Economic Growth, June 2014.

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A suggested solution to the problem set at the exam in Economic Growth, June 4, 2014

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (30 %)

The production function of firm $i = 1, 2, \ldots, N$ is

$$Y_{it} = F(K_{it}, T_t L_{it}), \tag{*}$$

where F is a neoclassical production function with CRS and satisfying the Inada conditions and T_t is the economy-wide technology level and satisfies

$$T_t = K_t^{\lambda}, \qquad 0 < \lambda \leq 1, \tag{**}$$

where $K_t = \sum_i K_{it}$. The size of the labor force is $L_t = L_0 e^{nt}$, where n is a constant ≥ 0. Aggregate output is Y_t per time unit and is used for consumption and investment in physical capital so that

$$\dot{K}_t \equiv \frac{dK_t}{dt} = Y_t - C_t - \delta K_t, \qquad \delta > 0, \qquad K_0 > 0 \text{ given.}$$

where $C_t \equiv c_t L_t$. Each firm is small relative to the economy as a whole and perceives it has no influence on aggregate variables, including K_t and thereby T_t.

a) We suppress the time index when not needed for clarity. Consider firm i. Its maximization of profits, $\Pi_i = F(K_i, TL_i) - (r + \delta)K_i - wL_i$, leads to the first-order conditions

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, TL_i) - (r + \delta) = 0, \\ \partial \Pi_i / \partial L_i &= F_2(K_i, TL_i)T - w = 0. \end{aligned} \tag{1.1}$$

[1] The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

By Euler's theorem, we can write (1.1) as

$$F_1(k_i, T) = r + \delta, \tag{1.2}$$

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (1.2) determines k_i uniquely.

From (1.2) follows that the chosen k_i will be the same for all firms, say $\bar{k}$. In equilibrium $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both pre-determined). Since $K = \sum_i K_i \equiv \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k} L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{1.3}$$

Substituting into (1.2) gives $r = F_1(k, T) - \delta$. Reintroducing explicit dating of the variables, the solution for the equilibrium interest rate at time t is

$$r_t = F_1(k_t, T_t) - \delta = F_1(k_t, K_t^{\lambda}) - \delta, \tag{1.4}$$

where both k_t and K_t are pre-determined.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k, T) L_i = F(k, T) \sum_i L_i = F(k, T) L \\ &= F(K, TL) = F(K, K^{\lambda} L). \end{aligned}$$

So

$$Y_t = F(K_t, K_t^{\lambda} L_t). \tag{1.5}$$

According to the general hypothesis of *learning-by-investing,* the economy-wide technology level in (*) is an increasing function of society's previous experience, proxied by cumulative aggregate net investment:

$$T_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1,$$

where I_s^n is aggregate net investment and λ is the "learning parameter". This effect of firms' investment on the economy-wide technology level is not taken into account by the individual firms when they maximize profit, but emerges as an externality implying that at the aggregate level there are increasing returns to scale w.r.t. capital and labor ($\lambda > 0$ in (1.5)). Moreover, whether $\lambda < 1$ or $\lambda = 1$ matters a lot because in the former case (the Arrow case), there are still diminishing returns to capital alone, whereas in the latter case (the Romer case), there are constant returns to capital alone. In the former case semi-endogenous growth is technically feasible while in the latter case fully endogenous growth is technically feasible.

b) From now on, $\lambda = 1$, $n = 0$, $L_t = L_0 = L$, and households are Ramsey-style with CRRA utility and a constant rate of time preference, ρ.

Inserting $\lambda = 1$ into (1.4) gives

$$r_t = F_1(k_t, K_t) - \delta = F_1(1, L) - \delta \equiv r,$$

where the second equal sign comes from dividing the arguments of F_1 by k_t and apply Euler's theorem and the fact that $K_t/k_t = L_t = L$. We see that the interest rate is constant "from the beginning" and independent of the historically given initial value of K.

