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RESEARCH ARTICLE

When economic growth is less than exponential

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Abstract This paper argues that growth theory needs a more general notion of "regularity" than that of exponential growth. We suggest that paths along which the rate of decline of the growth rate is proportional to the growth rate itself deserve attention. This opens up for considering a richer set of parameter combinations than in standard growth models. Moreover, it avoids the usual oversimplistic dichotomy of either exponential growth or stagnation. Allowing zero population growth in three different growth models (the Jones R&D-based model, a learning-by-doing model, and an embodied technical change model) serves as illustration that a continuum of "regular" growth processes fill the whole range between exponential growth and complete stagnation.

Keywords Quasi-arithmetic growth · Regular growth · Semi-endogenous growth · Knife-edge restrictions · Learning by doing · Embodied technical change

JEL Classification O31 · O40 · O41

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1 Introduction

The notion of balanced growth, generally synonymous with exponential growth, has proved extremely useful in the theory of economic growth. This is not only because of the historical evidence (Kaldor's "stylized facts"), but also because of its convenient simplicity. Yet, there may be a deceptive temptation to oversimplify and ignore other possible growth patterns. We argue that there is a need to allow for a richer set of parameter constellations than in standard growth models and to look for a more general regularity concept than that of exponential growth. The motivation is the following:

First, when setting up growth models researchers place severe restrictions on preferences and technology such that the resulting model is compatible with balanced growth (as pointed out by Solow 2000, Chaps. 8–9). In addition, population is either assumed to grow exponentially or to be constant. This paper demonstrates that regular long-run growth, in a sense specified below, can arise even when some of the archetype restrictions are left out.

Second, standard R&D-based semi-endogenous growth models imply that the long-run per-capita growth rate is proportional to the growth rate of the labor force (Jones 2005).[1] This class of models is frequently used for positive and normative analysis since it appears empirically plausible in many respects. And the models are consistent with more than a century of approximately exponential growth. If we employ this framework to evaluate the prospect of growth in the future, then we end up with the assertion that the growth *rate* will converge to zero. This is simply due to the fact that there must be limits to population growth, hence also to growth of human capital. The open question is then what this really implies for economic development in the future and thereby, for example, for the warranted discount rate for long-term environmental projects. This issue has not received much attention so far, and the answer is not that clear at first glance. Of course, there is an alternative to the semi-endogenous growth framework, namely that of fully endogenous growth as in the first-generation R&D-based growth models of Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992). This approach allows of exponential growth with zero population growth. However, in spite of their path-breaking nature these models rely on the simplifying knife-edge assumption of constant returns to scale (either exactly or asymptotically) with respect to producible factors in the invention production function.[2] As argued, for instance by McCallum (1996), the knife-edge assumption of constant returns to scale to producible inputs should be interpreted as a simplifying approximation to the case of slightly *decreasing* returns (increasing returns can be ruled out because they have the nonsensical implication of infinite output in finite time, see Solow 1994). But the case of decreasing returns to producible inputs is exactly the semi-endogenous growth case.

[1] Of course, if one digs a little deeper, it is not growth in population as such that matters. Rather, as Jones (2005) suggests, it is growth in human capital, but this ultimately depends on population growth.

[2] By "knife-edge assumption" is meant a condition imposed on a parameter value such that the set of values satisfying this condition has an empty interior in the space of all possible values for this parameter (see Growiec 2007).

A third reason for thinking about less than exponential growth is to open up for a perspective of *sustained* growth (in the sense of output per capita going to infinity for time going to infinity) in spite of the growth *rate* approaching zero. Everything less than exponential growth often seems interpreted as a fairly bad outcome and associated with economic stagnation. For instance, in the context of the Jones (1995) model with constant population, Young (1998, n. 10) states "*Thus, even if there are intertemporal spillovers, if they are not large enough to allow for constant growth, the development of the economy grinds to a halt*." However, to our knowledge, the case of zero population growth in the Jones model has not really been explored yet. We take the opportunity to let an analysis of this case serve as one of our illustrations that the usual dichotomy between either exponential growth or complete stagnation is too narrow. The analysis suggests that paths along which the rate of decline of the growth rate is proportional to the growth rate itself deserve attention. Indeed, this criterion will define our concept of *regular growth*. It turns out that exponential growth is the limiting case where the factor of proportionality, the "damping coefficient", is zero. And the "opposite" limiting case is stagnation which occurs when the "damping coefficient" is infinite.

To show the usefulness of this generalized regularity concept two further examples are provided. One of these is motivated by what seems to be a gap in the theoretical learning-by-doing literature. With the perspective of exponential growth, existing models either assume a very specific value of the learning parameter combined with zero population growth in order to avoid growth explosion (Barro and Sala-i-Martin 2004, Sect. 4.3) or allow for a range of values for the learning parameter below that specific value, but then combined with exponential population growth (Arrow 1962). There is an intermediate case, which to our knowledge has not been systematically explored. And this case leads to less-than-exponential, but sustained regular growth.

Our third example of regular growth is intended to show that the framework is easily applicable also to more realistic and complex models. As Greanwood et al. (1997) document, since World War II there has been a steady decline in the relative price of capital equipment and a secular rise in the ratio of new equipment investment to GNP. On this background we consider a model with investment-specific learning and embodied technical change, implying a persistent decline in the relative price of capital. When conditions do not allow of exponential growth, the same regularity emerges as in the two previous examples. We further sort out how and why the *source* of learning—be it *gross* or *net* investment—is decisive for this result.

