Short Note 3. Economic Growth.

May 9, 2016. Christian Groth

A Schumpeterian model of vertical innovations

This note presents the Schumpeterian model of vertical innovations from Chapter 5.3-4 of Jones and Vollrath (2013). The aim is to give a systematic presentation of the model and to clarify some of the technical issues.[1] The focus is on the *core* of the model, namely the production, inventions, and financing aspects. This core can be combined with alternative models of the household sector. In Section 5 we use the Ramsey-style representative agent description of the household sector.

The new element in the Schumpeterian model compared with the horizontal innovations model is the implication that innovations imply "creative destruction" − the process through which existing businesses and technologies are competed out of the market by new technologies.

We start with an overview of the production sectors.

1 Overview of the production sectors

The economy is closed and has population $L = L_0 e^{nt}$, $n \geq 0$. Labor is homogeneous. Each member of the population supplies one unit of labor per time unit. In contrast to the horizontal innovations model, there is only one type of capital good. But over time, better and better qualities − or "versions" in the terminology of Jones and Vollrath − are invented.

There are three production sectors:

Firms in *Sector 1* produce *final goods* (consumption goods and "raw capital" goods) in the amount $Y(t)$ per time unit, under perfect competition. The final good is the *numeraire.*

[1] The model, as presented in Jones and Vollrath as well as here, is in some respects a simplified version of the contribution by Aghion and Howitt (1992), e.g., by considering only one type of intermediate good. In other respects it is an extension of that contribution, e.g., by considering *durable* capital goods and allowing $\varphi < 1$ and $n > 0$.

In *Sector 2* there is at any point in time only one active firm, the incumbent monopolist. This firm supplies the leading edge quality of the economy's single kind of capital good *on a leasing basis* to firms in Sector 1 under conditions of monopoly and barriers to entry.

Firms in *Sector 3,* the R&D labs, perform R&D to develop *technical designs* ("blueprints") for improved qualities of the capital good under conditions of perfect competition and free entry.

The reason that the inputs from Sector 2 to Sector 1 are by Jones and Vollrath called "intermediate goods" is that they are delivered on a leasing basis. In national-income terminology this makes them "intermediate goods" (in the sense of non-human inputs that cannot be stored). As to the raw capital goods produced in sector 1, it is easiest to imagine that they are *sold* at the price 1 to either the incumbent sector-2 monopolist or to households that then *rent* them out to the incumbent sector-2 monopolist at the capital cost $r + \delta$ per unit of raw capital. To fix ideas, we choose the former interpretation.

There is a labor market and a market for risk-free loans. Both markets have perfect competition. We denote the real wage w_t and the risk-free real interest rate r_t. There is "ideosyncratic" uncertainty (to be defined below). The risk associated with R&D and "creative destruction" can be diversified via the equity-share market because the economy is "large", and there are "many" R&D labs in the economy. All firms are profit maximizers. Time is continuous.

2 The interaction between Sector 1 and Sector 2

2.1 Sector 1: Final goods

The representative firm in Sector 1 has the production function

$$Y(t) = x_i(t)^{\alpha}(A_i L_Y(t))^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{1}$$

where $Y(t)$ is the produced quantity of final goods per time unit at time t, $L_Y(t)$ is labor input, and $x_i(t)$ is input of the currently superior version of the capital good, version i. The version of the capital good that was in use from time $t = 0$ until a new innovation occurred is indexed 0, the version associated with that new innovation is indexed 1, the subsequent version is indexed 2 and so on up to the current version, i. Labor working with version i has efficiency A_i. It is assumed that A_i evolves stepwise from innovation to innovation:

$$A_i = (1+\gamma)A_{i-1}, \qquad \gamma > 0.$$

Observe that γ is the relative increase in A *per step*, not the growth rate of A per time unit. Not only may the number of steps per time unit be generally below one or generally above one, but this number is stochastic (uncertain, governed by a probability distribution). This reflects that the length of the time interval between successive innovations is stochastic.

The output of final goods is used partly for consumption, $C(t) \equiv c(t)L(t)$, partly for investment in raw capital, $I_K(t)$:

$$Y(t) = C(t) + I_K(t) = c(t)L(t) + \dot{K}(t) + \delta K(t), \qquad \delta \geq 0, \quad K(0) > 0 \text{ given}, \tag{2}$$

where $K(t)$ is the stock of raw capital goods in the economy at time t and δ is the capital depreciation rate.

From now on, the explicit dating of the time-dependent variables is omitted unless needed for clarity. With the final good as numeraire we let p_i denote the rental rate per time unit for using one unit of the capital good in its current version i.

Maximizing profit under perfect competition leads to the FOCs:

$$\frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y} = w, \tag{3}$$

$$\frac{\partial Y}{\partial x_i} = \alpha x_i^{\alpha-1} A_i^{1-\alpha} L_Y^{1-\alpha} = p_i. \tag{4}$$

2.2 Sector 2: The currently superior version of the capital good

Let the owner of the exclusive and perpetual[2] right to use technical design i commercially be called firm i. Given the technical design i, firm i can effortless transform raw capital goods into the specific version i simply by pressing a button on a computer, thereby activating a computer code. The following linear transformation rule applies:

it takes $x_i > 0$ units of raw capital to supply x_i units of capital of version i.

The reason that the model assumes that the capital good in version i is *rented out* to the users in Sector 1 is related to the IO problem known as the "durable-goods-monopoly

[2]Recall, patents are assumed to be perpetual – or at least durable enough so that they have not expired before the next innovation turns up.

problem". Selling the capital good implies a transfer of ownership to a durable good and thereby a risk that a second-hand market for the good arises. This could threaten the market power of the monopolist.

