Short Note 2. Economic Growth.

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The Romer-Jones horizontal innovations model

Below is a compact version of the Romer-Jones model of horizontal innovations in a closed industrialized economy. In contrast to Jones and Vollrath, Ch. 5.1-2, we specify the household sector to be of Ramsey type. There is no uncertainty and households have perfect foresight. The text is not meant to be a substitute to Jones and Vollrath's Ch. 5.1-2, but a complement to be read *after* Jones and Vollrath's introduction has been read. The aim is to give a systematic overview and to clarify some of the more technical issues. Our notation is as in exercises VII.10 - VII.14, thereby only in a few respects deviating from that in Jones and Vollrath.

1 The household sector

There is a fixed number of infinitely-lived households, all alike. Each household has $L(t) = L(0)e^{nt}$ members, $n \geq 0$, and each member supplies inelastically one unit of labor per time unit. We normalize the number of households to be one. Given $\theta > 0$ and $\rho > 0$, the representative household's problem is to choose a plan $(c(t))_{t=0}^{\infty}$ so as to

$$\begin{aligned}
\max U_0 &= \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \qquad (*)\\
c(t) &\geq 0,\\
\dot{a}(t) &= (r(t)-n)a(t) + w(t) - c(t), \qquad a(0) \text{ given},\\
\lim_{t\to\infty} a(t)e^{-\int_0^t (r(s)-n)ds} &\geq 0. \qquad \text{(NPG)}
\end{aligned}$$

Here $r(t)$ is the risk-free interest rate, and $a(t)$ is per capita financial wealth, which can be placed in "raw capital" or perpetual patents, as described below.

The solution to the problem (*) is given by the Keynes-Ramsey rule,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{1}$$

and the transversality condition,

$$\lim_{t\to\infty} a(t)e^{-\int_0^t (r(s)-n)ds} = 0. \tag{2}$$

This follows from applying Pontryagin's Maximum Principle to the problem.

2 The production side of the economy

There are three production sectors:

Firms in *Sector 1* produce *final goods* (consumption goods and "raw capital" goods) in the amount $Y(t)$ per time unit, under perfect competition. The final good is the numeraire.

Firms in *Sector 2* supply *specialized capital goods,* indexed by $j = 1, 2, \ldots, A(t)$. These specialized capital goods are *rented out* to firms in Sector 1, under conditions of monopolistic competition and barriers to entry. Like Jones and Vollrath, we sometimes refer to these specialized capital good services as "intermediate goods".[1]

Firms in *Sector 3* perform R&D to develop *technical designs* ("blueprints") for new specialized capital goods under conditions of perfect competition and free entry.

Labor is homogeneous, and also the labor market has perfect competition.

From now on, the explicit timing of the time-dependent variables is omitted unless needed for clarity; $\forall j$ means $j = 1, 2, \ldots, A$. The basic assumptions and conditions at the production side (technologies, behavior, use of Sector-1 output, no-arbitrage condition) can be presented the following way.

Sector 1: Final goods. The representative firm:

$$\begin{aligned}
Y &= L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha}, \qquad 0 < \alpha < 1, && (3) \\
\frac{\partial Y}{\partial L} &= (1-\alpha)\frac{Y}{L_Y} = w, && \text{(FOC1)} \\
\frac{\partial Y}{\partial x_j} &= \alpha L_Y^{1-\alpha} x_j^{\alpha-1} = p_j, \quad \forall j, && \text{(FOC2)}
\end{aligned}$$

[1] By definition, "intermediate goods" are non-human inputs that cannot be stored. Rental services cannot be stored.

