Lecture Notes in Economic Growth

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Preface

This is a slight extension of my lecture notes in Economic Growth from 2015. These notes have been used in recent years in the course Economic Growth within the Master's Program in Economics at the Department of Economics, University of Copenhagen. Discovered typos and similar have been corrected. In some of the chapters a terminal list of references is included, in some not.

The lecture notes contain many references to the textbook by Daron Acemoglu, *Introduction to Modern Economic Growth* (Princeton University Press, 2009). Parts of the lecture notes are alternative presentations of stuff also covered in the Acemoglu book, while other parts can be seen as complementary. Sections marked by an asterisk, *, can be skipped, at least in a first reading.

For constructive criticism I thank Niklas Brønager, class instructor since 2012, and plenty of earlier students. No doubt, obscurities remain. Hence, I very much welcome comments and suggestions of any kind relating to these lecture notes.

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Chapter 1

Introduction to economic growth

This introductory lecture note is a refresher on basic concepts.

Section 1.1 defines Economic Growth as a field of economics. In Section 1.2 formulas for calculation of compound average growth rates in discrete and continuous time are presented. Section 1.3 briefly presents two sets of what is by many considered as "stylized facts" about economic growth. Finally, Section 1.4 discusses, in an informal way, the different concepts of cross-country income convergence. In his introductory Chapter 1, §1.5, Acemoglu[1] briefly touches upon these concepts.

1.1 The field

Economic growth analysis is the study of what factors and mechanisms determine the time path of *productivity* (a simple index of productivity is output per unit of labor). The focus is on

- productivity levels and
- productivity growth.

1.1.1 Economic growth theory

Economic growth theory endogenizes productivity growth via considering human capital accumulation (formal education as well as learning-by-doing)

[1] Throughout these lecture notes, "Acemoglu" refers to Daron Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press: Oxford, 2009.

and endogenous research and development. Also the conditioning role of geography and juridical, political, and cultural institutions is taken into account.

For practical reasons, economic growth theory is often stated in terms of national income and product account variables like per capita GDP. Yet the term "economic growth" may be interpreted as referring to something deeper. We could think of "economic growth" as the widening of the opportunities of human beings to lead a freer and more worthwhile life (cf. Sen, ...).

To make our complex economic environment accessible for theoretical analysis we use economic models. What *is* an economic model? It is a way of organizing one's thoughts about the economic functioning of a society. A more specific answer is to define an economic model as a conceptual structure based on a set of mathematically formulated assumptions which have an economic interpretation and from which empirically testable predictions can be derived. In particular, an economic growth model is an economic model concerned with productivity issues. The union of connected and non-contradictory models dealing with economic growth and the propositions derived from these models constitute *economic growth theory*. Occasionally, intense controversies about the validity of alternative growth theories take place.

The terms "New Growth Theory" and "endogenous growth theory" refer to theory and models which attempt at explaining sustained per capita growth as an outcome of internal mechanisms in the model rather than just a reflection of exogenous technical progress as in "Old Growth Theory".

Among the themes addressed in this course are:

- How is the world income distribution evolving?
- Why do living standards differ so much across countries and regions? Why are some countries 50 times richer than others?
- Why do per capita growth rates differ over long periods?
- What are the roles of human capital and technology innovation in economic growth? Getting the questions right.
- Catching-up and increased speed of communication and technology diffusion.
- Economic growth, natural resources, and the environment (including the climate). What are the limits to growth?
- Policies to ignite and sustain productivity growth.

- The prospects of growth in the future.

The course concentrates on *mechanisms* behind the evolution of productivity in the industrialized world. We study these mechanisms as integral parts of dynamic models.

The exam is a test of the extent to which the student has acquired understanding of these models, is able to evaluate them, from both a theoretical and empirical perspective, and is able to use them to analyze specific economic questions. The course is calculus intensive.

1.1.2 Some long-run data

Let Y denote real GDP (per year) and let N be population size. Then Y/N is GDP per capita. Further, let g_Y denote the average (compound) growth rate of Y per year since 1870 and let $g_{Y/N}$ denote the average (compound) growth rate of Y/N per year since 1870. Table 1.1 gives these growth rates for four countries. (But we should not forget that data from before WWII should be taken with a grain of salt).

	g_Y	$g_{Y/N}$
Denmark	2,67	1,87
UK	1,96	1,46
USA	3,40	1,89
Japan	3,54	2,54

Table 1.1: Average annual growth rate of GDP and GDP per capita in percent, 1870–2006. Discrete compounding. Source: Maddison, A: The World Economy: Historical Statistics, 2006, Table 1b, 1c and 5c.

Figure 1.1 displays the time path of annual GDP and GDP per capita in Denmark 1870-2006 along with regression lines estimated by OLS (logarithmic scale on the vertical axis). Figure 1.2 displays the time path of GDP per capita in UK, USA, and Japan 1870-2006. In both figures the average annual growth rates are reported. In spite of being based on exactly the same data as Table 1.1, the numbers are slightly different. Indeed, the numbers in the figures are slightly lower than those in the table. The reason is that discrete compounding is used in Table 1.1 while continuous compounding is used in the two figures. These two alternative methods of calculation are explained in the next section.

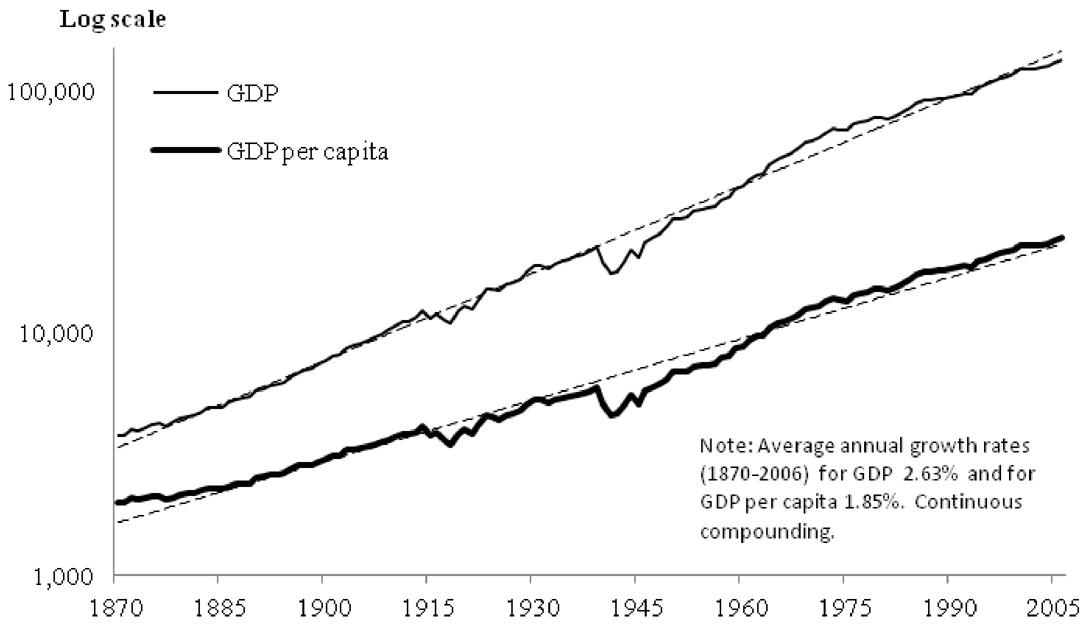


Figure 1.1: GDP and GDP per capita (1990 International Geary-Khamis dollars) in Denmark, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

1.2 Calculation of the average growth rate

1.2.1 Discrete compounding

Let y denote aggregate labor productivity, i.e., $y \equiv Y/L$, where L is employment. The average growth rate of y from period 0 to period t, with discrete compounding, is that G which satisfies

$$\begin{aligned} y_t &= y_0(1+G)^t, \qquad t = 1, 2, ... , \qquad \text{or} \qquad (1.1) \\ 1+G &= (\frac{y_t}{y_0})^{1/t}, \text{ i.e.,} \\ G &= (\frac{y_t}{y_0})^{1/t} - 1. \qquad (1.2) \end{aligned}$$

"Compounding" means adding the one-period "net return" to the "principal" before adding next period's "net return" (like with interest on interest, also called "compound interest"). The growth factor $1 + G$ will generally be less than the arithmetic average of the period-by-period growth factors. To

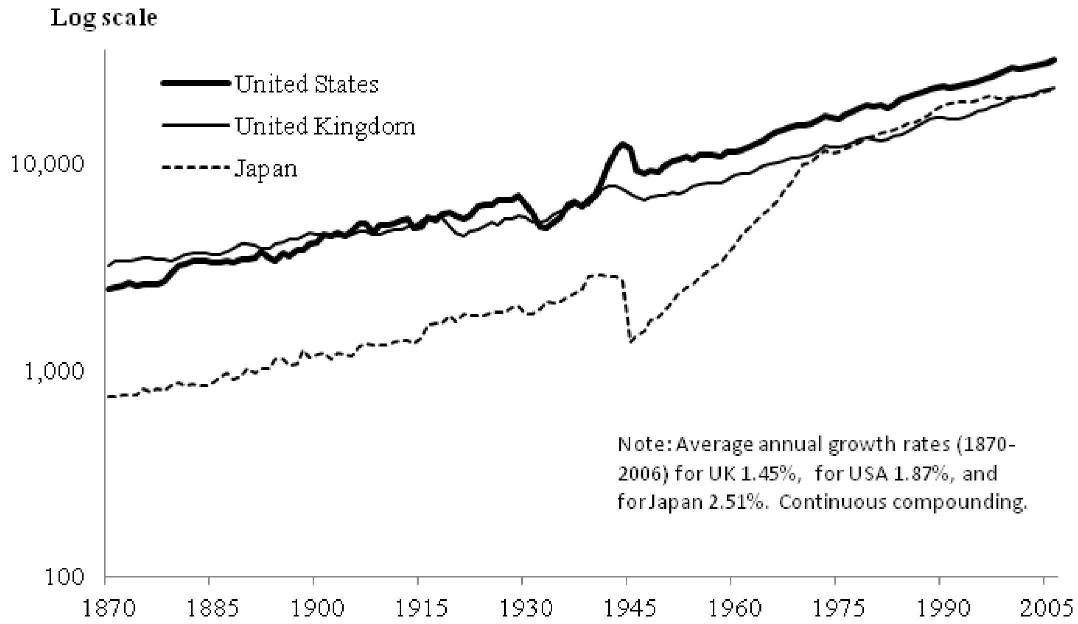


Figure 1.2: GDP per capita (1990 International Geary-Khamis dollars) in UK, USA and Japan, 1870-2006. Source: Maddison, A. (2009). Statistics on World Population, GDP and Per Capita GDP, 1-2006 AD, www.ggdc.net/maddison.

underline this difference, $1 + G$ is sometimes called the "compound average growth factor" or the "geometric average growth factor" and G itself then called the "compound average growth rate" or the "geometric average growth rate"

Using a pocket calculator, the following steps in the calculation of G may be convenient. Take logs on both sides of (1.1) to get

$$\begin{aligned} \ln \frac{y_t}{y_0} &= t \ln(1+G) \Rightarrow \\ \ln(1+G) &= \frac{\ln \frac{y_t}{y_0}}{t} \Rightarrow \qquad (1.3) \\ G &= \text{antilog}\left(\frac{\ln \frac{y_t}{y_0}}{t}\right) - 1. \qquad (1.4) \end{aligned}$$

Note that t in the formulas (1.2) and (1.4) equals the number of periods *minus 1.*

1.2.2 Continuous compounding

The average growth rate of y, with continuous compounding, is that g which satisfies

$$y_t = y_0 e^{gt}, \tag{1.5}$$

where e denotes the Euler number, i.e., the base of the natural logarithm.[2] Solving for g gives

$$g = \frac{\ln \frac{y_t}{y_0}}{t} = \frac{\ln y_t - \ln y_0}{t}. \tag{1.6}$$

The first formula in (1.6) is convenient for calculation with a pocket calculator, whereas the second formula is perhaps closer to intuition. Another name for g is the "exponential average growth rate".

Again, for discrete time data the t in the formula equals the number of periods minus 1.

Comparing with (1.3) we see that $g = \ln(1+G) < G$ for $G \neq 0$. Yet, by a first-order Taylor approximation of $\ln(1+G)$ about $G = 0$ we have

$$g = \ln(1+G) \approx G \text{ for } G \text{ "small"}. \tag{1.7}$$

For a given data set the G calculated from (1.2) will be slightly above the g calculated from (1.6), cf. the mentioned difference between the growth rates in Table 1.1 and those in Figure 1.1 and Figure 1.2. The reason is that a given growth force is more powerful when compounding is continuous rather than discrete. Anyway, the difference between G and g is usually unimportant. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$, the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference between G and g will be substantial. During hyperinflation the monthly inflation rate may be, say, $G = 100\%$, but the corresponding g will be only 69%.

Which method, discrete or continuous compounding, is preferable? To some extent it is a matter of taste or convenience. In period analysis discrete compounding is most common and in continuous time analysis continuous compounding is most common.

For calculation with a pocket calculator the continuous compounding formula, (1.6), is slightly easier to use than the discrete compounding formulas, whether (1.2) or (1.4).

[2] Unless otherwise specified, whenever we write $\ln x$ or $\log x$, the *natural* logarithm is understood.

To avoid too much sensitiveness to the initial and terminal observations, which may involve measurement error or depend on the state of the business cycle, one can use an OLS approach to the trend coefficient, g, in the following regression:

$$\ln Y_t = \alpha + gt + \varepsilon_t.$$

This is in fact what is done in Fig. 1.1.

1.2.3 Doubling time

How long time does it take for y to double if the growth rate with discrete compounding is G? Knowing G, we rewrite the formula (1.3):

$$t = \frac{\ln \frac{y_t}{y_0}}{\ln(1+G)} = \frac{\ln 2}{\ln(1+G)} \approx \frac{0.6931}{\ln(1+G)}.$$

With $G = 0.0187$, cf. Table 1.1, we find

$$t \approx 37.4 \text{ years},$$

meaning that productivity doubles every 37.4 years.

How long time does it take for y to double if the growth rate with continuous compounding is g? The answer is based on rewriting the formula (1.6):

$$t = \frac{\ln \frac{y_t}{y_0}}{g} = \frac{\ln 2}{g} \approx \frac{0.6931}{g}.$$

Maintaining the value 0.0187 also for g, we find

$$t \approx \frac{0.6931}{0.0187} \approx 37.1 \text{ years}.$$

Again, with a pocket calculator the continuous compounding formula is slightly easier to use. With a lower g, say $g = 0.01$, we find doubling time equal to 69.1 years. With $g = 0.07$ (think of China since the early 1980's), doubling time is about 10 years! Owing to the compounding, exponential growth is extremely powerful.

1.3 Some stylized facts of economic growth

1.3.1 The Kuznets facts

A well-known characteristic of modern economic growth is structural change: unbalanced sectorial growth. There is a massive reallocation of labor from

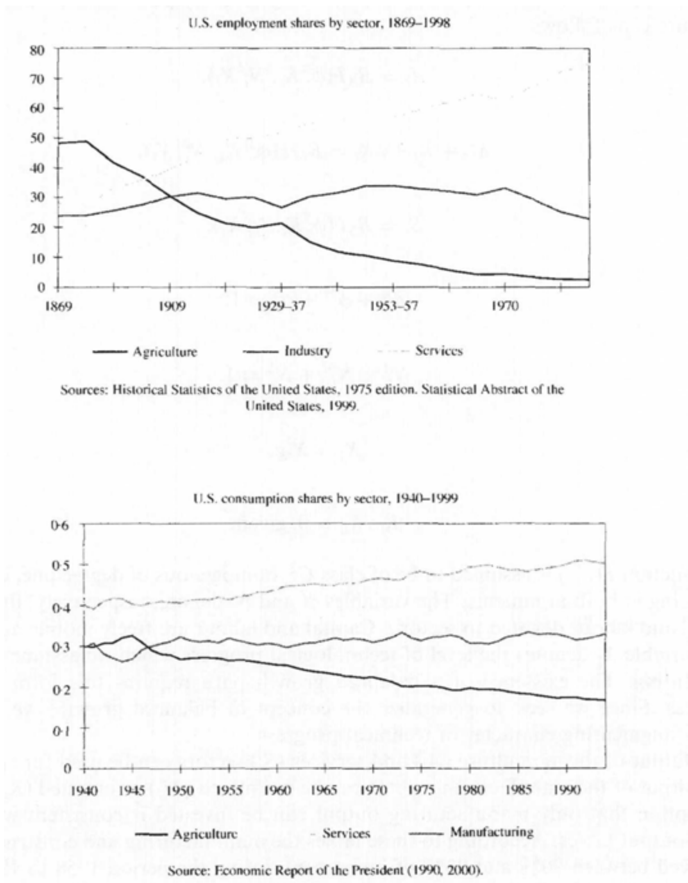


FIGURE 2
The Kuznets facts

Figure 1.3: The Kuznets facts. Source: Kongsamut et al., Beyond Balanced Growth, Review of Economic Studies, vol. 68, Oct. 2001, 869-82.

agriculture into industry (manufacturing, construction, and mining) and further into services (including transport and communication). The shares of total consumption expenditure going to these three sectors have moved similarly. Differences in the demand elasticities with respect to income seem the main explanation. These observations are often referred to as the *Kuznets facts* (after Simon Kuznets, 1901-85, see, e.g., Kuznets 1957).

The two graphs in Figure 1.3 illustrate the Kuznets facts.

1.3.2 Kaldor's stylized facts

Surprisingly, in spite of the Kuznets facts, the evolution at the *aggregate* level in developed countries is by many economists seen as roughly described by what is called Kaldor's "stylized facts" (after the Hungarian-British economist Nicholas Kaldor, 1908-1986, see, e.g., Kaldor 1957, 1961)[3]:

1. Real output per man-hour grows at a more or less constant rate over fairly long periods of time. (Of course, there are short-run fluctuations superposed around this trend.)

2. The stock of physical capital per man-hour grows at a more or less constant rate over fairly long periods of time.

3. The ratio of output to capital shows no systematic trend.

4. The rate of return to capital shows no systematic trend.

5. The income shares of labor and capital (in the national accounting sense, i.e., including land and other natural resources), respectively, are nearly constant.

6. The growth rate of output per man-hour differs substantially across countries.

These claimed regularities do certainly not fit all developed countries equally well. Although Solow's growth model (Solow, 1956) can be seen as the first successful attempt at building a model consistent with Kaldor's "stylized facts", Solow once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts" (Solow, 1970). Yet, for instance the study by Attfield and Temple (2010) of US and UK data since the Second World War concludes with support for Kaldor's "facts". Recently, several empiricists[4] have questioned "fact" 5, however, referring to the inadequacy of the methods which standard national income accounting applies to separate the income of entrepreneurs, sole proprietors, and unincorporated businesses into labor and capital income. It is claimed that these methods obscure a tendency in recent decades of the labor income share to fall.