Next we insert $\lambda = 1$ into (1.5) to get

$$Y = F(K, KL) = F(1, L)K. \tag{1.6}$$

Aggregate output is thus *proportional* to the aggregate capital stock. In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. This property, together with the constancy of the interest rate, tells us that the model belongs to the class of *reduced-form AK models*, that is, models where in general equilibrium the interest rate and the aggregate output-capital ratio are necessarily constant over time whatever the initial conditions.

c) Because there is a representative household, the Keynes-Ramsey rule holds at the aggregate level so that

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv g_c. \tag{1.7}$$

Thereby also g_c is constant "from the beginning".

From now on we assume (A1) $F_1(1, L) - \delta > \rho$ and (A2) $\rho > (1 - \theta)g_c$.

d) (A1) allows us to study a model with growth. (A2) is motivated by the fact that if it did not hold, the transversality condition of the representative household could not hold and so the model would have no equilibrium.

e) From the general theory of AK and reduced-form AK models in a Ramsey framework we know that the per capita "capital variable" of the model, here k_t, will "from the beginning" grow at the same rate as per capita consumption. Otherwise the TVC would be violated. Hence, $g_k = g_c$ for all $t \geq 0$. Moreover, by (1.6) follows that $y_t \equiv Y_t/L = F(1,L)k_t$ so that $g_y = g_k$ for all $t \geq 0$. The system thereby features balanced growth from the beginning (there is no transitional dynamics).

f) The policy proposal is to offer an investment subsidy σ to households' saving such that the after-subsidy rate of return on private saving is $(1+\sigma)r$. The subsidy is to be financed by a lump-sum tax on all households.

To find the appropriate value of σ, we set up the social planner's problem: choose $(c_t)_{=0}^{\infty}$ to

$$\begin{aligned} \max U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= F(1,L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, && (1.8) \\ k_t &\geq 0 \text{ for all } t > 0. && (1.9) \end{aligned}$$

The current-value Hamiltonian is

$$H(k,c,\eta,t) = \frac{c^{1-\theta}}{1-\theta} + \eta\left(F(1,L)k - c - \delta k\right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\begin{aligned} \frac{\partial H}{\partial c} &= c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, && (1.10) \\ \frac{\partial H}{\partial k} &= \eta(F(1,L) - \delta) = -\dot{\eta} + \rho\eta. && (1.11) \end{aligned}$$

The transversality condition is

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0. \tag{1.12}$$

Log-differentiating w.r.t. t in (1.10) and combining with (1.11) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv g_c^{SP}. \tag{1.13}$$

We see that $g_c^{SP} > g_c$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" rate of return, r^{SP}, equals $\partial y_t/\partial k_t = F(1,L) > F_1(1,L)$. To ensure bounded intertemporal utility, we sharpen (A2) to

$$\rho > (1-\theta)g_c^{SP}. \tag{A2'}$$

Because aggregate output is proportional to aggregate capital and r^{SP} is constant "from the beginning", the social planner's economy is an AK economy. Hence, from the general theory of AK models we know that $g_y = g_k = g_c^{SP}$ for all $t \geq 0$ (balanced growth from the beginning).

To replicate this allocation, σ should ensure that the net rate of return on saving faced by the consumer equals the net rate of return to capital investment implied by the aggregate production technology, $F(1, L)$. Hence, σ must satisfy

$$(1+\sigma)r = (1+\sigma)(F_1(1,L) - \delta) = F(1,L) - \delta,$$

that is,

$$\sigma = \frac{F(1,L) - \delta}{F_1(1,L) - \delta} - 1.$$

With this value of σ, the Keynes-rule for the decentralized economy becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}((1+\sigma)r - \rho) = \frac{1}{\theta}((1+\sigma)F_1(1,L) - \delta - \rho) = \frac{1}{\theta}(F(1,L) - \delta - \rho) \equiv g_c^{SP},$$

and we get $g_y = g_k = g_c^{SP}$ for all $t \geq 0$ as desired.

2. Solution to Problem 2 (55 %)

In Model I, in the *basic-goods* sector (sector 1):

$$Y_t = A\left(\sum_{i=1}^{N_t} {x_{it}}^{1-\beta}\right) L^{\beta}, \qquad A > 0,\ 0 < \beta < 1.$$

In the specialized *intermediate-goods* sector (sector 2) firms face marginal costs $\psi > 0$ and in the *R&D sector* (sector 3) there is ideosyncratic uncertainty and a constant research productivity $\eta > 0$. A defining characteristic of Model I is the assumption that once the technical design for intermediate good i has been invented in the R&D sector, the inventor can take out (free of charge) an *effective perpetual patent* on the commercial use of this design.