Uses of Y:

$$Y = C + I_K = cL + \dot{K} + \delta K, \qquad \delta \geq 0, \quad K(0) > 0 \text{ given.} \tag{4}$$

Sector 2: Specialized capital goods. Given the technical design j, firm j in Sector 2 can effortless transform x_j units of "raw capital" into x_j units of the specialized capital good j simply by pressing a button on a computer. Price-setting and accounting profit:

$$p_j = \frac{1}{\alpha}(r+\delta) \equiv p, \qquad \forall j, \tag{5}$$

$$\pi_j = (\frac{1}{\alpha} - 1)(r+\delta)x_j \equiv (\frac{1}{\alpha} - 1)(r+\delta)x \equiv \pi, \quad \forall j, \tag{6}$$

Sector 3: All R&D labs in Sector 3 face the same linear "research technology":

$$\# \text{ viable inventions per time unit } = \bar{\eta}\ell_A,$$

where ℓ_A is input of research labor, and $\bar{\eta}$ is productivity in R&D, which the individual R&D lab takes as given.[2] Let P_A denote the market value of the license to commercial utilization of a patent, j, forever. In brief, we may refer to P_A as the "market value of a patent", which in equilibrium turns out to be the same for all j, see below. Then the single lab's demand for research labor is

$$\ell_A = \begin{cases} \infty \text{ if } w < P_A\bar{\eta}, \\ \text{undetermined if } w = P_A\bar{\eta}, \\ 0 \text{ if } w > P_A\bar{\eta}. \end{cases} \tag{7}$$

This reflects that the value of the marginal product of research labor is $P_A\bar{\eta}$.

At the economy-wide level the accumulated stock of viable inventions, measured by the level of A, is treated as a continuous and differentiable function of time so that we can write the increase in A per time unit as

$$\dot{A} \equiv \frac{dA(t)}{dt} = \bar{\eta}L_A \equiv \eta A^{\varphi} L_A^{1-\xi}, \qquad \eta > 0, \varphi \leq 1, 0 \leq \xi < 1, \quad A(0) > 0 \text{ given,} \tag{8}$$

where $L_A \equiv \sum \ell_A$ is aggregate employment in Sector 3. Each R&D lab is "small" and therefore perceives, correctly, its contribution to aggregate $\dot{A}$, hence to $\bar{\eta}$, to be negligible.

While in (3) we consider j as a discrete variable taking values in $\{1, 2, \ldots, A\}$, at the aggregate level in (8) we "smooth out" the time path of A. This approximation seems acceptable when A is "large", and the increases in A per time unit are "small" relative to the size of A.

[2] By "viable" we mean non-duplicated.

3 General equilibrium

In general equilibrium with $L_A > 0$ we have:

$$(K^d =)\ Ax = K\ (= K^s), \tag{9}$$
$$L_Y + L_A = L, \tag{10}$$
$$Y = K^\alpha (AL_Y)^{1-\alpha}, \qquad \text{(by (3) and (9))} \tag{11}$$
$$\frac{1}{\alpha}(r+\delta) = \frac{\partial Y}{\partial x_j} = \alpha L_Y^{1-\alpha}(\frac{K}{A})^{\alpha-1} = \alpha \frac{Y}{K} = \frac{\partial Y}{\partial K}, \qquad \text{(by (5), (FOC2), (9))} \tag{12}$$
$$\pi = (1-\alpha)\alpha \frac{Y}{A}, \qquad \text{(by (6), (12), and (9))} \tag{13}$$
$$w = (1-\alpha)\frac{Y}{L_Y} = P_A \bar{\eta} = P_A \eta A^\varphi L_A^{-\xi}, \quad \text{(by (FOC1), (7), and (8))} \tag{14}$$
$$P_A r = \pi + \dot{P}_A. \tag{15}$$

The equation (15) is the no-arbitrage condition which the market value, P_A, of a patent must satisfy in equilibrium. Assuming absence of asset price bubbles, this condition is equivalent to a statement saying that the market value of the patent equals the *fundamental value* of the patent.[3] By fundamental value is meant the present value of the expected future accounting profits from commercial utilization of the technical design in question. That is,

$$P_A(t) = \int_t^\infty \pi(s) e^{-\int_t^s r(u)du} ds. \tag{16}$$

Indeed, in view of no uncertainty and perfect foresight, we may consider the no-arbitrage condition (15) as a differential equation for the function $P_A(t)$. The solution to this differential equation, presupposing that there are no bubbles, is given in (16) (as derived in Appendix A). The convenience of (16) is that, given the expected future profits and interest rates, the formula directly tells us the market value of a patent. If, for instance, π grows at a constant rate n, and r is constant, then (16) reduces to

$$P_A(t) = \int_t^\infty \pi(t) e^{n(s-t)} e^{-r(s-t)} ds = \pi(t) \int_t^\infty e^{-(r-n)(s-t)} ds = \pi(t) \frac{1}{r-n}. \tag{17}$$

This present-value formula is, among other things, useful for intuitive interpretation of the effects of a change in the interest rate in the economy (everything else equal: higher r implies lower present value).

