2.7 Appendix

A. Strict quasiconcavity

Consider a function $f : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A}$ is a convex set, $\mathcal{A} \subseteq \mathbb{R}^n$.[30] Given a real number a, if $f(x) = a$, the *upper contour set* is defined as $\{x \in \mathcal{A} | \, f(x) \geq a\}$ (the set of input bundles that can produce at least the amount a of output). The function $f(x)$ is called *quasiconcave* if its upper contour sets, for any constant a, are convex sets. If all these sets are strictly convex, $f(x)$ is called *strictly quasiconcave*.

Average and marginal costs To show that (2.14) holds with n production inputs, $n = 1, 2, \ldots$, we derive the cost function of a firm with a neoclassical production function, $Y = F(X_1, X_2, \ldots, X_n)$. Given a vector of strictly positive input prices $\mathbf{w} = (w_1, \ldots, w_n) >> 0$, the firm faces the problem of finding a cost-minimizing way to produce a given positive output level $\bar{Y}$ within the range of F. The problem is

$$\min \sum_{i=1}^{n} w_i X_i \quad \text{s.t. } F(X_1, \ldots, X_n) = \bar{Y} \text{ and } X_i \geq 0, \; i = 1, 2, \ldots, n.$$

An interior solution, $\mathbf{X}^* = (X_1^*, \ldots, X_n^*)$, to this problem satisfies the first-order conditions $\lambda F_i'(\mathbf{X}^*) = w_i$, where λ is the Lagrange multiplier, $i = 1, \ldots, n$.[31] Since F is neoclassical and thereby strictly quasiconcave in the interior of $\mathbb{R}_+^n$, the first-order conditions are not only necessary but also sufficient for the vector $\mathbf{X}^*$ to be a solution, and $\mathbf{X}^*$ will be unique[32] so that we can write it as a function, $\mathbf{X}^*(\bar{Y}) = (X_1^*(\bar{Y}), \ldots, X_n^*(\bar{Y}))$. This gives rise to the *cost function* $\mathcal{C}(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^*(\bar{Y})$. So *average cost* is $\mathcal{C}(\bar{Y})/\bar{Y}$. We find *marginal cost* to be

$$\mathcal{C}'(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^{*\prime}(\bar{Y}) = \lambda \sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^{*\prime}(\bar{Y}) = \lambda,$$

where the third equality comes from the first-order conditions, and the last equality is due to the constraint $F(\mathbf{X}^*(\bar{Y})) = \bar{Y}$, which, by taking the total derivative on both sides, gives $\sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^{*\prime}(\bar{Y}) = 1$. Consequently, the ratio of average to marginal costs is

$$\frac{\mathcal{C}(\bar{Y})/\bar{Y}}{\mathcal{C}'(\bar{Y})} = \frac{\sum_{i=1}^{n} w_i X_i^*(\bar{Y})}{\lambda \bar{Y}} = \frac{\sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^*(\bar{Y})}{F(\mathbf{X}^*)},$$

[30] Recall that a set S is said to be *convex* if $x, y \in S$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda) y \in S$.

[31] Since in this section we use a bit of vector notation, we exceptionally mark first-order partial derivatives by a prime in order to clearly distinguish from the elements of a vector (so we write F_i' instead of our usual F_i).

[32] See Sydsaeter et al. (2008), pp. 74, 75, and 125.

which in analogy with (2.13) is the elasticity of scale at the point $\mathbf{X}^*$. This proves (2.14).

Sufficient conditions for strict quasiconcavity The claim (iii) in Section 2.1.3 was that a continuously differentiable two-factor production function $F(K, L)$ with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, will automatically also be strictly quasi-concave in the interior of $\mathbb{R}^2$ and thus neoclassical.

To prove this, consider a function of two variables, $z = f(x, y)$, that is twice continuously differentiable with $f_1 \equiv \partial z/\partial x > 0$ and $f_2 \equiv \partial z/\partial y > 0$, everywhere. Then the equation $f(x, y) = a$, where a is a constant, defines an isoquant, $y = g(x)$, with slope $g'(x) = -f_1(x,y)/f_2(x,y)$. Substitute $g(x)$ for y in this equation and take the derivative w.r.t. x. By straightforward calculation we find

$$g''(x) = -\frac{f_1^2 f_{22} - 2f_1 f_2 f_{21} + f_2^2 f_{11}}{f_2^3} \tag{2.53}$$

If the numerator is negative, then $g''(x) > 0$; that is, the isoquant is strictly convex to the origin. And if this holds for all (x, y), then f is strictly quasi-concave in the interior of $\mathbb{R}^2$. A sufficient condition for a negative numerator is that $f_{11} < 0$, $f_{22} < 0$ and $f_{21} \geq 0$. All these conditions, including the last three are satisfied by the given function F. Indeed, F_K, F_L, F_{KK}, and F_{LL} have the required signs. And when F has CRS, F is homogeneous of degree 1 and thereby $F_{KL} > 0$, see Appendix B. Hereby claim (iii) in Section 2.1.3 is proved.