The paper is structured as follows. Section 2 introduces proportionality of the rate of decline of the growth rate and the growth rate itself as defining "regular growth". It is shown that this regularity concept nests, inter alia, exponential growth, arithmetic growth, and stagnation as special cases. Sections 3, 4, and 5 present our three economic examples which, by allowing for a richer set of parameter constellations than in standard growth models, give rise to growth patterns satisfying our regularity criterion, yet being non-exponential. Asymptotic stability of the regular growth pattern is established in all three examples. Finally, Sect. 6 summarizes the findings.

2 Regular growth

Growth theory explains long-run economic development as some pattern of regular growth. The most common regularity concept is that of exponential growth. Occasionally another regularity pattern turns up, namely that of arithmetic growth. Indeed, a Ramsey growth model with AK technology and CARA preferences features arithmetic GDP per capita growth (e.g., Blanchard and Fischer 1989, pp. 44–45). Similarly, under Hartwick's rule, a model with essential, non-renewable resources (but without population growth, technical change, and capital depreciation) features arithmetic growth of capital (Solow 1974; Hartwick 1977). In similar settings, Mitra (1983), Pezzey (2004), and Asheim et al. (2007) consider growth paths of the form $x(t) = x(0)(1 + \mu t)^{\omega}$, $\mu, \omega > 0$, which, by the last-mentioned authors, is called "quasi-arithmetic growth". In these analyses, the quasi-arithmetic growth pattern is associated with exogenous quasi-arithmetic growth in either population or technology. In this way, results by Dasgupta and Heal (1979, pp. 303–308) on optimal growth within a classical utilitarian framework with non-renewable resources, constant population, and constant technology are extended. Hakenes and Irmen (2007) also study exogenous quasi-arithmetic growth paths. Their angle is to evaluate the plausibility of equations of motion for technology on the basis of the ultimate forward-looking as well as backward-looking behavior of the implied path.

In our view there is a rationale for a concept of regular growth, subsuming exponential growth and arithmetic growth as well as the range between these two. Also some kind of less-than-arithmetic growth should be included. We label this general concept *regular growth*, for reasons that will become clear below. The example we consider in Sect. 3 illustrates that by varying one parameter (the elasticity of knowledge creation with respect to the level of existing knowledge), the whole range between complete stagnation and exponential growth of the knowledge stock is spanned. Furthermore, the example shows how a quasi-arithmetic growth pattern for knowledge, capital, output, and consumption may arise *endogenously* in a two-sector, knowledge-driven growth model. The second and third examples, discussed in Sects. 4 and 5, respectively, show that also models of learning by doing and learning by investing may endogenously generate quasi-arithmetic growth.

To describe our suggested concept of regular growth, a few definitions are needed. Let the variable $x(t)$ be a positively-valued differentiable function of time t. Then the growth rate of $x(t)$ at time t is

$$g_1(t) \equiv \frac{\dot{x}(t)}{x(t)},$$

where $\dot{x}(t) \equiv \mathrm{d}x(t)/\mathrm{d}t$. We call $g_1(t)$ the first-order growth rate. Since we seek a more general concept of regular growth than exponential growth, we allow $g_1(t)$ to be time-variant. Indeed, the regularity we look for relates precisely to the way growth rates change over time. Presupposing $g_1(t)$ is strictly positive within the time range considered, let $g_2(t)$ denote the second-order growth rate of $x(t)$ at time t, i.e.,

$$g_2(t) \equiv \frac{\dot{g}_1(t)}{g_1(t)}.$$

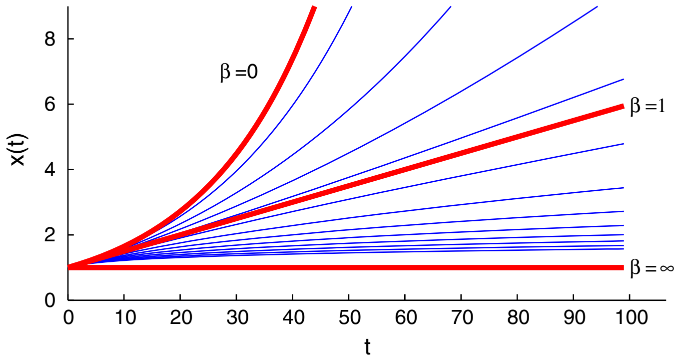

Fig. 1 A family of growth paths indexed by β

We suggest the following criterion as defining *regular growth*:

$$g_2(t) = -\beta g_1(t) \quad \text{for all } t \geq 0, \tag{1}$$

where $\beta \geq 0$. That is, the second-order growth rate is proportional to the first-order growth rate with a non-positive factor of proportionality. The coefficient β is called the *damping coefficient*, since it indicates the rate of damping in the growth process.

Let x_0 and α denote the initial values $x(0) > 0$ and $g_1(0) > 0$, respectively. The unique solution of the second-order differential equation (1) may then be expressed as:

$$x(t) = x_0 (1 + \alpha\beta t)^{\frac{1}{\beta}}. \tag{2}$$

Note that this solution has at least one well-known special case, namely $x(t) = x_0 e^{\alpha t}$ for $\beta = 0$.[3] Moreover, it should be observed that, given x_0, (2) is also the unique solution of the first-order equation:

$$\dot{x}(t) = \alpha x_0^{\beta} x(t)^{1-\beta}, \quad \alpha > 0, \beta \geq 0, \tag{3}$$

which is an autonomous Bernoulli equation. This gives an alternative and equivalent characterization of regular growth. The feature that $x(t)$ here has a constant exponent fits well with economists' preference for constant elasticity functional forms.