[3] Because accounting profits, π, per time unit is the same for all j, so is the market value P_A.

The size of per capita financial wealth is now given as

$$a(t) \equiv \frac{K(t) + P_A(t)A(t)}{L(t)}. \tag{18}$$

4 National income accounting

At this stage some national income accounting may be useful. From the final use side we have:

$$GNP = C + I_K + I_A = C + I_K + wL_A = C + \dot{K} + \delta K + P_A\dot{A} = Y + P_A\dot{A},$$

where we have applied (4) and the fact that, from (8) and (14), we have, in equilibrium, $P_A\dot{A} = P_A\bar{\eta}L_A = wL_A$ (no pure profits in R&D).

From the production (value added) side we have:

$$\begin{aligned} \text{value added in Sector 1} &= Y - pAx, \\ \text{value added in Sector 2} &= pAx, \\ \text{value added in Sector 3} &= P_A\dot{A}. \end{aligned}$$

So, total value added $= GNP = Y + P_A\dot{A}$.

From the income side:

$$\begin{aligned} GNP &= wL_Y + (r+\delta)K + A\pi + wL_A = wL_Y + (r+\delta)K + (1-\alpha)\alpha Y + wL_A \\ &= (1-\alpha)Y + \alpha^2 Y + (1-a)\alpha Y + wL_A = \left(1 - \alpha + \alpha^2 + \alpha - \alpha^2\right)Y + wL_A \\ &= Y + wL_A, \end{aligned}$$

where, as noted above, $wL_A = P_A\dot{A}$.

5 Balanced growth

Taking logs and then time derivatives in (11), we get

$$g_Y = \alpha g_K + (1-\alpha)(g_A + g_{L_Y}). \tag{19}$$

Now assume balanced growth. Since we have here two endogenous state variables, the capital stock, K, and the knowledge stock, A, we extend our definition from Lecture

Notes, Chapter 4, of a balanced growth path, BGP, to be a path along which $g_Y, g_C, g_K,$ *and* g_A are constant.[4] From the balanced growth equivalence theorem of Lecture Notes, Chapter 4, we know that, given the capital accumulation equation (4) and given that $I_K > 0$, a BGP will satisfy that

$$g_Y = g_K = g_C.$$

In view of $g_Y = g_K$, (19) implies that along a BGP

$$g_Y = g_A + g_{L_Y} = \text{ constant.} \tag{20}$$

Since g_A is constant along a BGP, so is g_{L_Y}.

In addition to $c \equiv C/L$, we define $y \equiv Y/L$ and $k \equiv K/L$. From now on we have to distinguish between two alternative cases, the Romer case and the Jones case.

5.1 The Romer case: $\varphi = 1$, $n = 0$, and $\xi = 0$

Since here $\varphi = 1$, we have $g_A = \eta L_A$. So, along the BGP, L_A must be constant and so must $L_Y = L - L_A$ since L is constant. Along the BGP, therefore,

$$g_Y = g_y = g_k = g_c = g_A = \eta L_A. \tag{21}$$

To determine L_A we need to take the household behavior, described in the Keynes-Ramsey rule (1) and the transversality condition (2), into account. Isolating r in (1) along a BGP immediately gives

$$r^* = \rho + \theta g_A^*, \tag{22}$$

using that $g_c = g_A$ by (21); an asterisk signifies that a value in steady state or balanced growth is considered. With this in mind, it can be shown (Exercise VII.14) that an equilibrium path featuring balanced growth with active R&D has

$$0 \; < \; L_A = \frac{\alpha\eta L - \rho}{(\theta + \alpha)\eta} \equiv L_A^*, \quad \text{and} \tag{23}$$

$$0 \; < \; g_A = \frac{\alpha\eta L - \rho}{\theta + \alpha} \equiv g_A^* \equiv g_c^*. \tag{24}$$

This is the "fully-endogenous growth" case.

