Y/K and C/K are constant. Then, also $C/Y = (Y/K)/(C/K)$ is constant, and so is S/Y since, in view of (1), $S/Y = (Y - C)/Y = 1 - C/Y$.

(ii) Suppose Y/K and C/K are constant. Then $g_Y = g_K = g_C$, so that C/Y is a constant. We can now show that g_K is constant. Indeed, from (1) $S/Y = 1 - C/Y$, so that also S/Y is constant. It follows that $g_S = g_Y = g_K$, so that S/K is constant. By (1),

$$\frac{S}{K} = \frac{\dot{K} + \delta K}{K} = g_K + \delta,$$

so that also g_K is constant. Consequently, also g_Y, g_C, and g_S are constant. □

The point is that this proposition can be used for *any* model for which (1) is valid. Further, we see that a model generating balanced growth is almost immediately a candidate for an explanation of Kaldor's stylized facts.

2 The crucial role of Harrod-neutrality

Here we present a fairly general theorem about the necessity and sufficiency of Harrod-neutral technical progress if balanced growth should be possible.

To the extent that one accepts Kaldor's stylized facts as a description of the past century's growth experience, a natural requirement of any growth model pretending to cover industrialized countries over the last century is that it should be able to generate balanced growth. Already Uzawa (1961) showed that given this requirement, technical progress must take the Harrod-neutral form (i.e., be labor-augmenting), at least along the balanced growth path. This is so whether technical progress is exogenous or endogenous.

Let the aggregate production function be

$$Y(t) = \tilde{F}(K(t), L(t); t) \tag{6}$$

where $\tilde{F}$ has CRS wrt. the first two arguments. As a representation of continuous technical progress, we assume $\partial\tilde{F}/\partial t \geq 0$ for all $t \geq 0$ (i.e., as time proceeds, unchanged inputs result in more and more output). We also assume that the labor force grows according to

$$L(t) = L(0)e^{nt}, \quad n \geq 0, \tag{7}$$

where n is constant. Further, non-consumed output is invested and (1) is the dynamic resource constraint of the economy.

Proposition 2 *(Uzawa's balanced growth theorem) Let* $(Y(t), K(t), C(t))_{t=0}^{\infty}$*, where* $0 < C(t) < Y(t)$ *for all* $t \geq 0$*, be a path satisfying the capital accumulation equation (1), given the CRS-production function (6) and labor force (7). Then:*

(i) a necessary condition for this path to be a balanced growth path is that along the path it holds that

$$Y(t) = \tilde{F}(K(t), L(t); t) = \tilde{F}(K(t), T(t)L(t); 0), \tag{8}$$

where $T(t) = e^{\gamma t}$ *with* $\gamma \equiv g_Y - n$*;*

(ii) for any $\gamma > 0$ *such that there is a* $q > \gamma + n + \delta$ *with the property that* $\tilde{F}(1, k^{-1}; 0) = q$ *for some* $k > 0$ *(i.e., at* $t = 0$ *the production function* $\tilde{F}$ *in (6) allows an output-capital ratio equal to* q*), a sufficient condition for the existence of a balanced growth path with output-capital ratio* q*, is that the technology can be written as in (8) with* $T(t) = e^{\gamma t}$*.*

Proof (i)[4] Suppose the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K(t) = K(0)e^{g_K t}$ and $Y(t) = Y(0)e^{g_Y t}$. With $t = 0$ in (6) we then have

$$Y(t)e^{-g_Y t} = Y(0) = \tilde{F}(K(0), L(0); 0) = \tilde{F}(K(t)e^{-g_K t}, L(t)e^{-nt}; 0). \tag{9}$$

In view of the assumption of positive saving $S(t) \equiv Y(t) - C(t)$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (9) then implies

$$Y(t) = \tilde{F}(K(t), e^{g_Y t}e^{-nt}; 0).$$

[4] This part draws heavily on Jones and Schlicht (2006). Note that the claim to be proved is more general than that in B & S, pp. 53 and 78-80. They start from $Y = F(BK, AL)$ instead of the more general (6).

We see that (8) holds for $T(t) = e^{\gamma t}$ with $\gamma \equiv g_Y - n$.

(ii) Suppose (8) holds with $T(t) = e^{\gamma t}$. Let $\gamma > 0$ be given such that there is a $q > \gamma + n + \delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$. We claim that with $K(0) = kL(0)$, $s \equiv (\gamma + n + \delta)/q$, and $S(t) = sY(t)$, (1), (7), and (8) imply $Y(t)/K(t) = q$ for all $t \geq 0$.

