Economic Growth, August 2012.

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A suggested solution to the problem set at the re-exam in Economic Growth, August 15, 2012

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (25 %)

We consider a closed economy with a manufacturing sector and an R&D sector. Aggregate output of the manufacturing sector at time t is

$$Y_t = \tilde{F}(K_t, L_{Yt}, t), \tag{1.1}$$

where K_t is capital input and L_{Yt} is labor input. The function $\tilde{F}$ is a neoclassical production function w.r.t. these two arguments. W.r.t. the third argument, we have $\partial \tilde{F}/\partial t \equiv \tilde{F}_t > 0$.

The only input in the R&D sector is research labor, L_{Rt}, and we have $L_{Yt} + L_{Rt} = L_t$, where L_t is aggregate employment. Time is continuous. We apply the convenient notation: $g_z \equiv \dot{z}/z$.

a) To decompose the output growth rate in the manufacturing sector into its three basic components we take logs in (1.1) and differentiate w.r.t. t to get

$$\begin{aligned} g_{Y,t} &\equiv \frac{\dot{Y}_t}{Y_t} = \frac{1}{Y_t}\left[\tilde{F}_K(K_t, L_t; t)\dot{K}_t + \tilde{F}_L(K_t, L_t; t)\dot{L}_t + \tilde{F}_t(K_t, L_t; t)\cdot 1\right] \\ &= \frac{K_t\tilde{F}_K(K_t, L_t; t)}{Y_t} g_{K,t} + \frac{L_t\tilde{F}_L(K_t, L_t; t)}{Y_t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t} \\ &\equiv \varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L,t} + \frac{\tilde{F}_t(K_t, L_t; t)}{Y_t}, \end{aligned} \tag{1.2}$$

[1] The solution below contains *more* details and more precision than can be expected at a three hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

where $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$ are the partial output elasticities w.r.t. capital and labor at time t, respectively, and $\tilde{F}_t(K_t, L_t; t)$ represents the partial derivative w.r.t. the third argument of the function $\tilde{F}$ (that is, K_t and L_t are kept fixed), evaluated at the point (K_t, L_t, t).

b) The TFP *growth rate* is defined as

$$x_t \equiv g_{Y,t} - (\varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} g_{L_Y,t}) = \frac{\tilde{F}_t(K_t, L_{Yt}; t)}{Y_t}, \tag{1.3}$$

So the TFP growth rate indicates what is left when from the output growth rate is subtracted the "direct contribution" from growth in the factor inputs weighted by the output elasticities w.r.t. these inputs. The interpretation is thus that the TFP growth rate reflects the direct contribution to output growth from current technical change (in a broad sense including learning by doing and organizational improvement).

The two key weights, $\varepsilon_{K,t}$ and $\varepsilon_{L,t}$, in the decomposition (1.2) are usually approximated the following way. The income share of labor in the manufacturing sector is the wage sum divided by value added, i.e.,

$$s_{L,t} \equiv \frac{w_t L_{Yt}}{Y_t}.$$

As growth accounting often subsumes the role of natural resources (land, oil wells etc.) into "capital", K, the income share of "capital" in the sector is the remainder, i.e.,

$$s_{K,t} \equiv 1 - \frac{w_t L_{Yt}}{Y_t},$$

These factor income shares are obtainable from national income accounts.[2] Then, in (1.3) the approximation

$$\begin{aligned} \varepsilon_{L,t} &\approx s_{L,t}, \\ \varepsilon_{K,t} &\approx s_{K,t}, \end{aligned}$$

is used.

The argument is that under CRS, perfect competition, and absence of externalities, the two partial output elasticities will in equilibrium equal the income shares of capital and labor, respectively.

c) From now on we have that

$$Y_t = F(K_t, A_t L_{Yt}), \tag{1.4}$$

[2] Whether we have L_{Yt} or L_t in the numerator and Y_t or GNP_t in the denominator of the factor income shares is not very important in this problem, since "in practice" the R&D sector of the economy tends to be relatively small.

where F has CRS and A_t represents the "technology level", which due to the activity in the R&D sector is growing over time. The increase in capital per time unit is given by

$$\dot{K}_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad \delta \geq 0,$$

where C_t is aggregate consumption; gross saving, S_t, is positive for all t.