In Model II the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain.* More precisely, cessation of monopoly power can be described as a Poisson process with an exogenous Poisson arrival rate $\lambda > 0$, the same for all monopolies. In Model II equilibrium output of basic goods is

$$Y_t = \left[1 + ((1-\beta)^{-(1-\beta)/\beta} - 1)\frac{N_t^{(c)}}{N_t}\right] Y_t^{(m)},$$

where $Y_t^{(m)} \equiv AN_t(x^{(m)})^{1-\beta}L^\beta$ is the equilibrium output of basic goods in Model I and $N_t^{(c)}$ denotes the number of intermediate-goods varieties that at time t are competitively supplied; moreover, $x^{(m)} \equiv (A(1-\beta)^2/\psi)^{1/\beta} L$.

a) As $0 < \beta < 1$, $(1-\beta)^{-(1-\beta)/\beta} = 1/\left[(1-\beta)^{(1-\beta)/\beta}\right] > 1$. So $Y_t > Y_t^{(m)}$.

Intuitively, with erosion of monopoly power, we have $N_t^{(c)} > 0$, and so a fraction of the intermediate goods are supplied at a price equal to marginal cost rather than above marginal cost. This reduces the monopolist price distortion and the demand for - and use of - specialized intermediate goods come closer to the efficient level. Thereby productivity is enhanced and we get larger output of basic goods in spite of employment in the sector being unchanged.

b) We are asked to give an intuitive explanation of the equilibrium condition

$$V_t\eta = 1. \tag{*}$$

Imagine the R&D input is one basic good (the numeraire) per time unit. Then there is an R&D cost of one unit of account per time unit, cf. the right-hand side of (*).

On the left-hand side we have the expected benefit per time unit. Indeed, V_t, is the market value of an invention. The Poisson arrival rate, η, is the approximate probability of a successful research outcome, i.e., an invention, in a time interval of unit length. Essentially, the alternative outcome is no invention, i.e., a payoff of nil. So $V_t\eta$ is the expected benefit which in equilibrium with active R&D and risk-neutral behavior due to ideosyncratic uncertainty must equal the cost, 1.

Alternatively, one can argue on the basis of $V_t = 1/\eta =$ expected cost per invention.

c) From (*) follows that $V_t = 1/\eta \equiv V$, a constant. Inserting into the no-arbitrage condition and solving for r_t gives

$$r_t = \eta\pi^{(m)} - \lambda \equiv r^*,$$

where $\pi^{(m)} = \frac{\beta}{1-\beta}\psi x^{(m)}$ is profit per time unit of a monopoly firm.

d) We see, first, that like in Model I, the equilibrium interest rate (= rate of return) is a constant from the beginning. Second, in view of $\lambda > 0$, we have $r^* \equiv r^{(m)} - \lambda < r^{(m)}$, where $r^{(m)} \equiv \eta\pi^{(m)}$ is the equilibrium interest rate in Model I which equals the (expected) rate of return on investing in R&D in *that* model (no erosion of monopoly power). The expected rate of return on investing in R&D in Model II is smaller because of the limited duration of monopoly power in Model II.

e) From the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r^* - \rho) = \frac{1}{\theta}(\eta\pi^{(m)} - \lambda - \rho) \equiv g_c^{II}.$$

The growth rate of per capita consumption is thus constant from the beginning, an implication of the constancy of the real interest rate.

f) A parameter condition ensuring that $g_c^{II} > 0$ is that

$$\eta\pi^{(m)} - \lambda > \rho. \tag{A1}$$

A parameter condition ensuring boundedness of the utility integral is

$$\rho > (1-\theta)g_c^{II}. \tag{A2}$$

We are told that for Model II it can be shown that:

(i) $s_t \equiv N_t^{(c)}/N_t$ approaches $s^* = \lambda/(g_N^* + \lambda)$ over time, where g_N^* is the long-run growth rate of N_t; and

