Supplement

to the paper

Medium-term Fluctuations and the "Great Ratios" of Economic Growth

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1. Introduction

This supplementary material gives an account of some details in the proof of (ii) of Proposition 1, omitted from the last paragraph of Appendix A of the paper. In addition, as a supplement to Appendix B of the paper, the mathematics behind the applied normalisation of the CES production function is explained. Finally, a list of data sources for Section 2 of the paper is provided.

2. The Jacobian Matrix

For convenience, we repeat here the entries of the Jacobian matrix of the three-dimensional dynamic system of the model, evaluated in steady state:

$$J = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix} =$$

$$\begin{bmatrix} \dfrac{\hat{c}(q^*,\tilde{w}^*)\left[\theta m'(q^*)q^* + \rho - n - (1-\theta)\gamma\right]}{h(q^*,\tilde{w}^*)} & \dfrac{\varepsilon(k^*)^2\hat{c}(q^*,\tilde{w}^*) + \theta q^*\sigma(k^*)^2\varphi'(\bar{v})\bar{v}}{\varepsilon(k^*)^2 k^* h(q^*,\tilde{w}^*)} & \dfrac{-\theta q^* \dfrac{\tilde{w}^*}{k^*}\sigma(k^*)\varphi'(\bar{v})}{\varepsilon(k^*)k^* h(q^*,\tilde{w}^*)} \\ 0 & -\dfrac{\sigma(k^*)}{\varepsilon(k^*)}\varphi'(\bar{v})\bar{v} & \dfrac{\tilde{w}^*}{k^*}\varphi'(\bar{v}) \\ m'(q^*)x^* & 0 & 0 \end{bmatrix}$$

where $h(q^*, \tilde{w}^*) \equiv \hat{c}(q^*, \tilde{w}^*) + \theta m'(q^*) q^{*2} > 0.$ In particular, we observe that $j_{21} = 0,\ j_{23} \neq 0$ and $j_{31} \neq 0.$

Let $\boldsymbol{x} = (x_1, x_2, x_3) = (q, \tilde{w}, x)$ and $\boldsymbol{x^*} = (x_1^*, x_2^*, x_3^*) = (q^*, \tilde{w}^*, x^*)$. The eigenvalues of $\boldsymbol{J}$ are denoted μ_1, μ_2 and μ_3. We know from Appendix A of the paper that one eigenvalue, say μ_3, is real and positive, and that the other two eigenvalues have negative real part, that is, $\mu_1 = a_1 + ib$ and $\mu_2 = a_2 - ib$, where $a_1 < 0$ and $a_2 < 0$. In case μ_1 and μ_2 are real, $b = 0$. Otherwise, μ_1 and μ_2 are complex, i.e., $b \neq 0$ and $a_1 = a_2 = a$.

3. The general convergent solution

There always exist two linearly independent vectors, $\mathbf{v}^1 = (v_1^1, v_2^1, v_3^1) \in R^3$ and $\mathbf{v}^2 = (v_1^2, v_2^2, v_3^2) \in R^3$, such that the *stable* linear subspace, M^s, is spanned by these, i.e. $M^s = Sp(v^1, v^2)$ (see, e.g., Braun, 1975).

In case μ_1 and μ_2 are real and distinct, any convergent solution is, in the neighbourhood of $\boldsymbol{x}^*$, approximately of the form

$$\boldsymbol{x}_t = c_1 \boldsymbol{s} e^{\mu_1 t} + c_2 \boldsymbol{u} e^{\mu_2 t} + \boldsymbol{x}^*, \tag{3.1}$$

where c_1 and c_2 denote constants that depend on initial conditions, whereas $\boldsymbol{s} = (s_1, s_2, s_3)$ and $\boldsymbol{u} = (u_1, u_2, u_3)$ are eigenvectors corresponding to μ_1 and μ_2, respectively; so, $\boldsymbol{s}$ and $\boldsymbol{u}$ are linearly independent and $M^s = Sp(\boldsymbol{s}, \boldsymbol{u})$. Alternatively, we may have $\mu_1 = \mu_2 = \mu < 0$, and then any convergent solution is of the form