The sixth Kaldor fact is, of course, generally accepted as a well documented observation (a nice summary is contained in Pritchett, 1997).

Kaldor also proposed hypotheses about the links between growth in the different sectors (see, e.g., Kaldor 1967):

a. Productivity growth in the manufacturing and construction sectors is enhanced by output growth in these sectors (this is also known as Verdoorn's Law). Increasing returns to scale and learning by doing are the main factors behind this.

[3] Kaldor presented his six regularities as "a stylised view of the facts".

[4] E.g., Gollin (2002), Elsby et al. (2013), and Karabarbounis and Neiman (2014).

b. Productivity growth in agriculture and services is enhanced by output growth in the manufacturing and construction sectors.

Kongsamut et al. (2001) and Foellmi and Zweimüller (2008) offer theoretical explanations of why the Kuznets facts and the Kaldor facts can coexist.

1.4 Concepts of income convergence

The two most popular across-country income convergence concepts are "β convergence" and "σ convergence".

1.4.1 β convergence vs. σ convergence

Definition 1 *We say that β convergence occurs for a given selection of countries if there is a tendency for the poor (those with low income per capita or low output per worker) to subsequently grow faster than the rich.*

By "grow faster" is meant that the growth rate of per capita income (or per worker output) is systematically higher.

In many contexts, a more appropriate convergence concept is the following:

Definition 2 *We say that σ convergence, with respect to a given measure of dispersion, occurs for a given collection of countries if this measure of dispersion, applied to income per capita or output per worker across the countries, declines systematically over time. On the other hand, σ divergence occurs, if the dispersion increases systematically over time.*

The reason that σ convergence must be considered the more appropriate concept is the following. In the end, it is the question of increasing or decreasing dispersion across countries that we are interested in. From a superficial point of view one might think that β convergence implies decreasing dispersion and vice versa, so that β convergence and σ convergence are more or less equivalent concepts. But since the world is not deterministic, but stochastic, this is not true. Indeed, β convergence is only a necessary, not a sufficient condition for σ convergence. This is because over time some reshuffling among the countries is always taking place, and this implies that there will always be some extreme countries (those initially far away from the mean) that move closer to the mean, thus creating a negative correlation between initial level and subsequent growth, in spite of equally many countries moving from a middle position toward one of the extremes.[5] In this way β convergence may be observed at the same time as there is no σ convergence; the mere presence of random measurement errors implies a bias in this direction because a growth rate depends negatively on the initial measurement and positively on the later measurement. In fact, β convergence may be consistent with σ *divergence* (for a formal proof of this claim, see Barro and Sala-i-Martin, 2004, pp. 50-51 and 462 ff.; see also Valdés, 1999, p. 49-50, and Romer, 2001, p. 32-34).

Hence, it is wrong to conclude from β convergence (poor countries tend to grow faster than rich ones) to σ convergence (reduced dispersion of per capita income) without any further investigation. The mistake is called "regression towards the mean" or "Galton's fallacy". Francis Galton was an anthropologist (and a cousin of Darwin), who in the late nineteenth century observed that tall fathers tended to have not as tall sons and small fathers tended to have taller sons. From this he falsely concluded that there was a tendency to averaging out of the differences in height in the population. Indeed, being a true aristocrat, Galton found this tendency pitiable. But since his conclusion was mistaken, he did not really have to worry.

Since σ convergence comes closer to what we are ultimately looking for, from now, when we speak of just "income convergence", σ convergence is understood.

In the above definitions of σ convergence and β convergence, respectively, we were vague as to what kind of selection of countries is considered. In principle we would like it to be a representative sample of the "population" of countries that we are interested in. The population could be all countries in the world. Or it could be the countries that a century ago had obtained a certain level of development.

One should be aware that historical GDP data are constructed retrospectively. Long time series data have only been constructed for those countries that became relatively rich during the after-WWII period. Thus, if we as our sample select the countries for which long data series exist, what is known as *selection bias* is involved which generates a spurious convergence. A country which was poor a century ago will only appear in the sample if it grew rapidly over the next 100 years. A country which was relatively rich a century ago will appear in the sample unconditionally. This selection bias problem was

[5] As an intuitive analogy, think of the ordinal rankings of the sports teams in a league. The dispersion of rankings is constant by definition. Yet, no doubt there will allways be some tendency for weak teams to rebound toward the mean and of champions to revert to mediocrity. (This example is taken from the first edition of Barro and Sala-i-Martin, *Economic Growth*, 1995; I do not know why, but the example was deleted in the second edition from 2004.)

pointed out by DeLong (1988) in a criticism of widespread false interpretations of Maddison's long data series (Maddison 1982).

1.4.2 Measures of dispersion

Our next problem is: *what* measure of dispersion is to be used as a useful descriptive statistics for σ convergence? Here there are different possibilities. To be precise about this we need some notation. Let

$$\begin{aligned} y &\equiv \frac{Y}{L}, \qquad \text{and} \\ q &\equiv \frac{Y}{N}, \end{aligned}$$

where Y = real GDP, L = employment, and N = population. If the focus is on living standards, Y/N, is the relevant variable.[6] But if the focus is on (labor) productivity, it is Y/L, that is relevant. Since most growth models focus on Y/L rather than Y/N, let os take y as our example.

One might think that the standard deviation of y could be a relevant measure of dispersion when discussing whether σ convergence is present or not. The *standard deviation* of y across n countries in a given year is

$$\sigma_y \equiv \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{1.8}$$

where

$$\bar{y} \equiv \frac{\sum_i y_i}{n}, \tag{1.9}$$

i.e., $\bar{y}$ is the average output per worker. However, if this measure were used, it would be hard to find *any* group of countries for which there is income convergence. This is because y tends to grow over time for most countries, and then there is an inherent tendency for the variance also to grow; hence also the square root of the variance, σ_y, tends to grow. Indeed, suppose that for all countries, y is doubled from time t_1 to time t_2. Then, automatically, σ_y is also doubled. But hardly anyone would interpret this as an increase in the income inequality across the countries.

Hence, it is more adequate to look at the standard deviation of *relative* income levels:

$$\sigma_{y/\bar{y}} \equiv \sqrt{\frac{1}{n}\sum_i(\frac{y_i}{\bar{y}} - 1)^2}. \tag{1.10}$$

[6] Or perhaps better, Q/N, where $Q \equiv GNP \equiv GDP - rD - wF$. Here, rD, denotes net interest payments on foreign debt and wF denotes net labor income of foreign workers in the country.

This measure is the same as what is called the *coefficient of variation*, CV_y, usually defined as

$$CV_y \equiv \frac{\sigma_y}{\bar{y}}, \tag{1.11}$$

that is, the standard deviation of y standardized by the mean. That the two measures are identical can be seen in this way:

$$\frac{\sigma_y}{\bar{y}} \equiv \frac{\sqrt{\frac{1}{n}\sum_i (y_i - \bar{y})^2}}{\bar{y}} = \sqrt{\frac{1}{n}\sum_i (\frac{y_i - \bar{y}}{\bar{y}})^2} = \sqrt{\frac{1}{n}\sum_i (\frac{y_i}{\bar{y}} - 1)^2} \equiv \sigma_{y/\bar{y}}.$$

The point is that the coefficient of variation is "scale free", which the standard deviation itself is not.

Instead of the coefficient of variation, another scale free measure is often used, namely the standard deviation of $\ln y$, i.e.,

$$\sigma_{\ln y} \equiv \sqrt{\frac{1}{n}\sum_i (\ln y_i - \ln y^*)^2}, \tag{1.12}$$

where

$$\ln y^* \equiv \frac{\sum_i \ln y_i}{n}. \tag{1.13}$$

Note that y^* is the geometric average, i.e., $y^* \equiv \sqrt[n]{y_1 y_2 \cdots y_n}$. Now, by a first-order Taylor approximation of $\ln y$ around $y = \bar{y}$, we have

$$\ln y \approx \ln \bar{y} + \frac{1}{\bar{y}}(y - \bar{y})$$

Hence, as a very rough approximation we have $\sigma_{\ln y} \approx \sigma_{y/\bar{y}} = CV_y$, though this approximation can be quite poor (cf. Dalgaard and Vastrup, 2001). It may be possible, however, to defend the use of $\sigma_{\ln y}$ in its own right to the extent that y tends to be approximately lognormally distributed across countries.

Yet another possible measure of income dispersion across countries is the *Gini index* (see for example Cowell, 1995).

1.4.3 Weighting by size of population

Another important issue is whether the applied dispersion measure is based on a *weighting of the countries by size of population*. For the world as a whole, when no weighting by size of population is used, then there is a slight tendency to income divergence according to the $\sigma_{\ln q}$ criterion (Acemoglu, 2009, p. 4), where q is per capita income ($\equiv Y/N$). As seen by Fig. 4 below, this tendency is not so clear according to the CV_q criterion. Anyway, when there *is* weighting by size of population, then in the last twenty years there has been a tendency to income convergence at the global level (Sala-i-Martin 2006; Acemoglu, 2009, p. 6). With weighting by size of population (1.12) is modified to

$$\sigma^w_{\ln q} \equiv \sqrt{\sum_i w_i(\ln q_i - \ln q^*)^2},$$

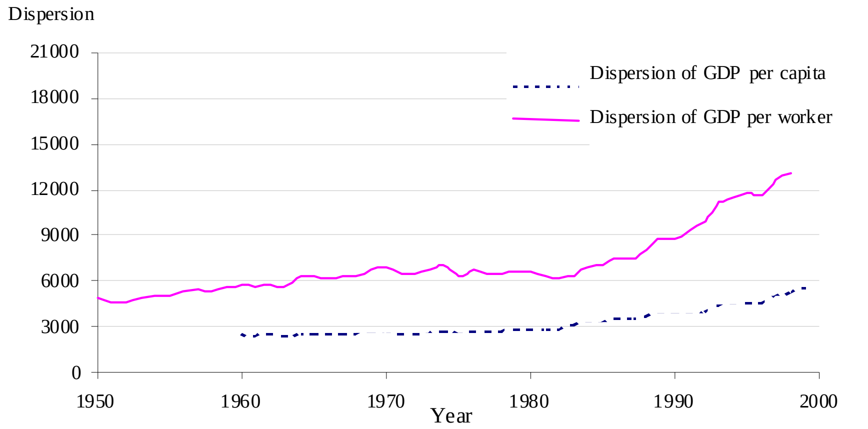
where

$$w_i = \frac{N_i}{N} \quad \text{and} \quad \ln q^* \equiv \sum_i w_i \ln q_i.$$

1.4.4 Unconditional vs. conditional convergence

Yet another distinction in the study of income convergence is that between unconditional (or absolute) and conditional convergence. We say that a large heterogeneous group of countries (say the countries in the world) show *unconditional* income convergence if income convergence occurs for the whole group without conditioning on specific characteristics of the countries. If income convergence occurs only for a subgroup of the countries, namely those countries that in advance share the same "structural characteristics", then we say there is *conditional* income convergence. As noted earlier, when we speak of just income "convergence", income "σ convergence" is understood. If in a given context there might be doubt, one should of course be explicit and speak of unconditional or conditional σ convergence. Similarly, if the focus for some reason is on β convergence, we should distinguish between unconditional and conditional β convergence.

What the precise meaning of "structural characteristics" is, will depend on what model of the countries the researcher has in mind. According to the Solow model, a set of relevant "structural characteristics" are: the aggregate production function, the initial level of technology, the rate of technical progress, the capital depreciation rate, the saving rate, and the population growth rate. But the Solow model, as well as its extension with human capital (Mankiw et al., 1992), is a model of a closed economy with exogenous technical progress. The model deals with "within-country" convergence in the sense that the model predicts that a closed economy being initially below or above its steady state path, will over time converge towards its steady state path. It is far from obvious that this kind of model is a good model of cross-country convergence in a globalized world where capital mobility and to some extent also labor mobility are important and some countries are



Remarks: Germany is not included in GDP per worker. GDP per worker is missing for Sweden and Greece in 1950, and for Portugal in 1998. The EU comprises Belgium, Denmark, Finland, France, Greece, Holland, Ireland, Italy, Luxembourg, Portugal, Spain, Sweden, Germany, the UK and Austria.
Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

Figure 1.4: Standard deviation of GDP per capita and per worker across 12 EU countries, 1950-1998.

pushing the technological frontier further out, while others try to imitate and catch up.

1.4.5 A bird's-eye view of the data

In the following no serious econometrics is attempted. We use the term "trend" in an admittedly loose sense.

Figure 1.4 shows the time profile for the standard deviation of y itself for 12 EU countries, whereas Figure 1.5 and Figure 1.6 show the time profile of the standard deviation of $\log y$ and the time profile of the coefficient of variation, respectively. Comparing the upward trend in Figure 1.4 with the downward trend in the two other figures, we have an illustration of the fact that the movement of the standard deviation of y itself does not capture income convergence. To put it another way: although there seems to be conditional income convergence with respect to the two scale-free measures, Figure 1.4 shows that this tendency to convergence is *not* so strong as to produce a narrowing of the absolute distance between the EU countries.[7]

[7]Unfortunately, sometimes misleading graphs or texts to graphs about across-country

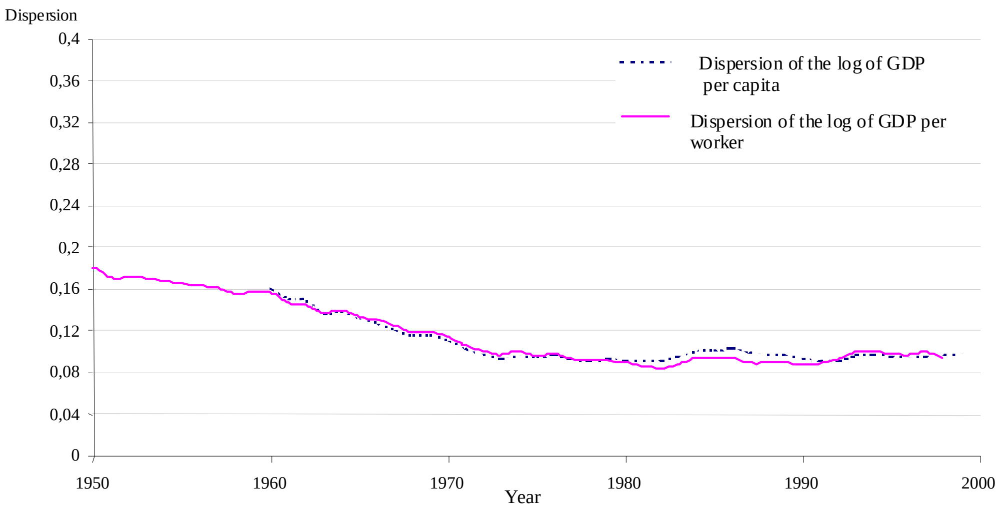


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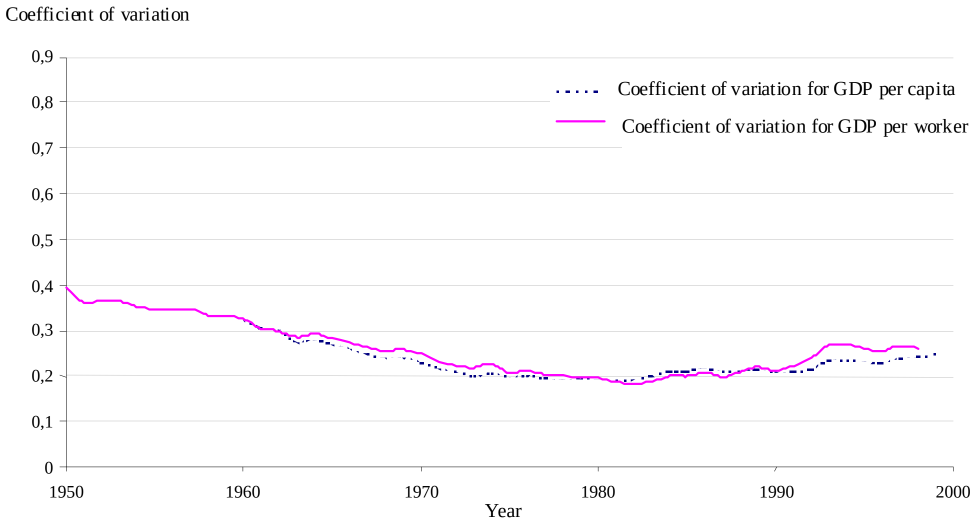
Figure 1.5: Standard deviation of the log of GDP per capita and per worker across 12 EU countries, 1950-1998.



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Source: Pwt6, OECD Economic Outlook No. 65 1999 via Eco Win and World Bank Global Development Network Growth Database.

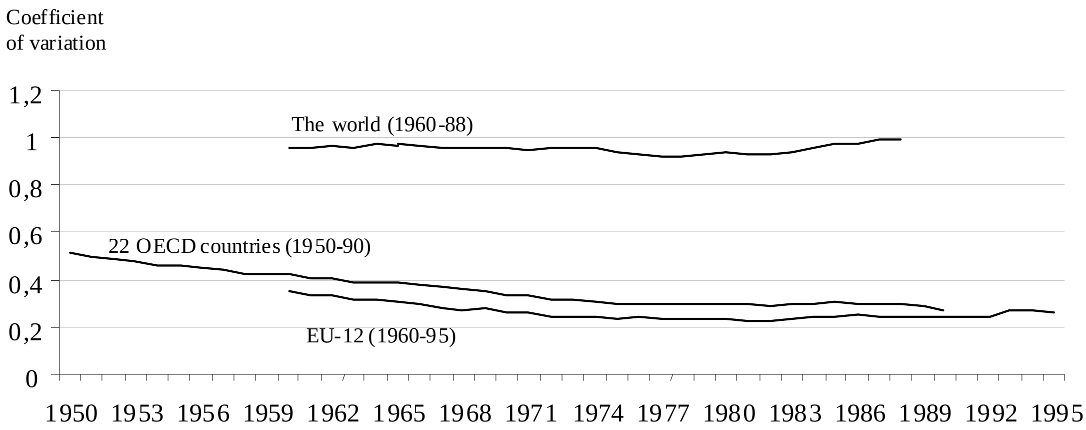
Figure 1.6: Coefficient of variation of GDP per capita and GDP per worker across 12 EU countries, 1950-1998.