[4] Recall that, on the one hand, the immediate interpretation of our symbol A is that it makes up an index for the most recently invented capital good type. On the other hand, we may also see A as an index of the stock of technical knowledge in society. In that context we treat A as a continuous and differentiable function of time.

The result is derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A *is* positive along the path. Let us check what the necessary and sufficient parameter conditions are for this to hold.

It can be shown (Exercise VII.14) that the transversality condition (2) with $n = 0$, in combination with (18), holds if and only if $\rho > (1-\theta)g_A^*$. By inserting (24) and isolating ρ, this inequality is equivalent to

$$\rho > \frac{(1-\theta)\alpha\eta L}{1+\alpha}. \tag{A1-R}$$

From (24) follows immediately that $L_A^* > 0$ if and only if

$$\rho < \alpha\eta L. \tag{A2}$$

Note that the right-hand side of (A1-R) is always smaller than the right-hand side of (A2) (since both θ and α are positive). Hence, (A1-R) and (A2) can hold at the same time. To assume both (A1-R) and (A2) is equivalent to assuming

$$\frac{(1-\theta)\alpha\eta L}{1+\alpha} < \rho < \alpha\eta L. \tag{**}$$

So for a BGP to be an equilibrium path in the Romer case, it is needed both that households are *not too patient* (in which case (A1-R) would be violated), *and* that they are *not too impatient* (in which case (A2) would be violated). On the one hand, being "too patient" means that households tend to save so much that the interest rate in the economy (implied by combining the result (24) with the Keynes-Ramsey rule along the BGP) would be larger than the growth rate of labor income. Because of the infinite time horizon of the households, this would imply that they had *infinite* human wealth, in which case it is a paradox that they do not consume much more than they do. When (A1-R) is violated, this paradox in unavoidable with Ramsey households. So general equilibrium within the Ramsey framework is in that case impossible.[5]

On the other hand, the meaning of being "too impatient", and thus violating (A2), is more straightforward. It simply means that households are not willing to deliver the saving needed to finance capital accumulation and R&D. Indeed, when $\rho \geq \alpha\eta L$, the willingness to save is so low that in the long run the economy will be in a *stationary state*

[5]This reflects one of the *limitations* of the Ramsey framework.

with just enough saving to maintain the capital stock and no saving left to finance R&D and net capital investment.[6]

It can be shown that the transitional dynamics of the model in the Romer case can be reduced to a three-dimensional dynamic system in $z_1 \equiv Y/K$, $z_2 \equiv C/K$, and $z_3 \equiv L_Y$. Under the assumptions (A1-R) and (A2), the system has a unique steady state, $z_1^* = (\rho+\theta g_A^*+\delta)/\alpha^2$, $z_2^* = z_1^*-g_A^*-\delta$, and $z_3^* = L_Y^*$, given in (23). In the steady state, y, k, c, and A follow the BGP described above. At least under realistic parameter values, the dynamic system can be shown to be saddle-point stable so that $(z_1(t), z_2(t), z_3(t)) \rightarrow (z_1^*, z_2^*, z_3^*)$ for $t \rightarrow \infty$ (Arnold, 2000). The transitional dynamics thus imply convergence towards the steady state which also means convergence towards balanced growth. Assuming (**), we thus know that, without recurrent disturbances, the system will in the long run be in balanced growth with a per capita growth rate equal to g_A^* given in (24).

Comments on the BGP solution in the Romer case Imposing both (A1-R) and (A2), in brief (**), there is in the Romer case a meaningful solution to the model. The solution features "fully endogenous" exponential growth. Exponential per capita growth is generated by an internal mechanism, through which labor is allocated to R&D; and this exponential per capita growth is maintained without support of growth in any exogenous factor.

Among other things, one can make comparative static analysis on the result in (24). For instance, we see that $\partial g_A^*/\partial L = \alpha\eta/(\theta+\alpha) > 0$. The Romer case thus implies a *scale effect on growth,* which is an empirically problematic feature.[7] In Exercise VII.14 the reader is asked to do further comparative static analysis on the result for g_A^*.