The simple formula (2) describes a family of growth paths, the members of which are indexed by the damping coefficient β. Figure 1 illustrates this family of regular growth paths.[4] There are three well-known special cases. For $\beta = 0$, we have $g_1(t) = \alpha$, a positive constant. This is the case of exponential growth. At the other extreme we have complete stagnation, i.e., the constant path $x(t) = x_0$. This can be

[3] To see this, use L'Hôpital's rule for "0/0" on $\ln(x(t)) = \ln(x_0) + \frac{1}{\beta}\ln(1 + \alpha\beta t)$.

[4] Figure 1 is based on $\alpha = 0.05$ and $x_0 = 1$. In this case, the time paths do not intersect. Intersections occur for $x_0 < 1$. However, for large t the picture always is as shown in Fig. 1.

Table 1 Regular growth paths: $g_2(t) = -\beta g_1(t)\ \forall t \geq 0, \beta \geq 0,\ g_1(0) = \alpha > 0$

Label	Damping coefficient	Time path
Limiting case 1: exponential growth	$\beta = 0$	$x(t) = x_0 e^{\alpha t}, \alpha > 0$
More-than-arithmetic growth	$0 < \beta < 1$	$x(t) = x_0(1 + \alpha\beta t)^{\frac{1}{\beta}}, \alpha > 0$
Arithmetic growth	$\beta = 1$	$x(t) = x_0(1 + \alpha t), \alpha > 0$
Less-than-arithmetic growth	$1 < \beta < \infty$	$x(t) = x_0(1 + \alpha\beta t)^{\frac{1}{\beta}}, \alpha > 0$
Limiting case 2: stagnation	$\beta = \infty$	$x(t) = x_0$

interpreted as the limiting case $\beta \to \infty$.[5] Arithmetic growth, i.e., $\dot{x}(t) = \alpha$, for all $t \geq 0$, is the special case $\beta = 1$.

Table 1 lists these three cases and gives labels also to the intermediate ranges for the value of the damping coefficient β. Apart from being written in another (and perhaps less "family-oriented") way, the "quasi-arithmetic growth" formula in Asheim et al. (2007) mentioned above, is subsumed under these intermediate ranges.

As to the case $\beta > 1$, notice that though the increase in x per time unit is falling over time, it remains positive; there is sustained growth in the sense that $x(t) \to \infty$ for $t \to \infty$.[6] Formally, also the case of $\beta < 0$ (more-than-exponential growth) could be included in the family of regular growth paths. However, this case should be considered as only relevant for a description of possible phases of *transitional* dynamics. A growth path (for, say, GDP per capita) with $\beta < 0$ is explosive in a very dramatic sense: it leads to infinite output in finite time (Solow 1994).

It is clear that with $0 < \beta < \infty$, the solution formula (2) cannot be extended, without bound, *backward* in time. For $t = -(\alpha\beta)^{-1} \equiv \bar{t}$, we get $x(t) = 0$, and thus, according to (3), $x(t) = 0$ for all $t \leq \bar{t}$. This should not, however, be considered a necessarily problematic feature. A certain growth regularity need not be applicable to all periods in history. It may apply only to specific historical epochs characterized by a particular institutional environment.[7]

By adding one parameter (the damping coefficient β), we have succeeded spanning the whole range of sustained growth patterns between exponential growth and complete stagnation. Our conjecture is that there are no other one-parameter extensions of exponential growth with this property (but we have no proof). In any case, as witnessed by the examples in the next sections, the extension has relevance for real-world economic problems. It is, of course, possible—and likely—that one will come across economic growth problems that will motivate adding a second parameter

[5] Use L'Hôpital's rule for "∞/∞" on $\ln x(t)$. If we allow $g_1(0) = 0$, stagnation can of course also be seen as the case $\alpha = 0$.

[6] Empirical investigation of post-WWII GDP per-capita data of a sample of OECD countries yields positive damping coefficients between 0.17 (UK) and 1.43 (Germany). The associated initial (annual) growth rates in 1951 are 2.3% (UK) and 12.4% (Germany), respectively. The fit of the regular growth formula is remarkable. This is not a claim, of course, that this data is better described as regular growth with damping than as transition to exponential growth. Yet, discriminating between the two should be possible in principle.

[7] Here, we disagree with Hakenes and Irmen (2007) who find a growth formula (for technical knowledge) implausible, if its unbounded extension backward in time implies a point where knowledge vanishes.

or introducing other functional forms. Exploring such extensions is beyond the scope of this paper.[8]

Before we discuss our economic examples of regular growth, a word on terminology is appropriate. Our reason for introducing the term “regular growth” for the described class of growth paths is that we want an inclusive name, whereas, for example, “quasi-arithmetic growth” will probably in general be taken to exclude the limiting cases of exponential growth and complete stagnation.

3 Example 1: R&D-based growth

As our first example of the regularity described above, we consider an optimal growth problem within the Romer (1990)–Jones (1995) framework. The labor force (= population), L, is governed by $L = L_0 e^{nt}$, where $n \geq 0$ is constant (this is a common assumption in most growth models whether $n = 0$, as with Romer, or $n > 0$, as with Jones). The idea of the example is to follow Jones' relaxation regarding Romer's value of the elasticity of knowledge creation (with respect to existing knowledge) but at the same time, contrary to Jones, allow $n = 0$ as well as a vanishing pure rate of time preference. We believe the case $n = 0$ is pertinent not only for theoretical reasons, but also because it is of practical interest in view of the projected stationarity of the population of developed countries as a whole already from 2005 (United Nations 2005).

