Economic Growth, May 31, 2016.

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A suggested solution to the problem set at the exam in Economic Growth, May 31, 2016

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (70 %)

We consider a horizontal-innovations model for a closed economy. The household sector consists of a fixed number of infinitely-lived households, all alike. Each household has $L_t = L_0 e^{nt}$ members, $n > 0$, and each member supplies inelastically one unit of labor per time unit. Households have perfect foresight. We normalize the number of households to be one. Given $\theta > 0$ and $\rho > 0$, the household's problem is to choose a plan $(c_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{{c_t}^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{a}_t &= (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given}, & (*) \\ \lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} &\geq 0, & (**) \end{aligned}$$

where c_t is per head consumption, a_t is per head financial wealth, r_t is the interest rate, and w_t is the real wage. For any variable $z > 0$ which is a differentiable function of time, t, we apply the notation $g_z \equiv \dot{z}/z$, where $\dot{z} \equiv dz/dt$.

a) (*) is the household's dynamic budget constraint, expressed in per capita terms. It gives the increase per time unit in per capita financial wealth, a_t. Let the household's total financial wealth be $W_t \equiv a_t L_t$. The dynamic book-keeping relation then reads:

$$\dot{W}_t = r_t W_t + w_t L_t - c_t L_t, \qquad W_0 \text{ given},$$

[1] The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

saying that the increase in financial wealth per time unit equals saving, which, by definition, equals total income minus consumption, where total income is the sum of capital income and labor income. When this is translated into per capita terms, we get (*). If $n > 0$, we have $r_t - n < r_t$, reflecting the dilution effect, coming from growth in family size, on the contribution of capital income to *per head* family income.

(**) is a NPG condition, i.e., a solvency condition that rules out long-run growth in per capita net debt at a rate as high as, or higher than, the growth-corrected "rate of return", $r - n$.

Given the CRRA utility function $u(c) \equiv c^{(1-\theta)}/(1-\theta)$, θ measures the absolute value of the elasticity of marginal utility of consumption, i.e.,$-(c/u'(c))u''(c) = -(c/c^{-\theta})(-\theta c^{-\theta-1})$ $= \theta > 0$. This elasticity reflects the strength of the preference for consumption smoothing over time.

ρ is the pure rate of time preference and measures the degree of impatience; $\rho - n$ acts as a per capita utility discount rate.

b) The Keynes-Ramsey rule is an optimality condition regarding the proportionate *rate of change* in planned consumption at any given point in time. In the present case, with CRRA utility, the rule is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \qquad \text{for all } t \geq 0. \tag{K-R}$$

Thereby, depending on whether $r_t \gtreqless \rho$, the household chooses a relatively low, a medium, or a relatively high level of current per capita consumption, so as to prepare for a rising, unchanged vs. declining per capita consumption. This is the strategy of "saving when the getting is good" and to "enjoy life when the getting is bad".

The transversality condition amounts to an optimality condition regarding the general *level* of the planned consumption path. In the present case, the transversality condition can be written

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0. \tag{TVC}$$

This requires that the solvency condition (NPG condition) is satisfied, but not "over-satisfied".

The transversality condition may alternatively be written in the general form for an optimal control problem with infinite horizon, state variable a_t, and effective discount rate $r - n$:

$$\lim_{t\to\infty} a_t \lambda_t e^{-(\rho-n)t} = 0,$$

where λ_t is the current-value co-state variable.

We are told that the production side of the economy is described by the following assumptions:

$$\begin{aligned}
Y &= L_Y^{1-\alpha}\sum_{j=1}^{A} x_j^{\alpha}, \qquad 0<\alpha<1, && (1)\\
\frac{\partial Y}{\partial L} &= (1-\alpha)\frac{Y}{L_Y} = w, \qquad \frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha}x_j^{\alpha-1} = p_j, \qquad \forall j, && (2)\\
Y &= C + I_K = cL + \dot{K} + \delta K,\ \delta \geq 0, \qquad K_0>0 \text{ given}, && (3)\\
p_j &= \frac{1}{\alpha}(r+\delta) \equiv p, \qquad \forall j, && (4)\\
\pi_j &= (\frac{1}{\alpha}-1)(r+\delta)x_j \equiv (\frac{1}{\alpha}-1)(r+\delta)x \equiv \pi, \qquad \forall j, && (5)\\
\dot{A} &= \bar{\eta}L_A \equiv \eta A^{\varphi}L_A^{1-\xi}, \qquad \eta>0,\ \varphi<1,\ 0\leq\xi<1, \qquad A_0>0 \text{ given}, && (6)\\
w &\geqq P_A \partial\dot{A}/\partial L_A = P_A\bar{\eta}, \text{ with "} = \text{" if } L_A>0. && (7)
\end{aligned}$$

(The examinee need not repeat what is said in the problem formulation. It is repeated here only for convenience.)

c) On the production side of the horizontal-innovations model we have three sectors, sector 1 (final goods), sector 2 (specialized capital-good services), and sector 3 (R&D). In the sectors 1 and 3, perfect competition and free entry rule. In sector 2 there is a monopolist for each specialized capital good type. The final good is the numeraire.

The symbol x_j indicates the input of the specialized capital good j, $j = 1, 2, \ldots, A$, into the production carried out by the representative firm in sector 1. The equations (2) are the corresponding first-order conditions in this firm's profit maximization problem. The firm demands labor up to the point where the value of the marginal product equals the wage. And the firm demands input of each capital-good type up to the point where the value of the marginal product equals the input price, p_j. As the capital goods are rented out to the users, the input price is a rental rate set by the monopolist in question. The marginal cost of the monopolist is her cost, $r+\delta$, per unit of capital per unit of time. But due to market power supported by a perpetual patent on the right to commercial use of the technical design j, the monopolist j charges $p_j = \alpha^{-1}(r+\delta)$.

Eq. (5) gives the resulting accounting profit, π_j.

At the economy-wide level, the accumulated stock of viable inventions, measured by the level of A, is treated as a continuous and differentiable function of time. Thereby, $\dot{A}$ in Eq. (6) indicates the increase in A per time unit. L_A is aggregate research labor.

From the point of view of each individual R&D lab, the productivity, $\bar{\eta}$, of its research input, say ℓ_A, is given. From the point of view of the economy as a whole, $\bar{\eta} = \eta A^{\varphi} L_A^{-\xi}$, where φ indicates the strength and the sign of the intertemporal "knowledge spillover" from the current stock of knowledge. If $\varphi > 0$, the spillover amounts to a "standing on the shoulders" effect. And if $\varphi < 0$, the spillover amounts to a "fishing out" effect. A positive value of the exponent, ξ, on $L_A \equiv \sum \ell_A$ reflects an additional externality coming from duplication of effort. This externality is often called the "standing on the toes" effect.

There are "many" R&D labs and so each of them rightly perceives its own influence on $\bar{\eta}$ as essentially negligible. The model assumes there is no uncertainty in the economy, not even in R&D.

(7) indicates a weak inequality for the *equilibrium* wage, w, vis-a-vis the product of P_A, which represents the market value of an invention (hence also of the associated perpetual patent) at time t, and $\bar{\eta}$, which is the marginal productivity of research from the point of view of the individual lab. Indeed, the right-hand side of (7) is the value of MPL in R&D. If the R&D labs choose to be active ($L_A > 0$), "$>$" can be ruled out in equilibrium. If the R&D labs choose to *not* be active ($L_A = 0$), both "$>$" and "$=$" are compatible with equilibrium, but "$<$" can be ruled out since it would make research infinitely profitable, thus creating excess demand for R&D labor.