Figure 1.7 shows the time path of the coefficient of variation across 121 countries in the world, 22 OECD countries and 12 EU countries, respectively. We see the lack of unconditional income convergence, but the presence of conditional income convergence. One should not over-interpret the observation of convergence for the 22 OECD countries over the period 1950-1990. It is likely that this observation suffer from the selection bias problem mentioned in Section 1.4.1. A country that was poor in 1950 will typically have become a member of OECD only if it grew relatively fast afterwards.

1.4.6 Other convergence concepts

Of course, just considering the time profile of the first and second moments of a distribution may sometimes be a poor characterization of the evolution of the distribution. For example, there are signs that the distribution has polarized into *twin peaks* of rich and poor countries (Quah, 1996a; Jones,

income convergence are published. In the collection of exercises, Chapter 1, you are asked to discuss some examples of this.



Remarks: 'The world' comprises 121 countries (not weighed by size) where complete time series for GDP per capita exist. The OECD countries exclude South Korea, Hungary, Poland, Iceland, Czech Rep., Luxembourg and Mexico. EU-12 comprises: Benelux, Germany, France, Italy, Denmark, Ireland, UK, Spain, Portugal og Greece.
Source: Penn World Table 5.6 and OECD Economic Outlook, Statistics on Microcomputer Disc, December 1998.

Figure 1.7: Coefficient of variation of income per capita across different sets of countries.

1997). Related to this observation is the notion of club convergence. If income convergence occurs *only* among a subgroup of the countries that to some extent share the same initial conditions, then we say there is *club-convergence.* This concept is relevant in a setting where there are *multiple* steady states toward which countries can converge. At least at the theoretical level multiple steady states can easily arise in overlapping generations models. Then the initial condition for a given country matters for which of these steady states this country is heading to. Similarly, we may say that *conditional club-convergence* is present, if income convergence occurs *only* for a subgroup of the countries, namely countries sharing similar structural characteristics (this may to some extent be true for the OECD countries) *and*, within an interval, similar initial conditions.

Instead of focusing on income convergence, one could study *TFP convergence* at aggregate or industry level.[8] Sometimes the less demanding concept of *growth rate convergence* is the focus.

The above considerations are only of a very elementary nature and are only about descriptive statistics. The reader is referred to the large existing literature on concepts and econometric methods of relevance for character-

[8] See, for instance, Bernard and Jones 1996a and 1996b.

izing the evolution of world income distribution (see Quah, 1996b, 1996c, 1997, and for a survey, see Islam 2003).

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Chapter 2

Review of technology and factor shares of income

The aim of this chapter is, first, to introduce the terminology concerning firms' technology and technological change used in the lectures and exercises of this course. At a few points I deviate somewhat from definitions in Acemoglu's book. Section 1.3 can be used as a formula manual for the case of CRS.

Second, the chapter contains a brief discussion of the notions of a representative firm and an aggregate production function. The distinction between long-run versus short-run production functions is also commented on. The last sections introduce the concept of elasticity of substitution between capital and labour and its role for the direction of movement over time of the income shares of capital and labor under perfect competition.

Regarding the distinction between discrete and continuous time analysis, most of the definitions contained in this chapter are applicable to both.

2.1 The production technology

Consider a two-factor production function given by

$$Y = F(K, L), \tag{2.1}$$

where Y is output (value added) per time unit, K is capital input per time unit, and L is labor input per time unit ($K \geq 0$, $L \geq 0$). We may think of (2.1) as describing the output of a firm, a sector, or the economy as a whole. It is in any case a very simplified description, ignoring the heterogeneity of output, capital, and labor. Yet, for many macroeconomic questions it may be a useful first approach. Note that in (2.1) not only Y but also K and L

represent *flows,* that is, quantities per unit of time. If the time unit is one year, we think of K as measured in machine hours per year. Similarly, we think of L as measured in labor hours per year. Unless otherwise specified, it is understood that the rate of utilization of the production factors is constant over time and normalized to one for each production factor. As explained in Chapter 1, we can then use the same symbol, K, for the *flow* of capital services as for the *stock* of capital. Similarly with L.

2.1.1 A neoclassical production function

By definition, K and L are non-negative. It is generally understood that a production function, $Y = F(K, L)$, is *continuous* and that $F(0, 0) = 0$ (no input, no output). Sometimes, when specific functional forms are used to represent a production function, that function may not be defined at points where $K = 0$ or $L = 0$ or both. In such a case we adopt the convention that the domain of the function is understood extended to include such boundary points whenever it is possible to assign function values to them such that continuity is maintained. For instance the function $F(K, L) = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$, is not defined at $(K, L) = (0, 0)$. But by assigning the function value 0 to the point $(0, 0)$, we maintain both continuity and the "no input, no output" property.

We call the production function *neoclassical* if for all (K, L), with $K > 0$ and $L > 0$, the following additional conditions are satisfied:

(a) $F(K, L)$ has continuous first- and second-order partial derivatives satisfying:

$$F_K > 0, \quad F_L > 0, \tag{2.2}$$
$$F_{KK} < 0, \quad F_{LL} < 0. \tag{2.3}$$

(b) $F(K, L)$ is strictly quasiconcave (i.e., the level curves, also called isoquants, are strictly convex to the origin).

In words: (a) says that a neoclassical production function has continuous substitution possibilities between K and L and the *marginal productivities* are positive, but diminishing in own factor. Thus, for a given number of machines, adding one more unit of labor, adds to output, but less so, the higher is already the labor input. And (b) says that every isoquant, $F(K, L) = \bar{Y}$, has a strictly convex form qualitatively similar to that shown in Figure 2.1.[1]

[1] For any fixed $\bar{Y} \geq 0$, the associated *isoquant* is the level set $\left\{(K, L) \in \mathbb{R}_+ | \ F(K, L) = \bar{Y}\right\}$.

When we speak of for example F_L as the marginal *productivity* of labor, it is because the "pure" partial derivative, $\partial Y/\partial L = F_L$, has the denomination of a productivity (output units/yr)/(man-yrs/yr). It is quite common, however, to refer to F_L as the marginal *product* of labor. Then a unit marginal increase in the labor input is understood: $\Delta Y \approx (\partial Y/\partial L)\Delta L = \partial Y/\partial L$ when $\Delta L = 1$. Similarly, F_K can be interpreted as the marginal *productivity* of capital or as the marginal *product* of capital. In the latter case it is understood that $\Delta K = 1$, so that $\Delta Y \approx (\partial Y/\partial K)\Delta K = \partial Y/\partial K$.

The definition of a neoclassical production function can be extended to the case of n inputs. Let the input quantities be $X_1, X_2, \ldots, X_n$ and consider a production function $Y = F(X_1, X_2, \ldots, X_n)$. Then F is called neoclassical if all the marginal productivities are positive, but diminishing, and F is strictly quasiconcave (i.e., the upper contour sets are strictly convex, cf. Appendix A).

Returning to the two-factor case, since $F(K, L)$ presumably depends on the level of technical knowledge and this level depends on time, t, we might want to replace (2.1) by

$$Y_t = F^t(K_t, L_t), \tag{2.4}$$

where the superscript on F indicates that the production function may shift over time, due to changes in technology. We then say that $F^t(\cdot)$ is a neoclassical production function if it satisfies the conditions (a) and (b) for all pairs (K_t, L_t). *Technological progress* can then be said to occur when, for K_t and L_t held constant, output increases with t.

For convenience, to begin with we skip the explicit reference to time and level of technology.

The marginal rate of substitution Given a neoclassical production function F, we consider the isoquant defined by $F(K, L) = \bar{Y}$, where $\bar{Y}$ is a positive constant. The *marginal rate of substitution*, MRS_{KL}, of K for L at the point (K, L) is defined as the absolute slope of the isoquant at that point, cf. Figure 2.1. The equation $F(K, L) = \bar{Y}$ defines K as an implicit function of L. By implicit differentiation we find $F_K(K, L)dK/dL + F_L(K, L) = 0$, from which follows

$$MRS_{KL} \equiv -\frac{dK}{dL}_{|Y=\bar{Y}} = \frac{F_L(K, L)}{F_K(K, L)} > 0. \tag{2.5}$$

That is, MRS_{KL} measures the amount of K that can be saved (approximately) by applying an extra unit of labor. In turn, this equals the ratio

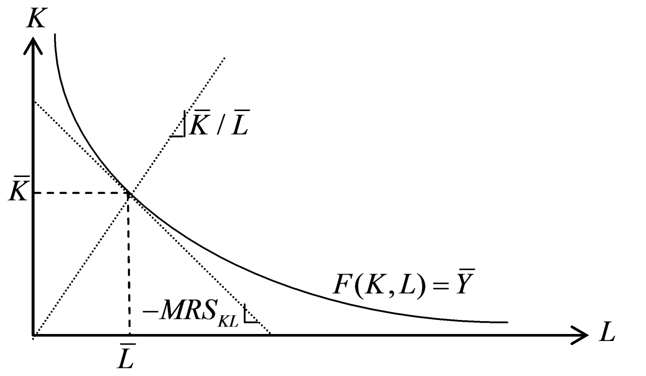


Figure 2.1: MRS_{KL} as the absolute slope of the isoquant.

of the marginal productivities of labor and capital, respectively.[2] Since F is neoclassical, by definition F is strictly quasi-concave and so the marginal rate of substitution is diminishing as substitution proceeds, i.e., as the labor input is further increased along a given isoquant. Notice that this feature characterizes the marginal rate of substitution for any neoclassical production function, whatever the returns to scale (see below).

When we want to draw attention to the dependency of the marginal rate of substitution on the factor combination considered, we write $MRS_{KL}(K, L)$. Sometimes in the literature, the marginal rate of substitution between two production factors, K and L, is called the *technical* rate of substitution (or the technical rate of transformation) in order to distinguish from a consumer's marginal rate of substitution between two consumption goods.

As is well-known from microeconomics, a firm that minimizes production costs for a given output level and given factor prices, will choose a factor combination such that MRS_{KL} equals the ratio of the factor prices. If $F(K, L)$ is homogeneous of degree q, then the marginal rate of substitution depends only on the factor proportion and is thus the same at any point on the ray $K = (\bar{K}/\bar{L})L$. That is, in this case the expansion path is a straight line.

The Inada conditions A continuously differentiable production function is said to satisfy the *Inada conditions*[3] if

$$\lim_{K\to 0} F_K(K,L) = \infty, \lim_{K\to\infty} F_K(K,L) = 0, \tag{2.6}$$

$$\lim_{L\to 0} F_L(K,L) = \infty, \lim_{L\to\infty} F_L(K,L) = 0. \tag{2.7}$$

[2] The subscript $|Y = \bar{Y}$ in (2.5) indicates that we are moving along a given isoquant, $F(K, L) = \bar{Y}$. Expressions like, e.g., $F_L(K, L)$ or $F_2(K, L)$ mean the partial derivative of F w.r.t. the second argument, evaluated at the point (K, L).

[3] After the Japanese economist Ken-Ichi Inada, 1925-2002.

In this case, the marginal productivity of either production factor has no upper bound when the input of the factor becomes infinitely small. And the marginal productivity is gradually vanishing when the input of the factor increases without bound. Actually, (2.6) and (2.7) express *four* conditions, which it is preferable to consider separately and label one by one. In (2.6) we have two *Inada conditions for MPK* (the marginal productivity of capital), the first being a *lower*, the second an *upper* Inada condition for MPK. And in (2.7) we have two *Inada conditions for MPL* (the marginal productivity of labor), the first being a *lower*, the second an *upper* Inada condition for MPL. In the literature, when a sentence like "the Inada conditions are assumed" appears, it is sometimes not made clear which, and how many, of the four are meant. Unless it is evident from the context, it is better to be explicit about what is meant.

The definition of a neoclassical production function we gave above is quite common in macroeconomic journal articles and convenient because of its flexibility. There are textbooks that define a neoclassical production function more narrowly by including the Inada conditions as a requirement for calling the production function neoclassical. In contrast, in this course, when in a given context we need one or another Inada condition, we state it explicitly as an additional assumption.

2.1.2 Returns to scale

If all the inputs are multiplied by some factor, is output then multiplied by the same factor? There may be different answers to this question, depending on circumstances. We consider a production function $F(K, L)$ where $K > 0$ and $L > 0$. Then F is said to have *constant returns to scale* (CRS for short) if it is homogeneous of degree one, i.e., if for all (K, L) and all $\lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

As all inputs are scaled up or down by some factor > 1, output is scaled up or down by the same factor.[4] The assumption of CRS is often defended by the *replication argument.* Before discussing this argument, lets us define the two alternative "pure" cases.

The production function $F(K, L)$ is said to have *increasing returns to scale* (IRS for short) if, for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

[4] In their definition of a neoclassical production function some textbooks add constant returns to scale as a requirement besides (a) and (b). This course follows the alternative terminology where, if in a given context an assumption of constant returns to scale is needed, this is stated as an additional assumption.

That is, IRS is present if, when all inputs are scaled up by some factor > 1, output is scaled up by *more* than this factor. The existence of gains by specialization and division of labor, synergy effects, etc. sometimes speak in support of this assumption, at least up to a certain level of production. The assumption is also called the *economies of scale* assumption.

Another possibility is *decreasing returns to scale* (DRS). This is said to occur when for all (K, L) and all $\lambda > 1$,

$$F(\lambda K, \lambda L) < \lambda F(K, L).$$

That is, DRS is present if, when all inputs are scaled up by some factor, output is scaled up by *less* than this factor. This assumption is also called the *diseconomies of scale* assumption. The underlying hypothesis may be that control and coordination problems confine the expansion of size. Or, considering the "replication argument" below, DRS may simply reflect that behind the scene there is an additional production factor, for example land or a irreplaceable quality of management, which is tacitly held fixed, when the factors of production are varied.

EXAMPLE 1 The production function

$$Y = AK^{\alpha}L^{\beta}, \qquad A > 0, 0 < \alpha < 1, 0 < \beta < 1, \tag{2.8}$$

where A, α, and β are given parameters, is called a *Cobb-Douglas production function*. The parameter A depends on the choice of measurement units; for a given such choice it reflects "efficiency", also called the "total factor productivity". As an exercise the reader may verify that (2.8) satisfies (a) and (b) above and is therefore a neoclassical production function. The function is homogeneous of degree $\alpha + \beta$. If $\alpha + \beta = 1$, there are CRS. If $\alpha + \beta < 1$, there are DRS, and if $\alpha + \beta > 1$, there are IRS. Note that α and β must be less than 1 in order not to violate the diminishing marginal productivity condition. □

EXAMPLE 2 The production function

$$Y = \min(AK, BL), \qquad A > 0, B > 0, \tag{2.9}$$

where A and B are given parameters, is called a *Leontief production function* or a *fixed-coefficients production function;* A and B are called the *technical coefficients*. The function is not neoclassical, since the conditions (a) and (b) are not satisfied. Indeed, with this production function the production factors are not substitutable at all. This case is also known as the case of *perfect complementarity* between the production factors. The interpretation is that already installed production equipment requires a fixed number of workers to operate it. The inverse of the parameters A and B indicate the required capital input per unit of output and the required labor input per unit of output, respectively. Extended to many inputs, this type of production function is often used in multi-sector input-output models (also called Leontief models). In aggregate analysis neoclassical production functions, allowing substitution between capital and labor, are more popular than Leontief functions. But sometimes the latter are preferred, in particular in short-run analysis with focus on the use of already installed equipment where the substitution possibilities are limited.[5] As (2.9) reads, the function has CRS. A generalized form of the Leontief function is $Y = \min(AK^\gamma, BL^\gamma)$, where $\gamma > 0$. When $\gamma < 1$, there are DRS, and when $\gamma > 1$, there are IRS. □

The replication argument The assumption of CRS is widely used in macroeconomics. The model builder may appeal to the *replication argument.* This is the argument saying that by doubling all the inputs, we should always be able to double the output, since we are just "replicating" what we are already doing. Suppose we want to double the production of cars. We may then build another factory identical to the one we already have, man it with identical workers and deploy the same material inputs. Then it is reasonable to assume output is doubled.

In this context it is important that the CRS assumption is about *technology* in the sense of functions linking outputs to inputs. Limits to the *availability* of input resources is an entirely different matter. The fact that for example managerial talent may be in limited supply does not preclude the thought experiment that *if* a firm could double all its inputs, including the number of talented managers, then the output level could also be doubled.

The replication argument presupposes, first, that *all* the relevant inputs are explicit as arguments in the production function; second, that these are changed equiproportionately. This, however, exhibits the weakness of the replication argument as a defence for assuming CRS of our present production function, $F(\cdot)$. One could easily make the case that besides capital and labor, also land is a necessary input and should appear as a separate argument.[6] If an industrial firm decides to duplicate what it has been doing, it needs a piece of land to build another plant like the first. Then, on the basis of the replication argument we should in fact expect DRS w.r.t. capital and labor alone. In manufacturing and services, empirically, this and other possible

[5] Cf. Section 2.4.

[6] We think of "capital" as producible means of production, whereas "land" refers to non-producible natural resources, including for example building sites.

sources for departure from CRS may be minor and so many macroeconomists feel comfortable enough with assuming CRS w.r.t. K and L alone, at least as a first approximation. This approximation is, however, less applicable to poor countries, where natural resources may be a quantitatively important production factor.

There is a further problem with the replication argument. Strictly speaking, the CRS claim is that by changing all the inputs equiproportionately by *any* positive factor, λ, which does not have to be an integer, the firm should be able to get output changed by the same factor. Hence, the replication argument requires that indivisibilities are negligible, which is certainly not always the case. In fact, the replication argument is more an argument *against* DRS than *for* CRS in particular. The argument does not rule out IRS due to synergy effects as size is increased.

Sometimes the replication line of reasoning is given a more subtle form. This builds on a useful *local* measure of returns to scale, named the *elasticity of scale*.

The elasticity of scale* To allow for indivisibilities and mixed cases (for example IRS at low levels of production and CRS or DRS at higher levels), we need a local measure of returns to scale. One defines the *elasticity of scale*, $\eta(K, L)$, of F at the point (K, L), where $F(K, L) > 0$, as

$$\eta(K,L) = \frac{\lambda}{F(K,L)} \frac{dF(\lambda K, \lambda L)}{d\lambda} \approx \frac{\Delta F(\lambda K, \lambda L)/F(K,L)}{\Delta\lambda/\lambda}, \text{ evaluated at } \lambda = 1. \tag{2.10}$$

So the elasticity of scale at a point (K, L) indicates the (approximate) percentage increase in output when both inputs are increased by 1 percent. We say that

$$\text{if } \eta(K,L) \begin{cases} > 1, & \text{then there are locally } \textit{IRS}, \\ = 1, & \text{then there are locally } \textit{CRS}, \\ < 1, & \text{then there are locally } \textit{DRS}. \end{cases} \tag{2.11}$$

The production function *may* have the same elasticity of scale everywhere. This is the case if and only if the production function is homogeneous. If F is homogeneous of degree h, then $\eta(K, L) = h$ and h is called the *elasticity of scale parameter*.

