VII

R&D and horizontal innovations

VII.1 *The production side of the Lab-Equipment Model (avoiding arbitrary parameter links)* Consider a closed economy with a given aggregate labor supply L, constant over time. There are three production sectors:

Firms in Sector 1 produce basic goods, in the amount Y_t per time unit, under perfect competition.

Firms in Sector 2 produce specialized intermediate goods, in the amount Q_t per time unit, under monopolistic competition and barriers to entry.

Firms in Sector 3 perform R&D to develop technical designs ("blueprints") for new specialized intermediate goods under conditions of perfect competition and free entry.

Basic goods and intermediate goods are nondurable goods. There is no physical capital in the economy. There is a labor market and a market for loans, both with perfect competition. All firms are profit maximizers. Time is continuous.

The representative firm in Sector 1 has the production function

$$Y_t = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right) L_t^{\beta}, \qquad A > 0, \quad 0 < \beta < 1, \tag{1}$$

where Y_t is output per time unit, x_{it} is input of intermediate good i $(i = 1, 2, ..., N)$, N_t is the number of different types of intermediate goods available at time t, and L_t is labor input. At any point in time, the firms in Sector 1 take this number of "varieties" as given. Aggregate labor supply equals a constant, L, and Sector 1 is the only sector that directly uses labor. In view of clearing in the labor market (where perfect competition is assumed to rule), $L_t = L$, which can be substituted into (1).

The output of basic goods is used partly for consumption, $C_t \equiv c_t L$, partly as input in sector 2, X_t, and partly for R&D investment in Sector 3, Z_t :

$$Y_t = X_t + C_t + Z_t, \tag{*}$$

where in equilibrium $X_t = \psi Q_t$, $\psi > 0$.

Unless needed for clarity, the dating of the time-dependent variables is from now on implicit. Let the basic good be the numeraire and let p_i denote the price of intermediate good i.

a) Find the demand for intermediate good i conditional on full employment. What is the price elasticity of this demand?

b) Suppose $p_i = p$, $\forall i$. Show that the assumed production function, (1), in this case is in conformity with the classical idea from Adam Smith that "there are gains by specialization and division of labor" or, with another formulation, "variety is productive". *Hint:* check how a rise in N affects Y for given L and given total input of intermediates, Q_t.

After having invented the technical design i, the inventor in Sector 3 has taken out (free of charge) a perpetual patent on the commercial use of this design. The inventor then entered Sector 2, starting to supply the new intermediate good corresponding to this design, that is, the intermediate good i. Performing this role, the inventor is called *firm* i. Given the technical design i, firm i can effortlessly transform basic goods into intermediate goods of type i simply by pressing a button on a computer, thereby activating a computer code. The following linear transformation rule applies to all $i = 1, \ldots, N$:

it takes $\psi x_i > 0$ units of the basic good to supply x_i units of intermediate good i,

that is, ψ is the marginal = average cost of supplying intermediate goods.

The market value of firm i in Sector 2 can be written

$$V_{it} = \int_t^\infty \pi_{is} e^{-\int_t^s r_\tau d\tau} ds,$$

where π_{is} is the profit at time s and r_τ is the discount rate at time τ. Since there is a time lag between R&D outlay and a successful R&D outcome and this time lag is stochastic, research is risky. It is assumed, however, that all risk is ideosyncratic and that the economy is "large" with "many" firms in all sectors. By holding their financial wealth in the form of balanced portfolios consisting of diversified equity shares in innovative firms in Sector 2 and 3, investors (the households) can thus essentially avoid risk. This allows the research labs to act in a risk-neutral manner.

c) Interpret the expression for V_{it}. What is the relevant discount rate, r_τ?

Being a monopolist, firm i is a price setter and thus chooses a time path $(p_{is})_{s=t}^{\infty}$ so as to maximize the market value of the firm.

d) This problem can be reduced to a series of static profit maximization problems. Why? Solve the problem. Comment.

e) Show that in general equilibrium,

$$\begin{aligned} x_{it} &= \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv \varphi(A,\beta,\psi)L \equiv x, \text{ for all } i,t, \\ \pi_{it} &= (p_{it}-\psi)x_{it} = (\frac{\psi}{1-\beta}-\psi)x = \frac{\beta}{1-\beta}\psi\varphi(A,\beta,\psi)L \equiv \pi \text{ for all } i,t, \\ V_{it} &= \pi\int_t^\infty e^{\int_t^s r_\tau d\tau} ds \equiv V_t \text{ for all } i, \end{aligned}$$

To simplify the formulas Acemoglu (pp. 434, 436) introduces two (not entirely innocent) parameter links:

$$A = \frac{1}{1-\beta}, \quad \text{and} \quad \psi = 1-\beta. \qquad (**)$$

f) Find x_{it} and π_{it} in this special case. In what sense may introducing parameter links in a model be "risky"?