We are told that in general equilibrium with $L_A > 0$:

$$
\begin{aligned}
Ax &= K, && (8)\\
L_Y + L_A &= L = L_0 e^{nt}, && (9)\\
Y &= K^{\alpha}(AL_Y)^{1-\alpha}, && (10)\\
\frac{1}{\alpha}(r+\delta) &= \frac{\partial Y}{\partial x_j} = \alpha \frac{Y}{K}, && (11)\\
\pi &= (1-\alpha)\alpha \frac{Y}{A}, && (12)\\
w &= (1-\alpha)\frac{Y}{L_Y} = P_A \eta A^{\varphi} L_A^{-\xi}, && (13)\\
a &= \frac{K + P_A A}{L}, && (14)\\
P_A r &= \pi + \dot{P}_A. && (15)
\end{aligned}
$$

d) (8): By (2) and (4), $x_j = x$, $\forall j$. Hence, aggregate demand for "raw capital" at time t is $K^d = Ax$, in view of the one-to-one technology regarding the transformation from "raw capital" to specialized capital. Aggregate supply of "raw capital" at time t is the available amount, K. So (8) is the equilibrium condition in the market for "raw capital".

(13): The first equality in (13) reflects that for equilibrium in the labor market to hold, the wage must equal the value of the marginal product of labor in sector 1. The second equality in (13) says that for equilibrium in the labor market with *active* R&D to hold, the wage must also equal the value of the marginal product of labor in sector 3.

(14): This equation comes from the fact that in this economy, aggregate financial wealth, W, at time t equals the total amount of net assets at time t, namely the amount of "raw capital", K, plus the market value of the stock of patents, $P_A A$.

(15) is a no-arbitrage condition. Suppose an investor wants to invest the amount P_A. One possibility is to invest in the loan market and get the return $P_A r$ per time unit. Another possibility is to invest in sector 2 by buying a patent and, per time unit, earn an accounting profit, π, plus an expected capital gain, $\dot{P}_A$. In equilibrium, P_A is such that the two alternatives give the same return.

Derivation of (10): As noted at d), from (2) and (4) follows $x_j = x$, $\forall j$. Substituting this into (1) and using (8) gives

$$Y = L_Y^{1-\alpha} A x^\alpha = L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^\alpha = K^\alpha (A L_Y)^{1-\alpha}. \tag{16}$$

Derivation of (11): In a similar way, from $x_j = x$, $\forall j$, (2), and (8) follows

$$\frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha} x^{\alpha-1} = \alpha L_Y^{1-\alpha} \left(\frac{K}{A}\right)^{\alpha-1} = \alpha K^{\alpha-1} (A L_Y)^{1-\alpha} = \alpha \frac{Y}{K} = p = \frac{1}{\alpha}(r+\delta), \tag{17}$$

where the last equality comes from (4).

Derivation of (12): From (5), (8), and (17) follows

$$\pi = (\frac{1}{\alpha} - 1)(r+\delta)x = (\frac{1}{\alpha} - 1)(r+\delta)\frac{K}{A} = (\frac{1}{\alpha} - 1)\alpha^2 \frac{Y}{K}\frac{K}{A} = (1-\alpha)\alpha \frac{Y}{A}.$$

We are now told that (15) allows us to write

$$P_{At} = \int_t^\infty \pi_s e^{-\int_t^s r_u du} ds. \tag{18}$$

e) (18) says that the market value of a patent is the present discounted value of expected future accounting profits.

If r is constant and π_s grows at the constant rate n, then (18) reduces to

$$P_{At} = \int_t^\infty \pi_t e^{n(s-t)} e^{-r(s-t)} ds = \pi_t \int_t^\infty e^{-(r-n)(s-t)} ds = \pi_t \frac{1}{r-n}, \tag{19}$$

as was to be shown.

f) In (6), divide through by A to get

$$g_A \equiv \frac{\dot{A}}{A} = \eta A^{\varphi-1} L_A^{1-\xi}.$$

In view of $L_A > 0$, we have $g_A > 0$ and can take logs and then time derivatives to get

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1)g_A + (1-\xi)g_{L_A} = 0$$

along a BGP where, by definition, g_A must be constant. Solving for g_A gives

$$g_A = \frac{1-\xi}{1-\varphi} g_{L_A} = \frac{1-\xi}{1-\varphi} n \equiv g_A^* > 0, \tag{20}$$

where the last equality comes from the fact that along a BGP with $L_A > 0$, g_{L_A} must be positive and equal to $n > 0$.