Note that the elasticity of scale at a point (K, L) will always equal the sum of the partial output elasticities at that point:

$$\eta(K,L) = \frac{F_K(K,L)K}{F(K,L)} + \frac{F_L(K,L)L}{F(K,L)}. \tag{2.12}$$

This follows from the definition in (2.10) by taking into account that

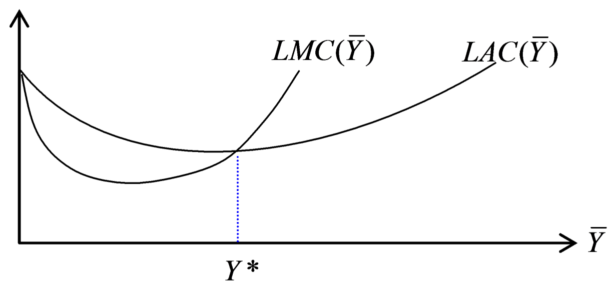


Figure 2.2: Locally CRS at optimal plant size.

$$\begin{aligned}\frac{dF(\lambda K, \lambda L)}{d\lambda} &= F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L \\ &= F_K(K, L)K + F_L(K, L)L, \text{ when evaluated at } \lambda = 1.\end{aligned}$$

Figure 2.2 illustrates a popular case from introductory economics, an average cost curve which from the perspective of the individual firm (or plant) is U-shaped: at low levels of output there are falling average costs (thus IRS), at higher levels rising average costs (thus DRS).[7] Given the input prices, w_K and w_L, and a specified output level, $\bar{Y}$, we know that the cost minimizing factor combination $(\bar{K}, \bar{L})$ is such that $F_L(\bar{K}, \bar{L})/F_K(\bar{K}, \bar{L}) = w_L/w_K$. It is shown in Appendix A that the elasticity of scale at $(\bar{K}, \bar{L})$ will satisfy:

$$\eta(\bar{K}, \bar{L}) = \frac{LAC(\bar{Y})}{LMC(\bar{Y})}, \tag{2.13}$$

where $LAC(\bar{Y})$ is average costs (the minimum unit cost associated with producing $\bar{Y}$) and $LMC(\bar{Y})$ is marginal costs at the output level $\bar{Y}$. The L in LAC and LMC stands for "long-run", indicating that both capital and labor are considered variable production factors within the period considered. At the optimal plant size, Y^*, there is equality between LAC and LMC, implying a unit elasticity of scale, that is, locally we have CRS. That the long-run average costs are here portrayed as rising for $\bar{Y} > Y^*$, is not essential for the argument but may reflect either that coordination difficulties are inevitable or that some additional production factor, say the building site of the plant, is tacitly held fixed.

Anyway, we have here a more subtle replication argument for CRS w.r.t. K and L at the aggregate level. Even though technologies may differ across plants, the surviving plants in a competitive market will have the same average costs at the optimal plant size. In the medium and long run, changes in

[7] By a "firm" is generally meant the company as a whole. A company may have several "manufacturing plants" placed at different locations.

aggregate output will take place primarily by entry and exit of optimal-size plants. Then, with a large number of relatively small plants, each producing at approximately constant unit costs for small output variations, we can without substantial error assume constant returns to scale at the aggregate level. So the argument goes. Notice, however, that even in this form the replication argument is not entirely convincing since the question of indivisibility remains. The optimal plant size may be large relative to the market − and is in fact so in many industries. Besides, in this case also the perfect competition premise breaks down.

2.1.3 Properties of the production function under CRS

The empirical evidence concerning returns to scale is mixed. Notwithstanding the theoretical and empirical ambiguities, the assumption of CRS w.r.t. capital and labor has a prominent role in macroeconomics. In many contexts it is regarded as an acceptable approximation and a convenient simple background for studying the question at hand.

Expedient inferences of the CRS assumption include:

(i) marginal costs are constant and equal to average costs (so the right-hand side of (2.13) equals unity);

(ii) if production factors are paid according to their marginal productivities, factor payments exactly exhaust total output so that pure profits are neither positive nor negative (so the right-hand side of (2.12) equals unity);

(iii) a production function known to exhibit CRS and satisfy property (a) from the definition of a neoclassical production function above, will automatically satisfy also property (b) and consequently *be* neoclassical;

(iv) a neoclassical two-factor production function with CRS has always $F_{KL} > 0$, i.e., it exhibits "direct complementarity" between K and L;

(v) a two-factor production function known to have CRS and to be twice continuously differentiable with positive marginal productivity of each factor everywhere in such a way that all isoquants are strictly convex to the origin, *must* have *diminishing* marginal productivities everywhere.[8]

[8] Proofs of these claims can be found in intermediate microeconomics textbooks and in the Appendix to Chapter 2 of my Lecture Notes in Macroeconomics.

A principal implication of the CRS assumption is that it allows a reduction of dimensionality. Considering a neoclassical production function, $Y = F(K, L)$ with $L > 0$, we can under CRS write $F(K, L) = LF(K/L, 1) \equiv Lf(k)$, where $k \equiv K/L$ is called the *capital-labor ratio* (sometimes the *capital intensity*) and $f(k)$ is the *production function in intensive form* (sometimes named the per capita production function). Thus output per unit of labor depends only on the capital intensity:

$$y \equiv \frac{Y}{L} = f(k).$$

When the original production function F is neoclassical, under CRS the expression for the marginal productivity of capital simplifies:

$$F_K(K, L) = \frac{\partial Y}{\partial K} = \frac{\partial [Lf(k)]}{\partial K} = Lf'(k)\frac{\partial k}{\partial K} = f'(k). \tag{2.14}$$

And the marginal productivity of labor can be written

$$\begin{aligned} F_L(K, L) &= \frac{\partial Y}{\partial L} = \frac{\partial [Lf(k)]}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} \\ &= f(k) + Lf'(k)K(-L^{-2}) = f(k) - f'(k)k. \end{aligned} \tag{2.15}$$

A neoclassical CRS production function in intensive form always has a positive first derivative and a negative second derivative, i.e., $f' > 0$ and $f'' < 0$. The property $f' > 0$ follows from (2.14) and (2.2). And the property $f'' < 0$ follows from (2.3) combined with

$$F_{KK}(K, L) = \frac{\partial f'(k)}{\partial K} = f''(k)\frac{\partial k}{\partial K} = f''(k)\frac{1}{L}.$$

For a neoclassical production function with CRS, we also have

$$f(k) - f'(k)k > 0 \text{ for all } k > 0, \tag{2.16}$$

in view of $f(0) \geq 0$ and $f'' < 0$. Moreover,

$$\lim_{k \to 0} [f(k) - f'(k)k] = f(0). \tag{2.17}$$

Indeed, from the *mean value theorem*[9] we know that for any $k > 0$ there exists a number $a \in (0, 1)$ such that $f'(ak) = (f(k) - f(0))/k$. For this a we thus have $f(k) - f'(ak)k = f(0) < f(k) - f'(k)k$, where the inequality

[9]This theorem says that if f is continuous in $[\alpha, \beta]$ and differentiable in (α, β), then there exists at least one point γ in (α, β) such that $f'(\gamma) = (f(\beta) - f(\alpha))/(\beta - \alpha)$.

follows from $f'(ak) > f'(k)$, by $f'' < 0$. In view of $f(0) \geq 0$, this establishes (2.16). And from $f(k) > f(k) - f'(k)k > f(0)$ and continuity of f (so that $\lim_{k\to 0^+} f(k) = f(0)$) follows (2.17).

Under CRS the Inada conditions for MPK can be written

$$\lim_{k\to 0} f'(k) = \infty, \qquad \lim_{k\to\infty} f'(k) = 0. \tag{2.18}$$

In this case standard parlance is just to say that "f satisfies the Inada conditions".

An input which must be positive for positive output to arise is called an *essential input*; an input which is not essential is called an *inessential input.* The second part of (2.18), representing the upper Inada condition for MPK under CRS, has the implication that *labor* is an essential input; but capital need not be, as the production function $f(k) = a + bk/(1 + k)$, $a > 0, b > 0$, illustrates. Similarly, under CRS the upper Inada condition for MPL implies that *capital* is an essential input. These claims are proved in Appendix C. Combining these results, when *both* the upper Inada conditions hold and CRS obtain, then both capital and labor are essential inputs.[10]

Figure 2.3 is drawn to provide an intuitive understanding of a neoclassical CRS production function and at the same time illustrate that the lower Inada conditions are more questionable than the upper Inada conditions. The left panel of Figure 2.3 shows output per unit of labor for a *CRS neoclassical production function* satisfying the Inada conditions for MPK. The $f(k)$ in the diagram could for instance represent the Cobb-Douglas function in Example 1 with $\beta = 1-\alpha$, i.e., $f(k) = Ak^{\alpha}$. The right panel of Figure 2.3 shows a non-neoclassical case where only two alternative *Leontief techniques* are available, technique 1: $y = \min(A_1k, B_1)$, and technique 2: $y = \min(A_2k, B_2)$. In the exposed case it is assumed that $B_2 > B_1$ and $A_2 < A_1$ (if $A_2 \geq A_1$ at the same time as $B_2 > B_1$, technique 1 would not be efficient, because the same output could be obtained with less input of at least one of the factors by shifting to technique 2). If the available K and L are such that $k < B_1/A_1$ or $k > B_2/A_2$, some of either L or K, respectively, is idle. If, however, the available K and L are such that $B_1/A_1 < k < B_2/A_2$, it is efficient to *combine* the two techniques and use the fraction μ of K and L in technique 1 and the remainder in technique 2, where $\mu = (B_2/A_2 - k)/(B_2/A_2 - B_1/A_1)$. In this way we get the "labor productivity curve" OPQR (the envelope of the two techniques) in Figure 2.3. Note that for $k \to 0$, MPK stays equal to $A_1 < \infty$, whereas for all $k > B_2/A_2$, $MPK = 0$. A similar feature remains true, when we consider *many,* say n, alternative efficient Leontief techniques available.

[10]Given a Cobb-Douglas production function, both production factors are essential whether we have DRS, CRS, or IRS.

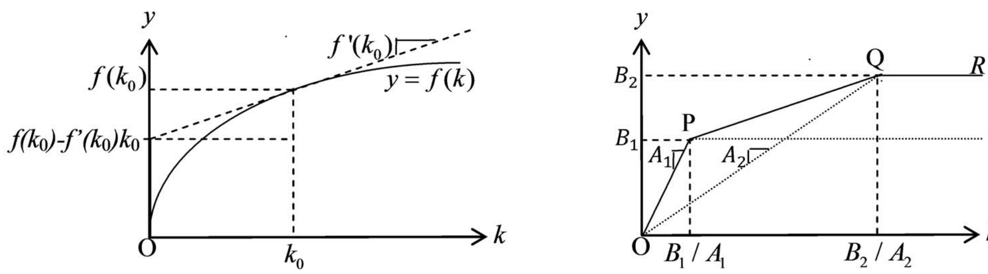


Figure 2.3: Two labor productivity curves based on CRS technologies. Left: neoclassical technology with Inada conditions for MPK satisfied; the graphical representation of MPK and MPL at $k = k_0$ as $f'(k_0)$ and $f(k_0) - f'(k_0)k_0$ are indicated. Right: a combination of two efficient Leontief techniques.

Assuming these techniques cover a considerable range w.r.t. the B/A ratios, we get a labor productivity curve looking more like that of a neoclassical CRS production function. On the one hand, this gives some intuition of what lies behind the assumption of a neoclassical CRS production function. On the other hand, it remains true that for all $k > B_n/A_n$, $MPK = 0$,[11] whereas for $k \to 0$, MPK stays equal to $A_1 < \infty$, thus questioning the lower Inada condition.

The implausibility of the lower Inada conditions is also underlined if we look at their implication in combination with the more reasonable upper Inada conditions. Indeed, the four Inada conditions taken *together* imply, under CRS, that output has no upper bound when either input goes to infinity for fixed amount of the other input (see Appendix C).

2.2 Technological change

When considering the movement over time of the economy, we shall often take into account the existence of *technological change*. When technological change occurs, the production function becomes time-dependent. Over time the production factors tend to become more productive: more output for given inputs. To put it differently: the isoquants move inward. When this is the case, we say that the technological change displays *technological progress*.

[11] Here we assume the techniques are numbered according to ranking with respect to the size of B.

Concepts of neutral technological change

A first step in taking technological change into account is to replace (2.1) by (2.4). Empirical studies often specialize (2.4) by assuming that technological change take a form known as *factor-augmenting* technological change:

$$Y_t = F(A_t K_t, B_t L_t), \tag{2.19}$$

where F is a (time-independent) neoclassical production function, Y_t, K_t, and L_t are output, capital, and labor input, respectively, at time t, while A_t and B_t are time-dependent "efficiencies" of capital and labor, respectively, reflecting technological change.

In macroeconomics an even more specific form is often assumed, namely the form of *Harrod-neutral technological change.*[12] This amounts to assuming that A_t in (2.19) is a constant (which we can then normalize to one). So only B_t, which is then conveniently denoted T_t, is changing over time, and we have

$$Y_t = F(K_t, T_t L_t). \tag{2.20}$$

The efficiency of labor, T_t, is then said to indicate the *technology level.* Although one can imagine natural disasters implying a fall in T_t, generally T_t tends to rise over time and then we say that (2.20) represents *Harrod-neutral technological progress.* An alternative name often used for this is *labor-augmenting* technological progress. The names "factor-augmenting" and, as here, "labor-augmenting" have become standard and we shall use them when convenient, although they may easily be misunderstood. To say that a change in T_t is labor-augmenting might be understood as meaning that more labor is required to reach a given output level for given capital. In fact, the opposite is the case, namely that T_t has risen so that less labor input is required. The idea is that the technological change affects the output level *as if* the labor input had been increased exactly by the factor by which T was increased, and nothing else had happened. (We might be tempted to say that (2.20) reflects "labor saving" technological change. But also this can be misunderstood. Indeed, keeping L unchanged in response to a rise in T implies that the same output level requires *less capital* and thus the technological change is "capital saving".)

If the function F in (2.20) is homogeneous of degree one (so that the technology exhibits CRS w.r.t. capital and labor), we may write

$$\tilde{y}_t \equiv \frac{Y_t}{T_t L_t} = F(\frac{K_t}{T_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0.$$

[12] After the English economist Roy F. Harrod, 1900-1978.

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$ (habitually called the "effective" capital intensity or, if there is no risk of confusion, just the capital intensity). In rough accordance with a general trend in aggregate productivity data for industrialized countries we often assume that T grows at a constant rate, g, so that in discrete time $T_t = T_0(1+g)^t$ and in continuous time $T_t = T_0 e^{gt}$, where $g > 0$. The popularity in macroeconomics of the hypothesis of labor-augmenting technological progress derives from its consistency with Kaldor's "stylized facts", cf. Chapter 4.

There exists two alternative concepts of neutral technological progress. *Hicks-neutral* technological progress is said to occur if technological development is such that the production function can be written in the form

$$Y_t = T_t F(K_t, L_t), \tag{2.21}$$

where, again, F is a (time-independent) neoclassical production function, while T_t is the growing technology level.[13] The assumption of Hicks-neutrality has been used more in microeconomics and partial equilibrium analysis than in macroeconomics. If F has CRS, we can write (2.21) as $Y_t = F(T_t K_t, T_t L_t)$. Comparing with (2.19), we see that in this case Hicks-neutrality is equivalent to $A_t = B_t$ in (2.19), whereby technological change is said to be *equally factor-augmenting*.

Finally, in a symmetric analogy with (2.20), what is known as *capital-augmenting* technological progress is present when

$$Y_t = F(T_t K_t, L_t). \tag{2.22}$$

Here technological change acts as if the capital input were augmented. For some reason this form is sometimes called *Solow-neutral* technological progress.[14] This association of (2.22) to Solow's name is misleading, however. In his famous growth model,[15] Solow assumed Harrod-neutral technological progress. And in another famous contribution, Solow generalized the concept of Harrod-neutrality to the case of *embodied* technological change and capital of *different vintages,* see below.

It is easily shown (Exercise I.9) that the Cobb-Douglas production function (2.8) (with time-independent output elasticities w.r.t. K and L) satisfies all three neutrality criteria at the same time, if it satisfies one of them (which it does if technological change does not affect α and β). It can also be shown that within the class of neoclassical CRS production functions the Cobb-Douglas function is the only one with this property (see Exercise ??).

[13] After the English economist and Nobel Prize laureate John R. Hicks, 1904-1989.
[14] After the American economist and Nobel Prize laureate Robert Solow (1924-).
[15] Solow (1956).

Note that the neutrality concepts do not say anything about the *source* of technological progress, only about the quantitative form in which it materializes. For instance, the occurrence of Harrod-neutrality should not be interpreted as indicating that the technological change emanates specifically from the labor input in some sense. Harrod-neutrality only means that technological innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* in the form (2.20), that is, *as if* an improvement in the quality of the labor input had occurred. (Even when improvement in the quality of the labor input is on the agenda, the result may be a reorganization of the production process ending up in a higher B_t along with, or instead of, a higher A_t in the expression (2.19).)

Rival versus nonrival goods

When a production function (or more generally a production possibility set) is specified, a given level of technical knowledge is presumed. As this level changes over time, the production function changes. In (2.4) this dependency on the level of knowledge was represented indirectly by the time dependency of the production function. Sometimes it is useful to let the knowledge dependency be explicit by perceiving knowledge as an additional production factor and write, for instance,

$$Y_t = F(X_t, T_t), \tag{2.23}$$

where T_t is now an index of the amount of knowledge, while X_t is a vector of ordinary inputs like raw materials, machines, labor etc. In this context the distinction between rival and nonrival inputs or more generally the distinction between rival and nonrival goods is important. A good is *rival* if its character is such that one agent's use of it inhibits other agents' use of it at the same time. A pencil is thus rival. Many production inputs like raw materials, machines, labor etc. have this property. They are elements of the vector X_t. By contrast, however, technical knowledge is a *nonrival* good. An arbitrary number of factories can simultaneously use the same piece of technical knowledge in the sense of a *list of instructions about how different inputs can be combined to produce a certain output*. An engineering principle or a farmaceutical formula are examples. (Note that the distinction rival-nonrival is different from the distinction excludable-nonexcludable. A good is *excludable* if other agents, firms or households, can be excluded from using it. Other firms can thus be excluded from commercial use of a certain piece of technical knowledge if it is patented. The existence of a patent concerns the legal status of a piece of knowledge and does not interfere with its economic character as a nonrival input.).