5.2 The Jones case: $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$

In this case, the "semi-endogenous growth" case, we can immediately determine g_A along a BGP with $L_A > 0$. We have

$$g_A \equiv \frac{\dot{A}}{A} = \eta A^{\varphi-1} L_A^{1-\xi}.$$

[6] This stationary state is in a sense still a BGP but with $g_y = g_k = g_c = g_A = 0$. Note that when $\rho > \alpha\eta L$, the formulas (23) and (24) cease to hold. This should be no surprise. Indeed, a path with $L_A < 0$ is obviously *impossible*; moreover, in the derivation of the two formulas we *relied* on the assumption that $L_A > 0$.

[7] If $n > 0$, the Romer case leads to a forever rising per capita growth rate, an implausible scenario.

Since, by assumption, $L_A > 0$, also $g_A > 0$, and so we can take logs on both sides and thereafter time derivatives, using the chain rule to get:

$$\frac{\dot{g}_A}{g_A} = (\varphi - 1)g_A + (1 - \xi)g_{L_A} = 0$$

along a BGP where, by definition, g_A must be constant. Hence

$$g_A = \frac{1-\xi}{1-\varphi} g_{L_A} = \frac{1-\xi}{1-\varphi} n \equiv g_A^*. \tag{25}$$

The last equality comes from the fact that since along a BGP with $L_A > 0$, g_{L_A} must be a positive constant at the same time as we know from (20) that g_{L_Y} is a constant along a BGP. Then, if either g_{L_A} or g_{L_Y} were smaller than n, the other would be larger than n and sooner or later violate $L_Y + L_A = L$. Hence, $g_{L_Y} = g_{L_A} = n$. From (20) then also follows that along a BGP,

$$g_Y = g_K = g_C = g_A^* + n. \tag{26}$$

The method of solving the model for $s_R \equiv L_A/L$ along the BGP is somewhat different from the method in the Romer case. To be able to pin down P_A under balanced growth, we first note that the no-arbitrage condition (15) can be written

$$P_A = \frac{\pi}{r - g_{P_A}}. \tag{27}$$

From (14) follows

$$g_{P_A} + \varphi g_A - \xi g_{L_A} = g_Y - g_{L_Y}. \tag{28}$$

We know that along the BGP, $g_{L_Y} = n = g_{L_A}$, so that (28) implies

$$g_{P_A} = g_Y - \varphi g_A - (1-\xi)n = g_A^* + n - \varphi g_A^* - (1-\xi)n = n, \tag{29}$$

where the second and third equalities build on (26) and (25). From the no-arbitrage condition (27) then follows that under balanced growth,

$$P_A = \frac{\pi}{r-n} = (1-\alpha)\alpha \frac{Y}{(r-n)A}, \tag{30}$$

the last equality following from (13).

By (12), $r = \alpha^2 Y/K - \delta$. Since under balanced growth, Y/K is a constant, so is r. Hence, (30) shows that $g_\pi = g_{P_A} = n$ under balanced growth. That is, the monopolies' accounting profit grow at the rate of population growth, n. This relationship reflects that a larger population growth rate means that the markets for the specialized intermediate goods grow faster, which in view of increasing returns makes R&D more profitable.

Now (14) gives

$$(1-\alpha)\frac{Y}{L_Y} = P_A\bar{\eta} = (1-\alpha)\alpha\frac{Y}{(r-n)A}\bar{\eta}.$$

Cancelling out $(1-\alpha)Y$ and multiplying through by L_A gives

$$\frac{L_A}{L_Y} = \alpha\frac{\bar{\eta}L_A}{(r-n)A} = \alpha\frac{g_A}{r-n},$$

where the last equality follows from (8). As $L_A/L_Y = s_R/(1-s_R)$, we get from this,

$$s_R = \frac{1}{1+\frac{r-n}{\alpha g_A^*}} \tag{31}$$

along a BGP with $L_A > 0$.