To find g_y along the BGP, we first take logs and then time derivatives in (10) to get

$$g_Y = \alpha g_K + (1-\alpha)(g_A + g_{L_Y}) = \alpha g_K + (1-\alpha)(g_A^* + n) \tag{21}$$

along the BGP. By definition, a BGP requires g_Y, g_C, g_K, and g_A to be constant. If $I_K > 0$, we know from the balanced growth equivalence proposition that the BGP also satisfies that $g_Y = g_C = g_K$, whereby (21) reduces to

$$g_Y = g_K = g_A^* + n = \text{ constant},$$

along the BGP. Since $y \equiv Y/L$, we get

$$g_y = g_Y - n = g_A^* \tag{22}$$

along the BGP.[2]

Finally, along the BGP, (12), together with (22), implies

$$g_\pi = g_Y - g_A^* = n. \tag{23}$$

From now on, we use "BGP" as name for the considered balanced growth path.

g) One approach is, following the hint, to use eq. (11) and the balanced growth equivalence proposition to argue that Y/K is constant along BGP, assuming $I_K > 0$.

[2] It is not wrong at this stage to point out that so far there has been given no argument to rule out that $I_K = 0$ along the BGP. In that case, $Y = C + I_K = C$, so that (21) gives $g_Y \ (= g_C) = -\alpha\delta + (1-\alpha)(g_A^* + n)$ $> g_K = -\delta$ along BGP. But as indicated in Remark at g), this alternative BGP *can* in fact be ruled out in view of the Keynes-Ramsey rule.

Another approach is to reason that, from (K-R) at b), we have, along BGP, $r = \rho + \theta g_c$, where $g_c = g_C - n$ must be constant since along BGP, by definition, g_C is constant. Then, so is r and thereby also Y/K, in view of (11).

Remark. By constancy of Y/K, (21) reduces to $g_Y = g_K = g_A^* + n =$ constant, along BGP, and so we can rule out $I_K = 0$ along BGP, cf. footnote 2.

h) Along BGP, according to g), r is constant, and from (23) we have $g_\pi = n$. This means that (15) via (18) implies (19) and thereby that $g_{P_A} = g_\pi$ along BGP. In combination with (23), this implies $g_{P_A} = n$ along BGP, as was to be shown. The conclusion can also be reached other ways. For instance, by taking growth rates in (13), we get, along a BGP,

$$\begin{aligned} g_Y - n &= g_{P_A} + \varphi g_A^* - \xi n, \text{ i.e.,} \\ g_{P_A} &= g_A^* - \varphi g_A^* + \xi n = (1-\xi)n + \xi n = n. \quad \text{(from (22) and (20), respectively)} \end{aligned}$$

We are now told that

$$s_R = \frac{1}{1 + \frac{r-n}{\alpha g_A^*}} \equiv s_R^*, \tag{***}$$

where g_A^* is the result for g_A in f).

i) We find:

$\partial s_R / \partial r < 0$. A higher interest rate implies lower P_A, cf. (19), hence lower profitability of R&D and thereby less resource allocation to R&D. The channel may be summarized this way:

$$P_A \downarrow \Rightarrow P_A \bar{\eta} \downarrow \Rightarrow \text{ (through (13)) } (1-\alpha)Y/L_Y \downarrow \Rightarrow L_Y/L_A \uparrow .$$

$\partial s_R / \partial n > 0$. A higher population growth rate implies faster growth in markets for intermediates, hence faster growth in monopoly profits, cf. (23), hence higher P_A and thereby higher profitability of R&D and thereby more resource allocation to R&D.

$\partial s_R / \partial g_A^* > 0$, when $r - n > 0$. A higher obtainable knowledge growth rate implies more R&D successes per time unit and higher profitability of R&D, hence more resource allocation to R&D.