$$\boldsymbol{x}_t = \left[c_1 \boldsymbol{s} + c_2 (\boldsymbol{u} + t\boldsymbol{s})\right] e^{\mu t} + \boldsymbol{x}^*, \tag{3.2}$$

where $\boldsymbol{s}$ is an eigenvector corresponding to μ, and $\boldsymbol{u}$ is a linearly independent eigenvector also corresponding to μ, if such an eigenvector exists; otherwise, $\boldsymbol{u}$ is a generalized eigenvector satisfying

$$J\boldsymbol{u} = \mu \boldsymbol{u} + \boldsymbol{s},\ \boldsymbol{u} \neq 0. \tag{3.3}$$

Finally, when μ_1 and μ_2 are complex, any convergent solution is of the form

$$\boldsymbol{x}_t = \left[c_1 (\boldsymbol{s} \cos bt - \boldsymbol{u} \sin bt) + c_2 (\boldsymbol{u} \cos bt + \boldsymbol{s} \sin bt)\right] e^{at} + \boldsymbol{x}^*, \tag{3.4}$$

where $\boldsymbol{s}$ and $\boldsymbol{u}$ are the real part and the imaginary part, respectively, of an eigenvector $\boldsymbol{w}$ corresponding to the eigenvalue $\mu_1 = a + ib$, that is, $\boldsymbol{w} = \boldsymbol{s} + i\boldsymbol{u}$.

So, in all three cases $\boldsymbol{s}$ and $\boldsymbol{u}$ are linearly independent and $M^s = Sp(\boldsymbol{s}, \boldsymbol{u})$.

4. Existence and uniqueness with given initial conditions

For $t = 0$ we have $\boldsymbol{x}_0 = c_1\boldsymbol{s} + c_2\boldsymbol{u} + \boldsymbol{x}^*$ in all three cases above. By coordinates,

$$\begin{aligned} x_{10} &= c_1 s_1 + c_2 u_1 + x_1^*, \\ x_{20} &= c_1 s_2 + c_2 u_2 + x_2^*, \\ x_{30} &= c_1 s_3 + c_2 u_3 + x_3^*. \end{aligned}$$

In our economic model $\tilde{w}$ and x are predetermined, whereas q is a jump variable. Hence, we should consider x_{20} and x_{30} as given and x_{10} as endogenous. Consequently, we rewrite the system as

$$\begin{aligned} s_1c_1 + u_1c_2 - x_{10} &= -x_1^*, \\ s_2c_1 + u_2c_2 \qquad &= x_{20} - x_2^*, \\ s_3c_1 + u_3c_2 \qquad &= x_{30} - x_3^*. \end{aligned} \tag{4.1}$$

This system has a unique solution for (c_1, c_2, x_{10}), if and only if the vector $\boldsymbol{z} \equiv (-1, 0, 0)$ does not belong to $Sp(\boldsymbol{s}, \boldsymbol{u})$. This condition is equivalent to the stable linear subspace M^s not being parallel to the x_1 axis (i.e., the q axis in Figure A1 in Appendix A of the paper). We now show that this condition is satisfied.

Lemma 1. Let the elements j_{21}, j_{23} and j_{31} in the 3×3 matrix $\boldsymbol{J}$ satisfy $j_{21} = 0,\ j_{23} \neq 0$ and $j_{31} \neq 0$. Let the two linearly independent vectors $\boldsymbol{s} \in R^3$ and $\boldsymbol{u} \in R^3$ be as defined in Section 3 above. Then the vector $\boldsymbol{z} \equiv (-1, 0, 0)$ does not belong to $Sp(\boldsymbol{s}, \boldsymbol{u})$.