This is not the final solution for s_R since r is endogenous. But again, reordering the Keynes-Ramsey rule gives, under balanced growth, $r = \rho + \theta g_A^* = r^*$ as in (22). Substituting this into (31) yields the solution for s_R along the BGP:

$$s_R = \frac{1}{1+\frac{1}{\alpha}(\frac{\rho-n}{g_A^*}+\theta)} = \frac{1}{1+\frac{1}{\alpha}(\frac{\rho-n}{\frac{1-\xi}{1-\varphi}n}+\theta)} \equiv s_R^*, \tag{32}$$

the second equality coming from (25).

Like the Romer results, the Jones results are derived under the pre-condition that the transversality condition of the representative household is satisfied along the BGP and that L_A (hence also g_A) *is* positive. Let us check what the necessary and sufficient parameter conditions (over and above the basic conditions $\varphi < 1$, $n > 0$, and $\xi \in [0,1)$) are for these conditions to hold.

First, as to the transversality condition (2), note that under balanced growth,

$$a(t) \equiv \frac{K(t)+P_A(t)A(t)}{L(t)} = \frac{K(0)e^{(g_A^*+n)t}+P_A(0)A(0)e^{(n+g_A^*)t}}{L(0)e^{nt}} = a(0)e^{g_A^* t},$$

where the second equality follows from (26) and (29). Consequently, along a BGP

$$a(t)e^{-(r^*-n)t} = a(0)e^{-(r^*-n-g_A^*)t} = a(0)e^{-(r^*-n-g_A^*)t} \to 0 \text{ if and only if } r^* > g_A^* + n,$$

where $r^* = \rho + \theta g_A^*$ by (22) which also holds here. So (2) holds along the BGP if and only if $\rho + \theta g_A^* > g_A^* + n$, that is, if and only if $\rho - n > (1-\theta)g_A^*$. By (25), this inequality is equivalent to

$$\rho > (1-\theta)\left(\frac{1-\xi}{1-\varphi}+1\right)n. \tag{A1-J}$$

Second, for $g_c = g_A > 0$ to be an outcome in balanced growth, we need $r^* > \rho$. In view of $r^* = \rho + \theta g_A^*$, this condition is equivalent to $\rho + \theta g_A^* > \rho$, which is automatically satisfied when $n > 0$, see (25).

We conclude that for a BGP to be an equilibrium path in the Jones case, it is just needed that households are *not too patient,* in the sense of violating the parameter condition (A1-J). For $0 < \theta < 1$, the right-hand side of (A1-J) defines a positive lower bound for the rate of impatience. For $\theta \geq 1$, the condition (A1-J) imposes only a mild constraint in that it is satisfied whenever just $\rho > 0$ ($\theta > 1$ even allows a negative ρ, although not "too large" in absolute value).

So, given the basic conditions $\varphi < 1$, $n > 0$, and $\xi \in [0, 1)$, we need only to add the assumption (A1-J) to ensure that in the Jones case there is a meaningful solution to the model. It can be shown that the transitional dynamics in the Jones case can be reduced to a *four*-dimensional dynamic system, that there is a unique steady state, equivalent to a balanced growth path, and that the dynamic system is saddle-point stable. Assuming (A1-J) we thus know that, without recurrent disturbances, the system will in the long run end up in balanced growth with per capita growth rate equal to g_A^*, given in (25).

Comments on the BGP solution in the Jones case From the result (25) we see that exponential growth is in the Jones case not "fully endogenous" since it can only be sustained if $n > 0$. In other words, exponential growth can only be sustained if the growth engine receives an inflow of "energy" from growth in the labor force, an exogenous source. In this sense the exponential growth in the Jones case is often referred to as "semi-endogenous". As mentioned in Short Note 1, p. 6, this terminology is somewhat seductive. The "semi-endogenous" Jones model sounds as something less deep than the "fully endogenous" Romer model. But nothing of that sort should be implied. It is just a matter of different parameter values (in fact, a matter of a "knife-edge" case versus a robust parameter case).