As to the stated condition $r - n > 0$, implicitly, the formulas (15) and (18) presuppose that $r > n$ since otherwise P_A would not be finite. Anyway, that $r > n$ is implied by the standard parameter restriction needed for existence of equilibrium in an economy with Ramsey households.[3]

[3]This is the restriction $\rho - n > (1-\theta) g_c^*$, which implies $\rho + \theta g_c^* > g_c^* + n = g_A^* + n > n$, where $\rho + \theta g_c^* = r^*$.

We now introduce an SP with the same criterion function as that of the representative household. SP's static optimization problem is:

$$\max_{x_1,\ldots,x_A} Y = \bar{L}_Y^{1-\alpha}\sum_{j=1}^{A} x_j^{\alpha} \text{ s.t.}$$
$$\sum_{j=1}^{A} x_j = \bar{K},$$

where $\bar{L}_Y$ and $\bar{K}$ are given positive numbers. SP's dynamic optimization problem is to choose a plan $(c_t, L_{Yt})_{t=0}^{\infty}$ so as to maximize U_0 above, subject to the constraints

$$\begin{aligned}
c_t &\geq 0, 0 \leq L_{Yt} \leq L_t, \\
\dot{K}_t &= K_t^{\alpha}(A_t L_{Yt})^{1-\alpha} - c_t L_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \\
\dot{A}_t &= \eta A_t^{\varphi}(L_t - L_{Yt})^{1-\xi}, \qquad A_0 > 0 \text{ given}, \\
K_t &\geq 0 \quad \text{for all} \quad t > 0.
\end{aligned}$$

j) Consider a fixed t. For any chosen L_Y and any given K, SP wants to maximize Y, which is total output available for consumption as well as capital accumulation. Because of the symmetric and strictly concave way in which the different capital good types enter the production function for Y, SP will choose the same amount of each. That is, $x_j = x$, $\forall j$, so that

$$Y = L_Y^{1-\alpha} A x^{\alpha} = L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^{\alpha} = K^{\alpha}\left(AL_Y\right)^{1-\alpha}.$$

This is the same Y as indirectly displayed in the second line of the dynamic problem.

An alternative approach to the question is to say that SP chooses x_j in accordance with the true marginal cost (i.e., without a markup):

$$\frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} = MC^{SP} = r^{SP} + \delta, \qquad \forall j,$$

which again indicates that $x_j = x$, $\forall j$, so that $Y = K^{\alpha}(AL_Y)^{1-\alpha}$.

We are now told that, assuming $\rho - n > (1-\theta)(1-\xi)n/(1-\varphi)$, the solution to the social planner's dynamic problem will, along a BGP, have $g_A = g_A^*$ from f) and satisfy

$$s_R = \frac{1}{1 + \frac{1}{1-\xi}\left(\frac{\rho - n}{g_A^*} + \theta - \varphi\right)} \equiv s_R^{SP}.$$

k) The solution to the social planner's problem, along a BGP, *must* in this economy have $g_A = g_A^*$ from (20) because there is no other way to have balanced growth with active

R&D and full capacity utilization. This follows from the derivation of g_A at f) and is a characteristic feature of "semi-endogenous" growth, which is what we have here.

(An exceptional good answer also comments that SP *will* choose to have active R&D. Indeed, we know from applying optimal control theory to SP's dynamic problem, that the solution will satisfy the Keynes-Ramsey rule, $g_c = \frac{1}{\theta}(r^{SP} - \rho)$, where $r^{SP} = \partial Y/\partial K - \delta$. A BGP with $g_c = g_A^* > 0$ is technically feasible in the long run. Along such a BGP, which must have $L_A > 0$, we have $r^{SP} = \rho + \theta g_A^* > \rho$. SP's consumption discount rate, r^{SP}, will thus, along a BGP, exceed SP's pure rate of time preference, ρ. Consequently, SP finds it worthwhile to invest in growth, which, in addition to $I_K > 0$, amounts to $L_A > 0$.)