B. Homogeneous production functions

The claim (iv) in Section 2.1.3 was that a two-factor production function with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, has always $F_{KL} > 0$, i.e., there is *direct complementarity* between K and L. This assertion is implied by the following observations on homogeneous functions.

Let $Y = F(K, L)$ be a twice continuously differentiable production function with $F_K > 0$ and $F_L > 0$ everywhere. Assume F is homogeneous of degree $h > 0$, that is, for all possible (K, L) and all $\lambda > 0$, $F(\lambda K, \lambda L) = \lambda^h F(K, L)$. According to Euler's theorem (see Math Tools) we then have:

CLAIM 1 For all (K, L), where $K > 0$ and $L > 0$,

$$KF_K(K, L) + LF_L(K, L) = hF(K, L). \tag{2.54}$$

Euler's theorem also implies the inverse:

CLAIM 2 If (2.54) is satisfied for all (K, L), where $K > 0$ and $L > 0$, then $F(K, L)$ is homogeneous of degree h.

Partial differentiation w.r.t. K and L, respectively, gives, after ordering,

$$KF_{KK} + LF_{LK} = (h-1)F_K \tag{2.55}$$
$$KF_{KL} + LF_{LL} = (h-1)F_L. \tag{2.56}$$

In (2.55) we can substitute $F_{LK} = F_{KL}$ (by Young's theorem). In view of Claim 2 this shows:

CLAIM 3 The marginal products, F_K and F_L, considered as functions of K and L, are homogeneous of degree $h-1$.

We see also that when $h \geq 1$ and K and L are positive, then

$$F_{KK} < 0 \text{ implies } F_{KL} > 0, \tag{2.57}$$
$$F_{LL} < 0 \text{ implies } F_{KL} > 0. \tag{2.58}$$

For $h = 1$ this establishes the direct complementarity result, (iv) in Section 2.1.3, to be proved. A by-product of the derivation is that also when a neoclassical production function is homogeneous of degree $h > 1$ (which implies IRS), does direct complementarity between K and L hold.

Remark. The terminology around complementarity and substitutability may easily lead to confusion. In spite of K and L exhibiting *direct complementarity* when $F_{KL} > 0$, K and L are still *substitutes* in the sense that cost minimization for a given output level implies that a rise in the price of one factor results in higher demand for the other factor.

The claim (v) in Section 2.1.3 was the following. Suppose we face a CRS production function, $Y = F(K, L)$, that has positive marginal products, F_K and F_L, everywhere and isoquants, $K = g(L)$, satisfying the condition $g''(L) > 0$ everywhere (i.e., F is strictly quasi-concave). Then the partial second derivatives must satisfy the neoclassical conditions:

$$F_{KK} < 0, F_{LL} < 0. \tag{2.59}$$

The proof is as follows. The first inequality in (2.59) follows from (2.53) combined with (2.55). Indeed, for $h = 1$, (2.55) and (2.56) imply $F_{KK} = -F_{LK}L/K = -F_{KL}L/K$ and $F_{KL} = -F_{LL}L/K$, i.e., $F_{KK} = F_{LL}(L/K)^2$ (or, in the notation of Appendix A, $f_{22} = f_{11}(x/y)^2$), which combined with (2.53) gives the conclusion $F_{KK} < 0$, when $g'' > 0$. The second inequality in (2.59) can be verified in a similar way.

Note also that for $h = 1$ the equations (2.55) and (2.56) entail

$$KF_{KK} = -LF_{LK} \text{ and } KF_{KL} = -LF_{LL}, \tag{2.60}$$

respectively. By dividing the left- and right-hand sides of the first of these equations with those of the second we conclude that $F_{KK}F_{LL} = F_{KL}^2$ in the CRS case. We see also from (2.60) that, under CRS, the implications in (2.57) and (2.58) can be turned round.

Finally, we asserted in § 2.1.1 that when the neoclassical production function $Y = F(K, L)$ is homogeneous of degree h, then the marginal rate of substitution between the production factors depends only on the factor proportion $k \equiv K/L$. Indeed,

$$MRS_{KL}(K,L) = \frac{F_L(K,L)}{F_K(K,L)} = \frac{L^{h-1}F_L(k,1)}{L^{h-1}F_K(k,1)} = \frac{F_L(k,1)}{F_K(k,1)} \equiv mrs(k), \tag{2.61}$$

where $k \equiv K/L$. The result (2.61) follows even if we only assume $F(K,L)$ is *homothetic*. When $F(K,L)$ is homothetic, by definition we can write $F(K, L) \equiv \varphi(G(K,L))$, where G is homogeneous of degree 1 and φ is an increasing function. In view of this, we get

$$MRS_{KL}(K,L) = \frac{\varphi' G_L(K,L)}{\varphi' G_K(K,L)} = \frac{G_L(k,1)}{G_K(k,1)},$$

where the last equality is implied by Claim 3 for $h = 1$.