The *pure profit* per time unit of firm i obtained by renting out x_i units of the capital good in version i can be written:

$$p_i(x_i)x_i - (r+\delta)x_i - rP_A \equiv \pi_i - rP_A,$$

where $p_i(x_i)$ denotes the maximum price at which the amount x_i can be rented out, r is the risk-free real interest rate, δ is the capital depreciation rate (and so $r+\delta$ is the capital cost per unit of raw capital held),[3] P_A is the market value of the right to use the technical design i, and π_i is the *accounting profit* in the sense of net revenue before subtraction of the imputed interest cost, rP_A. The latter is the opportunity cost of being in this business rather than for instance offering loans in the loan market. This interest cost is a *fixed* cost as long as the entrepreneur remains in the business. So, being in the business, maximizing pure profit is equivalent to maximizing the accounting profit π_i. The quantity x_i (or the price p_i) is thus set so as to maximize

$$\pi_i = p_i(x_i)x_i - (r+\delta)x_i.$$

The profit maximizing p_i $(= p_i(x_i))$ is such that marginal revenue, MR, equals marginal cost, MC :

$$\begin{aligned} MR &= \frac{dTR}{dx_i} = p_i(x_i) + x_i p_i'(x_i) = p_i\left(1 + \mathrm{E}\ell_{x_i}p_i\right) = p_i(1+\alpha-1) = p_i\alpha = MC = r+\delta \\ &\Rightarrow\ p_i = \frac{1}{\alpha}(r+\delta) \equiv p, \end{aligned} \tag{5}$$

where the third equality comes from (4). We observe that the profit maximizing price, p_i, is independent of what rung, i, on the quality ladder has been reached. Hence, we can just denote it p.

Can we be sure that the current technology leader can avoid being undercut by the previous incumbent when charging the monopoly price? No, only if the innovation is *drastic.* By this is meant that the step size, γ, is large enough so that even if the previous incumbent is ready to just charge the marginal cost, $r+\delta$, then she is competed out by the new firm i charging the monopoly price $\frac{1}{\alpha}(r+\delta)$ for supplying the more efficient version of the capital good.

[3] Jones and Vollrath implicitly assume $\delta = 0$ in Section 5.2-4, which is not in harmony with their Section 5.1 and the rest of the book.

To fix ideas, we simplifying assume that all innovations are drastic.[4] The price set by the monopolist is then the monopoly price and the accounting profit is

$$\pi_i = (\frac{1}{\alpha} - 1)(r + \delta)x_i. \tag{6}$$

From now, for simplicity we will refer to this as just the "profit" of firm i.

2.3 Preliminary observations regarding equilibrium

Before going into detail with the R&D sector, it is convenient to combine some elements from Sector 1 and 2 under the assumption of market clearing with perfectly flexible prices.

To supply x_i version-i units of capital, the monopolist in Sector 2 needs x_i units of raw capital. So the demand for raw capital goods is $K^d = x_i$. The supply of raw capital goods is simply the currently available stock of raw capital, i.e., $K^s = K$. For an arbitrary t, we thus have in equilibrium,

$$x_i = K. \tag{7}$$

Substituting this into (1) yields

$$Y = K^\alpha (A_i L_Y)^{1-\alpha}. \tag{8}$$

This is the aggregate production function in Sector 1 in equilibrium at time t where version i represents the leading-edge technology.

Starting with (4), we then have

$$\frac{\partial Y}{\partial x_i} = \alpha x_i^{\alpha-1}(A_i L_Y)^{1-\alpha} = \alpha K^{\alpha-1}(A_i L_Y)^{1-\alpha} = \alpha \frac{Y}{K} = \frac{\partial Y}{\partial K} = p = \frac{1}{\alpha}(r + \delta),$$

where the second equality comes from (7), the third and fourth from (8), and the last from (5) combined with (4). It follows that

$$r + \delta = \alpha^2 \frac{Y}{K} = \alpha \frac{\partial Y}{\partial K} < \frac{\partial Y}{\partial K} = \frac{\partial Y}{\partial x_i}. \tag{9}$$

Reading this from the right to the left, we see that, in equilibrium, the marginal productivity of capital of the currently superior quality, $\partial Y/\partial x_i$, is above the cost, $r+ \delta$, per unit

[4] If an innovation is *nondrastic,* then to discourage the incumbent from staying in the market, the newcommer has to charge a sufficiently low price, the *limit price.* Although this will be below the monopoly price, it will still be above the marginal cost (which is the same as for the incumbent). The reason is the higher efficiency associated with the new technology. So, also in case of non-drastic innovations will "creative" destruction take place.

of raw capital, by a factor $1/\alpha > 1$. This is due to the monopoly pricing of the capital input. Under perfect competition capital would be demanded up to the point where its marginal productivity equals the competitive cost, $r + \delta$, per unit of capital. In contrast, here capital is demanded only up to the point where its marginal productivity equals the capital cost dictated by a capital goods supplier with market power.

Substituting (9) and (7) into (6) gives

$$\pi_i = (\frac{1}{\alpha} - 1)(r + \delta)K = \frac{1-\alpha}{\alpha}\alpha^2 Y = (1-\alpha)\alpha Y \equiv \pi. \tag{10}$$

3 Sector 3: R&D

The model assumes, naturally, that there is uncertainty in R&D. Let t_i be the point in time at which the current leading-edge technology was invented and let t_{i+1} be the unknown future point in time where the next upward jump on the quality ladder takes place. Then the length of the time interval (t_i, t_{i+1}) – the "waiting time" – is a stochastic variable.