What the replication argument really says is that by, conceptually, doubling all the *rival* inputs, we should always be able to double the output, since we just "replicate" what we are already doing. This is then an argument for (at least) CRS w.r.t. the elements of X_t in (2.23). The point is that because of its nonrivalry, we do not need to increase the stock of knowledge. Now let us imagine that the stock of knowledge *is* doubled at the same time as the rival inputs are doubled. Then *more* than a doubling of output should occur. In this sense we may speak of IRS w.r.t. the rival inputs and T taken together.

Before proceeding, we briefly comment on how the capital stock, K_t, is typically measured. While data on gross investment, I_t, is available in national income and product accounts, data on K_t usually is not. One approach to the measurement of K_t is the *perpetual inventory method* which builds upon the accounting relationship

$$K_t = I_{t-1} + (1-\delta)K_{t-1}. \tag{2.24}$$

Assuming a constant capital depreciation rate δ, backward substitution gives

$$K_t = I_{t-1}+(1-\delta)\left[I_{t-2} + (1-\delta)K_{t-2}\right] = \ldots = \sum_{i=1}^{N}(1-\delta)^{i-1}I_{t-i}+(1-\delta)^N K_{t-N}. \tag{2.25}$$

Based on a long time series for I and an estimate of δ, one can insert these observed values in the formula and calculate K_t, starting from a rough conjecture about the initial value K_{t-N}. The result will not be very sensitive to this conjecture since for large N the last term in (2.25) becomes very small.

Embodied vs. disembodied technological progress

There exists an additional taxonomy of technological change. We say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The new technology is incorporated in the design of newly produced equipment, but this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Thus investment becomes an important driving force in productivity increases.

We way formalize embodied technological progress by writing capital accumulation in the following way:

$$K_{t+1} - K_t = q_t I_t - \delta K_t, \tag{2.26}$$

where I_t is gross investment in period t, i.e., $I_t = Y_t - C_t$, and q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising q so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in *efficiency units*. In aggregate models C and I are produced with the same technology, the aggregate production function. From this together with (2.26) follows that q capital goods can be produced at the same minimum cost as one consumption good. Hence, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of q, i.e., $p = 1/q$. The output-capital ratio in value terms is $Y/(pK) = qY/K$.

Note that even if technological change does not directly appear in the production function, that is, even if for instance (2.20) is replaced by $Y_t = F(K_t, L_t)$, the economy may experience a rising standard of living when q is growing over time.

In contrast, *disembodied technological change* occurs when new technical and organizational knowledge increases the combined productivity of the production factors independently of when they were constructed or educated. If the K_t appearing in (2.20), (2.21), and (2.22) above refers to the total, historically accumulated capital stock as calculated by (2.25), then the evolution of T in these expressions can be seen as representing disembodied technological change. All vintages of the capital equipment benefit from a rise in the technology level T_t. No new investment is needed to benefit.

Based on data for the U.S. 1950-1990, and taking quality improvements into account, Greenwood et al. (1997) estimate that embodied technological progress explains about 60% of the growth in output per man hour. So, empirically, *embodied* technological progress seems to play a dominant role. As this tends not to be fully incorporated in national income accounting at fixed prices, there is a need to adjust the investment levels in (2.25) to better take estimated quality improvements into account. Otherwise the resulting K will not indicate the capital stock measured in efficiency units.

2.3 The concepts of representative firm and aggregate production function*

Many macroeconomic models make use of the simplifying notion of a *representative firm.* By this is meant a fictional firm whose production "represents" aggregate production (value added) in a sector or in society as a whole.

Suppose there are n firms in the sector considered or in society as a whole. Let F^i be the production function for firm i so that $Y_i = F^i(K_i, L_i)$, where Y_i, K_i, and L_i are output, capital input, and labor input, respectively, $i = 1, 2, \ldots, n$. Further, let $Y = \Sigma_{i=1}^n Y_i$, $K = \Sigma_{i=1}^n K_i$, and $L = \Sigma_{i=1}^n L_i$. Ignoring technological change, suppose the aggregate variables are related through some function, F^*, such that we can write

$$Y = F^*(K, L),$$

and such that the choices of a single firm facing this production function coincide with the aggregate outcomes, $\Sigma_{i=1}^n Y_i$, $\Sigma_{i=1}^n K_i$, and $\Sigma_{i=1}^n L_i$, in the original economy. Then $F^*(K, L)$ is called the *aggregate production function* or the production function of the *representative* firm. It is *as if* aggregate production is the result of the behavior of such a single firm.

A simple example where the aggregate production function is well-defined is the following. Suppose that all firms have the *same* production function so that $Y_i = F(K_i, L_i)$, $i = 1, 2, \ldots, n$. If in addition F has CRS, we have

$$Y_i = F(K_i, L_i) = L_i F(k_i, 1) \equiv L_i f(k_i),$$

where $k_i \equiv K_i/L_i$. Hence, facing given factor prices, cost-minimizing firms will choose the same capital intensity $k_i = k$ for all i. From $K_i = kL_i$ then follows $\sum_i K_i = k \sum_i L_i$ so that $k = K/L$. Thence,

$$Y \equiv \sum Y_i = \sum L_i f(k_i) = f(k) \sum L_i = f(k)L = F(k, 1)L = F(K, L).$$

In this (trivial) case the aggregate production function is well-defined and turns out to be exactly the same as the identical CRS production functions of the individual firms. Moreover, given CRS and $k_i = k$ for all i, we have $\partial Y_i/\partial K_i = f'(k_i) = f'(k) = F_K(K, L)$ for all i. So each firm's marginal productivity of capital is the same as the marginal productivity of capital on the basis of the aggregate production function.

Allowing for the existence of *different* production functions at firm level, we may define the aggregate production function as

$$\begin{aligned} F(K, L) &= \max_{(K_1, L_1, \ldots, K_n, L_n) \geq 0} F^1(K_1, L_1) + \cdots + F^n(K_n, L_n) \\ \text{s.t.} \sum_i K_i &\leq K, \quad \sum_i L_i \leq L. \end{aligned}$$

Allowing also for existence of different output goods, different capital goods, and different types of labor makes the issue more intricate, of course.

Yet, if firms are price taking profit maximizers and there are nonincreasing returns to scale, we at least know that the aggregate outcome is *as if,* for given prices, the firms jointly maximize aggregate profit on the basis of their combined production technology (Mas-Colell et al., 1955). The problem is, however, that the conditions needed for this to imply existence of an aggregate production function which is *well-behaved* (in the sense of inheriting simple qualitative properties from its constituent parts) are restrictive.

Nevertheless macroeconomics often treats aggregate output as a single homogeneous good and capital and labor as being two single and homogeneous inputs. There was in the 1960s a heated debate about the problems involved in this, with particular emphasis on the aggregation of different kinds of equipment into one variable, the capital stock "K". The debate is known as the "Cambridge controversy" because the dispute was between a group of economists from Cambridge University, UK, and a group from Massachusetts Institute of Technology (MIT), which is located in Cambridge, USA. The former group questioned the theoretical robustness of several of the neoclassical tenets, including the proposition that rising aggregate capital intensity tends to be associated with a falling rate of interest. Starting at the disaggregate level, an association of this sort is not a logical necessity because, with different production functions across the industries, the relative prices of produced inputs tend to change, when the interest rate changes. While acknowledging the possibility of "paradoxical" relationships, the latter group maintained that in a macroeconomic context they are likely to cause devastating problems only under exceptional circumstances. In the end this is a matter of empirical assessment.[16]

To avoid complexity and because, for many important issues in growth theory, there is today no well-tried alternative, we shall in this course most of the time use aggregate constructs like "Y", "K", and "L" as simplifying devices, hopefully acceptable in a first approximation. There are cases, however, where some disaggregation is pertinent. When for example the role of imperfect competition is in focus, we shall be ready to disaggregate the production side of the economy into several product lines, each producing its own differentiated product. We shall also touch upon a type of growth models where a key ingredient is the phenomenon of "creative destruction" meaning that an incumbent technological leader is competed out by an entrant with a qualitatively new technology.

Like the representative firm, the *representative household* and the *aggre-*

[16] In his review of the Cambridge controversy Mas-Colell (1989) concluded that: "What the 'paradoxical' comparative statics [of disaggregate capital theory] has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it."

gate consumption function are simplifying notions that should be applied only when they do not get in the way of the issue to be studied. The importance of budget constraints may make it even more difficult to aggregate over households than over firms. Yet, *if* (and that is a big if) all households have the *same, constant* marginal propensity to consume out of income, aggregation is straightforward and the representative household is a meaningful concept. On the other hand, if we aim at understanding, say, the *interaction* between lending and borrowing households, perhaps via financial intermediaries, the representative household is not a useful starting point. Similarly, if the theme is conflicts of interests between firm owners and employees, the existence of *different* types of households should be taken into account.

2.4 Long-run vs. short-run production functions*

Is the substitutability between capital and labor the same "ex ante" and "ex post"? By ex ante is meant "when plant and machinery are to be decided upon" and by ex post is meant "after the equipment is designed and constructed". In the standard neoclassical competitive setup, of for instance the Solow or the Ramsey model, there is a presumption that also after the construction and installation of the equipment in the firm, the ratio of the factor inputs can be fully adjusted to a change in the relative factor price. In practice, however, when some machinery has been constructed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machinery. That is, after construction and installation of the machinery, the choice opportunities are no longer described by the neoclassical production function but by a Leontief production function,

$$Y = \min(Au\bar{K}, BL), \qquad A > 0, B > 0, \tag{2.27}$$

where $\bar{K}$ is the size of the installed machinery (a fixed factor in the short run) measured in efficiency units, u is its utilization rate ($0 \leq u \leq 1$), and A and B are given technical coefficients measuring efficiency.

So in the short run the choice variables are u and L. In fact, essentially only u is a choice variable since efficient production trivially requires $L = Au\bar{K}/B$. Under "full capacity utilization" we have $u = 1$ (each machine is used 24 hours per day seven days per week). "Capacity" is given as $A\bar{K}$ per week. Producing efficiently at capacity requires $L = A\bar{K}/B$ and the marginal product by increasing labor input is here nil. But if demand, Y^d, is *less* than capacity, satisfying this demand efficiently requires $u = Y^d/(A\bar{K}) < 1$ and $L = Y^d/B$. As long as $u < 1$, the marginal productivity of labor is a *constant*, B.

The various efficient input proportions that are possible *ex ante* may be approximately described by a neoclassical CRS production function. Let this function on intensive form be denoted $y = f(k)$. When investment is decided upon and undertaken, there is thus a choice between alternative efficient pairs of the technical coefficients A and B in (2.27). These pairs satisfy

$$f(k) = Ak = B. \tag{2.28}$$

So, for an increasing sequence of k's, $k_1, k_2, \ldots, k_i, \ldots$, the corresponding pairs are $(A_i, B_i) = (f(k_i)/k_i, f(k_i))$, $i = 1, 2, \ldots$[17] We say that ex ante, depending on the relative factor prices as they are "now" and are expected to evolve in the future, a suitable technique, (A_i, B_i), is chosen from an opportunity set described by the given neoclassical production function. But ex post, i.e., when the equipment corresponding to this technique is installed, the production opportunities are described by a Leontief production function with $(A, B) = (A_i, B_i)$.

In the picturesque language of Phelps (1963), technology is in this case *putty-clay*. Ex ante the technology involves capital which is "putty" in the sense of being in a malleable state which can be transformed into a range of various machinery requiring capital-labor ratios of different magnitude. But once the machinery is constructed, it enters a "hardened" state and becomes "clay". Then factor substitution is no longer possible; the capital-labor ratio at full capacity utilization is fixed at the level $k = B_i/A_i$, as in (2.27). Following the terminology of Johansen (1972), we say that a putty-clay technology involves a "long-run production function" which is neoclassical and a "short-run production function" which is Leontief.

In contrast, the standard neoclassical setup assumes the same range of substitutability between capital and labor ex ante and ex post. Then the technology is called *putty-putty*. This term may also be used if ex post there is at least *some* substitutability although less than ex ante. At the opposite pole of putty-putty we may consider a technology which is *clay-clay*. Here neither ex ante nor ex post is factor substitution possible. Table 2.1 gives an overview of the alternative cases.

Table 2.1. Technologies classified according to

[17] The points P and Q in the right-hand panel of Fig. 2.3 can be interpreted as constructed this way from the neoclassical production function in the left-hand panel of the figure.

factor substitutability ex ante and ex post

Ex ante substitution	Ex post substitution	
	possible	impossible
possible	putty-putty	putty-clay
impossible		clay-clay

The putty-clay case is generally considered the realistic case. As time proceeds, technological progress occurs. To take this into account, we may replace (2.28) and (2.27) by $f(k_t, t) = A_t k_t = B_t$ and $Y_t = \min(A_t u_t \bar{K}_t, B_t L_t)$, respectively. If a new pair of Leontief coefficients, (A_{t_2}, B_{t_2}), efficiency-dominates its predecessor (by satisfying $A_{t_2} \geq A_{t_1}$ and $B_{t_2} \geq B_{t_1}$ with at least one strict equality), it may pay the firm to invest in the new technology at the same time as some old machinery is scrapped. Real wages tend to rise along with technological progress and the scrapping occurs because the revenue from using the old machinery in production no longer covers the associated labor costs.

The clay property ex-post of many technologies is important for short-run analysis. It implies that there may be non-decreasing marginal productivity of labor up to a certain point. It also implies that in its investment decision the firm will have to take expected future technologies and future factor prices into account. For many issues in long-run analysis the clay property ex-post may be less important, since over time adjustment takes place through new investment.

2.5 The neoclassical theory of factor income shares

To begin with, we ignore technological progress and write aggregate output as $Y = F(K, L)$, where F is neoclassical with CRS. From Euler's theorem follows that $F(K, L) = F_1 K + F_2 L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta = f'(k) \equiv \hat{r}(k)$ is the cost per unit of capital input and $w = f(k) - kf'(k) \equiv w(k)$ is the real wage, i.e., the cost per unit of labor input. We have $\hat{r}'(k) = f''(k) < 0$ and $w'(k) = -kf''(k) > 0$.

The *labor income share* is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}}, \tag{2.29}$$

where the function $SL(\cdot)$ is the income share of labor function, $w/\hat{r}$ is the *factor price ratio*, and $(w/\hat{r})/k = w/(\hat{r}k)$ is the *factor income ratio.* As $\hat{r}'(k) = f''(k) < 0$ and $w'(k) = -kf''(k) > 0$, the factor price ratio, $w/\hat{r}$, is an increasing function of k.

Suppose that capital tends to grow faster than labor so that k rises over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. By (2.29) we see that the labor income share moves in the same direction as the factor *income* ratio, $(w/\hat{r})/k$. The latter goes up (down) depending on whether the percentage rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than the percentage rise in k. So, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, we can only say that

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k \frac{w}{\hat{r}} \gtreqless 1, \tag{2.30}$$

respectively. In words: if the production function is such that the ratio of the marginal productivities of the two production factors is strongly (weakly) sensitive to the capital-labor ratio, then the labor income share rises (falls) along with a rise in K/L.

Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}}k$ ($= 1/\mathrm{E}\ell_k \frac{w}{\hat{r}}$). This elasticity indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$. Under perfect competition $\mathrm{E}\ell_{w/\hat{r}}k$ coincides with what is known as the *elasticity of factor substitution* (for a general definition, see below). The latter is often denoted σ. In the CRS case, σ will be a function of only k so that we can write $\mathrm{E}\ell_{w/\hat{r}}k = \sigma(k)$. By (2.30), we therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively.

The size of the elasticity of factor substitution is a property of the production function, hence of the technology. In special cases the elasticity of factor substitution is a constant, i.e., independent of k. For instance, if F is Cobb-Douglas, i.e., $Y = K^\alpha L^{1-\alpha}, 0 < \alpha < 1$, we have $\sigma(k) \equiv 1$, as shown in Section 2.7. In this case variation in k does not change the labor income share under perfect competition. Empirically there is not agreement about the "normal" size of the elasticity of factor substitution for industrialized economies, but the bulk of studies seems to conclude with $\sigma(k) < 1$ (see below).

Adding Harrod-neutral technical progress We now add Harrod-neutral technical progress. We write aggregate output as $Y = F(K, TL)$, where F

is neoclassical with CRS, and $T = T_t = T_0(1+g)^t$. Then the labor income share is

$$\frac{wL}{Y} = \frac{w/T}{Y/(TL)} \equiv \frac{\tilde{w}}{\tilde{y}}.$$

The above formulas still hold if we replace k by $\tilde{k} \equiv K/(TL)$ and w by $\tilde{w} \equiv w/T$. We get

$$SL'(\tilde{k}) \gtreqless 0 \text{ for } \sigma(\tilde{k}) \lesseqgtr 1,$$

respectively. We see that if $\sigma(\tilde{k}) < 1$ in the relevant range for $\tilde{k}$, then market forces tend to *increase* the income share of the factor that is becoming relatively more scarce, which is efficiency-adjusted labor, TL, if $\tilde{k}$ is increasing. And if instead $\sigma(\tilde{k}) > 1$ in the relevant range for $\tilde{k}$, then market forces tend to *decrease* the income share of the factor that is becoming relatively more scarce.

While k empirically is clearly growing, $\tilde{k} \equiv k/T$ is not necessarily so because also T is increasing. Indeed, according to Kaldor's "stylized facts", apart from short- and medium-term fluctuations, $\tilde{k}$ − and therefore also $\hat{r}$ and the labor income share − tend to be more or less constant over time. This can happen whatever the sign of $\sigma(\tilde{k}^*) - 1$, where $\tilde{k}^*$ is the long-run value of the effective capital-labor ratio $\tilde{k}$. Given CRS and the production function f, the elasticity of substitution between capital and labor does not depend on whether $g = 0$ or $g > 0$, but only on the function f itself and the level of $K/(TL)$.