ℓ) From the Keynes-Ramsey rule (K-R) we find the solution, in the laissez-faire market economy, for r along a BGP to be

$$r = \rho + \theta g_c^* = \rho + \theta g_A^* \equiv r^*.$$

Substituting into (***) gives

$$s_R^* = \frac{1}{1 + \frac{\rho + \theta g_A^* - n}{\alpha g_A^*}} = \frac{1}{1 + \frac{1}{\alpha}\left(\frac{\rho - n}{g_A^*} + \theta\right)}.$$

m) We are now told that $\xi = 0$ and $0 < \varphi < 1$. We then have

$$s_R^* = \frac{1}{1 + \frac{1}{\alpha}\left(\frac{\rho - n}{g_A^*} + \theta\right)} = \frac{1}{1 + \frac{1}{\alpha}(\frac{\rho - n}{n}(1 - \varphi) + \theta)}$$

to be compared with

$$s_R^{SP} = \frac{1}{1 + \frac{\rho - n}{g_A^*} + \theta - \varphi} = \frac{1}{1 + \frac{\rho - n}{n}(1 - \varphi) + \theta - \varphi}.$$

We see that

$$s_R^{SP} - s_R^* > 0$$

for two reasons. One reason is the monopoly markup $1/\alpha > 1$, which creates a wedge between the economic benefit (the marginal productivity) of the services of specialized capital goods and their marginal cost, thus giving too little incentive to do R&D. A second reason is "$-\varphi$" in the denominator of s_R^{SP}. This reflects that the laissez-faire market economy does not internalize the positive intertemporal spillover deriving from $\varphi > 0$.

n) The distortion due to monopoly pricing can be remedied by a subsidy, σ, to buyers of the services of specialized capital goods. The purchase price from the point of view of the firms in sector 1 will then be $(1-\sigma)(r+\delta)/\alpha$. By choosing $\sigma = 1-\alpha$, this price comes down to marginal cost, $r+\delta$.

The distortion due to the positive intertemporal spillover can be remedied by a subsidy, s, to cover part of the R&D cost. Then the cost per unit of research labor from the point of view of the single R&D lab will be $(1-s)w$, and (13) is, for $\xi = 0$, replaced by

$$(1-s)w = (1-s)(1-\alpha)\frac{Y}{L_Y} = P_A \eta A^{\varphi}.$$

So, for given P_A and A, the larger is the subsidy s, the larger is the ratio Y/L_Y, which in turn requires a low L_Y, hence a high $L_A = L - L_Y$.

The subsidies should be financed by a lump-sum tax or a tax equivalent to a lump-sum tax, at least along a BGP. Since labor supply is assumed inelastic, a tax on labor income or consumption is appropriate.

2. Solution to Problem 2 (30 %)

a) Yes we can! Poor countries differ from rich countries w.r.t. several factors that are complementary to physical capital, not least the technology level and the human capital level.

It is OK to answer in purely verbal terms. But here is an answer using simple formulas.

Consider a set of countries, $j = 1, 2, ..., N$. Country j has the aggregate production function

$$Y_j = F(K_j, A_j h_j L_j) = A_j h_j L_j F(\frac{K_j}{A_j h_j L_j}, 1) \equiv A_j h_j L_j f(\tilde{k}_j), \quad f' > 0, f'' < 0,$$

where F is neoclassical with CRS (standard notation). Let r_j denote the equilibrium net rate of return on capital in country j. Then, under perfect competition,

$$r_j = \frac{\partial Y_j}{\partial K_j} - \delta = f(\tilde{k}_j) - \delta.$$

Can the countries have the same r in spite of widely differing K_j/L_j? Yes. Differing K_j/L_j does not rule out that $\tilde{k}_j = \tilde{k}$ for all j. Indeed, as

$$\tilde{k}_j \equiv \frac{K_j}{A_j h_j L_j} = \frac{K_j/L_j}{A_j h_j},$$

and as countries with low K_j/L_j (the poor countries) also tend to have low A_j and h_j, the $\tilde{k}_j$'s - and therefore also the r_j's - may be more or less of the same size.

b) The three kinds of capital have in common that they are *producible* production factors.

Physical capital is a *non-human* production factor perceived as a stock, i.e., it is a store of productive power. It is a *rival good* in the sense that by its very nature, its use by one agent at a given point in time precludes its use by other agents at the same time. It is also an *excludable good* in the sense that the right (the title) to use it can, via the institutions (rules of the game) of society, be assigned to a particular agent, say by ownership, thereby excluding others from this right.