3.1 The "research technology"

The R&D process is modelled as an *inhomogeneous Poisson process.*

3.1.1 The single R&D lab

Consider a single R&D lab which is active in the time interval (t_i, t_{i+1}). By definition, *within* this time interval the lab does not face the event of another lab "coming first". Let $\ell_A(t)$ denote the input of R&D labor per time unit at time $t \in (t_i, t_{i+1})$ and let arrival of a "success" mean arrival of the event that the considered lab makes a "viable" invention (by "viable" we mean "not duplicated"). The model then introduces four assumptions:

(i) The *success arrival rate* (per time unit) at time t is $\bar{\eta}(t)\ell_A(t)$, where $\bar{\eta}(t)$ is an economy-wide "research productivity", which by the lab is perceived as exogenous.[5]

This means that the probability of success within a short time interval "from now", conditional on no other labs "coming first", is approximately proportional to the length

[5] To get conformity with notation in the exercises, we have replaced the textbook's $\bar{\mu}$ by $\bar{\eta}$.

of this time interval:

$$P(\text{success}\,|(t,t+\Delta t)) = \bar{\eta}(t)\ell_A(t)\Delta t + o(\Delta t) \approx \bar{\eta}(t)\ell_A(t)\Delta t, \tag{11}$$

where $o(\Delta t)$ is standard symbol for a function, the value of which declines faster than its argument, here Δt, when the latter approaches zero, that is, $\lim_{\Delta t\to 0} o(\Delta t)/\Delta t = 0$. Thereby $\lim_{\Delta t\to 0} P(\text{success}|(t,t+\Delta t))\,/\Delta t = \bar{\eta}(t)\ell_A(t)$. We may say: the difference between $P(\text{success}|(t,t+\Delta t))$ and $\bar{\eta}(t)\ell_A(t)\Delta t$ has "order of magnitude less than Δt".

(ii) There is stochastic independence across time within the time interval (t_i, t_{i+1}).

Digression on Poisson processes If $\bar{\eta}(t)\ell_A(t)$ were a *constant*, equal to $\lambda > 0$, say, then the R&D process would be a *homogeneous Poisson process* with arrival rate λ. With T denoting the waiting time from time t and onward until a success arrives, then, again conditional on no other labs "coming first", the probability that T exceeds $\tau > 0$ would be $P(T > \tau) = e^{-\lambda\tau}$. Moreover, assuming the lab, in case of success, continues researching for yet another quality improvement, the number, m, of success arrivals within a time interval of length Δt would follow a *Poisson distribution,* that is,

$$P(m = a\,|(t,t+\Delta t)) = e^{-\lambda\Delta t}(\lambda\Delta t)^a/a!, \tag{*}$$

where $a = 0, 1, 2, \ldots$, and $a! \equiv a \cdot (a-1)\cdot(a-2)\cdot\cdots\cdot 1$, $0! = 1$. The expectation of m is $\lambda\Delta t$, and the variance is the same.

In the present model, however, both $\bar{\eta}(t)$ and $\ell_A(t)$ will generally be time dependent. The R&D process is assumed to be an *inhomogeneous Poisson process* with arrival rate $\lambda(t) = \bar{\eta}(t)\ell_A(t)$. This means that the probability of the event $m = a$ is as in (*) except that $\lambda\Delta t$ should be replaced by $\int_t^{t+\Delta t}\lambda(s)ds$. If Δt is "small", the expectation of m thus equals

$$\int_t^{t+\Delta t}\lambda(s)ds \approx \lambda(t)\Delta t, \tag{**}$$

and the same holds true for the variance. □

In accordance with (*) and (**), under the assumption that the lab continues its research throughout the time interval $(t, t+\Delta t)$, the expected number of success arrivals is

$$E(m\,|\bar{\eta}(t)\ell_A(t), (t,t+\Delta t)\,) \approx \bar{\eta}(t)\ell_A(t)\Delta t.$$

In case $\bar{\eta}(t)\ell_A(t)$ were a *constant* during the considered time interval, we could replace the "$\approx$" by "$=$".

Before proceeding, a reservation seems appropriate. The assumption (ii) is a kind of "no memory" assumption since it ignores learning over time within the lab. This seems problematic. Indeed, R&D should be considered a cumulative process. The only excuse for assumption (ii) is the need for simplicity in a first approach.

3.1.2 Aggregate R&D and the evolution of technology

The third assumption concerning R&D deals with the economy-wide R&D where many labs are involved:

(iii) Research outcomes are stochastically independent across R&D labs.

Let $L_A(t)$ denote the aggregate input of research labor at time t, i.e., $L_A(t) \equiv \sum \ell_A(t)$. We then have

$$P(\text{success}\,|(t, t+\Delta t)) \approx \bar{\eta}(t)L_A(t)\Delta t. \tag{12}$$

Thus, with $M(t)$ denoting the aggregate number of success arrivals in the time interval $(t, t+1)$, and letting our time unit be "small", the following approximation holds for the expected aggregate number of success arrivals over the time interval $(t, t+1)$ is:

$$E(M(t)\,|\bar{\eta}(t)L_A(t), (t, t+1)\,) \approx \bar{\eta}(t)L_A(t). \tag{13}$$

Finally, the fourth assumption is about how economy-wide "research productivity" is determined:

(iv) $\bar{\eta}(t) = \eta A_i^{\varphi-1}L_A(t)^{-\xi}$, $\eta > 0, \varphi \leq 1, 0 \leq \xi < 1$. Here $\varphi - 1$ is the elasticity of research productivity w.r.t. the accumulated "stock of knowledge" at time t, measured by A_i, and ξ is the degree of R&D overlap in the economy.[6] There are "many" labs in the economy, and the individual labs rightly perceive their influence on $\bar{\eta}(t)$ to be negligible.

The only uncertainty assumed present in the economy is the uncertainty related to research outcomes in the individual labs. According to Assumption (iii) these research

[6] The motivation for choosing the exponent on A to be $\varphi - 1$ rather than just φ, as in the horizontal innovations model, is that each innovation in the present model generates a rise in A that is proportionate to A and thus becomes larger and larger.

We have replaced the textbook's θ by η in order to reserve θ to denote a preference parameter when specifying the household sector in the model. We have further replaced the textbook's $1 - \lambda$ by ξ.

outcomes are stochastically independent across labs.[7] In economists' jargon, uncertainty is thus "idiosyncratic", allowing investors to reduce their risk by diversification, as we shall see below.