As alluded to earlier, there are empiricists who reject Kaldor's "facts" as a general tendency. For instance Piketty (2014) essentially claims that in the very long run the effective capital-labor ratio $\tilde{k}$ has an upward trend, temporarily braked by two world wars and the Great Depression in the 1930s. If so, the sign of $\sigma(\tilde{k}) - 1$ becomes decisive for in what direction wL/Y will move. Piketty interprets the econometric literature as favoring $\sigma(\tilde{k}) > 1$, which means there should be downward pressure on wL/Y. This particular source behind a falling wL/Y can be questioned, however. Indeed, $\sigma(\tilde{k}) > 1$ contradicts the more general empirical view referred to above.[18]

Immigration

Here is another example that illustrates the importance of the size of $\sigma(\tilde{k})$. Consider an economy with perfect competition and a given aggregate capital stock K and technology level T (entering the production function in the labor-augmenting way as above). Suppose that for some reason, immigration,

[18] According to Summers (2014), Piketty's interpretation relies on conflating gross and net returns to capital.

say, aggregate labor supply, L, shifts up and full employment is maintained by the needed real wage adjustment. Given the present model, in what direction will aggregate labor income $wL = \tilde{w}(\tilde{k})TL$ then change? The effect of the larger L is to some extent offset by a lower w brought about by the lower effective capital-labor ratio. Indeed, in view of $d\tilde{w}/d\tilde{k} = -\tilde{k}f''(\tilde{k}) > 0$, we have $\tilde{k} \downarrow$ implies $w \downarrow$ for fixed T. So we cannot apriori sign the change in wL. The following relationship can be shown (Exercise ??), however:

$$\frac{\partial(wL)}{\partial L} = (1 - \frac{\alpha(\tilde{k})}{\sigma(\tilde{k})})w \gtreqless 0 \text{ for } \alpha(\tilde{k}) \lesseqgtr \sigma(\tilde{k}), \tag{2.31}$$

respectively, where $a(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/f(\tilde{k})$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the larger L will not be fully offset by the lower w as long as the elasticity of factor substitution, $\sigma(\tilde{k})$, exceeds the gross capital income share, $\alpha(\tilde{k})$. This condition seems confirmed by most of the empirical evidence (see Section 2.7).

The next section describes the concept of the elasticity of factor substitution at a more general setting. The subsequent section introduces the special case known as the CES production function.

2.6 The elasticity of factor substitution*

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 2.4 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e., $MRS = F_L(K, L)/F_K(K, L)$.[19] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio K/L, as a function of MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

[19] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more precise expression MRS_{KL}.

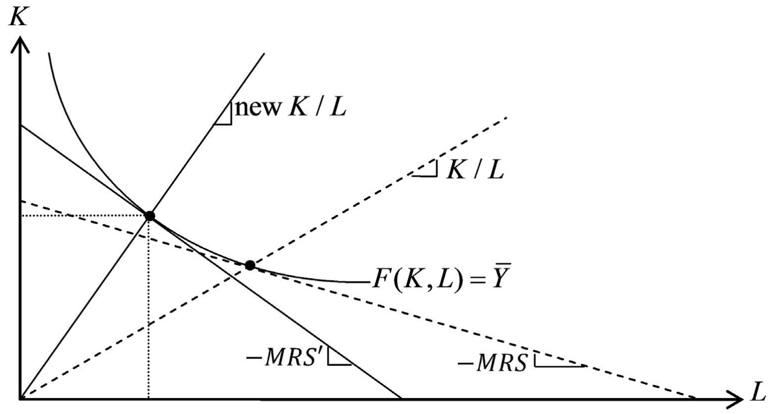


Figure 2.4: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

The *elasticity of substitution* between capital and labor is defined as the elasticity of the ratio K/L with respect to MRS when we move along a given isoquant, evaluated at the point (K, L). Let this elasticity be denoted $\tilde{\sigma}(K, L)$. Thus,

$$\tilde{\sigma}(K,L) = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} = \frac{\frac{d(K/L)}{K/L}}{\frac{dMRS}{MRS}}_{|Y=\bar{Y}}. \tag{2.32}$$

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[20] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to an increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitution. In an analogue way, in consumer theory one considers the elasticity of substitution between two consumption goods or between consumption today and consumption tomorrow. In that context the role of the given isoquant

[20]This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

is taken over by an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. the next chapter.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\tilde{\sigma}(K, L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{2.33}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$ has CRS, the formula (2.33) simplifies to

$$\tilde{\sigma}(K, L) = \frac{F_K(K, L) F_L(K, L)}{F_{KL}(K, L) F(K, L)} = -\frac{f'(k)\left(f(k) - f'(k)k\right)}{f''(k) k f(k)} \equiv \sigma(k), \tag{2.34}$$

where $k \equiv K/L$.[21] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level. We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

2.7 The CES production function

It can be shown[22] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^\beta + (1 - \alpha) L^\beta\right]^{\frac{1}{\beta}}, \tag{2.35}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or what is known as *total factor productivity*) and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (16.32) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain (see Appendix E).

[21] The formulas (2.33) and (2.34) are derived in Appendix D of Chapter 4 of Groth, Lecture Notes in Macroeconomics..

[22] See, e.g., Arrow et al. (1961).

By taking partial derivatives in (16.32) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta} \left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta} \left(\frac{Y}{L}\right)^{1-\beta}, \tag{2.36}$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$. The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} > 0.$$

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into (16.34) gives

$$\tilde{\sigma}(K, L) = \frac{1}{1-\beta} \equiv \sigma, \tag{2.37}$$

confirming the constancy of the elasticity of substitution. Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of factor substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (16.32), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (16.32) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[23] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0.$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (16.35) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution whatever the capital-labor ratio.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A \min(K, L), \text{ for } \beta \to -\infty. \tag{2.38}$$

[23] For proofs of this and the further claims below, see Appendix E of Chapter 4 of Groth, Lecture Notes in Macroeconomics.

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Fig. 2.5. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (16.35) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \rightarrow A\left[\alpha K + (1-\alpha)L\right], \text{ for } \beta \rightarrow 1.$$

Here the elasticity of substitution calculated from (16.35) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

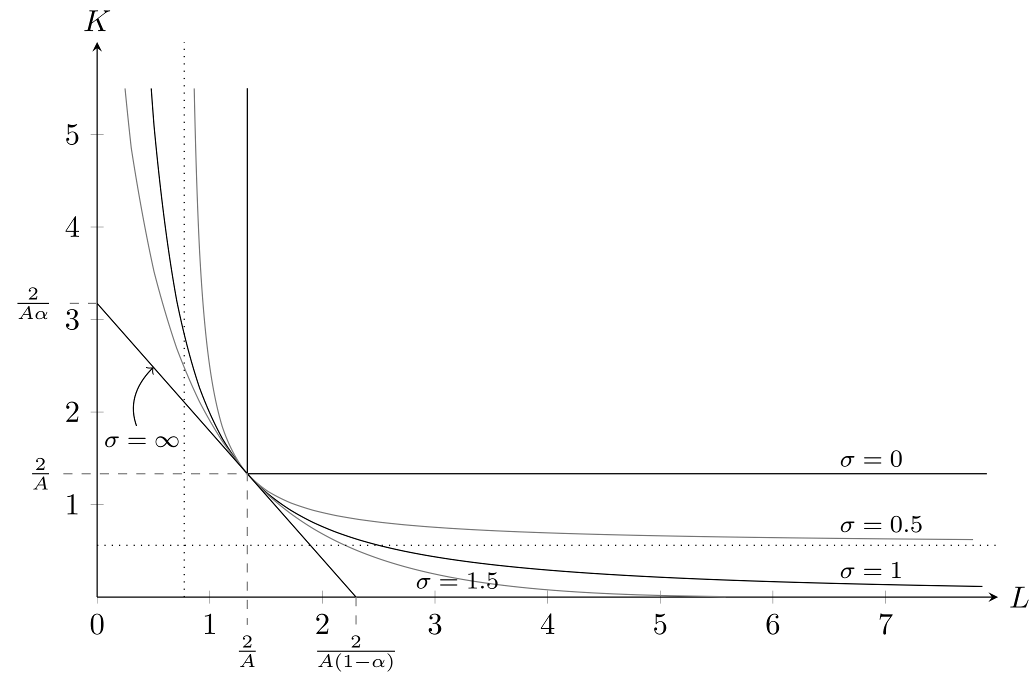


Figure 2.5: Isoquants for the CES function for alternative values of σ ($A = 1.5$, $\bar{Y} = 2$, and $\alpha = 0.42$).

Fig. 2.5 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes. When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for industrialized economies. The elasticity also differs across the production sectors. A thorough econometric study (Antràs, 2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$. The survey by Chirinko (2008) concludes with the interval $(0.4, 0.6)$. Starting from micro data, a recent study by Oberfield and Raval (2014) finds that the elasticity of factor substitution for the US manufacturing sector as a whole has been stable since 1970 at about 0.7.

The CES production function in intensive form

Dividing through by L on both sides of (16.32), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{2.39}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta}\left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y/\partial K$ in (16.33). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \to 0$ both y/k and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the lower Inada condition for MPK (see the right-hand panel of Fig. 2.3). It is also noteworthy that in this case, for $k \to \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \to \infty$ both y/k and $dy/dk \to A\alpha^{1/\beta} > 0$, thus violating the upper Inada condition for MPK (see right panel of Fig. 2.6). It is also noteworthy that for $k \to 0$, y approaches a positive lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, in this case capital is not essential. At the same time $dy/dk \to \infty$ for $k \to 0$ (so the lower Inada condition for the marginal productivity of capital holds). Details are in Appendix E.

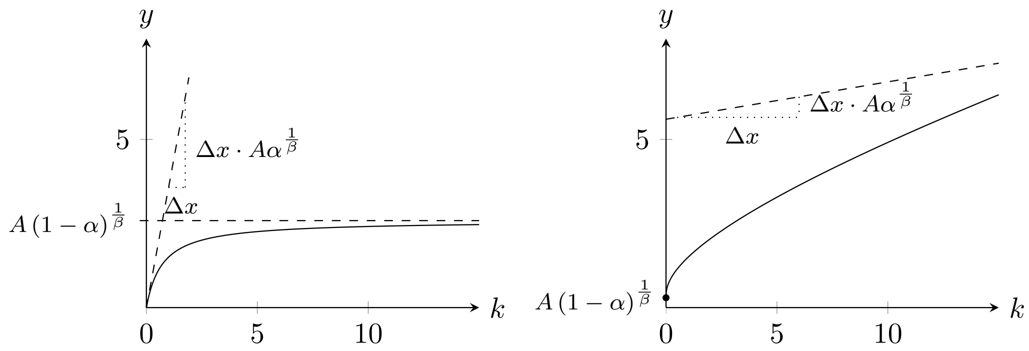
The marginal productivity of labor is

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}y^{1-\beta} = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} \equiv w(k),$$

from (16.33).

Since (16.32) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\beta < 0$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\beta > 0$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \to 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \to 0$ for $k \to 0$, and $y \to \infty$ for $k \to \infty$.



a) The case of $\sigma < 1$. a) The case of $\sigma > 1$.

Figure 2.6: The CES production function in intensive form, $\sigma = 1/(1-\beta)$, $\beta < 1$.

Generalizations

The CES production function considered above has CRS. By adding an elasticity of scale parameter, γ, we get the generalized form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{2.40}$$

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L$ $= A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 2.2.

Table 2.2 The family of production functions with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (16.38) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 X_1{}^{\beta} + \alpha_2 X_2{}^{\beta} + ...\alpha_n X_n{}^{\beta}\right]^{\frac{\gamma}{\beta}}, \; \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{2.41}$$

In analogy with (16.34), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j} \frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i, j$, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (16.39), all the partial elasticities of substitution take the same value, $1/(1-\beta)$.

2.8 Literature notes

As to the question of the empirical validity of the constant returns to scale assumption, Malinvaud (1998) offers an account of the econometric difficulties associated with estimating production functions. Studies by Basu (1996) and Basu and Fernald (1997) suggest returns to scale are about constant or decreasing. Studies by Hall (1990), Caballero and Lyons (1992), Harris and Lau (1992), Antweiler and Treffler (2002), and Harrison (2003) suggest there are quantitatively significant increasing returns, either internal or external. On this background it is not surprising that the case of IRS (at least at industry level), together with market forms different from perfect competition, has in recent years received more attention in macroeconomics and in the theory of economic growth.

Macroeconomists' use of the value-laden term "technological progress" in connection with technological change may seem suspect. But the term should be interpreted as merely a label for certain types of shifts of isoquants in an abstract universe. At a more concrete and disaggregate level analysts of course make use of more refined notions about technological change, recognizing for example not only benefits of new technologies, but also the risks, including risk of fundamental mistakes (think of the introduction and later abandonment of asbestos in the construction industry).

Informative history of technology is contained in Ruttan (2001) and Smil (2003). For more general economic history, see, e.g., Clark (2008) and Persson (2010). Forecasts of technological development in the next decades are contained in, for instance, Brynjolfsson and McAfee (2014).

Embodied technological progress, sometimes called investment-specific technological progress, is explored in, for instance, Solow (1960), Greenwood et al. (1997), and Groth and Wendner (2014). Hulten (2001) surveys the literature and issues related to measurement of the direct contribution of capital accumulation and technological change, respectively, to productivity growth.

Conditions ensuring that a representative household is admitted and the concept of Gorman preferences are discussed in Acemoglu (2009). Another useful source, also concerning the conditions for the representative firm to be a meaningful notion, is Mas-Colell et al. (1995). For general discussions of the limitations of representative agent approaches, see Kirman (1992) and Gallegati and Kirman (1999). Reviews of the "Cambridge Controversy" are contained in Mas-Colell (1989) and Felipe and Fisher (2003). The last-mentioned authors find the conditions required for the well-behavedness of these constructs so stringent that it is difficult to believe that actual economies are in any sense close to satisfy them. For a less distrustful view, see for instance Ferguson (1969), Johansen (1972), Malinvaud (1998), Jorgenson et al. (2005), and Jones (2005).

It is often assumed that capital depreciation can be described as geometric (in continuous time exponential) evaporation of the capital stock. This formula is popular in macroeconomics, more so because of its simplicity than its realism. An introduction to more general approaches to depreciation is contained in, e.g., Nickell (1978).

2.9 References

(incomplete)

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Chapter 3

Continuous time analysis

Because dynamic analysis is generally easier in continuous time, growth models are often stated in continuous time. This chapter gives an account of the conceptual aspects of continuous time analysis. Appendix A considers simple growth arithmetic in continuous time. And Appendix B provides solution formulas for linear first-order differential equations.

3.1 The transition from discrete time to continuous time

We start from a discrete time framework. The run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

3.1.1 Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t + \Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t + \Delta t)$

approach zero, i.e., let $\Delta t \to 0$. When a is a differentiable function of t, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t + \Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{3.1}$$

where $s(t)$ is the saving flow at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time, if wealth grows at a constant rate $g > 0$ per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1 + g)^i, \quad i = 0, 1, 2, \ldots . \tag{3.2}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period,

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{3.3}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (3.3) gives

$$a(t) = a_{nt} = a_0(1 + \frac{g}{n})^{nt} = a_0 \left[(1 + \frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$. Thus $m \to \infty$. Then, in the limit there is continuous compounding and it can be shown that

$$a(t) = a_0 e^{gt}, \tag{3.4}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.7182818285....$

The formula (3.4) is the continuous-time analogue to the discrete time formula (3.2) with annual compounding. A geometric growth factor is replaced by an exponential growth factor, e^{gt}, and this growth factor is valid for any t in the time interval $(-\tau_1, \tau_2)$ for which the growth rate of a equals the constant g (τ_1 and τ_2 being some positive real numbers).

We can also view the formulas (3.2) and (3.4) as the solutions to a difference equation and a differential equation, respectively. Thus, (3.2) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value

a_0. And (3.4) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{3.5}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (3.5) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

3.1.2 Compound interest and discounting

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{3.6}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (3.5) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. We reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time

t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt} dt, \tag{3.7}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (3.7) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau} dt.$$

In (3.7) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (**??**) where it is c_{t-1} that is discounted by $(1+r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t-1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

3.2 The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{3.8}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (3.8) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

3.3 Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these elementary units we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t)-\delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0,\ T)$ or perhaps

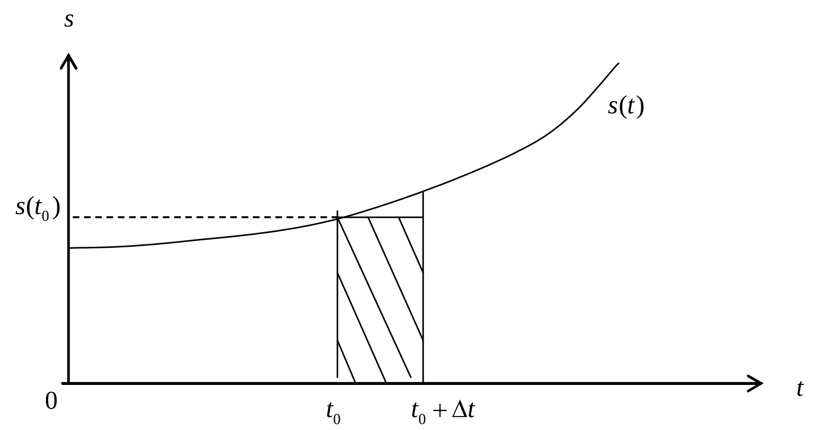

Figure 3.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0+\Delta t$ is $\approx$ the hatched area.

$[0,\ \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Figure 3.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

3.4 The choice between discrete and continuous time formulation

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And, as for instance Allen (1967) argued, it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And centralized asset markets equilibrate very fast and respond immediately to new information. For such markets a formulation in continuous time seems a better approximation.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{3.9}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1 + \alpha)x_t$, where $1 + \alpha < 0$. The solution is $x_t = (1 + \alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1 + \alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise (together with divergence if $\alpha < -2$) in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (3.9) the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (3.9). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. Yet, everything else equal, the clearer distinction between stocks and flows in continuous time than in discrete time speaks for the former. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.

3.5 Appendix A: Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small.

3.6 Appendix B: Solution formulas for linear differential equations of first order

For a general differential equation of first order, $\dot{x}(t) = \varphi(x(t), t)$, with $x(t_0) = x_{t_0}$ and where φ is a continuous function, we have, at least for t in an interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$,

$$x(t) = x_{t_0} + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau. \qquad (*)$$

To get a confirmation, calculate $\dot{x}(t)$ from (*).