(ii) $g_N^* = g_c^{II} \equiv g^*$.

g) The information in (i) and (ii) tells us that the system converges and that $g^* = g_c^{II}$ is thus the long-run growth rate. We have

$$\begin{aligned}
&\text{(i)} & \frac{\partial g^*}{\partial A} &= \frac{\eta}{\theta}\frac{\partial \pi^{(m)}}{\partial A} = \frac{\eta}{\theta}\frac{\beta}{1-\beta}\psi\frac{\partial x^{(m)}}{\partial A} > 0,\\
&\text{(ii)} & \frac{\partial g^*}{\partial \eta} &= \frac{\pi^{(m)}}{\theta} > 0,\\
&\text{(iii)} & \frac{\partial g^*}{\partial \lambda} &= -\frac{1}{\theta} < 0.
\end{aligned}$$

Comments:

(i) Higher TFP $\Rightarrow$ higher return on saving $\Rightarrow$ more saving at the aggregate level (the negative substitution effect and wealth effect on consumption dominates the positive income effect) $\Rightarrow$ more investment in R&D. As usual the constant A need not have a narrow technical interpretation. It can reflect the quality of the institutions in society (rule of law etc.) and the level of "social capital".

(ii) Higher R&D productivity results in more R&D investment and higher growth.

(iii) Faster erosion of monopoly power over commercial use of an invention implies lower present value of the invention, hence lower equilibrium rate of return in the economy and less incentive to save and invest in R&D.

Because of the long-run AK structure of the model, these effects come out as not only long-run level affects but also long-run growth effects.

h) There are many dilemmas regarding how to design patent systems. Model II above illustrates one of them, namely the question what the period length of patents should be. The inverse of λ can be interpreted as a measure of the average duration of patents. A larger λ (shorter duration) reduces static inefficiency in the economy but it also aggravates the underinvestment in R&D and thereby increases the dynamic inefficiency in the economy. We could more generally interpret λ as reflecting strictness of antitrust policy and the conclusion would be similar.

i) To avoid that the effective price of intermediate goods supplied by the monopolies is above marginal cost, the subsidy, σ, should be such that

$$(1-\sigma)p_i = (1-\sigma)\psi/(1-\beta) = \psi.$$

The solution is

$$\sigma = \beta.$$

By choosing this value for the subsidy, we obtain a situation where basic-goods firms are fully compensated for having to pay a monopoly price above the marginal cost of supplying intermediate goods. So the distortion of demand resulting from this source is eliminated (as long as the government is able to finance the subsidy by a lump-sum tax).

j) Yes, it should do that. A subsidy, s, to reduce research cost, $1/\eta$, per expected invention *is* relevant for implementing the social planner's solution. The limited duration of monopoly power makes the *dynamic* distortion of incentives more serious than in Model I where compensating for the *static* distortion due to monopoly power is sufficient to simultaneously solve the dynamic problem of underinvestment in R&D. This is not so in Model II, and a subsidy to reduce research costs is thus motivated, in addition to the subsidy σ.

k) Yes, it is possible to finance the optimal subsidy policy in a non-distortionary way by taxing consumption. Since the Ramsey households supply labor inelastically, a *constant* consumption tax, τ, will be non-distortionary from a static as well as dynamic point of view (a time-varying consumption tax will be intertemporally distortionary). Will a constant consumption tax be able to maintain a balanced budget forever? Yes, this follows from the fact that by appropriate choice of the value of τ, the economy behaves as a reduced-form AK economy. Thereby there will be balanced growth forever (no transitional dynamics) and no need for adjusting the tax rate during transitional dynamics in order to maintain a balanced budget.

3. Solution to Problem 3 (15 %)

a) Yes, we can. The explanation relies on the fact that poor countries differ from rich countries in many respects, including w.r.t. factors that are complementary to physical capital, for instance the technology level and the human capital level.

Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function

$$Y_j = F(K_j, A_j h_j L_j) = A_j h_j L_j F(\frac{K_j}{A_j h_j L_j}, 1) \equiv A_j h_j L_j f(\tilde{k}_j), \quad f' > 0, f'' < 0,$$

where F is neoclassical with CRS (standard notation). Let r_j denote the equilibrium net rate of return on capital in country j. Then, under perfect competition,

$$r_j = \frac{\partial Y_j}{\partial K_j} - \delta = f'(\tilde{k}_j) - \delta.$$

Can the countries have the same r in spite of widely differing K_j/L_j? Yes! A difference w.r.t. K_j/L_j does not rule out that $\tilde{k}_j \approx \tilde{k}$ for all j. Indeed,

$$\tilde{k}_j \equiv \frac{K_j}{A_j h_j L_j} = \frac{K_j/L_j}{A_j h_j}.$$

As countries with low K_j/L_j (the poor countries) also tend to have low A_j and h_j, the $\tilde{k}_j$'s - and therefore also the r_j's - may be more or less of the same size. The K_j/L_j ratios may even be negatively correlated with the $\tilde{k}_j$'s.

b) Consider the "expanding input variety" models with "knowledge-spillovers" in our syllabus (the R&D-based growth models by Romer and Jones). The expected output of new technical designs in R&D lab j at time t is

$$E_t \dot{N}_{jt} = \tilde{\eta} L_{jt}, \qquad \tilde{\eta} > 0. \tag{*}$$

The individual R&D labs are "small" and take the research productivity, $\tilde{\eta}$, as given. Nevertheless, at the economy-wide level this productivity is determined according to

$$\tilde{\eta} = \eta N_t^{\varphi}, \qquad \eta > 0, \varphi \leq 1, \tag{**}$$

where N_t is the aggregate number of existing input varieties in the economy at time t.

When $\varphi \neq 0$, the factor N_t^{φ} represents an *inter-temporal* externality (of the "standing on the shoulders" variety if $\varphi > 0$ and the "fishing out" variety if $\varphi < 0$).

Remark. Because the uncertainty is assumed ideosyncratic and the economy is assumed "large", at the aggregate level the actual number of new technical designs invented per time unit coincide with the expected number, i.e., $\dot{N}_t = E_t\dot{N}_t$ (by the "law of large numbers"). Hence it is not an error if the student's answer is based not on (*) but on an aggregate form like

$$\dot{N}_t = \tilde{\eta}L_{Rt},$$

where $L_{Rt} = \sum_j L_{jt}$, together with (**).

As emphasized by Jones in his original article (Jones 1995), presence of an *intra-temporal* externality is also likely. Jones' point is that because of "overlap" ("stepping on toes") in R&D, we may have $\dot{N}_t < \sum_j \dot{N}_{jt}$, where $\dot{N}_t$ refers to the aggregate arrival of truly original (not duplicate) inventions. Indeed, according to Jones, one should allow for the possibility that

$$\tilde{\eta} = \eta N_t^{\varphi} L_{Rt}^{\lambda-1}, \qquad 0 < \lambda < 1. \tag{***}$$

Then, at the aggregate level

$$\dot{N}_t = \tilde{\eta}L_{Rt} = \eta N_t^{\varphi} L_{Rt}^{\lambda}.$$

The "degree of overlapping" in research is then measured by $1 - \lambda$. If the "degree of overlapping" is denoted δ, (***) takes the form

$$\tilde{\eta} = \eta N_t^{\varphi} L_{Rt}^{-\delta}, \qquad 0 < \delta < 1, \tag{***}$$

so that

$$\dot{N}_t = \eta N_t^{\varphi} L_{Rt}^{1-\delta}.$$

c) Let the expected output of new technical designs in lab j at time t be

$$E_t\dot{N}_{jt} = \tilde{\eta}Z_{jt}, \qquad \tilde{\eta} > 0.$$

To allow for a *negative* intra-temporal externality in R&D we may assume that at the economy-wide level

$$\tilde{\eta} = \eta Z_t^{-\delta}, \qquad 0 \leq \delta < 1,$$

where $Z_t = \sum_j Z_{jt}$ and δ is the "degree of overlapping" (duplicate R&D activities). Instead, to allow for a *positive* intra-temporal externality in R&D, we may assume that

$$\tilde{\eta} = \eta Z_t^{-\delta}, \qquad -\delta \geq 0.$$

Think of "knowledge clusters" and "knowledge sharing" as in Silicon Valley.

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