The technology of the economy is described by constant elasticity functional forms:[9]

$$Y = A^{\sigma} K^{\alpha} (uL)^{1-\alpha}, \quad \sigma > 0, 0 < \alpha < 1, \tag{4}$$

$$\dot{K} = Y - cL, \quad K(0) = K_0 > 0 \text{ given}, \tag{5}$$

$$\dot{A} = \gamma A^{\varphi} (1-u) L, \quad \gamma > 0, \varphi \leq 1, A(0) = A_0 > 0 \text{ given}, \tag{6}$$

where Y is aggregate manufacturing output (net of capital depreciation), A society's stock of “knowledge”, K society's capital, u the fraction of the labor force employed in manufacturing, and c per-capita consumption; σ, α, γ, and φ are constant parameters. The criterion functional of the social planner is:

$$U_0 = \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} L e^{-\rho t} \mathrm{d}t,$$

where $\theta > 0$ and $\rho \geq n$. In the spirit of Ramsey (1928) we include the case $\rho = 0$, since giving less weight to future than to current generations might be deemed “ethi-

8 However, an interesting paper by Growiec (2008) takes steps in this direction. We may add that this paper, as well as the constructive comments by its author on the working paper version of the present article, has taught us that *reducing* the number of problematic knife-edge restrictions is not the same as “getting rid of” knife-edge assumptions concerning parameter values and/or functional forms.

9 From now, the explicit timing of the variables is suppressed when not needed for clarity.

cally indefensible". When $\rho = n$, there exist feasible paths for which the integral U_0 does not converge. In that case our optimality criterion is the catching-up criterion (see Case 4 below). The social planner chooses a plan $(c(t), u(t))_{t=0}^{\infty}$, where $c(t) > 0$ and $u(t) \in [0, 1]$, to optimize U_0 under the constraints (4), (5) and (6) as well as $K \geq 0$, and $A \geq 0$, for all $t \geq 0$. From now, the (first-order) growth rate of any positive-valued variable v will be denoted g_v.

Case 1: $\varphi = 1, \rho > n = 0$. This is the fully endogenous growth case considered by Romer (1990).[10] An interior optimal solution converges to exponential growth with growth rate $g_c = (1/\theta)\left[\sigma\gamma L/(1-\alpha) - \rho\right]$ and $u = 1 - (1-\alpha)g_c/(\sigma\gamma L)$.[11]

Case 2: $\varphi < 1, \rho > n > 0$. This is the semi-endogenous growth case considered by Jones (1995). An interior optimal solution converges to exponential growth with growth rate $g_c = n/(1-\varphi)$ and $u = \frac{(\sigma/(1-\alpha))(\theta-1)n+(1-\varphi)\rho}{(\sigma/(1-\alpha))\theta n+(1-\varphi)\rho}$.[12]

Case 3: $\varphi < 1, \rho > n = 0$. In this case, the economy ends up in complete stagnation (constant c) with all labor in the manufacturing sector, as is indicated by setting $n = 0$ in the formula for u in Case 2. The explanation is the combination of a) no population growth to countervail the diminishing marginal returns to knowledge ($\partial \dot{A}/\partial A \to 0$ for $A \to \infty$), and b) a positive constant rate of time preference.

Case 4: $\varphi < 1, \rho = n = 0$. This is the canonical Ramsey case. Depending on the values of φ, σ, α and θ, a continuum of dynamic processes for A, K, Y, and c emerges which fill the whole range between stagnation and exponential growth. Since this case does not seem investigated in the literature, we shall spell it out here. The optimality criterion is the *catching-up criterion:* a feasible path $(\hat{K}, \hat{A}, \hat{c}, \hat{u})_{t=0}^{\infty}$ is catching-up optimal if

$$\lim_{t\to\infty} \inf \left(\int_0^t \frac{\hat{c}^{1-\theta} - 1}{1-\theta} \mathrm{d}\tau - \int_0^t \frac{c^{1-\theta} - 1}{1-\theta} \mathrm{d}\tau \right) \geq 0$$

for all feasible paths $(K, A, c, u)_{t=0}^{\infty}$.

Let p be the shadow price of knowledge in terms of the capital good. Then, the value ratio $x \equiv pA/K$ is capable of being stationary in the long run. Indeed, as shown in Appendix A, the first-order conditions of the problem lead to

$$\dot{x} = \frac{\gamma L A^{\varphi-1}}{1-\alpha} \left\{(\alpha - s)xu - [\sigma + (1-\alpha)(1-\varphi)]\, u + (1-\alpha)(1-\varphi)\right\} x, \quad (7)$$

[10] Contrary to Romer (1990), though, we permit $\sigma \neq 1-\alpha$ since that still allows stable, fully endogenous growth and, in addition, avoids blurring countervailing effects (see Alvarez-Pelaez and Groth 2005).

[11] With $\varphi = 1$, an $n > 0$ would generate an implausible ever-increasing growth rate.