Before proceeding, note the striking simplicity of the result (25). The growth rate in income per capita under balanced growth depends only on three parameters: the growth rate of the labor force, n, the elasticity of research productivity with respect to the stock of knowledge, A, and the degree of duplication, ξ, in economy-wide research. Neither household preferences, represented by the parameters ρ and θ, nor for instance an R&D subsidy that raises the share of labor allocated to R&D, affect g_A^*. There will be a temporarily higher growth rate of A, but in the long run g_A will return to the same g_A^*

as before, namely that given in (25), cf. Jones and Vollrath, p. 109-110.

On the basis of the formula (32), long-run level effects on s_R of different parameter shifts can be studied (exercises VII.12 and VII.13). While for instance the preference parameters ρ and θ do not here have long-run growth effects, they affect the share of labor allocated to R&D. They thus have *level* effects on $L_A^*(t) = s_R^* L(t)$ along a BGP. As expected, both a rise in ρ and a rise in θ affect $L_A^*(t)$ *negatively.* The intuition is as follows. A rise in impatience, ρ, implies reduced saving, hence less R&D can be financed by the saving. We could also say that a rise in impatience means greater scarcity of finance, which in turn tends to raise the interest rate. This implies lower present value of expected future accounting profits to be obtained by an invention, cf. (16). In turn, this means that R&D is less rewarding.

Likewise, a rise in θ (the desire for consumption smoothing) implies reduced saving in the normal case where $r > \rho$, cf. the Keynes-Ramsey rule. The level effect on $L_A^*(t)$ of a rise in θ has thus similarity with that of a rise in ρ.

The level effects on $L_A^*(t)$ will not affect g_A in the long run, since (25) shows that g_A^* only depends on n and φ, not on s_R. A higher s_R will temporarily increase both the growth rate of A and that of y. But the fact that $\varphi < 1$ ("diminishing returns to knowledge" in the growth engine) makes it impossible to maintain the higher growth rate in A forever. The growth rate will, after a possibly quite *durable* adjustment process[8] return to the same g_A^* as before. But the level of the growth path will generally be permanently affected. This is like in the Solow model or the original Ramsey model, where an increase in the propensity to save raises the growth rate only temporarily due to the falling marginal productivity of capital.

While level effects of shifts in s_R on $L_A^*(t)$ are straightforward to analyze, level effects on $y^*(t)$ and $c^*(t)$ are a bit more complicated. Indeed, a shift in s_R has ambiguous effects on both $y^*(t)$ and $c^*(t)$ along a BGP. If s_R initially is "low", a "small" increase in s_R will have a positive level effect on y via the productivity-enhancing effect of more knowledge creation. But if s_R is already quite large initially, L_Y will be small, which implies that $\partial Y/\partial L_Y$ is large. This large marginal productivity constitutes the opportunity cost of increasing s_R further and dominates the benefit of a higher s_R, when $s_R > 1/(2 - \varphi)$ (in the case $\xi = 0$), cf. Exercise VII.7e).

[8] See Jones (1995).

6 Economic policy

The presented version of the Romer-Jones model implies in the Romer case that under laissez-faire, the decentralized market equilibrium unambiguously leads to too little R&D. This is due to three circumstances: (a) the positive externality generated by the intertemporal knowledge spillover, represented by $\varphi = 1$; (b) the "surplus appropriability problem" illustrated in Jones and Vollrath, p. 134; and (c) the demand-reducing monopoly pricing over and above marginal cost of intermediates. All three circumstances contribute to too little R&D. And there are no externalities going in the opposite direction. It can be shown that in combination with a subsidy to R&D, a subsidy to purchases of specialized capital good services can solve the problem, if these subsidies are financed by lump-sum taxes or lump-sum-equivalent taxes like, in the present framework, a labor income tax (recall that the model's labor supply is inelastic).

In the Jones case, the "stepping-on-toes" effect ($\xi > 0$) is a negative externality pointing in the opposite direction. And so is the intertemporal knowledge spillover *if* $\varphi < 0$. The different calibrations made by Jones and coauthors use a positive value of both φ and ξ. Even taking to some extent creative destruction into account, Jones and Williams (1998) estimate the resource allocation to R&D in USA to be only a fourth of the social optimum, given that discounted utility of the representative household is the optimality criterion.