Before detailing that aspect, some observations about the aggregate research outcome *per time unit* are pertinent. Let $A(t)$ indicate the labour efficiency associated with the leading-edge technology at time t. Thus, in the present situation $A(t) = A_i$. With $M(t)$ success arrivals in the time interval $(t, t+1)$, we then have

$$\begin{aligned}
A(t+1) &= A(t)(1+\gamma)^{M(t)} \Rightarrow \\
\ln A(t+1) &= \ln A(t) + M(t)\ln(1+\gamma) \Rightarrow \\
\frac{A(t+1)-A(t)}{A(t)} &\approx \ln A(t+1) - \ln A(t) = M(t)\ln(1+\gamma) \Rightarrow \\
E_t\frac{A(t+1)-A(t)}{A(t)} &= E_t(M(t))\ln(1+\gamma) \approx \bar{\eta}(t)L_A(t)\ln(1+\gamma), \qquad (14)
\end{aligned}$$

where E_t is the expectation operator conditional on the current Poisson arrival rate $\bar{\eta}(t)L_A(t)$, and $E_tM(t)$ is a shorthand for the left-hand side of (13). Besides, "$\approx$" in the last line follows from the approximation in (13). (One should not here introduce γ as an approximation to $\log(1+\gamma)$ because that would require γ to be "small" which need not be true here. Imagine for instance that the focus is on a series of "big" communication innovations: electrical telegraphs, telephone, cell phone, internet, Skype. The time elapsed between the innovations may be many years, but each new innovation is "large".)

3.2 The economics of R&D

3.2.1 Demand for R&D labor

As noted in Section 2.1, P_A is the market value of the right to use the technical design i corresponding to innovation i. In other words, P_A is the market value of a successful research outcome. Let us consider the situation from the point of view of an R&D lab which is active in the time interval between innovation $i-1$ and innovation i. The lab's demand for R&D labor is

$$\ell_A^d = \begin{cases} \infty \text{ if } w < P_A\bar{\eta}, \\ \text{undetermined if } w = P_A\bar{\eta}, \\ 0 \text{ if } w > P_A\bar{\eta}. \end{cases} \qquad (15)$$

[7] There are no economy-wide risk factors in the model (say earthquakes, economic recession, shocks to terms of trade).

Here $P_A\bar{\eta}$ can be viewed as the value of the expected payoff per worker per time unit = value of "marginal product" of R&D labor = $P_A\partial(\bar{\eta}L_A)/\partial L_A$ from (13). For the lab to be willing to hire R&D workers, we must have $w \leq P_A\partial(\bar{\eta}L_A)/\partial L_A = P_A\bar{\eta}$, *if* R&D firms behave in a risk-neutral manner. As the next sub-section will argue, that is what they will do.

3.2.2 The financing of R&D

There is a time lag of *random* length between a research lab's outlay on R&D and the arrival of a successful research outcome, an invention. During this period, which in principle has no upper bound, the individual R&D lab is incurring sunk costs and has no revenue at all. R&D is thus risky, and continuous refinancing is needed until the research is successful.

However, since the uncertainty is "ideosyncratic", and the economy is "large" and has "many" R&D labs, the risk can be diversified. R&D labs as well as the monopolist in Sector 2 can behave in a risk-neutral manner. In equilibrium all investors will receive a rate of return equal to the risk-free interest rate.

The easiest approach to the financing issue is to assume that R&D labs finance their current expense, wL_A, by issuing equity shares that pay no dividend until success arrives. A part of households' saving is via mutual funds (that are assumed to have no administration costs) channeled to the many different R&D labs. When success arrives, the mutual funds collect a return which can take two alternative forms. Either the return is in the form of a share of the sales price, P_A, of the patent (which the successful lab receives free of charge). Or the return is in the form of shares in the profit, π, if the R&D lab decides itself to enter Sector 2 and supply the new version of the capital good services as a monopolist. For simplicity we assume that the mutual funds manage the *total* household saving and thus allocate only a part of it to R&D. The remaining part is used to buy equity shares issued by the incumbent monopolist to finance the purchases of raw capital goods in the market for these. Finally, the mutual funds pay out to their risk-averse investors, the households, a rate of return equal to the risk-free rate of interest.

3.2.3 No-arbitrage condition regarding P_A

How is under these (idealized) conditions the market value, $P_A(t)$, of a patent at time t determined in equilibrium? In view of the risk-neutral behavior by the participants in the financial markets, equilibrium requires that $P_A(t)$ satisfies the no-arbitrage condition

$$P_A(t)r(t) = \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t). \tag{16}$$

Here $\dot{P}_A(t)$ ($\equiv dP_A(t)/dt$) is the incumbent monopolist's expected capital gain per time unit *conditional* on the monopoly position remaining in place also in the next moment. The alternative possible situation is that the monopoly position is lost due to the arrival of an innovating firm with a more productive version of the capital good. In that case the total value $P_A(t)$ is lost.

The whole right-hand side of (16) indicates the expected return per time unit on holding the patent instead of selling it and investing in the loan market. To understand this, consider a small time interval $(t, t+\Delta t)$. As seen from time t, two outcomes are possible. Either the monopoly position, and hence $P_A(t)$, is lost. According to (12), the probability of that event is approximately $\bar{\eta}(t)L_A(t)\Delta t$. Alternatively, the incumbent's monopoly remains in place over the time interval, in which case the total revenue is $(\pi(t) + \dot{P}_A(t))\Delta t$. The probability of that event is approximately $1 - \bar{\eta}(t)L_A(t)\Delta t$.