C. The Inada conditions combined with CRS

We consider a neoclassical production function, $Y = F(K,L)$, exhibiting CRS. Defining $k \equiv K/L$, we can then write $Y = LF(k,1) \equiv Lf(k)$, where $f(0) \geq 0, f' > 0$, and $f'' < 0$.

Essential inputs In Section 2.1.2 we claimed that the upper Inada condition for MPL together with CRS implies that without capital there will be no output:

$$F(0,L) = 0 \qquad \text{for any } L > 0.$$

In other words: in this case capital is an essential input. To prove this claim, let $K > 0$ be fixed and let $L \to \infty$. Then $k \to 0$, implying, by (2.16) and (2.18), that $F_L(K,L) = f(k) - f'(k)k \to f(0)$. But from the upper Inada condition for MPL we also have that $L \to \infty$ implies $F_L(K,L) \to 0$. It follows that

$$\text{the upper Inada condition for } MPL \text{ implies } f(0) = 0. \tag{2.62}$$

Since under CRS, for any $L > 0$, $F(0,L) = LF(0,1) \equiv Lf(0)$, we have hereby shown our claim.

Similarly, we can show that the upper Inada condition for MPK together with CRS implies that labor is an essential input. Consider the output-capital ratio $x \equiv Y/K$. When F has CRS, we get $x = F(1, \ell) \equiv g(\ell)$, where $\ell \equiv L/K$, $g' > 0$, and $g'' < 0$. Thus, by symmetry with the previous argument, we find that under CRS, the upper Inada condition for MPK implies $g(0) = 0$. Since under CRS $F(K, 0) = KF(1, 0) \equiv Kg(0)$, we conclude that the upper Inada condition for MPK together with CRS implies

$$F(K, 0) = 0 \qquad \text{for any } K > 0,$$

that is, without labor, no output.

Sufficient conditions for output going to infinity when either input goes to infinity Here our first claim is that when F exhibits CRS and satisfies the upper Inada condition for MPL and the lower Inada condition for MPK, then

$$\lim_{L \to \infty} F(K, L) = \infty \qquad \text{for any } K > 0.$$

To prove this, note that Y can be written $Y = Kf(k)/k$, since $K/k = L$. Here,

$$\lim_{k \to 0} f(k) = f(0) = 0,$$

by continuity and (2.62), presupposing the upper Inada condition for MPL. Thus, for any given $K > 0$,

$$\lim_{L \to \infty} F(K, L) = K \lim_{L \to \infty} \frac{f(k)}{k} = K \lim_{k \to 0} \frac{f(k) - f(0)}{k} = K \lim_{k \to 0} f'(k) = \infty,$$

by the lower Inada condition for MPK. This verifies the claim.

Our second claim is symmetric with this and says: when F exhibits CRS and satisfies the upper Inada condition for MPK and the lower Inada condition for MPL, then

$$\lim_{K \to \infty} F(K, L) = \infty \qquad \text{for any } L > 0.$$

The proof is analogue. So, in combination, the four Inada conditions imply, under CRS, that output has no upper bound when either input goes to infinity.

D. Concave neoclassical production functions

Two claims made in Section 2.4 are proved here.

CLAIM 1 When a neoclassical production function $F(K, L)$ is concave, it has non-increasing returns to scale everywhere.

Proof. We consider a concave neoclassical production function, F. Let $\mathbf{x} = (x_1, x_2) = (K, L)$. Then we can write $F(K, L)$ as $F(\mathbf{x})$. By concavity, for all pairs $\mathbf{x}^0, \mathbf{x} \in \mathbb{R}^2_+$, we have $F(\mathbf{x}^0) - F(\mathbf{x}) \leq \sum_{i=1}^2 F_i'(\mathbf{x})(x_i^0 - x_i)$. In particular, for $\mathbf{x}^0 = (0, 0)$, since $F(\mathbf{x}^0) = F(0, 0) = 0$, we have

$$-F(\mathbf{x}) \leq -\sum_{i=1}^{2} F_i'(\mathbf{x})x_i. \tag{2.63}$$

Suppose $\mathbf{x} \in \mathbb{R}^2_{++}$. Then $F(\mathbf{x}) > \mathbf{0}$ in view of F being neoclassical so that $F_K > 0$ and $F_L > 0$. From (2.63) we now find the elasticity of scale to be

$$\sum_{i=1}^{2} F_i'(\mathbf{x})x_i / F(\mathbf{x}) \leq 1. \tag{2.64}$$

In view of (2.13) and (2.12), this implies non-increasing returns to scale everywhere. □

CLAIM 2 When a neoclassical production function $F(K, L)$ is strictly concave, it has decreasing returns to scale everywhere.

Proof. The argument is analogue to that above, but in view of strict concavity the inequalities in (2.63) and (2.64) become strict. This implies that F has DRS everywhere. □

2.8 Exercises

2.1