We are told that $g_{A,t} = g > 0$, constant. In view of CRS, (1.4) implies

$$1 = F(\frac{K_t}{Y_t}, \frac{A_t L_{Yt}}{Y_t}) = F(\frac{K_t}{Y_t}, \frac{A_t}{y_t}), \tag{1.5}$$

where $y_t \equiv Y_t/L_{Yt}$. From the balanced growth equivalence theorem we know that under balanced growth in the manufacturing sector and with $S_t > 0$ for all t, K_t/Y_t will be constant. It then follows from (1.5) that A_t/y_t will be constant. Hence,

$$g_y = g_{A,t} = g.$$

d) With L_{Yt} growing at a constant rate n, (1.3) gives

$$\begin{aligned} x_t &= g_{Y,t} - (\varepsilon_{K,t} g_{K,t} + \varepsilon_{L,t} n) = g_Y - \varepsilon_{K,t} g_K - \varepsilon_{L,t} n \\ &= g + n - \varepsilon_{K,t} g_K - (1 - \varepsilon_{K,t}) n = g - \varepsilon_{K,t}(g_K - n) \\ &= g - \varepsilon_{K,t} g_k = (1 - \varepsilon_{K,t}) g, \end{aligned}$$

where the second equality (constant $g_{K,t}$) follows from balanced growth, the third from $g_Y = g_y + n$ together with the hint, and the two last ones from $k \equiv K/L_Y$ and $g_k = g_y = g$.

The only thing that remains is an argument that $\varepsilon_{K,t}$ is constant over time. Since F is homogeneous of degree one, by Euler's theorem follows that F_K is homogeneous of degree zero. Thereby,

$$\varepsilon_{K,t} = \frac{K_t F_K(K_t, A_t L_{Yt})}{Y_t} = \frac{K_t F_K(\frac{K_t}{Y_t}, \frac{A_t}{y_t})}{Y_t},$$

where K_t/Y_t and A_t/y_t are constant, due to balanced growth. Hence, the $\varepsilon_{K,t} = \varepsilon_K$, a constant $\in (0, 1)$, so that

$$x_t = (1 - \varepsilon_K) g \equiv x, \tag{1.6}$$

a constant.

e) Since A_t is growing due to the activity in the R&D sector, considering the economy as a whole, there is a sense in which technical progress explains more than what the growth accounting suggests. Indeed, in view of balanced growth $g_y = g_k = g$. Hence, (1.6) can be written

$$g_y = x + \varepsilon_K g_k = x + \varepsilon_K g = g,$$

indicating that not only x but the whole of g_y is due to technical progress.

2. Solution to Problem 2 (50 %)

We consider an economy described by the Lab-Equipment Model.

a) Under laissez-faire the economy suffers from a certain kind of inefficiency, due to the invented specialized intermediate goods being priced above marginal costs. In turn this reflects the monopoly position of the inventors, due either to patenting or secrecy. Consequently, "too little" of these goods is demanded, that is, the market for each specialized intermediate good is "too small". This results in "too little" remuneration of the R&D activity, which invents the new types of intermediate goods, the new varieties.

b) Considering a social planner, we are told that the static problem of the planner is to ensure that Sector 1 uses the "right" quantity of intermediate goods. Output in the sector is

$$Y_t = A\left(\sum_{i=1}^{N_t} x_{it}^{1-\beta}\right)L^{\beta}, \qquad A > 0, \quad 0 < \beta < 1, \tag{2.1}$$

where Y_t is output per time unit, x_{it} is input of intermediate good i, N_t is the number of different types of intermediate goods available at time t, and L is labor input = the exogenous and constant labor supply. The output of basic goods is used partly as input, X_t, in Sector 2, partly for consumption, $C_t \equiv c_t L$, and partly for R&D investment, Z_t, in Sector 3:

$$Y_t = X_t + C_t + Z_t = X_t + C_t + \frac{\dot{N}_t}{\eta}, \qquad \eta > 0, \tag{2.2}$$

and

$$X_t = \sum_{i=1}^{N_t} \psi x_{it}, \qquad \psi > 0.$$

The parameter η is seen to be a measure of research productivity. And the parameter ψ is seen to be marginal cost (in terms of basic goods) of supplying an intermediate good (after its design has been invented).

From now, for convenience the dating of the variables is suppressed when not needed.

c) It is *net* output of basic goods that is relevant for *final* use, i.e., use for consumption or for investment in R&D. Indeed,

$$Y - X = cL + Z.$$

So it is relevant for the social planner to choose quantities $x_1, ..., x_N$ so as to

$$\max Y - X = A\left(\sum_{i=1}^{N} x_i^{1-\beta}\right) L^{\beta} - \sum_{i=1}^{N} \psi x_i.$$

The first-order conditions for an interior solution is

$$\frac{\partial(Y-X)}{\partial x_i} = \frac{\partial Y}{\partial x_i} - \psi = A(1-\beta)x_i^{-\beta}L^{\beta} - \psi = 0, \qquad i = 1, ..., N.$$