Consequently, if $z(t)$ denotes the total return per time unit on holding the patent instead of selling it, the expected return over the time interval $(t, t+\Delta t)$ is approximately

$$\begin{aligned} E_t(z(t)\Delta t) &\approx \bar{\eta}(t)L_A(t)\Delta t(-P_A(t)) + (1 - \bar{\eta}(t)L_A(t)\Delta t)\left[\pi(t) + \dot{P}_A(t)\right]\Delta t \\ &= \left[\pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t)\right]\Delta t - \bar{\eta}(t)L_A(t)\left[\pi(t) + \dot{P}_A(t)\right](\Delta t)^2. \end{aligned}$$

Dividing through by Δt, we get

$$\begin{aligned} \frac{E_t(z(t)\Delta t)}{\Delta t} &= E_t(z(t)) = \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t) - \bar{\eta}(t)L_A(t)\left[\pi(t) + \dot{P}_A(t)\right]\Delta t \\ &\rightarrow \pi(t) + \dot{P}_A(t) - \bar{\eta}(t)L_A(t)P_A(t) \quad \text{for} \quad \Delta t \rightarrow 0. \end{aligned}$$

Thus, the right-hand side of (16) does indeed represent the expected return per time unit on holding the patent instead of selling it. And the left-hand side of (16) is the return obtained by selling the patent and investing in a safe loan market. Under risk neutrality, for given expectations, the market price $P_A(t)$ adjusts so as to equalize the two sides of (16).

The no-arbitrage condition (16) plays a key role in the determination of the risk-free interest rate in general equilibrium, cf. point (v) of Lemma 1 below. Before proceeding, for purposes of intuition, it may be useful to consider the no-arbitrage condition from additional angles.

We may rewrite the no-arbitrage condition (16) in "required rate of return" form:

$$\frac{\pi(t) + \dot{P}_A(t)}{P_A(t)} = r(t) + \bar{\eta}(t)L_A(t). \tag{17}$$

Here, the instantaneous *conditional* rate of return per time unit on shares in the monopoly firm is equalized to the "required rate of return" in the sense of the minimum expected rate of return justifying staying in the Sector-2 business. This minimum rate of return is the sum of the risk-free interest rate and a premium reflecting the risk that the monopoly position expires within the next instant.

Yet another useful way of thinking about the no-arbitrage condition is in the form of the present value of expected future accounting profits:

$$P_A(t) = \int_t^\infty \pi(s) e^{-\int_t^s (r(\tau)+\bar{\eta}(\tau)L_A(\tau))d\tau} ds. \tag{18}$$

The right-hand side here makes up the *fundamental value* of the patent at time t, given the expected future risk-adjusted interest rates, $r(\tau) + \bar{\eta}(\tau)L_A(\tau)$. Indeed, (16) can be considered a differential equation for the function $P_A(t)$. The solution to this differential equation, presupposing that there are no bubbles, *is* (18) (the proof is similar to that in the appendix of Short Note 2). The convenience of (18) is that, given the expected future accounting profits and risk-adjusted interest rates, the formula directly tells us the market value of the incumbent monopolist's patent. If, for instance, π grows at a constant rate g_π, and r and $\bar{\eta}L_A$ are constant, then (18) can be written

$$\begin{aligned} P_A(t) &= \int_t^\infty \pi(t) e^{g_\pi(s-t)} e^{-(r+\bar{\eta}L_A)(s-t)} ds = \pi(t) \int_t^\infty e^{-(r+\bar{\eta}L_A-g_\pi)(s-t)} ds \\ &= \pi(t) \frac{1}{r + \bar{\eta}L_A - g_\pi}. \end{aligned} \tag{19}$$

This present-value formula is useful for intuitive interpretation of effects of everything-else-equal shifts in the interest rate, r, in the expected number of innovations per time unit, $\bar{\eta}L_A$, and in the growth rate of the profit:

$$\begin{aligned} r \ \uparrow \ &\Rightarrow P_A(t) \downarrow \text{ due to stronger discounting,} \\ \bar{\eta}L_A \ \uparrow \ &\Rightarrow P_A(t) \downarrow \text{ due to lower expected duration of monopoly,} \\ g_\pi \ \uparrow \ &\Rightarrow P_A(t) \uparrow \text{ because investors like fast-growing dividends.} \end{aligned}$$

A final comment: We have throughout presumed that a new technological breakthrough means that the monopoly position of the incumbent is lost. Could the incumbent not bid for the patent offered to the market by the successful R&D lab? Yes, it could. But new potential entrepreneurs will always (in this model) be willing to bid *more.* The incumbent faces the problem that the gain by investing in the new technology is partly destroyed since she looses the existing profits earned. This point is known as *Arrow's replacement effect* (Arrow, 1962).

4 Equilibrium in the labor market

The labor market is competitive. There is an inelastic labor supply of size $L = L_0 e^{nt}$. Equilibrium in the labor market thus requires that

$$L_Y + L_A = L = L_0 e^{nt}.$$

In equilibrium with *active* R&D ($L_A > 0$), we must have

$$w = P_A \bar{\eta},$$

in view of (15). Since labor is homogeneous, the equilibrium wage, w, must also equal marginal productivity of labor in Sector 1 at full employment:

$$w = \frac{\partial Y}{\partial L_Y} = (1-\alpha)\frac{Y}{L_Y}.$$

Combining the two last equations gives

$$P_A \bar{\eta} \equiv P_A \eta A^{\varphi - 1} L_A^{-\xi} = (1-\alpha)\frac{Y}{L_Y}. \tag{20}$$

5 Balanced growth

In the non-stochastic Romer-Jones model of horizontal innovations with Ramsey households, cf. Short Note 2, we have, under certain parameter restrictions that in the long run, the system converges to a BGP with the property that $g_y = g_c = g_k = g_A = constant$ > 0. In analogy with this, we may think of the present model as portraying a system which, in a stochastic sense, in the long run approaches a path with the property that the *average* growth rates of $y, c, k,$ and A over long time horizons are both constant and equal:

$$Eg_y = Eg_c = Eg_k = Eg_A = constant > 0. \tag{21}$$

The constancy of Eg_A means that on average there is exponential growth in input quality. This corresponds to "Moore's law": the observation that, over the history of computing hardware, the efficiency of microprocessors has approximately doubled every two years.[8] Indeed, a constant doubling time is equivalent to exponential growth. But a *two* years' doubling time is, of course, much *faster* exponential growth than what we see anywhere regarding productivity at a more aggregate level.[9]

5.1 An approximating deterministic BGP

We now take a bird's eye view and look at the long-run evolution *as if* the level of labor efficiency, A, evolves in a "smooth" deterministic way as a function of time and has actual growth rate, $g_A \equiv dA(t)/dt$, equal to the expected constant long-run growth rate, Eg_A.[10] By (14) we see that this amounts to

$$g_A = Eg_A = \bar{\eta}(t) L_A(t) \ln(1+\gamma) = \eta A(t)^{\varphi-1} L_A(t)^{1-\xi} \ln(1+\gamma) = \textit{constant} > 0, \qquad (22)$$

where the last equality comes from Assumption (iv) (with $A_i = A(t)$) in Section 3.1.2 about how $\bar{\eta}(t)$ is determined.