[12] The Jones (1995) model also includes a negative duplication externality in R&D, which is not of importance for our discussion. Convergence of this model is shown in Arnold (2006). In both Case 1 and Case 2 boundedness of the utility integral U_0 requires that parameters are such that $(1-\theta)g_c < \rho - n$.

where $s = 1 - cL/Y$ is the saving rate; further,

$$\dot{u} = \frac{\gamma L A^{\varphi-1}}{1-\alpha}\left[-(1-s)xu + \sigma u + \frac{1-\alpha}{\alpha}\sigma\right]u, \text{ and} \tag{8}$$

$$\dot{s} = \frac{\gamma L A^{\varphi-1}}{1-\alpha}\left[-\left(\frac{1-\theta}{\theta}\alpha + 1 - s\right)xu + \frac{1-\alpha}{\alpha}\sigma\right](1-s). \tag{9}$$

Provided $\theta > 1$, this dynamic system has a unique steady state:

$$\begin{aligned} x^* &= \frac{\sigma\theta}{\alpha(\theta-1)} > \frac{\sigma}{\alpha}, \quad u^* = \frac{(\theta-1)\left[\sigma + \alpha(1-\varphi)\right]}{\theta\sigma + (\theta-1)\alpha(1-\varphi)} \in (0,1), \\ s^* &= \frac{\alpha(\sigma+1-\varphi)}{\theta\left[\sigma+\alpha(1-\varphi)\right]} \in \left(\frac{\alpha}{\theta}, \frac{1}{\theta}\right). \end{aligned} \tag{10}$$

The resulting paths for A, K, Y, and c feature regular growth with positive damping. This is seen in the following way. First, given $u = u^*$, the innovation equation (6) is a Bernoulli equation of form (3) and has the solution

$$A(t) = \left[{A_0}^{1-\varphi} + (1-\varphi)\gamma(1-u^*)Lt\right]^{\frac{1}{1-\varphi}} = A_0\left(1+\mu t\right)^{\frac{1}{1-\varphi}}, \tag{11}$$

where $\mu \equiv (1-\varphi)\gamma(1-u^*)L{A_0}^{\varphi-1} > 0$. Second, the optimality condition saying that at the margin, time must be equally valuable in its two uses, implies the same value of the marginal product of labor in the two sectors, that is, $p\gamma A^{\varphi} = (1-\alpha)Y/(uL)$. Substituting (4) into this equation, we see that

$$x \equiv \frac{pA}{K} = \frac{(1-\alpha)A^{\sigma+1-\varphi}}{\gamma K^{1-\alpha}(uL)^{\alpha}}. \tag{12}$$

Thus, solving for K yields, in the steady state,

$$K(t) = (u^*L)^{\frac{-\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\gamma x^*}\right)^{\frac{1}{1-\alpha}} {A_0}^{\frac{\sigma+1-\varphi}{1-\alpha}}(1+\mu t)^{\frac{\sigma+1-\varphi}{(1-\alpha)(1-\varphi)}}. \tag{13}$$

The resultant path for Y is

$$\begin{aligned} Y(t) &= A(t)^{\sigma}K(t)^{\alpha}(u^*L)^{1-\alpha} \\ &= (u^*L)^{\frac{1-2\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\gamma x^*}\right)^{\frac{\alpha}{1-\alpha}} {A_0}^{\frac{\sigma+\alpha(1-\varphi)}{1-\alpha}}(1+\mu t)^{\frac{\sigma+\alpha(1-\varphi)}{(1-\alpha)(1-\varphi)}}. \end{aligned} \tag{14}$$

Finally, per capita consumption is given by $c(t) = (1-s^*)Y(t)/L$. The assumption that $\theta > 1$ (which seems to be consistent with the microeconometric evidence, see

Attanasio and Weber 1995) is needed to avoid postponement *forever* of the consumption return to R&D.[13]

When $0 < \varphi < 1$ (the "standing on the shoulders" case), the damping coefficient for knowledge growth equals $1-\varphi < 1$, i.e., knowledge features more-than-arithmetic growth. When $\varphi < 0$ (the "fishing out" case), the damping coefficient is $1 - \varphi > 1$, and knowledge features less-than-arithmetic growth. In the intermediate case, $\varphi = 0$, knowledge features arithmetic growth. The coefficient μ, which equals the initial growth rate times the damping coefficient, could be called the *growth momentum*. It is seen to incorporate a *scale effect* from L. This is as expected, in view of the non-rival character of technical knowledge.

The time paths of K and Y also feature regular growth, though with a damping coefficient different from that of technology. The time path of Y, to which the path of c is proportional, features more-than-arithmetic growth if and only if $\sigma > (1-2\alpha)(1-\varphi)$. A sufficient condition for this is that $\frac{1}{2} \leq \alpha < 1$. It is interesting that $\varphi > 0$ is not needed; the reason is that even if knowledge exhibits less-than-arithmetic growth ($\varphi < 0$), this may be compensated by high-enough production elasticities with respect to knowledge or capital in the manufacturing sector. Notice also that the capital–output ratio features exactly arithmetic growth *always* along the regular growth path of the economy, i.e., independently of the size relation between the parameters. Indeed, $K/Y = [K(0)/Y(0)]\,(1 + \mu t)$. This is like in Hartwick's rule (Solow 1974). A mirror image of this is that the marginal product of capital always approaches zero for $t \to \infty$, a property not surprising in view of $\rho = 0$.

Is the regular growth path robust to small disturbances in the initial conditions? The answer is yes: the regular growth path is locally saddle-point stable. That is, if the pre-determined initial value of the ratio, $A^{\sigma+1-\varphi}/K^{1-\alpha}$, is in a small neighborhood of its steady-state value (which is $\gamma L^{\alpha} x^* u^{*\alpha}/(1-\alpha)$), then the dynamic system (7), (8), and (9) has a unique solution $(x_t, u_t, s_t)_{t=0}^{\infty}$ and this solution converges to the steady state (x^*, u^*, s^*) for $t \to \infty$ (see Appendix A). Thus, the time paths of A, K, Y, and c approach regular growth in the long run.