From now we ignore (**) and return to the general case where A and ψ are independent of β.

All the R&D firms in Sector 3 face the same simple "research technology". The rate at which successful research outcomes arrive is proportional to the flow investment of basic goods into research. Consider R&D firm j. Let z_{jt} be the amount of basic goods per time unit the firm devotes at time t in its endeavor to make an invention. With η_{jt} denoting the instantaneous success arrival rate, we have

$$\eta_{jt} = \eta z_{jt}, \qquad \eta > 0,$$

where η is a given parameter reflecting "research productivity".

g) Give a verbal intuitive argument for the claim that the expected payoff per unit of basic goods devoted to R&D per time unit is $V_t\eta$, where V_t is the market value of an arbitrary firm in Sector 2.

h) At time t, let there be J_t R&D firms, indexed by $j = 1, 2, \ldots, J_t$. So aggregate research input is $Z_t \equiv \sum_{j=1}^{J_t} z_{jt}$. "In equilibrium with $Z_t > 0$, we must have $V_t\eta = 1$." True or false? Why?

i) Show that the risk-free real interest rate in equilibrium is a constant and equals $\eta\pi$. *Hint:* consider the no-arbitrage condition for the asset markets.

Under the simplifying assumption of independence, no memory, and no overlap in research, the expected aggregate number of inventions per time unit at time t is ηZ_t.

j) Ignoring indivisibilities and appealing to the law of large numbers, relate $\dot{N}_t$ ($\equiv dN_t/dt$) to Z_t.

Problem VII.2 below considers this economy from a national income accounting perspective. Problem VII.3 introduces a household sector into the model and considers the growth rate of c_t and N_t in general equilibrium.

VII.2 *National income accounting in the lab-equipment model* Here we consider the same model and use the same notation as in Problem VII.1 (it is an advantage if you have already solved at least e) and h) of that problem). We assume that general equilibrium obtains in the economy.

a) A correct answer to e) of Problem VII.1 implies that, the total quantity, Q_t, of intermediate goods produced per time unit at time t can be written $Q_t = xN_t$. Why?

b) Referring to (*), we have $X_t = \psi x N_t$. Why?

c) Show that

$$GDP_t = Y_t - \psi x N_t.$$

Hint: add up the value added in the three sectors and apply the conclusion to h) of Problem VII.1.

d) We also have

$$GDP_t = C_t + S_t,$$

and

$$GDP_t = w_t L + \pi N_t,$$

where S_t is aggregate net saving, w_t is the real wage, and π is profit per firm in Sector 2. Explain these two equations.

VII.3 *R&D-driven fully endogenous growth* We consider the same model and use the same notation as in Problem VII.1 (it is an advantage if you have already solved that problem). We "close" the model by specifying the household sector.

Suppose there are L infinitely-lived households (L "large"), all alike. Each household supplies inelastically one unit of labor per time unit. Given $\theta > 0$ and $\rho > 0$, each household chooses a plan $(c_t)_{t=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= r a_t + w_t - c_t, \qquad a_0 \text{ given,} \\ \lim_{t\to\infty} a_t e^{-rt} &\geq 0, \end{aligned}$$

where a_t is financial wealth.

a) Express a_t in terms of V and N_t as defined in Problem VII.1; comment on the absence of a time subscript on r.

b) Find the growth rate of c_t in general equilibrium; comment on your result. *Hint:* results in i) and e) of Problem VII.1 are useful here.

We assume that the parameter values are such that there is positive consumption growth.

c) Write down the required parameter restriction.

d) Write down the parameter restriction needed to ensure that the utility integral U_0 is bounded.

e) By defining $\hat{A}$ appropriately, in an equilibrium with $Z_t > 0$ we have the following relationship (which is useful in many contexts):

$$Y_t = \hat{A} L N_t = \psi x N_t + C_t + \eta^{-1} \dot{N}_t. \tag{Δ}$$

Show that

$$\hat{A} = A\left(\frac{A(1-\beta)^2}{\psi}\right)^{(1-\beta)/\beta}$$

and derive the second equality in (Δ). *Hint:* as to the second equality in (Δ), the conclusion to j) of Problem VII.1 may be of help.

f) Find the growth rate of N_t and Y_t; comment on your result. *Hint:* there are two features of the model that indicates it is a kind of reduced-form AK model; this allows you to give a quick answer.

g) How does the growth rate of c depend on η and L, respectively? Comment on the intuition.

h) "The resource allocation in the economy is not Pareto optimal'". True or false? Why?