When the evolution of A is "smooth", so is that of y, c, and k. In the present context we define an "approximating deterministic BGP" as a deterministic path along which

$$g_y = g_c = g_k = g_A, \qquad (23)$$

where g_A is constant and satisfies (22). It is well-known that if $\varphi = 1$ and $n > 0$ or $\varphi < 1$ and $n = 0$, no deterministic BGP can exist (in the first case because the growth rates will continue to be rising over time, in the latter case because the needed sustained growth in L_A to compensate for the declining $A^{\varphi-1}$ will be absent). In the following lemma we therefore only need consider the combinations $\varphi = 1$ together with $n = 0$ and $\varphi < 1$ together with $n > 0$.

LEMMA 1. Let $\varphi \leq 1, n \geq 0, \eta > 0, 0 \leq \xi < 1$. Consider an approximating deterministic BGP. Let the associated g_A have the value $g_A^* > 0$. It holds that:

[8] Gordon E. Moore was co-founder of the micro-electronics industry firm Intel in the late 1960s.

[9] Two years' doubling time is equivalent to a constant growth rate of 35 percent per year ($g = (\ln 2)/2 = 0.35$).

[10] Given the time unit, say one year, and given the proportionate size, γ, of the step increases, this "even out" of the growth path of A seems more acceptable, the "larger" is A (the denominator in the calculation of the growth rate), and the more frequent are the step increases, cf. the law of large numbers.

(i) If $\varphi = 1$ and $n = 0$, then $L_A(t) = L_A$, a positive constant, $\bar{\eta} = \eta$, and $g_A^* = \eta L_A^{1-\xi} \ln(1+\gamma)$.

(ii) If $\varphi < 1$ and $n > 0$, then $g_{L_Y} = g_{L_A} = n$ and $g_A^* = \frac{1-\xi}{1-\varphi} n$.

(iii) $\bar{\eta} L_A = g_A^* / \ln(1+\gamma)$.

(iv) $g_{P_A} = g_Y = g_A^* + n = g_\pi$.

(v) $r = \alpha \bar{\eta} L_Y - (1 - \ln(1+\gamma))\bar{\eta} L_A + n$.

Proof. (i) Apply (22). (ii) That $g_{L_Y} = g_{L_A} = n$ follows by the same reasoning as in Short Note 2, Section 5.2. As to g_A^*, "take logs and time derivatives" in (22) and then solve for g_A. (iii) In (22), let $g_A = g_A^*$, and solve for $\bar{\eta} L_A$. (iv) Multiplying through by L_A in (20) gives

$$P_A \bar{\eta} L_A = (1-\alpha) Y \frac{L_A}{L_Y}, \tag{24}$$

where, along the BGP, by (iii), $\bar{\eta} L_A$ is constant, and, by (i) and (ii), so is L_A / L_Y. Hence, $g_{P_A} = g_Y = g_\pi$, where the last equality comes from (10). Moreover, $Y = yL$ so that $g_Y = g_y + n = g_A^* + n$, where the last equality follows from (23) in combination with $g_A = g_A^*$. (v) From the no-arbitrage condition (16), we have along the BGP that $r = (\pi + \dot{P}_A)/P_A - \bar{\eta} L_A = \alpha \bar{\eta} L_Y + g_A^* + n - \bar{\eta} L_A$, by (10), (20), (iii), and (iv). □

5.2 The representative household

To determine L_A and g_A along the BGP, we need more knowledge of the real interest rate, which in turn requires taking household behavior into account.

As in connection with the horizontal innovations model in Short Note 2, we assume a representative household with infinite horizon, rate of time preference equal to ρ, and CRRA instantaneous utility with parameter $\theta > 0$. The household's per head consumption will thus satisfy the Keynes-Ramsey rule

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{25}$$

and the per head financial wealth, $a(t)$, of the household will satisfy the transversality condition

$$\lim_{t \to \infty} a(t) e^{-\int_0^t (r(s)-n) ds} = 0. \tag{26}$$

Per head financial wealth is

$$a(t) \equiv \frac{K(t) + P_A(t)}{L(t)} = \frac{(K(0) + P_A(0)) e^{(g_A^* + n)t}}{L(0) e^{nt}} = \frac{(K(0) + P_A(0)) e^{g_A^* t}}{L(0)} \equiv a(0) e^{g_A^* t} \tag{27}$$

along the BGP, in view of (23) and (iv) of Lemma 1.

Now to the final solution of the model, along the approximating deterministic BGP. Here we have to distinguish between two alternative cases, the fully-endogenous growth case versus the semi-endogenous growth case.

5.3 The fully-endogenous growth case: $\varphi = 1$ and $n = 0$

This is the case studied in the pioneering article by Aghion and Howitt (1992). For simplicity we ignore the duplication externality and set $\xi = 0$ (as also Aghion and Howitt do).

As in Short Note 2 on the horizontal innovations model, a first step in the analysis is to pin down the relationship between the interest rate and the constant employment in R&D along the approximating deterministic BGP, as this relationship is manifested at the production side. From (v) of Lemma 1, with $n = 0$, we have along the BGP,

$$r = \alpha\eta L - \alpha\eta L_A + (\ln(1+\gamma) - 1)\eta L_A, \qquad \text{(by (i) and (iii) of Lemma 1)} \tag{28}$$

where, in addition to (v) and (i) from the lemma, we have used that $L_Y + L_A = L$.