Of course, exactly constant population is an abstraction but, for example, logistic population growth should over time lead to approximately the same pattern. Admittedly, also the nil time-preference rate is a particular case, but in our opinion not the least interesting one in view of its benchmark character as an expression of a canonical ethical principle.[14]

4 Example 2: learning by doing

In the first example, regular non-exponential growth arose in the Ramsey case with a zero rate of time preference. Are there examples with a *positive* rate of time preference?

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[13] The conjectured necessary and sufficient transversality conditions (see Appendix A) require $\theta > (\sigma+1-\phi)/[\sigma + \alpha(1-\phi)]$, which we assume to be satisfied. This condition is a little stronger than the requirement $\theta > 1$.

[14] The entire spectrum of regular growth patterns can also be obtained in an elementary version of the Jones (1995) model with no capital, but two types of (immobile) labor, i.e., unskilled labor in final goods production and skilled labor in R&D.

This question was raised by Chad Jones (private correspondence), who kindly suggested us to look at learning by doing. The answer to the question turns out be a yes.

Assume there is learning by doing in the following form:

$$\dot{A} = \gamma A^{\varphi} L, \qquad \gamma > 0, \varphi < 1, \; A(0) = A_0 > 0 \text{ given}, \tag{15}$$

where, as before, A is an index of productivity at time t and L is the labor force (= population).[15] As noted in the introduction, the case $\varphi = 1$, combined with constant L, and the case $\varphi < 1$ combined with exponential growth in L, are well understood. And the case $\varphi > 1$ leads to explosive growth. But the remaining case, $\varphi < 1$, combined with constant L, has to our knowledge not received much attention, possibly because of the absence of a conceptual framework for the kind of regularity which arises in this case. Moreover, this case is also of interest because its dynamics turn out to reappear as a sub-system of the more elaborate example with embodied technical change in the next section.

The Bernoulli equation (15) has the solution

$$A(t) = \left[A_0^{1-\varphi} + (1-\varphi)\gamma L t \right]^{1/(1-\varphi)}. \tag{16}$$

Thus, A features regular growth. We wish to see whether, in the problem below, also Y, K, and c feature regular growth when $\rho > 0$.[16]

The social planner chooses a plan $(c(t))_{t=0}^{\infty}$ so as to maximize

$$U_0 = \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} L e^{-\rho t} \mathrm{d}t \quad \text{s.t.}$$

$$\dot{K} = Y - cL - \delta K, \qquad \delta \geq 0, \; K(0) = K_0 > 0 \text{ given}, \tag{17}$$

where

$$Y = A^{\sigma} K^{\alpha} L^{1-\alpha}, \qquad \sigma > 0, 0 < \alpha < 1, \tag{18}$$

with the time path of A given by (16). Whereas the previous example assumed that *net* output was described by a Cobb–Douglas production function, here it can be *gross* output as well. The current-value Hamiltonian is

$$H(K, c, \lambda, t) = \frac{c^{1-\theta} - 1}{1-\theta} L + \lambda (A^{\sigma} K^{\alpha} L^{1-\alpha} - cL - \delta K),$$

[15] As an alternative to our "learning-by-doing" interpretation of (15), one might invoke a "population-breeds-ideas" hypothesis. In his study of the very-long run history of population Kremer (1993) combines such an interpretation of (15) with a Malthusian story of population dynamics.

[16] In order to allow potential scale effects to be visible, we do not normalize L to 1.

where λ is the co-state variable associated with physical capital. Necessary first-order conditions for an interior solution are:

$$\frac{\partial H}{\partial c} = c^{-\theta}L - \lambda L = 0, \tag{19}$$

$$\frac{\partial H}{\partial K} = \lambda\left(\alpha\frac{Y}{K} - \delta\right) = -\dot{\lambda} + \rho\lambda. \tag{20}$$

These conditions, combined with the transversality condition,

$$\lim_{t\to\infty} \lambda(t)\mathrm{e}^{-\rho t}K(t) = 0, \tag{21}$$

are *sufficient* for an optimal solution. Owing to strict concavity of the Hamiltonian with respect to (K, c) this solution will be unique, if it exists (see Appendix B).

It remains to show the existence of such a path. Combining (19) and (20) gives the Keynes–Ramsey rule

$$g_c = \frac{1}{\theta}\left(\alpha\frac{Y}{K} - \delta - \rho\right). \tag{22}$$

Let $v \equiv cL/K$ and log-differentiate v with respect to time to get

$$g_v = \frac{1}{\theta}(\alpha z - \delta - \rho) - (z - v - \delta),$$

where

$$z \equiv \frac{Y}{K} = A^{\sigma}K^{\alpha-1}L^{1-\alpha}.$$

Log-differentiating z with respect to time gives

$$g_z = \sigma\gamma A^{\varphi-1}L + (\alpha - 1)(z - v - \delta).$$

Thus, we have a system in v and z:

$$\dot{v} = \left[\frac{1}{\theta}(\alpha z - \delta - \rho) - (z - v - \delta)\right]v,$$

$$\dot{z} = \left[\sigma\gamma A^{\varphi-1}L - (1 - \alpha)(z - v - \delta)\right]z,$$

