where $\beta \equiv (1+\rho)^{-1}$. The function $x = g(c_1, c_2) \equiv (c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}$ is homogeneous of degree one and the function $G(x) \equiv (1/(1-\theta))x^{1-\theta} - (1+\beta)/(1-\theta)$ is an increasing function, given $\theta > 0$, $\theta \neq 1$. *Case 2:* $\theta = 1$. Here we start from $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$. This can be written

$$U(c_1, c_2) = (1+\beta)\ln\left[(c_1 c_2^{\beta})^{1/(1+\beta)}\right],$$

where $x = g(c_1, c_2) = (c_1 c_2^{\beta})^{1/(1+\beta)}$ is homogeneous of degree one and $G(x) \equiv (1+\beta)\ln x$ is an increasing function. □

D. General formulas for the elasticity of factor substitution

We here prove (4.30) and (4.31). Given the neoclassical production function $F(K, L)$, the slope of the isoquant $F(K, L) = \bar{Y}$ at the point $(\bar{K}, \bar{L})$ is

$$\frac{dK}{dL}_{|Y=\bar{Y}} = -MRS = -\frac{F_L(\bar{K}, \bar{L})}{F_K(\bar{K}, \bar{L})}. \tag{4.40}$$

We consider this slope as a function of the value of $k \equiv K/L$ as we move along the isoquant. The derivative of this function is

$$\begin{aligned} -\frac{dMRS}{dk}_{|Y=\bar{Y}} &= -\frac{dMRS}{dL}_{|Y=\bar{Y}} \frac{dL}{dk}_{|Y=\bar{Y}} \\ &= -\frac{(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}}{F_K^3} \frac{dL}{dk}_{|Y=\bar{Y}} \end{aligned} \tag{4.41}$$

by (**??**) of Chapter 2. In view of $L \equiv K/k$ we have

$$\frac{dL}{dk}_{|Y=\bar{Y}} = \frac{k\frac{dK}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{k\frac{dK}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{-kMRS\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2}.$$

From this we find

$$\frac{dL}{dk}_{|Y=\bar{Y}} = -\frac{K}{(k+MRS)k},$$

to be substituted into (4.41). Finally, we substitute the inverse of (4.41) together with (4.40) into the definition of the elasticity of factor substitution:

$$\begin{aligned} \sigma(K, L) &\equiv \frac{MRS}{k}\frac{dk}{dMRS}_{|Y=\bar{Y}} \\ &= -\frac{F_L/F_K}{k}\frac{(k+F_L/F_K)k}{K}\frac{F_K^3}{[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}]} \\ &= -\frac{F_K F_L (F_K K + F_L L)}{KL\,[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}]}, \end{aligned}$$

which is the same as (4.30).

Under CRS, this reduces to

$$\begin{aligned}
\sigma(K,L) &= -\frac{F_K F_L F(K,L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]} \text{ (from (??) with } h=1) \\
&= -\frac{F_K F_L F(K,L)}{KLF_{KL}\left[-(F_L)^2 L/K - 2F_K F_L - (F_K)^2 K/L\right]} \text{ (from (??))} \\
&= \frac{F_K F_L F(K,L)}{F_{KL}(F_L L + F_K K)^2} = \frac{F_K F_L}{F_{KL} F(K,L)}, \text{ (using (??) with } h=1)
\end{aligned}$$

which proves the first part of (4.31). The second part is an implication of rewriting the formula in terms of the production in intensive form.

E. Properties of the CES production function

The generalized CES production function is

$$Y = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]^{\frac{\gamma}{\beta}}, \tag{4.42}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0, \gamma > 0$. If $\gamma < 1$, there is DRS, if $\gamma = 1$, CRS, and if $\gamma > 1$, IRS. The elasticity of substitution is always $\sigma = 1/(1-\beta)$. Throughout below, k means K/L.