For the special case of a linear differential equation of first order, $\dot{x}(t) + a(t)x(t) = b(t)$, we can specify the solution. Three sub-cases of rising complexity are:

1. $\dot{x}(t) + ax(t) = b$, with $a \neq 0$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = (x_{t_0} - x^*)e^{-a(t-t_0)} + x^*, \text{ where } x^* = \frac{b}{a}.$$

If $a = 0$, we get, directly from (*), the solution $x(t) = x_{t_0} + bt$.[1]

2. $\dot{x}(t) + ax(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)} \int_{t_0}^{t} b(s)e^{a(s-t_0)} ds.$$

Special case: $b(t) = ce^{ht}$, with $h \neq -a$ and initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-a(t-t_0)} + e^{-a(t-t_0)} c \int_{t_0}^{t} e^{(a+h)(s-t_0)} ds = (x_{t_0} - \frac{c}{a+h})e^{-a(t-t_0)} + \frac{c}{a+h}e^{h(t-t_0)}.$$

3. $\dot{x}(t) + a(t)x(t) = b(t)$, with initial condition $x(t_0) = x_{t_0}$. Solution:

$$x(t) = x_{t_0}e^{-\int_{t_0}^{t} a(\tau)d\tau} + e^{-\int_{t_0}^{t} a(\tau)d\tau} \int_{t_0}^{t} b(s)e^{\int_{t_0}^{s} a(\tau)d\tau} ds.$$

[1] Some non-linear differential equations can be transformed into this simple case. For simplicity let $t_0 = 0$. Consider the equation $\dot{y}(t) = \alpha y(t)^{\beta}$, $y_0 > 0$ given, $\alpha \neq 0, \beta \neq 1$ (a *Bernoulli equation*). To find the solution for $y(t)$, let $x(t) \equiv y(t)^{1-\beta}$. Then, $\dot{x}(t) = (1-\beta)y(t)^{-\beta}\dot{y}(t) = (1-\beta)y(t)^{-\beta}\alpha y(t)^{\beta} = (1-\beta)\alpha$. The solution for this is $x(t) = x_0 + (1-\beta)\alpha t$, where $x_0 = y_0^{1-\beta}$. Thereby the solution for $y(t)$ is $y(t) = x(t)^{1/(1-\beta)} = \left(y_0^{1-\beta} + (1-\beta)\alpha t\right)^{1/(1-\beta)}$, which is defined for $t > -y_0^{1-\beta}/((1-\beta)\alpha$.

Special case: $b(t) = 0$. Solution:

$$x(t) = x_{t_0} e^{-\int_{t_0}^{t} a(\tau) d\tau}.$$

Even more special case: $b(t) = 0$ and $a(t) = a$, a constant. Solution:

$$x(t) = x_{t_0} e^{-a(t-t_0)}.$$

Remark 1 For $t_0 = 0$, most of the formulas will look simpler.

Remark 2 To check whether a suggested solution *is* a solution, calculate the time derivative of the suggested solution and add an arbitrary constant. By appropriate adjustment of the constant, the final result should be a replication of the original differential equation together with its initial condition.

Chapter 4

Skill-biased technical change. Balanced growth theorems

This chapter is both an alternative and a supplement to the pages 60-64 in Acemoglu, where the concepts of neutral technical change and balanced growth, including Uzawa's theorem, are discussed.

Since "neutral" technical change should be seen in relation to "biased" technical change, Section 1 below introduces the concept of "biased" technical change. Also concerning biased technical change do three different definitions, Hicks', Harrod's, and what the literature has dubbed "Solow's". Below we concentrate on Hick's definition − with an application to how technical change affects the evolution of the skill premium. So the focus is on the production factors skilled and unskilled labor rather than capital and labor. While regarding capital and labor it is Harrod's classifications that are most used in macroeconomics, regarding skilled and unskilled labor it is Hicks'.

The remaining sections discuss the concept of balanced growth and present three fundamental propositions about balanced growth. In view of the generality of the propositions, they have a broad field of application. Our propositions 1 and 2 are slight extensions of part 1 and 2, respectively, of what Acemoglu calls Uzawa's Theorem I (Acemoglu, 2009, p. 60). Our Proposition 3 essentially corresponds to what Acemoglu calls Uzawa's Theorem II (Acemoglu, 2009, p. 63).

4.1 The rising skill premium

4.1.1 Skill-biased technical change in the sense of Hicks: An example

Let aggregate output be produced through a differentiable three-factor production function $\tilde{F}$:

$$Y = \tilde{F}(K, L_1, L_2, t),$$

where K is capital input, L_1 is input of unskilled labor (also called blue-collar labor below), and L_2 is input of skilled labor. Suppose technological change is such that the production function can be rewritten

$$\tilde{F}(K, L_1, L_2, t) = F(K, H(L_1, L_2, t)), \tag{4.1}$$

where the "nested" function $H(L_1, L_2, t)$ represents input of a "human capital" aggregate. Let F be CRS-neoclassical w.r.t. K and H and let H be CRS-neoclassical w.r.t. (L_1, L_2). Finally, let $\partial H/\partial t > 0$. So "technical change" amounts to "technical progress".

In equilibrium under perfect competition in the labor markets the relative wage, often called the "skill premium", will be

$$\frac{w_2}{w_1} = \frac{\partial Y/\partial L_2}{\partial Y/\partial L_1} = \frac{F_H \partial H/\partial L_2}{F_H \partial H/\partial L_1} = \frac{H_2(L_1, L_2, t)}{H_1(L_1, L_2, t)} = \frac{H_2(1, L_2/L_1, t)}{H_1(1, L_2/L_1, t)}, \tag{4.2}$$

where we have used Euler's theorem (saying that if H is homogeneous of degree one in its first two arguments, then the partial derivatives of H are homogeneous of degree zero w.r.t. these arguments).

Time is continuous (nevertheless the time argument of a variable, x, is in this section written as a subscript t). Hicks' definitions are now: If for all $L_2/L_1 > 0$,

$$\frac{d\left(\frac{H_2(1,L_2/L_1,t)}{H_1(1,L_2/L_1,t)}\right)}{dt}\Bigg|_{\frac{L_2}{L_1}\text{ constant}} \gtreqqless 0, \text{ then technical change is}$$

$$\left\{\begin{array}{c} \text{skill-biased in the sense of Hicks,} \\ \text{skill-neutral in the sense of Hicks.} \\ \text{blue collar-biased in the sense of Hicks,} \end{array}\right. \tag{4.3}$$

respectively.

In the US the skill premium (measured by the wage ratio for college grads vis-a-vis high school grads) has had an upward trend since 1950 (see for instance Jones and Romer, 2010).[1] If in the same period the relative supply of skilled labor had been roughly constant, by (4.3) in combination with (4.2), a possible explanation could be that technological change has been skill-biased in the sense of Hicks. In reality, in the same period also the relative supply of skilled labor has been rising (in fact even faster than the skill premium). Since in spite of this the skill premium has *risen,* it suggests that the extend of "skill-biasedness" has been even stronger.

We may alternatively put it this way. As the H function is CRS-neoclassical w.r.t. L_1 and L_2, we have $H_{22} < 0$ *and* $H_{12} > 0$, cf. Chapter 2. Hence, by (4.2), a rising L_2/L_1 without technical change would imply a *declining* skill premium. That the opposite has happened must, within our simple model, be due to (a) there *has* been technical change, and (b) technical change has *favoured skilled labor* (which means that technical change has been skill-biased in the sense of Hicks).

An additional aspect of the story is that skill-biasedness helps *explain* the observed increase in the relative *supply* of skilled labor. If for a constant relative supply of skilled labor, the skill premium is increasing, this increase strengthens the incentive to go to college. Thereby the relative supply of skilled labor (reflecting the fraction of skilled labor in the labor force) tends to increase.

4.1.2 Capital-skill complementarity

An additional potential source of a rising skill premium is *capital-skill complementarity.* Let the aggregate production function be

$$Y = \tilde{F}(K, L_1, L_2, t) = F(K, A_{1t}L_1, A_{2t}L_2) = (K + A_{1t}L_1)^{\alpha}(A_{2t}L_2)^{1-\alpha}, \ 0 < \alpha < 1,$$

where A_{1t} and A_{2t} are technical coefficients that may be rising over time. In this production function capital and unskilled labor are perfectly substitutable (the partial elasticity of factor substitution between them is $+\infty$). On the other hand there is *direct complementarity* between capital and skilled labor, i.e., $\partial^2 Y/(\partial L_2 \partial K) > 0$.

Under perfect competition the skill premium is

$$\begin{aligned} \frac{w_2}{w_1} &= \frac{\partial Y/\partial L_2}{\partial Y/\partial L_2} = \frac{(K + A_{1t}L_1)^{\alpha}(1-\alpha)(A_{2t}L_2)^{-\alpha}A_{2t}}{\alpha(K + A_{1t}L_1)^{\alpha-1}A_{1t}(A_{2t}L_2)^{1-\alpha}} \\ &= \frac{1-\alpha}{\alpha}\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)\frac{A_{2t}}{A_{1t}}. \end{aligned} \tag{4.4}$$

[1] On the other hand, over the years 1915 - 1950 the skill premium had a downward trend (Jones and Romer, 2010).

Here, if technical change is absent (A_{1t} and A_{2t} constant), a rising capital stock will, *for fixed* L_1 *and* L_2, raise the skill premium.

A more realistic scenario is, however, a situation with an approximately constant real interest rate, cf. Kaldor's stylized facts. We have, again by perfect competition,

$$\frac{\partial Y}{\partial K} = \alpha(K + A_{1t}L_1)^{\alpha-1}(A_{2t}L_2)^{1-\alpha} = \alpha\left(\frac{K + A_{1t}L_1}{A_{2t}L_2}\right)^{\alpha-1} = r_t + \delta, \quad (4.5)$$

where r_t is the real interest rate at time t and δ is the (constant) capital depreciation rate. For $r_t = r$, a constant, (4.5) gives

$$\frac{K + A_{1t}L_1}{A_{2t}L_2} = \left(\frac{r+\delta}{\alpha}\right)^{-\frac{1}{1-\alpha}} \equiv c, \quad (4.6)$$

a constant. In this case, (4.4) shows that capital-skill complementarity is *not sufficient* for a rising skill premium. A rising skill premium requires that technical change brings about a rising A_{2t}/A_{1t}. So again an observed rising skill premium, along with a more or less constant real interest rate, suggests that technical change is skill-biased.

We may rewrite (4.6) as

$$\frac{K}{A_{2t}L_2} = c - \frac{A_{1t}L_1}{A_{2t}L_2},$$

where the conjecture is that $A_{1t}L_1/(A_{2t}L_2) \to 0$ for $t \to \infty$. The analysis suggests the following story. Skill-biased technical progress generates rising productivity as well as a rising skill premium. The latter induces more and more people to go to college. The rising level of education in the labor force raises productivity further. This is a basis for further capital accumulation, continuing to replace unskilled labor, and so on.

In particular since the early 1980s the skill premium has been sharply increasing in the US (see Acemoglu, p. 498). This is also the period where ICT technologies took off.

4.2 Balanced growth and constancy of key ratios

The focus now shifts to homogeneous labor vis-a-vis capital.

We shall state general definitions of the concepts of "steady state" and "balanced growth", concepts that are related but not identical. With respect to "balanced growth" this implies a minor deviation from the way Acemoglu briefly defines it informally on his page 57. The main purpose of the present chapter is to lay bare the connections between these two concepts as well as their relation to the hypothesis of Harrod-neutral technical progress and Kaldor's stylized facts.

4.2.1 The concepts of steady state and balanced growth

A basic equation in many one-sector growth models for a closed economy in continuous time is

$$\dot{K} = I - \delta K = Y - C - \delta K \equiv S - \delta K, \tag{4.7}$$

where K is aggregate capital, I aggregate gross investment, Y aggregate output, C aggregate consumption, S aggregate gross saving ($\equiv Y - C$), and $\delta \geq 0$ is a constant physical capital depreciation rate.

Usually, in the theoretical literature on dynamic models, a *steady state* is defined in the following way:

Definition 3 *A steady state of a dynamic model is a stationary solution to the fundamental differential equation(s) of the model.*

Or briefly: a steady state is a stationary point of a dynamic process.

Let us take the Solow growth model as an example. Here gross saving equals sY, where s is a constant, $0 < s < 1$. Aggregate output is given by a neoclassical production function, F, with CRS and Harrod-neutral technical progress: $Y = F(K, AL) = ALF(\tilde{k}, 1) \equiv ALf(\tilde{k})$, where L is the labor force, A is the level of technology, and $\tilde{k} \equiv K/(AL)$ is the (effective) capital intensity. Moreover, $f' > 0$ and $f'' < 0$. Solow assumes $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$, where n and $g \geq 0$ are the constant growth rates of the labor force and technology, respectively. By log-differentiating $\tilde{k}$ w.r.t. t,[2] we end up with the *fundamental differential equation* ("law of motion") of the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}. \tag{4.8}$$

Thus, in the Solow model, a (non-trivial) steady state is a $\tilde{k}^* > 0$ such that, if $\tilde{k} = \tilde{k}^*$, then $\dot{\tilde{k}} = 0$. In passing we note that, by (4.8), such a $\tilde{k}^*$ must satisfy the equation $f(\tilde{k}^*)/\tilde{k}^* = (\delta + g + n)/s$, and in view of $f'' < 0$, it is unique and globally asymptotically stable if it exists. A sufficient condition

[2] Or by directly using the fraction rule, see Appendix A to Chapter 3.

for its existence is that $\delta + g + n > 0$ and f satisfies the Inada conditions $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$.

The most common definition in the literature of *balanced growth* for an aggregate economy is the following:

Definition 4 *A balanced growth path is a path $(Y, K, C)_{t=0}^{\infty}$ along which the quantities $Y, K,$ and C are positive and grow at constant rates (not necessarily positive and not necessarily the same).*

Acemoglu, however, defines (Acemoglu, 2009, p. 57) balanced growth in the following way: "balanced growth refers to an allocation where output grows at a constant rate and capital-output ratio, the interest rate, and factor shares remain constant". My problem with this definition is that it mixes growth of aggregate quantities with income distribution aspects (interest rate and factor income shares). And it is not made clear what is meant by the output-capital ratio if the relative price of capital goods is changing over time. So I stick to the definition above which is quite standard and is known to function well in many different contexts.

Note that in the Solow model (as well as in many other models) we have that if the economy is in a steady state, $\tilde{k} = \tilde{k}^*$, then the economy features balanced growth. Indeed, a steady state of the Solow model implies by definition that $\tilde{k} \equiv K/(AL)$ is constant. Hence K must grow at the same *constant* rate as AL, namely $g + n$. In addition, $Y = f(\tilde{k}^*)AL$ in a steady state, showing that also Y must grow at the constant rate $g + n$. And so must then $C = (1 - s)Y$. So in a steady state of the Solow model the path followed by $(Y, K, C)_{t=0}^{\infty}$ is a balanced growth path.

As we shall see in the next section, in the Solow model (and many other models) the reverse also holds: if the economy features balanced growth, then it is in a steady state. But this equivalence between steady state and balanced growth does not hold in all models.

4.2.2 A general result about balanced growth

An interesting fact is that, given the dynamic resource constraint (4.7), we have *always* that if there is balanced growth with positive gross saving, then the ratios Y/K and C/Y are constant (by "*always*" is meant: independently of how saving is determined and how the labor force and technology evolve). And also the other way round: as long as gross saving is positive, constancy of the Y/K and C/Y ratios is enough to ensure balanced growth. So balanced growth and constancy of certain key ratios are essentially equivalent.

This is a very practical general observation. And since Acemoglu does not state any balanced growth theorem at this general level, we shall do it here, together with a proof. Letting g_x denote the growth rate of the (positively valued) variable x, i.e., $g_x \equiv \dot{x}/x$, we claim:

Proposition 1 *(the balanced growth equivalence theorem). Let $(Y, K, C)_{t=0}^{\infty}$ be a path along which Y, K, C, and $S \equiv Y - C$ are positive for all $t \geq 0$. Then, given the accumulation equation (4.7), the following holds:*

(i) if there is balanced growth, then $g_Y = g_K = g_C$, and the ratios Y/K and C/Y are constant;

(ii) if Y/K and C/Y are constant, then Y, K, and C grow at the same constant rate, i.e., not only is there balanced growth, but the growth rates of Y, K, and C are the same.

Proof Consider a path $(Y, K, C)_{t=0}^{\infty}$ along which Y, K, C, and $S \equiv Y - C$ are positive for all $t \geq 0$. (i) Assume there is balanced growth. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.7), we have that $S/K = g_K + \delta$ is constant, implying

$$g_S = g_K. \tag{*}$$

Further, since $Y = C + S$,

$$\begin{aligned} g_Y &= \frac{\dot{Y}}{Y} = \frac{\dot{C}}{Y} + \frac{\dot{S}}{Y} = g_C\frac{C}{Y} + g_S\frac{S}{Y} = g_C\frac{C}{Y} + g_K\frac{S}{Y} \qquad \text{(by (*))} \\ &= g_C\frac{C}{Y} + g_K\frac{Y - C}{Y} = \frac{C}{Y}(g_C - g_K) + g_K. \qquad (**) \end{aligned}$$

Now, let us provisionally assume that $g_K \neq g_C$. Then (**) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{***}$$

which is a constant since g_Y, g_K, and g_C are constant. Constancy of C/Y requires that $g_C = g_Y$, hence, by (***), $C/Y = 1$, i.e., $C = Y$. In view of $Y = C + S$, however, this outcome contradicts the given condition that $S > 0$. Hence, our provisional assumption and its implication, (***), are falsified. Instead we have $g_K = g_C$. By (**), this implies $g_Y = g_K = g_C$, but now without the condition $C/Y = 1$ being implied. It follows that Y/K and C/Y are constant.

(ii) Suppose Y/K and C/Y are constant. Then $g_Y = g_K = g_C$, so that C/K is a constant. We now show that this implies that g_K is constant. Indeed, from (4.7), $S/Y = 1 - C/Y$, so that also S/Y is constant. It follows that $g_S = g_Y = g_K$, so that S/K is constant. By (4.7),

$$\frac{S}{K} = \frac{\dot{K} + \delta K}{K} = g_K + \delta,$$

so that g_K is constant. This, together with constancy of Y/K and C/Y, implies that also g_Y and g_C are constant. $\square$

Remark. It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of g_C and g_Y had a constant growth rate. This growth rate, however, could easily differ from that of K. Suppose $Y = AK^{\alpha}L^{1-\alpha}$, $g_A = \gamma$ and $g_L = n$ (γ and n constants). Then we would have $g_C = g_Y = \gamma - \alpha\delta + (1-\alpha)n$, which could easily be strictly positive and thereby different from $g_K = -\delta \leq 0$ so that (i) no longer holds. $\square$

The nice feature is that this proposition holds for *any* model for which the simple dynamic resource constraint (4.7) is valid. No assumptions about for example CRS and other technology aspects or about market form are involved. Note also that Proposition 1 suggests a link from balanced growth to steady state. And such a link *is* present in for instance the Solow model. Indeed, by (i) of Proposition 1, balanced growth implies constancy of Y/K, which in the Solow model implies that $f(\tilde{k})/\tilde{k}$ is constant. In turn, the latter is only possible if $\tilde{k}$ is constant, that is, if the economy is in steady state.