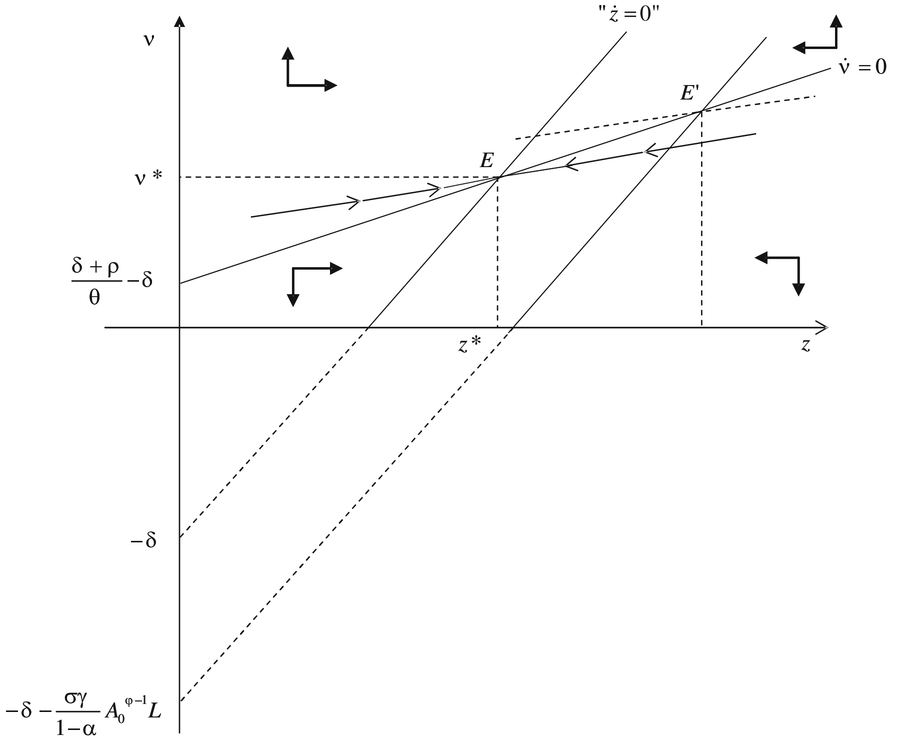

Fig. 2 Phase diagram for the learning-by-doing model

where v is a jump variable and z a pre-determined variable. We have $\sigma\gamma A^{\varphi-1}L \to 0$ for $t \to \infty$. There is an asymptotic steady state, (v^*, z^*), where

$$v^* = \frac{\rho}{\alpha} + \frac{1-\alpha}{\alpha}\delta,$$

$$z^* = v^* + \delta = \frac{\rho+\delta}{\alpha}.$$

The investment–capital ratio, $(Y - cL)/K \equiv z - v$, in this asymptotic steady state is $z^* - v^* = \delta$. The associated Jacobian is

$$J = \begin{bmatrix} v^* & (\frac{\alpha}{\theta}-1)v^* \\ (1-\alpha)z^* & -(1-\alpha)z^* \end{bmatrix},$$

with determinant $\det J = -(1-\alpha)v^*z^* - (\frac{\alpha}{\theta}-1)(1-\alpha)v^*z^* = -\frac{\alpha}{\theta}(1-\alpha)v^*z^* < 0$. The eigenvalues of J are, thus, of opposite sign.

Figure 2 contains an illustrating phase diagram. The line marked by "$\dot{z} = 0$" is the locus for $\dot{z} = 0$ only in the long run. The path (with arrows) through the point E is the "long-run saddle path". If the level of $A^{\varphi-1}$ remained at its initial value, $A_0^{\varphi-1}$, the point E' would be a steady state and have a saddle path going through it (as illustrated by the dashed line through E'). But over time, $A^{\varphi-1}$ decreases and approaches zero. Hence, the point E' shifts and approaches the long-run steady state, E.[17]

The following relations must hold asymptotically:

$$\frac{Y}{K} = \frac{A^\sigma K^\alpha L^{1-\alpha}}{K} = z^* \text{ so that}$$

$$K^{1-\alpha} = \frac{A^\sigma L^{1-\alpha}}{z^*} \text{ or}$$

$$K(t) = z^{*-\frac{1}{1-\alpha}} A(t)^{\frac{\sigma}{1-\alpha}} L = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} L \left[A_0^{1-\varphi} + (1-\varphi)\gamma L t\right]^{\frac{\sigma}{(1-\alpha)(1-\varphi)}}$$

$$= \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} L A_0^{\frac{\sigma}{1-\alpha}} (1+\mu t)^{\frac{\sigma}{(1-\alpha)(1-\varphi)}}, \quad \text{where } \mu \equiv (1-\varphi)\gamma A_0^{\varphi-1} L > 0.$$

Thus, in the long run, K features regular growth with positive damping. The damping coefficient is $\frac{(1-\alpha)(1-\varphi)}{\sigma}$, which may be above or below one, depending on σ. In the often considered benchmark case, $\sigma = 1-\alpha$, the damping coefficient is less than one if $\varphi > 0$. Then K features more-than-arithmetic growth. The growth momentum is μ and is seen to incorporate a scale effect (reflecting the non-rival character of learning). Although K is growing, the growth rate of K tends to zero. The investment–capital ratio, $(Y - cL)/K$, tends to δ; thus, the saving rate, $s \equiv 1 - cL/Y$, tends to $\delta K/Y = \delta/z^*$.