The second step in the analysis is to pin down a second relationship between the interest rate and the constant employment in R&D, this time involving households' behavior. Isolating r in (25) along a BGP immediately gives

$$r = \rho + \theta g_c^* = \rho + \theta g_A^* = \rho + \theta\eta L_A \ln(1+\gamma), \tag{29}$$

where the second equality comes from (23) and the third from (iii) of Lemma 1; an asterisk signifies that a value along the BGP is considered. Equalizing the right-hand sides of (28) and (29) and rearranging gives

$$L_A = \frac{\alpha\eta L - \rho}{[(\theta - 1)\ln(1+\gamma) + 1 + \alpha]\,\eta} \equiv L_A^*. \tag{30}$$

By (i) of Lemma 1, with $\xi = 0$, we finally get

$$g_A^* = \eta L_A^* \ln(1+\gamma) = \frac{(\alpha\eta L - \rho)\ln(1+\gamma)}{(\theta - 1)\ln(1+\gamma) + 1 + \alpha}. \tag{31}$$

These results have been derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A is positive.

To ensure that the transversality condition (26) with $n = 0$, in combination with (27), holds along the BGP, we need the assumption that $\rho > (1-\theta)g_A^*$. Inserting (31), and rearranging, gives the requirement

$$\rho > \frac{(1-\theta)\alpha\eta L}{1+\alpha}\ln(1+\gamma). \tag{A1-f}$$

To ensure $L_A^* > 0$, we assume

$$\rho < \alpha\eta L \quad \text{and, if } \theta < 1, \text{ then } 0 < \gamma \leq e - 1. \tag{A2}$$

Empirics generally find $\theta \geq 1$.

Imposing both (A1-f) and (A2) in present case where $\varphi = 1$ and $n = 0 = \xi$, there is a meaningful BGP solution to the model. The solution features "fully endogenous" exponential growth. This per capita growth is generated by an internal mechanism, through which labor is allocated to R&D. And the exponential per capita growth is maintained without support of growth in any exogenous factor.

Among other things, one can make comparative static analysis on the result in (31). For instance, not surprisingly, $\partial g_A^*/\partial\rho < 0$, $\partial g_A^*/\partial\theta < 0$, and $\partial g_A^*/\partial\eta > 0$, $\partial g_A^*/\partial\gamma > 0$.

We also see that $\partial g_A^*/\partial L > 0$. The "fully endogenous" growth case thus implies a *scale effect on growth,* which is an empirically problematic feature.

5.4 The semi-endogenous growth case: $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$

The order in which we find g_A^* and L_A^*/L is now reversed. The growth rate of A along the approximating deterministic BGP was found already in (ii) of Lemma 1, which displays the standard semi-endogenous growth result emphasized by Jones. We repeat the result here:

$$g_A^* = \frac{(1-\xi)n}{1-\varphi} > 0, \tag{32}$$

as $n > 0$.

Contrary to the fully-endogenous growth case, here the relative step increase, γ, does *not* affect the expected growth rate of A. This is due to Assumption (iv) in Section 3.1.2 about how the economy-wide research productivity, $\bar{\eta}$, is determined. In view of the exponent $\varphi - 1$ on A_i being negative when $\varphi < 1$, in Assumption (iv), a larger γ implies that the upward jumps in A reduce the economy-wide research productivity, $\bar{\eta}$, by more than otherwise. In the long run this means a larger expected waiting time before the next technological breakthrough.

It remains to solve for the fraction of labor in research, $s_R \equiv L_A/L$, along the approximating deterministic BGP. The solution for s_R is important for the analysis of how the *level* of y and c along the BGP depends on parameters and economic policy. From (v) of Lemma 1,

$$\begin{aligned}
r - n + (1 - \ln(1+\gamma))\bar{\eta}L_A &= \alpha\bar{\eta}L_Y \Rightarrow \frac{r-n}{\alpha\bar{\eta}L_A} + \frac{1-\ln(1+\gamma)}{\alpha} = \frac{L_Y}{L_A} \Rightarrow \\
\frac{L_A/L}{L_Y/L} &\equiv \frac{s_R}{1-s_R} = \frac{1}{\frac{r-n}{\alpha\bar{\eta}L_A} + \frac{1-\ln(1+\gamma)}{\alpha}} \Rightarrow \\
s_R &= \frac{1}{1 + \frac{r-n}{\alpha\bar{\eta}L_A} + \frac{1-\ln(1+\gamma)}{\alpha}}. \qquad (33)
\end{aligned}$$

This result is essentially the same as (5.33) in Jones and Vollrath, since they have $\mu \equiv \bar{\eta}L_A$ and implicitly use the "approximation" $\gamma \approx \ln(1+\gamma)$ (which we have avoided because γ may be "large" as argued at the end of Section 3.1). Anyway, the result is only a step towards a solution because both $\mu \equiv \bar{\eta}L_A$ and r are endogenous variables in the general equilibrium of the model. Fortunately, however, we have (iii) of Lemma 1, so that (33) can be written

$$s_R = \frac{1}{1 + \ln(1+\gamma)\frac{r-n}{\alpha g_A^*} + \frac{1-\ln(1+\gamma)}{\alpha}}. \qquad (34)$$

Given our household description, along the approximating deterministic BGP, r must equal $\rho + \theta g_A^*$, which, inserted into (34), gives the final solution for s_R :

$$s_R = \frac{1}{1 + \frac{1}{\alpha}\Big(\ln(1+\gamma)(\frac{\rho-n}{g_A^*} + \theta - 1) + 1\Big)} \equiv s_R^*, \qquad (35)$$

where (32) can be inserted.

To ensure that the transversality condition (26), in combination with (27), holds along the BGP, we need the same parameter restriction as in the "fully-endogenous growth" case above and in the horizontal innovations model of Short Note 2, namely that

$$\rho - n > (1-\theta)g_A^*, \qquad \text{(A1-s)}$$

with g_A^* given by (32). Moreover, with this parameter restriction we automatically have $\rho + \theta g_A^*$ $(= r^*) > \rho$ which, according to the Keynes-Ramsey rule, is needed for $g_c^* > 0$ to be an outcome in balanced growth. In addition, given $g_A^* > 0$, (A1-s) is equivalent to the factor $((\rho - n)/g_A^* + \theta - 1)$ in (35) being positive.