There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technological change, see Groth et al., 2010). Therefore, it is recommendable always to maintain a distinction between the terms steady state and balanced growth.

4.3 The crucial role of Harrod-neutrality

Proposition 1 suggests that if one accepts Kaldor's stylized facts (see Chapter 1) as a characterization of the past century's growth experience, and if one wants a model consistent with them, one should construct the model such that it can generate balanced growth. For a model to be capable of generating balanced growth, however, technological progress must be of the Harrod-neutral type (i.e., be labor-augmenting), at least in a neighborhood of the balanced growth path. For a fairly general context (but of course not as general as that of Proposition 1), this was shown already by Uzawa (1961). We now present a modernized version of Uzawa's contribution.

Let the aggregate production function be

$$Y(t) = \tilde{F}(K(t), BL(t), t), \qquad B > 0, \tag{4.9}$$

where B is a constant that depends on measurement units. The only technology assumption needed is that $\tilde{F}$ has CRS w.r.t. the first two arguments

($\tilde{F}$ need not be neoclassical for example). As a representation of technical progress, we assume $\partial\tilde{F}/\partial t > 0$ for all $t \geq 0$ (i.e., as time proceeds, unchanged inputs result in more and more output). We also assume that the labor force evolves according to

$$L(t) = L(0)e^{nt}, \tag{4.10}$$

where n is a constant. Further, non-consumed output is invested and so (4.7) is the dynamic resource constraint of the economy.

Proposition 2 *(Uzawa's balanced growth theorem) Let* $P = (Y(t), K(t), C(t))_{t=0}^{\infty}$, *where* $0 < C(t) < Y(t)$ *for all* $t \geq 0$, *be a path satisfying the capital accumulation equation (4.7), given the CRS-production function (4.9) and the labor force path in (4.10). Then:*

(i) a necessary condition for this path to be a balanced growth path is that along the path it holds that

$$Y(t) = \tilde{F}(K(t), BL(t), t) = \tilde{F}(K(t), A(t)L(t), 0), \tag{4.11}$$

where $A(t) = Be^{gt}$ *with* $g \equiv g_Y - n$;

(ii) for any $g > 0$ *such that there is a* $q > \delta + g + n$ *with the property that the production function* $\tilde{F}$ *in (4.9) allows an output-capital ratio equal to* q *at* $t = 0$ *(i.e.,* $\tilde{F}(1, \tilde{k}^{-1}, 0) = q$ *for some real number* $\tilde{k} > 0$*), a sufficient condition for* $\tilde{F}$ *to be compatible with a balanced growth path with output-capital ratio* q*, is that* $\tilde{F}$ *can be written as in (4.11) with* $A(t) = Be^{gt}$.

Proof (i)[3] Suppose the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path. By definition, g_K and g_Y are then constant, so that $K(t) = K(0)e^{g_K t}$ and $Y(t) = Y(0)e^{g_Y t}$. We then have

$$Y(t)e^{-g_Y t} = Y(0) = \tilde{F}(K(0), BL(0), 0) = \tilde{F}(K(t)e^{-g_K t}, BL(t)e^{-nt}, 0), \quad (*)$$

where we have used (4.9) with $t = 0$. In view of the precondition that $S(t) \equiv Y(t) - C(t) > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (*) then implies

$$Y(t) = \tilde{F}(K(t)e^{g_Y t}e^{-g_K t}, BL(t)e^{g_Y t}e^{-nt}, 0) = \tilde{F}(K(t), Be^{(g_Y - n)t}L(t), 0).$$

[3]This part draws upon Schlicht (2006), who generalized a proof in Wan (1971, p. 59) for the special case of a constant saving rate.

We see that (4.11) holds for $A(t) = Be^{gt}$ with $g \equiv g_Y - n$.

(ii) Suppose (4.11) holds with $A(t) = Be^{gt}$. Let $g \geq 0$ be given such that there is a $q > g + n + \delta > 0$ with the property that

$$\tilde{F}(1, \tilde{k}^{-1}, 0) = q \tag{**}$$

for some constant $\tilde{k} > 0$. Our strategy is to prove the claim in (ii) by construction of a path $P = (Y(t), K(t), C(t))_{t=0}^{\infty}$ which satisfies it. We let P be such that the saving-income ratio is a constant $s \equiv (\delta+g+n)/q \in (0,1)$, i.e., $Y(t) - C(t) \equiv S(t) = sY(t)$ for all $t \geq 0$. Inserting this, together with $Y(t) = f(\tilde{k}(t))A(t)L(t)$, where $f(\tilde{k}(t)) \equiv \tilde{F}(\tilde{k}(t), 1, 0)$ and $\tilde{k}(t) \equiv K(t)/(A(t)L(t))$, into (4.7), rearranging gives the Solow equation (4.8). Hence $\tilde{k}(t)$ is constant if and only if $\tilde{k}(t)$ satisfies the equation $f(\tilde{k}(t))/\tilde{k}(t) = (\delta + g + n)/s \equiv q$. By (**) and the definition of f, the required value of $\tilde{k}(t)$ is $\tilde{k}$, which is consequently the unique steady state for the constructed Solow equation. Letting $K(0)$ satisfy $K(0) = \tilde{k}BL(0)$, where $B = A(0)$, we thus have $\tilde{k}(0) = K(0)/(A(0)L(0)) = \tilde{k}$. So that the initial value of $\tilde{k}(t)$ equals the steady state value. It follows that $\tilde{k}(t) = \tilde{k}$ for all $t \geq 0$, and so $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t) = f(\tilde{k})/\tilde{k} = q$ for all $t \geq 0$. In addition, $C(t) = (1 - s)Y(t)$, so that $C(t)/Y(t)$ is constant along the path P. As both Y/K and C/Y are thus constant along the path P, by (ii) of Proposition 1 follows that P is a balanced growth path, as was to be proved. □

The form (4.11) indicates that along a balanced growth path, technical progress must be purely "labor augmenting", that is, Harrod-neutral. It is in this case convenient to define a new CRS function, F, by $F(K(t), A(t)L(t)) \equiv \tilde{F}(K(t), A(t)L(t), 0)$. Then (i) of the proposition implies that at least along the balanced growth path, we can rewrite the production function this way:

$$Y(t) = \tilde{F}(K(t), A(0)L(t), t) = F(K(t), A(t)L(t)), \tag{4.12}$$

where $A(0) = B$ and $A(t) = A(0)e^{gt}$ with $g \equiv g_Y - n$.

It is important to recognize that the occurrence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality should not be interpreted as indicating that the technological progress emanates specifically from the labor input. Harrod-neutrality only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* at the aggregate level in the form (4.12).[4]

[4]For a CRS Cobb-Douglas production function with technological progress, Harrod-neutrality is present whenever the output elasticity w.r.t capital (often denoted α) is constant over time.

What is the intuition behind the Uzawa result that for balanced growth to be possible, technical progress must have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, in simple macro models labor is a non-produced production factor which (at least in the context of (4.10)) grows in an exogenous way. Second, because of CRS, the original formulation, (4.9), of the production function implies that

$$1 = \tilde{F}(\frac{K(t)}{Y(t)}, \frac{L(t)}{Y(t)}, t). \tag{4.13}$$

Now, since capital is accumulated non-consumed output, it tends to inherit the trend in output such that $K(t)/Y(t)$ must be constant along a balanced growth path (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio $L(t)/Y(t)$ is free to adjust as time proceeds. When there is technical progress ($\partial\tilde{F}/\partial t > 0$) along a balanced growth path, this progress must manifest itself in the form of a changing $L(t)/Y(t)$ in (13.5) as t proceeds, precisely because $K(t)/Y(t)$ *must* be constant along the path. In the "normal" case where $\partial\tilde{F}/\partial L > 0$, the needed change in $L(t)/Y(t)$ is a *fall* (i.e., a rise in $Y(t)/L(t)$). This is what (13.5) shows. Indeed, the fall in $L(t)/Y(t)$ must exactly offset the effect on $\tilde{F}$ of the rising t, when there is a fixed capital-output ratio.[5] It follows that along the balanced growth path, $Y(t)/L(t)$ is an increasing implicit function of t. If we denote this function $A(t)$, we end up with (4.12) with specified properties (given by g and q).

The generality of Uzawa's theorem is noteworthy. The theorem assumes CRS, but does not presuppose that the technology is neoclassical, not to speak of satisfying the Inada conditions.[6] And the theorem holds for exogenous as well as endogenous technological progress. It is also worth mentioning that the proof of the sufficiency part of the theorem is *constructive*. It provides a method to construct a hypothetical balanced growth path (BGP from now).[7]

A simple implication of the Uzawa theorem is the following. Interpreting the $A(t)$ in (4.11) as the "level of technology", we have:

COROLLARY Along a BGP with positive gross saving and the technology level, $A(t)$, growing at the rate $g \geq 0$, output grows at the rate $g + n$ while labor productivity, $y \equiv Y/L$, and consumption per unit of labor, $c \equiv C/L$, grow at the rate g.

[5] This way of presenting the intuition behind the Uzawa result draws upon Jones and Scrimgeour (2008).

[6] Many accounts of the Uzawa theorem, including Jones and Scrimgeour (2008), presume a neoclassical production function, but the theorem is much more general.

[7] Part (ii) of Proposition 2 is left out in Acemoglu's book.

Proof That $g_Y = g + n$ follows from (i) of Proposition 2. As to the growth rate of labor productivity we have

$$y_t = \frac{Y(0)e^{g_Y t}}{L(0)e^{nt}} = y(0)e^{(g_Y - n)t} = y(0)e^{gt}.$$

Finally, by Proposition 1, along a BGP with $S > 0$, $c \equiv C/L$ must grow at the same rate as y. $\square$

We shall now consider the implication of Harrod-neutrality for the income shares of capital and labor when the technology is neoclassical and markets are perfectly competitive.

4.4 Harrod-neutrality and the functional income distribution

There is one facet of Kaldor's stylized facts we have so far not related to Harrod-neutral technical progress, namely the long-run "approximate" constancy of both the income share of labor, wL/Y, and the rate of return to capital. At least with neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these properties are inherent in the combination of constant returns to scale, balanced growth, and the assumption that the relative price of capital goods (relative to consumption goods) is constant over time. The latter condition holds in models where the capital good is nothing but non-consumed output, cf. (4.7).[8]

To see this, we start out from a neoclassical CRS production function with Harrod-neutral technological progress,

$$Y(t) = F(K(t), A(t)L(t)). \tag{4.14}$$

With $w(t)$ denoting the real wage at time t, in equilibrium under perfect competition the labor income share will be

$$\frac{w(t)L(t)}{Y(t)} = \frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = \frac{F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)}. \tag{4.15}$$

In this simple model, without natural resources, (gross) capital income equals non-labor income, $Y(t) - w(t)L(t)$. Hence, if $r(t)$ denotes the (net) rate of return to capital at time t, then

$$r(t) = \frac{Y(t) - w(t)L(t) - \delta K(t)}{K(t)}. \tag{4.16}$$

[8] The reader may think of the "corn economy" example in Acemoglu, p. 28.

Denoting the (gross) capital income share by $\alpha(t)$, we can write this $\alpha(t)$ (in equilibrium) in three ways:

$$\begin{aligned}
\alpha(t) &\equiv \frac{Y(t) - w(t)L(t)}{Y(t)} = \frac{(r(t)+\delta)K(t)}{Y(t)}, \\
\alpha(t) &= \frac{F(K(t), A(t)L(t)) - F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)} = \frac{F_1(K(t), A(t)L(t))K(t)}{Y(t)}, \\
\alpha(t) &= \frac{\frac{\partial Y(t)}{\partial K(t)}K(t)}{Y(t)}, \qquad (4.17)
\end{aligned}$$

where the first row comes from (4.16), the second from (4.14) and (4.15), the third from the second together with Euler's theorem.[9] Comparing the first and the last row, we see that in equilibrium

$$\frac{\partial Y(t)}{\partial K(t)} = r(t) + \delta.$$

In this condition we recognize one of the first-order conditions in the representative firm's profit maximization problem under perfect competition, since $r(t) + \delta$ can be seen as the firm's required gross rate of return.[10]

In the absence of uncertainty, the equilibrium real interest rate in the bond market must equal the rate of return on capital, $r(t)$. And $r(t) + \delta$ can then be seen as the firm's cost of disposal over capital per unit of capital per time unit, consisting of interest cost plus capital depreciation.

Proposition 3 *(factor income shares and rate of return under balanced growth) Let the path $(K(t), Y(t), C(t))_{t=0}^{\infty}$ be a BGP in a competitive economy with the production function (4.14) and with positive saving. Then, along the BGP, the $\alpha(t)$ in (4.17) is a constant, $\alpha \in (0,1)$. The labor income share will be $1-\alpha$ and the (net) rate of return on capital will be $r = \alpha q - \delta$, where q is the constant output-capital ratio along the BGP.*

Proof By CRS we have $Y(t) = F(K(t), A(t)L(t)) = A(t)L(t)F(\tilde{k}(t), 1) \equiv A(t)L(t)f(\tilde{k}(t))$. In view of part (i) of Proposition 2, by balanced growth, $Y(t)/K(t)$ is some constant, q. Since $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t)$ and $f'' < 0$, this implies $\tilde{k}(t)$ constant, say equal to $\tilde{k}^*$. But $\partial Y(t)/\partial K(t) = f'(\tilde{k}(t))$, which

[9] From Euler's theorem, $F_1K + F_2AL = F(K, AL)$, when F is homogeneous of degree one.

[10] With natural resources, say land, entering the set of production factors, the formula, (4.16), for the rate of return to capital should be modified by subtracting land rents from the numerator.

then equals the constant $f'(\tilde{k}^*)$ along the BGP. It then follows from (4.17) that $\alpha(t) = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, where $0 < \alpha$ follows from $f' > 0$ and $\alpha < 1$ from the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. Then, by the first equality in (4.17), $w(t)L(t)/Y(t) = 1 - \alpha(t) = 1 - \alpha$. Finally, by (4.16), the (net) rate of return on capital is $r = (1 - w(t)L(t)/Y(t))Y(t)/K(t) - \delta = \alpha q - \delta$. □

This proposition is of interest by displaying a link from balanced growth to constancy of factor income shares and the rate of return, that is, some of the "stylized facts" claimed by Kaldor. Note, however, that although the proposition implies constancy of the income shares and the rate of return, it does not *determine* them, except in terms of α and q. But both q and, generally, α are endogenous and depend on $\tilde{k}^*$,[11] which will generally be unknown as long as we have not specified a theory of saving. This takes us to theories of aggregate saving, for example the simple Ramsey model, cf. Chapter 8 in Acemoglu's book.

4.5 What if technological change is embodied?

In our presentation of technological progress above we have implicitly assumed that all technological change is *disembodied*. And the way the propositions 1, 2, and 3, are formulated assume this.

As noted in Chapter 2, *disembodied technological change* occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again the aggregate dynamic resource constraint (4.7) and the production function (4.9):

$$\dot{K}(t) = I(t) - \delta K(t), \tag{4.18}$$
$$Y(t) = \tilde{F}(K(t), BL(t), t), \qquad \partial\tilde{F}/\partial t > 0. \tag{4.19}$$

Here $Y(t) - C(t)$ is aggregate gross investment, $I(t)$. For a given level of $I(t)$, the resulting amount of new capital goods per time unit $(\dot{K}(t) + \delta K(t))$, measured in efficiency units, is independent of *when* this investment occurs. It is thereby not affected by technological progress. Similarly, the interpretation of $\partial\tilde{F}/\partial t > 0$ in (4.19) is that the higher technology level obtained as time proceeds results in higher productivity of *all* capital and labor. Thus also

[11] As to α, there is of course a trivial exception, namely the case where the production function is Cobb-Douglas and α therefore is a given parameter.

firms that have only old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.[12]

In contrast, we say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.

As also noted in Chapter 2, we may represent embodied technological progress by writing capital accumulation in the following way,

$$\dot{K}(t) = q(t)I(t) - \delta K(t), \tag{4.20}$$

where $I(t)$ is gross investment at time t and $q(t)$ measures the "quality" (productivity) of newly produced investment goods. The increasing level of technology implies increasing $q(t)$ so that a given level of investment gives rise to a greater and greater additions to the capital stock, K, measured in efficiency units. As in our aggregate framework, q capital goods can be produced at the same minimum cost as one consumption good, we have $p \cdot q = 1$, where p is the equilibrium price of capital goods in terms of consumption goods. So embodied technological progress is likely to result in a steady decline in the relative price of capital equipment, a prediction confirmed by the data (see, e.g., Greenwood et al., 1997).

This raises the question how the propositions 1, 2, and 3 fare in the case of embodied technological progress. The answer is that a generalized version of Proposition 1 goes through. Essentially, we only need to replace (4.7) by (13.13) and interpret K in Proposition 1 as the *value* of the capital stock, i.e., we have to replace K by $\tilde{K} = pK$.

But the concept of Harrod-neutrality no longer fits the situation without further elaboration. Hence to obtain analogies to Proposition 2 and Proposition 3 is a more complicated matter. Suffice it to say that with embodied technological progress, the class of production functions that are consistent with balanced growth is smaller than with disembodied technological progress.

[12] In the standard versions of the Solow model and the Ramsey model it is assumed that all technological progress has this form - for no other reason than that this is by far the simplest case to analyze.

4.6 Concluding remarks

In the Solow model as well as in many other models with disembodied technological progress, a steady state and a balanced growth path imply each other. Indeed, they are in that model, as well as many others, two sides of the same process. There *exist* exceptions, however, that is, cases where steady state and a balanced growth are not equivalent (some open economy models and some models with *embodied* technical change). So the two concepts should be held apart.[13]

Note that the definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, *structural change* may occur. That is, a changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009, Chapter 20).

In view of the key importance of Harrod-neutrality, a natural question is: has growth theory uncovered any *endogenous* tendency for technical progress to converge to Harrod-neutrality? Fortunately, in his Chapter 15 Acemoglu outlines a theory about a mechanism entailing such a tendency, the theory of "directed technical change". Jones (2005) suggests an alternative mechanism.

4.7 References

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[13] Here we deviate from Acemoglu, p. 65, where he says that he will use the two terms "interchangingly". We also deviate from Barro and Sala-i-Martin (2004, pp. 33-34) who *define* a steady state as synonymous with a balanced growth path as the latter was defined above.

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