7 Concluding remarks

A weakness of the presented Romer-Jones model is the unrealistic feature that obsolescence of specialized capital goods never occurs. The mentioned Jones and Williams (1998) paper attempts to surmount that problem.

Another weakness is that there are two hidden arbitrary parameter links in the specification of the production function for final goods. One is related to the way the variety index A enters the production function. The parameter reflecting "gains to variety", sometimes called the "gains to specialization" parameter, below denoted μ, is arbitrarily identified with the output elasticity w.r.t. labor, $1 - \alpha$. Another arbitrary parameter link is that the elasticity of substitution between the different capital good types calculated from the production function (3) is $1/(1 - \alpha)$ and thus implies market power equal to $1/\alpha$, the monopoly markup. Thereby, effects of a rise in monopoly power can not be studied independently of a fall in the output elasticity w.r.t. capital, α. This arbitrary parameter link has in the Romer case the implication that a rise in market power *reduces* g_c^*, an effect arising solely because the positive effect on growth of the rise in the markup is *blurred* by a negative effect coming from a diminished output elasticity w.r.t. capital.

A specification of the production function free of these two arbitrary parameter links, but maintaining power functions throughout, is the following:

$$Y = A^{\mu} X^{\alpha} {N_Y}^{1-\alpha}, \qquad 0 < \alpha < 1, \quad \mu > 0,$$

where X is a CES aggregate (with constant returns to scale) of quantities, x_j, of specialized capital goods:

$$X = A(\frac{1}{A}\sum_{j=1}^{A} {x_j}^{\varepsilon})^{\frac{1}{\varepsilon}}, \qquad 0 < \varepsilon < 1.$$

Here the existing specialized capital goods exhibit an elasticity of substitution equal to $1/(1-\varepsilon)$, implying that the market power, or the monopoly markup, is given by $1/\varepsilon > 1$. Now g_{c^*} generally differs from g_A^* and the formulas become more complicated. But a rise in market power, $1/\varepsilon$, can be shown to unambiguously *raise* g_c^*. This is the opposite of what we got above, where market power was arbitrarily linked to the output elasticity w.r.t. capital, α. For details, see Alvarez-Pelaez and Groth (2005).

8 Appendix: Solving the no-arbitrage equation for P_A(t) in the absence of asset price bubbles

In Section 3 we claimed that in the absence of bubbles, the differential equation implied by the no-arbitrage equation (15) has the solution

$$P_A(t) = \int_t^{\infty} \pi(s) e^{-\int_t^s r(u)du} ds. \tag{*}$$

To prove this, we write the no-arbitrage equation on the standard form for a linear differential equation:

$$\dot{P}_A(t) - r(t)P_A(t) = -\pi(t).$$

The general solution to this (see Appendix B to Chapter 3 of Lecture Notes) is

$$P_A(t) = P_A(t_0)e^{\int_{t_0}^t r(u)du} - e^{\int_{t_0}^t r(u)du} \int_{t_0}^t \pi(s)e^{-\int_{t_0}^s r(u)du} ds.$$

Multiplying through by $e^{-\int_{t_0}^{t} r(u)du}$ gives

$$P_A(t)e^{-\int_{t_0}^{t} r(u)du} = P_A(t_0) - \int_{t_0}^{t} \pi(s)e^{-\int_{t_0}^{s} r(u)du}ds.$$

Rearranging and letting $t \to \infty$, we get

$$P_A(t_0) = \int_{t_0}^{\infty} \pi(s)e^{-\int_{t_0}^{s} r(u)du}ds + \lim_{t\to\infty} P_A(t)e^{-\int_{t_0}^{t} r(u)du}. \tag{33}$$

The first term on the right-hand side is the fundamental value of the patent, i.e., the present value of the expected future accounting profits on using the patent commercially. The second term on the right-hand side thus amounts to the difference between the market value, $P_A(t_0)$, of the patent and its fundamental value. By definition, this difference represents a bubble. In the absence of bubbles, the difference is nil, and the market price, $P_A(t_0)$, coincides with the fundamental value. So (*) holds (in (33) replace t by T and t_0 by t), as was to be shown.

9 References

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