By construction

$$\frac{Y(0)}{K(0)} = \frac{\tilde{F}(K(0), L(0); 0)}{K(0)} = \tilde{F}(1, k^{-1}; 0) = q = \frac{\delta + \gamma + n}{s}. \tag{10}$$

Now, by (1), $\dot{K}(0)/K(0) = sY(0)/K(0) - \delta = \gamma + n$, so that K initially grows at the same rate as effective labor input, $T(t)L(t)$. Then, in view of $\tilde{F}$ being homogeneous of degree one wrt. its first two arguments, also Y grows initially at this rate. As an implication, the ratio Y/K does not change, but remains equal to the right-hand side of (10) for all $t \geq 0$. Consequently, K, Y and C continue to grow at the same constant rate. □

The form (8) indicates that along a balanced growth technical progress must be purely "labor augmenting", that is, Harrod-neutral. It is important to recognize that the occurrence of Harrod-neutrality says nothing about what the *source* of the technical progress is. Harrod-neutrality should not be interpreted as indicating that the technical change emanates specifically from the labor input. The crux of the matter is just that an invention has been made such that labor and physical capital together are more productive and that this happens to *manifest itself* in the form (8).

Note the generality of Uzawa's theorem. It does not presuppose the technology is neoclassical, not to speak of the Inada conditions. A simple implication of the theorem is the following. Let output per unit of labor ("labor productivity") be denoted $y(t) \equiv Y(t)/L(t)$. Interpreting the $T(t)$ in (8) as the "level of technology", we have:

COROLLARY In a balanced growth path with positive gross saving and the technology level growing at the rate γ, output grows at the rate $\gamma + n$ and labor productivity at the rate γ.

Proof That $g_Y = \gamma + n$ follows from (i) of the proposition. As to g_y we have

$$y_t = \frac{Y(0)e^{g_Y t}}{L(0)e_0^{nt}} = y(0)e^{(g_Y - n)t} = y(0)e^{\gamma t}. \qquad \square$$

We shall now consider the implication of Harrod-neutrality for the income shares of capital and labor under perfect competition.

3 Harrod-neutrality and the functional income distribution

There is one facet of Kaldor's stylized facts we have not yet related to Harrod-neutral technical progress, namely the long-run "approximate" constancy of the income share of labor, wL/Y, and the rate of return to capital, $(Y - wL - \delta K)/K$. But at least with neoclassical technology and perfect competition in the output and factor markets these properties are inherent in the combination of constant returns to scale and balanced growth.

To see this, let $Y(t) = F(K(t), T(t)L(t)) \equiv \tilde{F}(K(t), T(t)L(t); 0)$ where F is *neoclassical* and has CRS. Let the (gross) capital income share in period t be denoted $\alpha(t)$, that is,

$$\alpha(t) = \frac{(r(t)+\delta)K(t)}{Y(t)} = \frac{(\partial Y(t)/\partial K(t))K(t)}{Y(t)}.$$

The labor income share will be

$$\frac{w(t)L(t)}{Y(t)} = \frac{(\partial Y(t)/\partial L(t))L(t)}{Y(t)}.$$

Proposition 3 *(factor income shares) Suppose the path $(K(t), Y(t), C(t))_{t=0}^{\infty}$ is a balanced growth path in this economy with positive saving. Then $\alpha(t) = \alpha$, a constant $\in (0,1)$. The labor income share will be $1-\alpha$ and the rate of return on capital $\alpha q - \delta$, where q is the constant output-capital ratio along the balanced growth path.*

Proof We have $Y(t) = F(K(t), T(t)L(t)) = T(t)L(t)F(\tilde{k}(t), 1) \equiv T(t)L(t)f(\tilde{k}(t))$. By balanced growth, $Y(t)/K(t)$ is some constant, q. Since $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t)$ and $f'' < 0$, this implies $\tilde{k}(t)$ constant, say equal to $\tilde{k}^*$. But $\partial Y(t)/\partial K(t) = f'(\tilde{k}(t))$, which then equals the constant $f'(\tilde{k}^*)$ along the balanced growth path. It follows that $\alpha(t) = f'(\tilde{k}^*)/q$ $\equiv \alpha \in (0,1)$ in view of $f'' < 0$. By Euler's theorem,[5] then $(\partial Y(t)/\partial L(t))L(t)/Y(t) = 1-\alpha(t) = 1-\alpha \in (0,1)$. Finally, the rate of return on capital is $(1-w(t)L(t))Y(t)/K(t)-\delta$ $= \alpha q - \delta$. □

Among other things, this proposition is of interest in relation to one of the "stylized facts" claimed by Kaldor, namely the one saying that the functional income distribution has no trend. Note, however, that the proposition implies constancy of the income shares,

[5] From Euler's theorem, $KF_K + LF_L = F(K, L)$, when F is homogeneous of degree one.

but does not *determine* them, except in terms of q. But q (or $\tilde{k}^*$) will generally be unknown as long as we have not specified a theory of saving. This takes us to theories of aggregate saving, for example the Ramsey model.

4 Appendix: Growth formulas in continuous time

Let the variables z, x, and y be continuous and differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE:

$$z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}.$$

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation wrt. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* wrt. t.

FRACTION RULE

$$z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}.$$

The proof is similar.

POWER FUNCTION RULE

$$z(t) = x(t)^\alpha \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha \frac{\dot{x}(t)}{x(t)}.$$

The proof is similar.

An advantage of continuous time analysis is that these simple rules are exactly true in continuous time. In discrete time, the analogue formulas are only approximately true and the approximation can be quite bad unless $\dot{x}(t)/x(t)$ and $\dot{y}(t)/y(t)$ are "small".

5 References

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