VII.4 In an economy described by the Lab-Equipment Model it is likely that inefficiency problems are present under laissez-faire. Hence we introduce a "social planner". The technologies and households are as described in the problems VII.1 and VII.3. The social planner's criterion function is the same as that of the representative household.

a) Why is it likely that inefficiency problems are present under laissez-faire?

The social planner faces both a static problem and a dynamic problem. The static problem is to ensure that Sector 1 uses the "right" quantities of intermediate goods. Output in Sector 1 (basic goods) is

$$Y_t = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right)L^{\beta}, \qquad A>0, \quad 0<\beta<1,$$

where Y_t is output per time unit, x_{it} is input of intermediate good i $(i = 1, 2, ..., N)$, N_t is the number of different types of intermediate goods available at time t, and L is labor input = the exogenous and constant labor supply. The output of basic goods is used partly as input in sector 2, X_t, partly for consumption, $C_t \equiv c_t L$, and partly for R&D investment in Sector 3, Z_t :

$$Y_t = X_t + C_t + Z_t = X_t + C_t + \frac{\dot{N}_t}{\eta},$$

where $\eta > 0$ and

$$X_t = \sum_{i=1}^{N_t} \psi x_{it}, \qquad \psi > 0.$$

For every t the social planner solves the following static problem (the timing is suppressed for convenience):

$$\max_{x,\ldots,x_N} Y - X = A\left(\sum_{i=1}^{N} x_i^{1-\beta}\right) L^{\beta} - \sum_{i=1}^{N} \psi x_i.$$

b) Why is this problem of relevant? Find the solution to the problem. Let x_{SP} denote the solution for x_i.

c) Compare with the outcome under laissez-faire given in e) of Problem VII.1. What is the intuition behind the difference?

d) Show that net output of basic goods can be written $Y - X = \tilde{A}N$, where $\tilde{A}$ is a positive constant.

The dynamic problem faced by the social planner is to choose $(c_t)_{t=0}^{\infty}$ so as to:

$$\max U_0 = \int_0^{\infty} \frac{c_t{}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$0 \le c_t \le \frac{\tilde{A}N_t}{L}, \tag{*}$$

$$\dot{N}_t = \eta(\tilde{A}N_t - Lc_t), \qquad N_0 > 0 \text{ given and “large”}, \tag{**}$$

$$N_t \ge 0 \text{ for all } t \ge 0. \tag{***}$$

In (**) indivisibilities are ignored and N is regarded as a continuous and differentiable function of time t. In view of (*), (***) will automatically hold and can be deleted.

e) Explain why the control region is bounded and how the dynamic constraint, (**), arises.

f) Assuming there is an interior solution, derive the first-order conditions and the transversality condition. Determine the implied growth rate of c_t. Next, apply your general knowledge about reduced-form AK models to determine the time paths of N_t and Y_t (a brief verbal account is enough).

g) Write down the required parameter restrictions for positive growth and boundedness of the utility integral.

h) Recalling that the equilibrium interest rate in the laissez-faire market economy is $r^* = \eta\pi = \eta(\frac{\psi}{1-\beta} - \psi)\left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L$, compare the social planner's growth rate with that of the laissez-faire market economy. *Hint:* an answer can be reached by showing that $\tilde{A} = (\frac{\psi}{1-\beta} - \psi)x_{SP}$ and bearing in mind the result from c).

i) Now consider a government that attempts to obtain the social planner's allocation in a decentralized way. The government pays a subsidy at constant rate, σ, to purchases of intermediate goods such that the net price of intermediate good i is $(1-\sigma)p_i$, where $p_i = \psi/(1-\beta)$ is the price set by the monopolist supplier of good i. The government finances this subsidy by taxing consumption at the constant rate τ. It can be shown that a proper choice of σ and τ is sufficient to obtain the social planner's allocation in a decentralized way. Derive the required value of the subsidy rate σ. *Hint:* the size of σ must ensure that the private cost of using intermediates equals the social marginal cost.