In contrast, *human capital* is a *human* production factor. The term refers to the stock of productive skills *embodied in an individual* and lost upon death. Increases in the stock of human capital occurs through formal education and on-the-job-training. In this sense human capital is also producible. Since human capital is embodied in individuals and can only be used one place at a time, it is a *rival* good. It is also an *excludable* good.

We think of *technical knowledge* as a list of instructions about how different inputs can be combined to produce a certain output. A principle of chemical engineering is an example of a piece of technical knowledge. In contrast to both physical and human capital, technical knowledge is a *non-rival* good. The same principle of chemical engineering can, by its very nature, be used at the same time by arbitrarily many agents.

By its nature, technical knowledge is only a *partially excludable good*. Basic science is not excludable, but the right to use a particular "technical design" commercially may via the institutions of society (patents, copyright) be made excludable for some duration.

c) We are asked to briefly evaluate the following kind of model for advanced economies:

$$Y = F(K, H) \equiv F(K, hL), \qquad (*)$$

combined with the relations

$$\begin{aligned} Y &= cL + I_K + I_H, \\ \dot{K} &= I_K - \delta_K K, \qquad (**) \\ \dot{H} &= I_H - \delta_H H, \end{aligned}$$

where perfect competition in all markets is assumed.

Evaluation:

At a purely theoretical level the approach can be seen as a pedagogically simple way of constructing a model leading to a reduced-form AK model after some initial period of full specialization with regard to what to invest in. The approach can thereby illustrate that *if* production has CRS w.r.t. two producible inputs (and thereby absence of diminishing returns to "broad capital"), then "fully-endogenous" exponential growth is technically feasible.

There are several weaknesses of the approach, however:

1. It ignores technical knowledge as well as the productive *complementarity* between human capital and technical knowledge. This complementarity, which is for instance visible in the formulas at question a) but absent in (*), is empirically well documented (skill-biased technical change).

2. The approach arbitrarily imposes CRS w.r.t. two producible inputs, K and H, produced by means of the technology (*) and (**). The empirical support for this seems absent. Recall, for instance, the cross-country regression analysis by Mankiw, Romer, and Weil (1992). Based on (**) they conclude that the following production function for a country's GDP is an acceptable approximation:

$$Y = BK^{1/3}H^{1/3}L^{1/3},$$

where B stands for the total factor productivity of the country and is generally growing over time. In view of $H \equiv hL$, this can be written:

$$Y = BK^{1/3}(hL)^{1/3}L^{1/3} = K^{1/3}(Ah^{1/2}L)^{2/3}, \qquad (***)$$

where $A = B^{3/2}$. In contrast to (*), first, there are *diminishing returns to scale* to K and H, taking together. Second, the technology level appears with an independent role along with that of human capital.

3. While MRW (1992), and many others, disagree with (*), MRW (**) seem satisfied with the human capital *formation* approach given in (**). But also this part of the above approach may be questioned, at least if one wants a measure of human capital, $\tilde{h}$, allowing us, under perfect competition, to write the real wage per man-hour as

$$w = \frac{\partial Y}{\partial L} = \tilde{F}_2(K, \tilde{h}L, t)\tilde{h} = \hat{w} \cdot \tilde{h}, \qquad \frac{\partial \tilde{F}}{\partial t} > 0,$$

where $\hat{w}$ is the real wage *per unit of human capital* per time unit, and $\frac{\partial \tilde{F}}{\partial t} > 0$ represents technical progress. As (***) shows, this convenient relationship, where the real wage is proportional to human capital (which many analysts in fact implicitly assume), does not, empirically, arise when human capital is assumed formed the way given in (**), that is, as similar to the way physical capital is formed.

Several empirical studies find that this proportionality arises approximately when $\tilde{h}$ is modelled as $\tilde{h} = h(S) = \alpha S^{\beta}$, $a > 0, \beta > 0$, S being a measure of average schooling in the labor force.

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