Solving for x_i gives

$$x_i = \left(\frac{A(1-\beta)}{\psi}\right)^{1/\beta} L \equiv x_{SP}, \qquad i = 1, ..., N. \tag{2.3}$$

d) The market outcome under laissez-faire is $x_i = \left(\frac{A(1-\beta)^2}{\psi}\right)^{1/\beta} L \equiv x$ for all i. We see that

$$x = (1-\beta)^{1/\beta} x_{SP} < x_{SP},$$

where the inequality is due to $0 < \beta < 1$. The economic explanation of the inequality is that the monopoly pricing under laissez-faire results in a markup, $(1-\beta)^{-1} > 1$, on marginal costs, ψ. Cf. the answer to a).

e) We have

$$\begin{aligned} Y - X &= ANx_{SP}^{1-\beta}L^{\beta} - N\psi x_{SP} = (Ax_{SP}^{-\beta}L^{\beta} - \psi)x_{SP}N \\ &= \left(A\left(\frac{x_{SP}}{L}\right)^{-\beta} - \psi\right) x_{SP}N \equiv \tilde{A}N, \end{aligned} \tag{2.4}$$

where $\tilde{A}$ is seen to be a constant. That $\tilde{A} > 0$ follows from (2.3) according to which

$$A\left(\frac{x_{SP}}{L}\right)^{-\beta} - \psi = A\left(\frac{A(1-\beta)}{\psi}\right)^{-1} - \psi = A\frac{\psi}{A(1-\beta)} - \psi = \frac{\psi}{1-\beta} - \psi > 0, \tag{2.5}$$

since $0 < \beta < 1$.

f) The dynamic problem faced by the social planner is to choose $(c_t)_{t=0}^{\infty}$ so as to:

$$\max U_0 = \int_0^{\infty} \frac{{c_t}^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \qquad \text{s.t.}$$

$$0 \leq c_t \leq \frac{\tilde{A}N_t}{L}, \tag{*}$$

$$\dot{N}_t = \eta(\tilde{A}N_t - Lc_t), \qquad N_0 > 0 \text{ given}, \tag{**}$$

$$N_t \geq 0 \text{ for all } t \geq 0, \tag{***}$$

where θ and ρ are given parameters, $\theta > 0$.

The reason that in (*) the control variable c is bounded above is that the "knowledge capital stock", N_t, can not be eaten. So the maximum aggregate consumption level, $c_t L$, per time unit equals net output, $Y_t - X_t$, of basic goods per time unit which in turn equals $\tilde{A}N_t$.

The dynamic constraint (**) arises the following way. The state variable in the problem is the stock of "knowledge", N_t. From (2.2) follows that

$$\dot{N}_t = \eta Z_t = \eta(Y_t - X_t - c_t L) = \eta(\tilde{A}N_t - c_t L),$$

which is the same as (**).

g) The current-value Hamiltonian is

$$H = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda\eta(\tilde{A}N - Lc).$$

The first-order conditions for an interior solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \lambda\eta L = 0 \Rightarrow c^{-\theta} = \lambda\eta L, \tag{FOC1}$$

$$\frac{\partial H}{\partial N} = \lambda\eta\tilde{A} = -\dot{\lambda} + \rho\lambda. \tag{FOC2}$$

Log-differentiating w.r.t. t in (FOC1) and combining with (FOC2) gives the Keynes-Ramsey rule

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(\eta\tilde{A} - \rho) \equiv \frac{1}{\theta}(r_{SP} - \rho) \equiv g_{SP}, \tag{2.6}$$

a constant.

By proportionality between Y and N and constancy of the required rate of return, r_{SP}, follows that the model is a reduced-form AK model where N has the usual role of K. We know that in such models there will be no transitional dynamics; the transversality condition, $\lim_{t\to\infty} N_t\lambda_t e^{-\rho t} = 0$, ensures that N from the beginning will grow at the same rate as cL, that is, the rate g_{SP}. Consequently, so will $Y = \tilde{A}N$.

h) To ensure a positive growth rate we need that $r_{SP} > \rho$, i.e.,

$$\eta\tilde{A} > \rho. \tag{A1}$$

To ensure boundedness of the utility integral we need that

$$\rho > (1-\theta)g_{SP} \text{ and therefore } g_{SP} < \theta g_{SP} + \rho = r_{SP}. \tag{A2}$$

i) The consumption growth rate in the laissez-faire market economy is

$$g^* = \frac{1}{\theta}(r^* - \rho),$$

where

$$r^* = \eta(\frac{\psi}{1-\beta} - \psi)x.$$

We have $g_{SP} \gtreqless g^*$ if and only if $r_{SP} \gtreqless r^*$. Now, by (2.6), (2.4), and (2.5),

$$\begin{aligned} r_{SP} &= \eta\tilde{A} = \eta\left(A\left(\frac{x_{SP}}{L}\right)^{-\beta} - \psi\right)x_{SP} \\ &= \eta\left(\frac{\psi}{1-\beta} - \psi\right)x_{SP} > \eta\left(\frac{\psi}{1-\beta} - \psi\right)x = r^*, \end{aligned}$$

where the inequality is due to $x_{SP} > x$, by the answer to d). It follows that $g_{SP} > g^*$.