Proof. We prove this by showing that the opposite leads to a contradiction. Suppose $\boldsymbol{z} \equiv (-1, 0, 0)$ belongs to $Sp(\boldsymbol{s}, \boldsymbol{u})$. Then there exist constants α_1 and α_2, so that

$$\alpha_1\boldsymbol{s} + \alpha_2\boldsymbol{u} = \boldsymbol{z} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

Multiplying from the left by $\boldsymbol{J}$ gives

$$\alpha_1 J\boldsymbol{s} + \alpha_2 J\boldsymbol{u} = J\boldsymbol{z} = -\begin{pmatrix} j_{11} \\ j_{21} \\ j_{31} \end{pmatrix}, \tag{4.3}$$

There are three cases to consider.

Case 1: μ_1 and μ_2 real, and $\mu_1 \neq \mu_2$, both negative. In this case, $\boldsymbol{s}$ and $\boldsymbol{u}$ are eigenvectors corresponding to μ_1 and μ_2, respectively. Hence, (4.3) gives

$$\alpha_1\mu_1\boldsymbol{s}+\alpha_2\mu_2\boldsymbol{u}=-\begin{pmatrix} j_{11} \\ j_{21} \\ j_{31} \end{pmatrix}. \tag{4.4}$$

If $\alpha_1 = 0$, then (4.2) implies $\alpha_2 \neq 0$ and therefore $u_2 = u_3 = 0$, so that $j_{31} = 0$, in view of (4.4). But this contradicts the presupposition that $j_{31} \neq 0$. Suppose $\alpha_1 \neq 0$. Since $j_{21} = 0$, (4.4) implies $\alpha_1\mu_1 s_2 + \alpha_2\mu_2 u_2 = 0$, which, by (4.2), yields $(\mu_1 - \mu_2)\alpha_1 s_2 = 0$, implying $s_2 = 0$. But $\boldsymbol{s}$ is an eigenvector corresponding to μ_1, so that, in particular,

$$j_{22}s_2 + j_{23}s_3 = \mu_1 s_2, \tag{4.5}$$

in view of $j_{21} = 0$. Hence, from $s_2 = 0$ and $j_{23} \neq 0$ follows $s_3 = 0$, and in view of (4.2) this gives $\alpha_2 u_3 = 0$, implying, by (4.4), $j_{31} = 0$, which again contradicts the presupposition that $j_{31} \neq 0$.

Case 2: μ_1 and μ_2 real, and $\mu_1 = \mu_2 = \mu < 0$. Then, at least $\boldsymbol{s}$ is an eigenvector corresponding to μ. If there exists a linearly independent eigenvector also corresponding to μ, $\boldsymbol{u}$ may be taken to be that vector, and then, from (4.4) with $\mu_1 = \mu_2 = \mu$, we get $j_{31} = \alpha_1\mu s_3 + \alpha_2\mu u_3 = \mu(\alpha_1 s_3 + \alpha_2 u_3) = 0$, in view of (4.2); but this contradicts the presupposition that $j_{31} \neq 0$. Otherwise, $\boldsymbol{u}$ is a generalized eigenvector satisfying (3.3), which together with (4.3) implies

$$\alpha_1\mu\boldsymbol{s}+\alpha_2(\mu\boldsymbol{u}+\boldsymbol{s})=\mu(\alpha_1\boldsymbol{s}+\alpha_2\boldsymbol{u})+\alpha_2\boldsymbol{s}=-\begin{pmatrix} j_{11} \\ j_{21} \\ j_{31} \end{pmatrix}. \tag{4.6}$$

By (4.2), this gives, in particular,

$$\alpha_2 s_2 = j_{21} = 0, \tag{4.7}$$

in view of the presupposition that $j_{21} = 0$, and

$$\alpha_2 s_3 = j_{31}. \tag{4.8}$$

If $\alpha_2 = 0$, then, by (4.8), $j_{31} = 0$, which is a contradiction. On the other hand, if $\alpha_2 \neq 0$, (4.7) gives $s_2 = 0$. Since $\boldsymbol{s}$ is an eigenvector and $j_{21} = 0$, (4.5) still holds, so that we now have $s_3 = 0$, in view of $j_{23} \neq 0$. Then, by (4.8), $j_{31} = 0$, which is a contradiction.