VII.5 *Hidden parameter links in the simple Lab-Equipment Model* Here we consider an extension of what we call the simple Lab-Equipment Model, namely the model of Problem VII.1 (the notation is the same). In the simple Lab-Equipment Model, even without Acemoglu's simplification (**) in Problem VII.1, the aggregate production function (1) contains three simplifying, but arbitrary parameter links.

In what we shall call the *extended Lab-Equipment Model*, (1) is replaced by

$$Y_t = AN_t^{\gamma}(CES_t)^{1-\beta}L^{\beta}, \qquad A > 0,\ \gamma > 0, 0 < \beta < 1, \tag{Y}$$

where the parameter γ reflects "gains to specialization" (see d) below), and CES_t is a CES aggregate of the quantities $x_{1t}, ..., x_{N_t t}$:

$$CES_t \equiv N_t \left(N_t^{-1} \sum_{i=1}^{N_t} x_{it}{}^{\varepsilon} \right)^{\frac{1}{\varepsilon}}, \qquad 0 < \varepsilon < 1, \tag{CES}$$

This is the standard definition of a CES aggregate. The parameter ε is called the *substitution parameter* in that the elasticity of substitution between the different specialized input goods is $1/(1-\varepsilon) > 1$ and thereby an increasing function of ε.

In (Y) it is understood that employment in Sector 1 equals the constant labor supply, L. The institutional setting is a laissez-faire market economy.

a) Show that the right-hand side of (CES) has CRS with respect to the inputs $x_{1t}, ..., x_{N_t t}$.

This CRS property is convenient because it opens up for "gains to specialization" to be represented by an independent parameter, γ, appearing explicitly *outside* the CES index as in (Y).

b) "The specification (1) in Problem VII.1 is a special case of (Y)-(CES), namely the case $\gamma = \beta$ together with $\varepsilon = 1 - \beta$". True or false? Comment.

In equilibrium, because of symmetry and the fact that the prices of intermediate goods will all be set at the same level, the representative firm in Sector 1 chooses $x_{it} = x_t$, for all i.

c) With $x_{it} = x_t$ for all i, (CES) reduces to $CES_t = N_t x_t$. Show this.

Ignoring for a moment the issue whether the specialized input goods are durable or non-durable, we may think of $N_t x_t$ as the total input of physical capital, K_t, in the representative firm of Sector 1.

d) Write (CES) as a Cobb-Douglas production function with CRS to the rival inputs, labor and capital. In this interpretation, if N_t grows at the constant rate g_N, what is then the growth rate of total factor productivity?

Let us for a while consider a fixed point in time and suppress the explicit timing of the variables. The representative firm in Sector 1 faces given input prices, $p_1, \ldots, p_N$, and w. The demand for the specialized input good i can be shown[1] to have price elasticity equal to $-1/(1-\varepsilon) < -1$.

Given the technical design corresponding to intermediate good i, the marginal cost of supplying this good is $\psi > 0$ for all i.

e) Show that the monopoly price is $p_i = \psi/\varepsilon$ for all i. *Hint:* MR $= p_i + x_i dp_i/dx_i$ $= (1 + (x_i/p_i)dp_i/dx_i)p_i =$ MC.

f) Does the monopoly power (defined by the markup on marginal cost) depend on the output elasticity w.r.t. labor input? Compare with the simple Lab-Equipment Model.

[1] See the appendix.

g) In equilibrium, $x_{it} = x_t$ and $\pi_{it} = \pi_t$ for all i. Why? Express x_t and π_t, respectively, in terms of N_t. *Hint:* It can be shown (see appendix) that in general equilibrium with $p_i = \psi/\varepsilon$ for all i,

$$K = Nx = ((1-\beta)AN^{\gamma})^{1/\beta}(\frac{\psi}{\varepsilon})^{-1/\beta}L.$$

h) Show that in general equilibrium

$$Y_t = A\left(\frac{(1-\beta)\varepsilon}{\psi}\right)^{(1-\beta)/\beta} LN_t^{\gamma+\gamma(1-\beta)/\beta}.$$

i) What is the necessary and sufficient condition for π_t being independent of N_t as in Problem VII.1? And what is the necessary and sufficient condition for Y_t being proportional to N_t? Relate your answers to your answer to b).

j) Suppose gains to specialization is less than the elasticity of Y w.r.t. L. In this case, would you think the economy is capable of generating fully endogenous growth? Why or why not?

k) Suppose gains to specialization is larger than the elasticity of Y w.r.t. L. This case has an implication that makes it implausible. What implication could that be?