On the basis of the formula (35), long-run level effects on s_R^* of different parameter shifts can be studied. The roles of the parameters ρ, θ, n, φ, and ξ are qualitatively similar to their roles in the horizontal innovations model. A new feature compared with the horizontal innovations model is the appearance of the relative step increase, γ, in the formula $-$ and with a *negative* effect on the equilibrium allocation of labor to R&D. The explanation is related to that of the absence of an effect on g_A^* from γ given above.

Like in the horizontal innovations model (cf. Exercise VII.7), level effects on $y^*(t)$ and $c^*(t)$ of parameter shifts are a bit more complicated than the level effects on s_R^*. Indeed, a shift in s_R has ambiguous effects on both $y^*(t)$ and $c^*(t)$ along a BGP.

6 Concluding remarks

In extended versions of the Schumpeterian model, there are many different types of capital goods. Each of these types are produced in its own product line represented by a point on a horizontal axis. For each of these points there is then a vertical "quality ladder" along which the quality improvements of each capital good type take place, based on new technical designs developed in corresponding specific subsets of R&D labs. Overall labor efficiency, A, then becomes an average of the leading-edge qualities in the different product lines. As an implication of this "averaging" across *many* product lines, it is common in the literature to completely "smooth out" the evolution of A and, appealing to the law of large numbers, assume away any uncertainty at the aggregate level. Thereby, a deterministic streamlined description of the economy, with $g_A = Eg_A$ at the aggregate level, is upheld.

Obviously, the present model is in many respects very abstract. For instance, it does not consider the mutual relationship between private R&D and the evolution of basic science and higher education at universities.

Another limitation is the simplifying assumption that the innovator has *perpetual* monopoly over the production and sale of the new version of the capital good. In practice, by legislation, patents are of limited duration, 15-20 years. Moreover, it may be difficult to codify exactly the technical aspects of innovations, hence not even within such a limited period do patents give 100% effective protection. While the pharmaceutical industry rely quite much on patents, in many other branches innovative firms use other protection strategies such as *concealment* of the new technical design. In ICT industries copyright to new software plays a significant role. Still, whatever the protection strategy used, imitators sooner or later find out how to make very close substitutes.

To better accommodate such facts, models have been developed where the duration of monopoly power over the commercial use of an invention is *limited* and *uncertain*. For instance, Barro and Sala-i-Martin (20014, Ch. 6.2) present a model with stochastic erosion of the innovator's monopoly power. The model exposes the policy dilemma regarding the design of patents. Both *static* and *dynamic* distortions are involved. Compared with perpetual monopoly, shorter duration of patents *mitigates* the *static inefficiency problem* arising from prices above marginal cost. Shorter duration of patents also make it easier and less expensive to build on previous discoveries. On the other hand, there is the problem that shorter duration of patents may *aggravate* the *dynamic distortion* deriving from the "surplus appropriability problem" illustrated in Jones and Vollrath, p. 134: there may be too little private incentive to invest in R&D.

At the empirical level, Jones and Williams (1998) estimate that R&D investment in the U.S. economy is only about a fourth of the social optimum. So government intervention seems motivated. But how should it be done? According to Paul Romer (2000) it may be a better growth policy strategy to support education in science and engineering than to support specific R&D activities.

There are many further aspects to take into account, e.g., spill-over effects of R&D and intensional knowledge sharing, which we shall not consider here. A survey is contained in Hall and Harhoff (2012). We end this Short Note by a citation from Wikipedia (07-05-2015):

> Legal scholars, economists, scientists, engineers, activists, policymakers, industries, and trade organizations have held differing views on patents and engaged in contentious debates on the subject. Recent criticisms primarily from the scientific community focus on the core tenet of the intended utility of patents, as now some argue they are retarding innovation. Critical perspectives emerged in the nineteenth century, and recent debates have discussed the merits and faults of software patents, nanotechnology patents, and biological patents. These debates are part of a larger discourse on intellectual property protection which also reflects differing perspectives on copyright.

7 Literature

Aghion, P., and P. Howitt, 1992, A model of growth through creative destruction, *Econometrica,* vol. 60, 323-351.

Aghion, P., and P. Howitt, 1998, *Endogenous Growth Theory*, MIT Press.

Aghion, P., and P. Howitt, 2009, *The Economics of Growth*, MIT Press.

Arrow, K. J., 1962,

Barro, R. J., and X. Sala-i-Martin, 2004, *Economic Growth,* 2nd edition, MIT Press.

Eaton, J., E. Gutierrez, and S. Kortum, 1998, European Research Policy, *Economic Policy,* vol. 27, 405-438.

Eaton, J., and S. Kortum, 1999, International technology diffusion: Theory and measurement, *International Economic Review*, vol. 40 (3), 537-570.

Hagedorn, J., and J. Schakenraad, 1992, Leading companies and networks of strategic alliances in information technologies, *Research Policy*, vol. 21, 163-190.

Hall and Harhoff, 2012,

Katz, M. L., and J. A. Ordover, 1990, R&D cooperation and competition, BPEA, vol. 3, 137-2004.

Klemperer, P., 1995, Competition when consumers have switching costs: An overview and applications to industrial organization, macroeconomics, and international trade, RES, vol. 62, 515-539.

Reinganum, J. F., 1989, The timing of innovation: Research, development, and diffusion. In: R. Schmalensee and R.D. Willig, eds., *The Handbook of Industrial Organization*, N.Y.: North-Holland.

Romer, P., 2000, Should the government subsidize supply or demand in the market for scientists and engineers?, NBER WP No. 7723.

V. W. Ruttan, *Technology, Growth, and Development: An Induced Innovation Perspective*, Oxford University Press, 2001.