Case 3: μ_1 and μ_2 complex, i.e., $\mu_1 = a + ib$ and $\mu_2 = a - ib$, where $b \neq 0$ and $a < 0$. In this case, $\boldsymbol{s}$ and $\boldsymbol{u}$ are the real part and the imaginary part, respectively, of an eigenvector $\boldsymbol{w}$ corresponding to the eigenvalue μ_1, that is, $\boldsymbol{w} = \boldsymbol{s} + i\boldsymbol{u}$. Let $\bar{\boldsymbol{w}}$ denote the complex conjugate of $\boldsymbol{w}$, i.e., $\bar{\boldsymbol{w}} = \boldsymbol{s} - i\boldsymbol{u}$. Then $\boldsymbol{w} + \bar{\boldsymbol{w}} = 2\boldsymbol{s}$ and $\boldsymbol{w} - \bar{\boldsymbol{w}} = i2\boldsymbol{u}$. Since $\bar{\boldsymbol{w}}$ is an eigenvector corresponding to μ_2, we get

$$J\boldsymbol{s} = ½J(\boldsymbol{w} + \bar{\boldsymbol{w}}) = ½(J\boldsymbol{w} + J\bar{\boldsymbol{w}}) = ½(\mu_1\boldsymbol{w} + \mu_2\bar{\boldsymbol{w}}) = ½(2a\boldsymbol{s} - 2b\boldsymbol{u}) = a\boldsymbol{s} - b\boldsymbol{u}, \tag{4.9}$$

$$iJ\boldsymbol{u} = ½J(\boldsymbol{w} - \bar{\boldsymbol{w}}) = ½(J\boldsymbol{w} - J\bar{\boldsymbol{w}}) = ½(\mu_1\boldsymbol{w} - \mu_2\bar{\boldsymbol{w}}) = ½i(2a\boldsymbol{u} + 2b\boldsymbol{s}) = i(a\boldsymbol{u} + b\boldsymbol{s}). \tag{4.10}$$

Hence, (4.3) yields $\alpha_1(a\boldsymbol{s} - b\boldsymbol{u}) + \alpha_2(a\boldsymbol{u} + b\boldsymbol{s}) = J\boldsymbol{z}$, which can be written $a(\alpha_1\boldsymbol{s} + \alpha_2\boldsymbol{u}) + b(\alpha_2\boldsymbol{s} - \alpha_1\boldsymbol{u}) = J\boldsymbol{z}$. In view of (4.2) and the definition of $\boldsymbol{z}$, this implies, in particular,