The limiting functional forms We claimed in the text that, for fixed $K > 0$ and $L > 0$, (4.42) implies:

$$\begin{aligned}
\lim_{\beta\to 0} Y &= A(K^\alpha L^{1-\alpha})^\gamma = AL^\gamma k^{\alpha\gamma}, && (*) \\
\lim_{\beta\to -\infty} Y &= A\min(K^\gamma, L^\gamma) = AL^\gamma \min(k^\gamma, 1). && (**)
\end{aligned}$$

Proof. Let $q \equiv Y/(AL^\gamma)$. Then $q = (\alpha k^\beta + 1 - \alpha)^{\gamma/\beta}$ so that

$$\ln q = \frac{\gamma \ln(\alpha k^\beta + 1 - \alpha)}{\beta} \equiv \frac{m(\beta)}{\beta}, \tag{4.43}$$

where

$$m'(\beta) = \frac{\gamma\alpha k^\beta \ln k}{\alpha k^\beta + 1 - \alpha} = \frac{\gamma\alpha \ln k}{\alpha + (1-\alpha)k^{-\beta}}. \tag{4.44}$$

Hence, by L'Hôpital's rule for "0/0",

$$\lim_{\beta\to 0} \ln q = \lim_{\beta\to 0} \frac{m'(\beta)}{1} = \gamma\alpha \ln k = \ln k^{\gamma\alpha},$$

so that $\lim_{\beta\to 0} q = k^{\gamma\alpha}$, which proves (*). As to (**), note that

$$\lim_{\beta\to-\infty} k^{\beta} = \lim_{\beta\to-\infty} \frac{1}{k^{-\beta}} \to \begin{cases} 0 \text{ for } k > 1, \\ 1 \text{ for } k = 1, \\ \infty \text{ for } k < 1. \end{cases}$$

Hence, by (4.43),

$$\lim_{\beta\to-\infty} \ln q = \begin{cases} 0 \text{ for } k \geq 1, \\ \lim_{\beta\to-\infty} \frac{m'(\beta)}{1} = \gamma \ln k = \ln k^{\gamma} \text{ for } k < 1, \end{cases}$$

where the result for $k < 1$ is based on L'Hôpital's rule for "∞/-∞". Consequently,

$$\lim_{\beta\to-\infty} q = \begin{cases} 1 \text{ for } k \geq 1, \\ k^{\gamma} \text{ for } k < 1, \end{cases}$$

which proves (**). □

Properties of the isoquants of the CES function The absolute value of the slope of an isoquant for (4.42) in the (L, K) plane is

$$MRS_{KL} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} \to \begin{cases} 0 \text{ for } k \to 0, \\ \infty \text{ for } k \to \infty. \end{cases} \quad (*)$$

This holds whether $\beta < 0$ or $0 < \beta < 1$.

Concerning the asymptotes and terminal points, if any, of the isoquant $Y = \bar{Y}$ we have from (4.42) $\bar{Y}^{\beta/\gamma} = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]$. Hence,

$$K = \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A\alpha} - \frac{1-\alpha}{\alpha} L^{\beta}\right)^{\frac{1}{\beta}},$$

$$L = \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A(1-\alpha)} - \frac{\alpha}{1-\alpha} K^{\beta}\right)^{\frac{1}{\beta}}.$$

From these two equations follows, when $\beta < 0$ (i.e., $0 < \sigma < 1$), that

$$K \to (A\alpha)^{-\frac{1}{\beta}} \bar{Y}^{\frac{1}{\gamma}} \text{ for } L \to \infty,$$

$$L \to [A(1-\alpha)]^{-\frac{1}{\beta}} \bar{Y}^{\frac{1}{\gamma}} \text{ for } K \to \infty.$$

When instead $\beta > 0$ (i.e., $\sigma > 1$), the same limiting formulas obtain for $L \to 0$ and $K \to 0$, respectively.

Properties of the CES function on intensive form Given $\gamma = 1$, i.e., CRS, we have $y \equiv Y/L = A(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$ from (4.42). Then

$$\frac{dy}{dk} = A\frac{1}{\beta}(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}-1}\alpha\beta k^{\beta-1} = A\alpha\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}}.$$

Hence, when $\beta < 0$ (i.e., $0 < \sigma < 1$),

$$\begin{aligned} y &= \frac{A}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \to \begin{cases} 0 \text{ for } k \to 0, \\ A(1-\alpha)^{1/\beta} \text{ for } k \to \infty. \end{cases} \\ \frac{dy}{dk} &= \frac{A\alpha}{[\alpha + (1-\alpha)k^{-\beta}]^{(\beta-1)/\beta}} \to \begin{cases} A\alpha^{1/\beta} \text{ for } k \to 0, \\ 0 \text{ for } k \to \infty. \end{cases} \end{aligned}$$