j) To neutralize the distortion from the monopoly pricing of intermediate goods we need that

$$(1-\sigma)p_i = MC_i = \psi,$$

where $p_i = \psi/(1-\beta)$. So the optimal σ satisfies

$$(1-\sigma)\frac{\psi}{1-\beta} = \psi,$$

that is, the required value of σ is

$$\sigma = \beta.$$

Comment: This subsidy to purchases of intermediate goods makes the effective price of these goods equal to the social marginal cost of supplying them. So the static distortion from monopoly pricing is neutralized. It can be shown that this increases the market for intermediate goods and thereby the remuneration of the R&D activity just enough to obtain the "right" incentive to do R&D so that the social planner's solution is implemented. The financing of the subsidy by a *constant* consumption tax is not distortionary in *this* model since the model assumes an inelastic labor supply.

3. Solution to Problem 3 (25 %)

a) Acemoglu makes an important distinction between proximate and fundamental determinants of differences in economic performance. *Proximate determinants* include accumulation of physical and human capital and rises in technical knowledge through R&D and learning by doing. These elements taken together constitute what Acemoglu calls the "mechanics of growth".

Digging deeper we may ask *why* some countries do much more capital accumulation and knowledge creation than others. Answering this requires considering such factors as:

- geography (natural resources, soil quality, the amount of coast line, disease burden);
- culture (beliefs, religion, values, norms, trust, social capital);
- institutions (formal rules of the game, laws, policies).

Acemoglu calls such factors *fundamental determinants* of economic performance. He considers institutions, in particular property rights protection, the most basic among these.

b) Recalling the notation $y \equiv Y/L$ and $g_y \equiv \dot{y}/y$, in this course we use the following definitions.

Endogenous growth is present if there is a positive long-run per capita growth rate (i.e., $g_y > 0$) and the source of this is some internal mechanism in the model (so that exogenous technology growth is not deeded).

There are two basic kinds of endogenous growth:

Fully endogenous growth (sometimes called *strictly endogenous growth*) is present if there is a positive long-run per capita growth rate and this occurs without the support by growth in any exogenous factor (for example exogenous growth in the labor force).

Semi-endogenous growth is present if growth is endogenous but a positive long-run per capita growth rate can not be sustained without the support by growth in some exogenous factor (for example exogenous growth in the labor force).

A knife-edge condition is present in a model if (a) a particular value is imposed for a parameter which *apriori* can take any value within an interval; (b) the imposed value is such that non-robust results arise.

Here are five examples of fully endogenous growth models that rely on at least one important knife-edge condition:

1. The simple AK model with aggregate production function $Y_t = AK_t$. This production function can be seen as an extreme version of for instance $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, $0 < \alpha < 1$, namely the limiting case $\alpha = 1$.

2. The reduced-form AK model with physical and human capital presented in Acemoglu's Chapter 11.2. The aggregate production function of the model is

$$Y_t = F(K_t, H_t) \equiv F(K_t, h_t L_t), \tag{3.1}$$

where F is neoclassical with CRS. At the same time it is assumed that human capital, H, is formed in a way similar to physical capital, K. In that case a more natural and general specification would be $Y_t = F(K_t, h_t^{\varphi} L_t)$, $\varphi > 0$.

3. Paul Romer's learning-by-investing growth model where $Y_t = F(K_t, A_t L_t)$ and $A_t = K_t$. The more natural and general specification would be $Y_t = F(K_t, A_t L_t)$ and $A_t = K_t^{\lambda}$, $0 < \lambda \leq 1$.

4. The Lab-Equipment Model considered in Problem 2 above where (2.1) is arbitrarily specified such that in the laissez-faire market economy output of Sector 1 ends up as $Y = ANx^{1-\beta}L^{\beta}$ whereby the exponent on N is arbitrarily equal to one and the elasticity of substitution, β^{-1}, between the intermediate input goods (which gives rise to the monopolist markup) arbitrarily ends up equal to the inverse of the output elasticity w.r.t. labor.[3]

5. Paul Romer's version of the knowledge-spillover R&D-based growth model.

c) "Economic policy will have no effect on the long-term economic performance of an economy described by a semi-endogenous growth model". False. Although well-designed economic policy in such a model can not raise the long-run per capita growth rate, it can raise the per capita output level and per capita consumption level forever.

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[3] Only a very independent answer will be aware of these knife-edge conditions which are not mentioned in our syllabus and have not been discussed in exercise class.