$$b(\alpha_2 s_2 - \alpha_1 u_2) = j_{21} = 0, \tag{4.11}$$

by assumption, and

$$b(\alpha_2 s_3 - \alpha_1 u_3) = j_{31}. \tag{4.12}$$

In view of $j_{21} = 0$ the second element of $J\boldsymbol{s}$ is

$$j_{22}s_2 + j_{23}s_3 = as_2 - bu_2, \tag{4.13}$$

by (4.9), and the second element of $J\boldsymbol{u}$ is

$$j_{22}u_2 + j_{23}u_3 = bs_2 + au_2, \tag{4.14}$$

by (4.10). If $\alpha_1 = 0$, then (4.2) implies $\alpha_2 \neq 0$ and thereby $u_2 = u_3 = 0$, so that, by (4.11), $s_2 = 0$. Then (4.13) gives $s_3 = 0$, in view of $j_{23} \neq 0$. This implies, by (4.12), $j_{31} = 0$, which contradicts the presupposition that $j_{31} \neq 0$. Now, suppose $\alpha_1 \neq 0$. From (4.2) follows $s_2 = -\alpha_2 u_2/\alpha_1$, which substituted into (4.11) gives $\alpha_2(-\alpha_2 u_2/\alpha_1) - \alpha_1 u_2 = 0$ or $-(\alpha_2{}^2 + \alpha_1{}^2)u_2 = 0$, implying $u_2 = 0$ and thereby $s_2 = 0$. Then, (4.14) gives $u_3 = 0$, implying, by (4.2), $s_3 = 0$. From (4.12) then follows $j_{31} = 0$, contradicting the presupposition that $j_{31} \neq 0$. Q.E.D.

5. Normalization of the CES function[1]

The “normalisation” of the CES production function described in Appendix C of the paper is based on the following facts. Expressed in the classical way, as in Arrow et al. (1961), the CES production function reads:

$$y = f(k) = B(\alpha k^{\psi} + 1 - \alpha)^{1/\psi}, \qquad \psi\ (\equiv 1 - \sigma^{-1}) < 1,\ 0 < \alpha < 1,\ B > 0. \tag{5.1}$$

Suppose that to begin with we have not specified the parameters ψ, α, and B. Instead, for alternative values of $\psi \in (-\infty, 1)$ we want to adjust the (not dimensionless) parameters α and

[1] This section essentially builds on La Grandville (1989) and Klump and Saam (2008).

B so that at some baseline point $\bar{k} > 0$, the output elasticity with respect to capital, $\varepsilon(k)$, and output per unit of effective labour, y, are and remain equal to some pre-specified values, $\bar{\varepsilon} \in (0,1)$ and $\bar{y} > 0$, respectively.

For any $k > 0$,

$$\varepsilon(k) \equiv \frac{kf'(k)}{f(k)} = \frac{\alpha}{(1-\alpha)k^{-\psi} + \alpha}, \tag{5.2}$$

where the second equality comes from (5.1). Requiring $\varepsilon(\bar{k}) = \bar{\varepsilon}$, we find α as a function of ψ, $\bar{k}$ and $\bar{\varepsilon}$:

$$\alpha = \frac{\bar{\varepsilon}}{(1-\bar{\varepsilon})\bar{k}^{\psi} + \bar{\varepsilon}} \equiv \alpha(\psi, \bar{k}, \bar{\varepsilon}). \tag{5.3}$$

Substituting this value of α into (5.1), we find the required value of B to be

$$B = \bar{y}(1-\bar{\varepsilon} + \bar{\varepsilon}\bar{k}^{-\psi})^{1/\psi} \equiv B(\psi, \bar{k}, \bar{\varepsilon}, \bar{y}). \tag{5.4}$$

We end up with a CES function in "family" form, also called "normalised" form:

$$y = B(\psi, \bar{k}, \bar{\varepsilon}, \bar{y})\left[\alpha(\psi, \bar{k}, \bar{\varepsilon})k^{\psi} + 1 - \alpha(\psi, \bar{k}, \bar{\varepsilon})\right]^{1/\psi}. \tag{5.5}$$

So (5.3) and (5.4) are necessary conditions for α and B to be such that $\varepsilon(\bar{k}) = \bar{\varepsilon}$ and $f(\bar{k})$ $= \bar{y}$.

On the other hand, when α and B in (5.1) equal $\alpha(\psi, \bar{k}, \bar{\varepsilon})$ and $B(\psi, \bar{k}, \bar{\varepsilon}, \bar{y})$, respectively, then it is easily verified that (5.2) implies $\varepsilon(\bar{k}) = \bar{\varepsilon}$ and (5.1) implies $f(\bar{k}) = \bar{y}$. We conclude that (5.3) and (5.4) are not only necessary but also sufficient conditions for α and B to be such that $\varepsilon(\bar{k}) = \bar{\varepsilon}$ and $f(\bar{k}) = \bar{y}$. Thereby the formula (5.5) identifies the family of CES production functions that are distinguished by the elasticity of substitution but at the point $k = \bar{k}$ have output elasticity with respect to capital equal to $\bar{\varepsilon}$ and output per unit of effective labour equal to $\bar{y}$.