If instead $\beta > 0$ (i.e., $\sigma > 1$),

$$\begin{aligned} y &\to \begin{cases} A(1-\alpha)^{1/\beta} \text{ for } k \to 0, \\ \infty \text{ for } k \to \infty. \end{cases} \\ \frac{dy}{dk} &\to \begin{cases} \infty \text{ for } k \to 0, \\ A\alpha^{1/\beta} \text{ for } k \to \infty. \end{cases} \end{aligned}$$

The output-capital ratio is $y/k = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and has the same limiting values as dy/dk, when $\beta > 0$.

Continuity at the boundary of $\mathbb{R}^2_+$ When $0 < \beta < 1$, the right-hand side of (4.42) is defined and continuous also on the boundary of $\mathbb{R}^2_+$. Indeed, we get

$$Y = F(K,L) = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}} \to \begin{cases} A\alpha^{\frac{\gamma}{\beta}}K^{\gamma} \text{ for } L \to 0, \\ A(1-\alpha)^{\frac{\gamma}{\beta}}L^{\gamma} \text{ for } K \to 0. \end{cases}$$

When $\beta < 0$, however, the right-hand side is not defined on the boundary. We circumvent this problem by redefining the CES function in the following way when $\beta < 0$:

$$Y = F(K,L) = \begin{cases} A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}} \text{ when } K > 0 \text{ and } L > 0, \\ 0 \text{ when either } K \text{ or } L \text{ equals } 0. \end{cases} \tag{4.45}$$

We now show that continuity holds in the extended domain. When $K > 0$ and $L > 0$, we have

$$Y^{\frac{\beta}{\gamma}} = A^{\frac{\beta}{\gamma}}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right] \equiv A^{\frac{\beta}{\gamma}}G(K,L). \tag{4.46}$$

Let $\beta < 0$ and $(K, L) \to (0,0)$. Then, $G(K, L) \to \infty$, and so $Y^{\beta/\gamma} \to \infty$. Since $\beta/\gamma < 0$, this implies $Y \to 0 = F(0,0)$, where the equality follows from the definition in (4.45). Next, consider a fixed $L > 0$ and rewrite (4.46) as

$$
\begin{aligned}
Y^{\frac{1}{\gamma}} &= A^{\frac{1}{\gamma}} \left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} = A^{\frac{1}{\gamma}} L(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}} \\
&= \frac{A^{\frac{1}{\gamma}} L}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \to 0 \text{ for } k \to 0,
\end{aligned}
$$

when $\beta < 0$. Since $1/\gamma > 0$, this implies $Y \to 0 = F(0, L)$, from (4.45). Finally, consider a fixed $K > 0$ and let $L/K \to 0$. Then, by an analogue argument we get $Y \to 0 = F(K, 0)$, (4.45). So continuity is maintained in the extended domain.

4.10 Exercises

4.1 *(the aggregate saving rate in steady state)*

a) In a well-behaved Diamond OLG model let n be the rate of population growth and k^* the steady state capital-labor ratio (until further notice, we ignore technological progress). Derive a formula for the long-run aggregate net saving rate, S^N/Y, in terms of n and k^*. *Hint:* use that for a closed economy $S^N = K_{t+1} - K_t$.

b) In the Solow growth model without technological change a similar relation holds, but with a different interpretation of the causality. Explain.

c) Compare your result in a) with the formula for S^N/Y in steady state one gets in *any* model with the same CRS-production function and no technological change. Comment.

d) Assume that $n = 0$. What does the formula from a) tell you about the level of net aggregate savings in this case? Give the intuition behind the result in terms of the aggregate saving by any generation in two consecutive periods. One might think that people's rate of impatience (in Diamond's model the rate of time preference ρ) affect S^N/Y in steady state. Does it in this case? Why or why not?

e) Suppose there is Harrod-neutral technological progress at the constant rate $g > 0$. Derive a formula for the aggregate net saving rate in the long run in a well-behaved Diamond model in this case.

f) Answer d) with "from a)" replaced by "from e)". Comment.

g) Consider the statement: "In Diamond's OLG model any generation saves as much when young as it dissaves when old." True or false? Why?

4.2 (*increasing returns to scale and balanced growth)*