VII.6 *Knowledge-spillover models* The bulk of empirical evidence suggests that market economies do too little R&D investment compared to the optimal level as defined from the perspective of a social planner respecting the preferences of an assumed representative infinitely-lived household.

a) Is the "lab-equipment" version of the expanding input variety model consistent with this evidence? Briefly discuss.

b) What kind of subsidy and taxation scheme is capable of implementing the social planner's allocation in the "lab-equipment" model?

c) Our syllabus describes two other versions of the expanding input variety model. The aggregate invention production functions in these two versions are two alternative cases within the common form

$$\dot{N}_t = \eta N_t^{\varphi} L_{Rt}, \qquad \eta > 0, \varphi \leq 1,$$

where N_t is the number of existing different varieties of intermediate goods (indivisibilities are ignored) and L_{Rt} the input of research labor at time t (time is continuous). Briefly interpret.

d) Are these versions consistent with the mentioned evidence? Why or why not?

e) Are there features in these versions that may call for additional policy measures compared with b)? Briefly discuss.

f) The patent-R&D ratio is defined as the number of new patents per year divided by aggregate R&D expenditures (in real terms). With w_t denoting the real wage, write down an expression for the patent-R&D ratio according to the model versions mentioned under c).

g) What predictions concerning the time path of the patent-R&D ratio can we derive from the two alternative model versions mentioned under c), assuming balanced growth? Why?

h) Since the late fifties, in the US a systematic decline in the empirical patent-R&D ratio has taken place. Briefly relate to your result in g).

Appendix to d) of Problem VII.5

We claimed that the demand for the specialized input good i has price elasticity equal to $-1/(1-\varepsilon) < -1$. This follows from microeconomic duality theory. Here we give a brief account of how the demand for intermediate good i is determined. There are two steps:

Step 1. For a given size $K > 0$ of CES, choose $(x_1, \ldots, x_N)$ so as to minimize the cost of obtaining K. That is, solve the problem:

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} p_i x_i \quad \text{s.t.} \quad N\left(N^{-1}\sum_{i=1}^{N} x_i{}^{\varepsilon}\right)^{\frac{1}{\varepsilon}} = K.$$

This problem can be shown to have the solution

$$x_i = \frac{K}{N}\left(\frac{p_i}{\lambda}\right)^{-\frac{1}{1-\varepsilon}} \equiv x_i^*, \tag{i}$$

where

$$\lambda = \left(N^{-1}\sum_{i=1}^{N} p_i^{\frac{\varepsilon}{\varepsilon-1}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \equiv \bar{p}$$

is the Lagrange multiplier, $i = 1, \ldots, N$. We see that the solution, $(x_1^*, \ldots, x_N^*)$, is proportional to K (as expected in view of CES being homogeneous of degree one in $x_1, \ldots, x_N$. It can be shown that $\sum_{i=1}^{N} p_i x_i^* = \bar{p}K$ and so $\bar{p}$ can be interpreted as the minimum cost per unit of K. (Note that $\bar{p}$ is a kind of average of the p_i's in the sense that (a) if $p_i = p$ for all i, then $\bar{p} = p$; and (b) if for any $\mu > 0$, p_i is replaced by $p_i' = \mu p_i$, then $\bar{p}' = \mu\bar{p}$.)

Step 2. Choose K and L so as to maximize $\Pi = AN^{\gamma}K^{1-\beta}L^{\beta} - \bar{p}K - wL$. The first-order condition w.r.t. K is

$$\frac{\partial \Pi}{\partial K} = (1-\beta)AN^{\gamma}K^{-\beta}L^{\beta} - \bar{p} = 0,$$

so that, given $\bar{p}$ and N, the profit maximizing $k \equiv K/L$ is $k = ((1-\beta)AN^{\gamma})^{1/\beta}\bar{p}^{-1/\beta}$ (in view of CRS, $\bar{p}$ determines only the profit maximizing factor *ratio*). If moreover L is considered given, we have

$$K = ((1-\beta)AN^{\gamma})^{1/\beta}\bar{p}^{-1/\beta}L. \tag{ii}$$

The supplier of intermediate good i is "small" relative to the economy as a whole and takes $\bar{p}$, and thereby K, as given. Hence the perceived price elasticity of the demand for intermediate good i is given by (i) as $-1/(1-\varepsilon) < -1$.