This claim includes even the Cobb-Douglas case $\psi = 0$ (i.e., $\sigma = 1$). To see this, note that when $\psi = 0$, (5.1) above should be interpreted as $f(k) = Bk^{\alpha}$. In this case $\varepsilon(k)$ $= \alpha$ for *all* $k > 0$. Hence, to require $\varepsilon(\bar{k}) = \bar{\varepsilon}$ immediately means that α must equal $\bar{\varepsilon}$. This is also what inserting $\psi = 0$ into the formula (5.3) gives, since $\alpha(0, \bar{k}, \bar{\varepsilon}) = \bar{\varepsilon}$. The additional requirement $f(\bar{k}) = \bar{y}$ is seen to imply $B = B(0, \bar{k}, \bar{\varepsilon}, \bar{y}) = \bar{y}\bar{k}^{-\bar{\varepsilon}}$ (in (5.4), apply L'Hôpital's rule for "0/0"). So we end up with the Cobb-Douglas function $y = \bar{y}\bar{k}^{-\bar{\varepsilon}}k^{\bar{\varepsilon}}$, which indeed satisfies both requirements since it has $\varepsilon(k) = \bar{\varepsilon}$ for all $k > 0$ (hence also for $k = \bar{k}$) and $y = \bar{y}$ for $k = \bar{k}$.

One may interpret the original Arrow et al. (1961) form as having an implicit baseline point at $\bar{k} = 1$ in the sense that α in the formula (5.1) equals the output elasticity with respect to capital at $k = 1$ while B equals output per unit of effective labour at $k = 1$. Indeed, from (5.2) follows that $\varepsilon(1) = \alpha$ and from (5.1) follows that $f(1) = B.$ Moreover, a convenient way of rewriting the normalized CES function is as

$$\frac{y}{\bar{y}} = \left(\bar{\varepsilon} \left(\frac{k}{\bar{k}} \right)^{\psi} + 1 - \bar{\varepsilon} \right)^{1/\psi}, \qquad \psi < 1,\ 0 < \bar{\varepsilon} < 1, \bar{y} > 0,\ \bar{k} > 0.$$

Here the capital input and output are measured in a dimensionless way as index numbers, $k / \bar{k}$ and $y / \bar{y}$, respectively.

As Appendix C of the paper describes, in the context of our complete model we let the role as baseline constellation $(\bar{k}, \bar{\varepsilon}, \bar{y})$ be taken by the steady-state triple $(k^*, k^* f'(k^*) / f(k^*), f(k^*))$ obtained, given the baseline values of the background parameters, the baseline value of the investment flexibility, β , and the requirement that $f(k^*) / k^*$ is consistent with an investment-GDP ratio of 0.19.

6. Data sources

Data are compiled for the following 13 OECD countries: Canada, the US, Australia, Belgium, Denmark, Finland, France, Germany, the Netherlands, Norway, Spain, Sweden and the UK.

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Economy-wide real GDP. Madsen *et al*. (2012), *op **cit.*** Updated using OECD, *National Accounts*, Volume 2, Paris: OECD.

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Tobin’s *q*. The log of consumer-price-deflated stock prices is regressed on a time trend and a constant and Tobin’s *q* is presented by the residuals from these regressions except for the US before 2004 where Tobin’s *q* for the US is from S Wright, 2004, “Measures of Stock Market Value and Returns for the U.S. Nonfinancial Corporate Sector”, 1900-2002, *Review of Income and Wealth*, 50, 561-584. For other data see Madsen, J. B. and E. P. Davis (2006), "Equity Prices, Productivity Growth and the New Economy”, *The Economic Journal*, 116 (513), 791-811. The data are updated from Datastream.

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