As to manufacturing output, we have in the long run

$$Y(t) = z^* K(t) = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}} L A_0^{\frac{\sigma}{1-\alpha}} (1+\mu t)^{\frac{\sigma}{(1-\alpha)(1-\varphi)}},$$

which is, of course, also regular growth with positive damping. A similar pattern is then true for the marginal product of labor $w(t) = (1-\alpha)Y(t)/L$. The output–capital ratio tends to a constant in the long run. Per capita consumption, $c(t) = (1 - s(t))Y(t)/L$, tends to $(1-\delta/z^*)Y(t)/L$. Finally, the net marginal product of capital, $\alpha Y(t)/K(t) - \delta$, tends to

$$\alpha z^* - \delta = \rho.$$

This explains why the growth rate of consumption tends to zero.

Although the asymptotic steady state is never reached, the conclusion is that K, Y, and c in the long run are arbitrarily close to a regular growth pattern with a damping coefficient, $\frac{(1-\alpha)(1-\varphi)}{\sigma}$, and a growth momentum, μ, the same for all three variables. In spite of the absence of exponential growth, key ratios such as Y/K and wL/Y tend to be constant in the long run.

[17] We shall not here pursue the potentially interesting dynamics going on *temporarily*, if z_0 is above z^* but below the value associated with the point E'.

The purpose of this example was to show that a positive rate of time preference, ρ, is no hindrance to such an outcome.[18] Given that the regular growth pattern was inherited from the independent technology path described by (16), this conclusion is perhaps no surprise. In the next section, we consider an example where there is mutual dependence between the development of technology and the remainder of the economy.

5 Example 3: investment-specific learning and embodied technical change

Motivated by the steady decline of the relative price of capital equipment and the secular rise in the ratio of new equipment investment to GNP, Greanwood et al. (1997) developed a tractable model with *embodied* technical change. The framework has afterwards been applied and extended in different directions. One such application is that of Boucekkine et al. (2003).[19] They show that a relative shift from general to investment-specific learning externalities may explain the simultaneous occurrence of a faster decline in the price of capital equipment and a productivity slowdown in the 1970s after the first oil price shock.

In this section we present a related model and show that regular, but less-than-exponential growth may arise. To begin with, we allow for population growth in order to clarify the role of this aspect for the long-run results. Notation is as above, unless otherwise indicated. The technology of the economy is described by

$$Y = K^{\alpha} L^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{23}$$

$$\dot{K} = qI - \delta K, \qquad \delta > 0, K(0) = K_0 \text{ given}, \tag{24}$$

$$q = \tilde{\gamma} \left(\int_{-\infty}^{t} I(\tau) \mathrm{d}\tau \right)^{\beta}, \tilde{\gamma} > 0, 0 < \beta < (1-\alpha)/\alpha, q(0) = q_0 \text{ given}, \tag{25}$$

where $L = L_0 \mathrm{e}^{nt}, n \geq 0$, and K_0, q_0, and L_0 are positive. The new variables are $I \equiv Y - cL$, i.e., gross investment, and q which denotes the quality (productivity) of newly produced investment goods. There is learning by investing, but new learning is incorporated only in newly produced investment goods (this is the embodiment hypothesis). Thus, over time each new investment good gives rise to a greater and greater addition to the capital stock, K, measured in constant efficiency units. The quality q of investment goods of the current vintage is determined by cumulative aggregate gross investment as indicated by (25). The parameter β is named the "learning parameter". The upper bound on β is brought in to avoid explosive growth (infinite output in finite time). We assume capital goods cannot be converted back into consumption goods. So gross investment, I, is always non-negative.

18 Presupposing $\delta > 0$, qualitatively the same outcome—asymptotic regular growth—emerges for $\rho = 0$ (although in this case we have to use catching-up as optimality criterion).

19 We are thankful to Solow for suggesting that embodied technical change might fit our approach and to a referee for suggesting in particular a look at the Boucekkine et al. (2003) paper.

As we will see, with this technology and the same preferences as in the previous example, including a positive rate of time preference, the following holds. (a) If $n > 0$, the social planner's solution features exponential growth. (b) If $n = 0$, the solution features asymptotic quasi-arithmetic growth; in the limiting case $\beta = (1-\alpha)/\alpha$, asymptotic exponential growth arises, whereas the case $\beta > (1-\alpha)/\alpha$ implies explosive growth. Before proceeding further, it is worth pointing out two key differences between the present model and that of Boucekkine et al. (2003). In their paper q is determined by cumulative *net* investment. We find it more plausible to have learning associated with *gross* investment. And, in fact, this difference turns out to be crucial for whether $n = 0$ leads to quasi-arithmetic growth or merely stagnation. Another difference is that in the spirit of our general endeavor, we impose no knife-edge condition on the learning parameter.[20]

Since not even the exponential growth case of this model seems explored in the literature, our exposition will cover that case as well as the less-than-exponential growth case. Many of the basic formulas are common but imply different conclusions depending on the value of n.

5.1 The general context

By taking the time derivative on both sides of (25) we get the more convenient differential form

$$\dot{q} = \gamma q^{(\beta-1)/\beta} I = \gamma q^{(\beta-1)/\beta}(Y - cL), \qquad \gamma \equiv \tilde{\gamma}^{1/\beta}\beta. \tag{26}$$

Given $\rho > n$ and initial positive $K(0)$ and $q(0)$, the social planner chooses a plan $(c(t))_{t=0}^{\infty}$, where $0 < c(t) \leq Y(t)/L(t)$, so as to maximize

$$U_0 = \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta} L e^{-\rho t} dt$$

subject to (24), (26), and non-negativity of K for all t. From the first-order conditions for an interior solution we find (see Appendix C) that the Keynes–Ramsey rule takes the form

$$g_c = \frac{1}{\theta}(\alpha z - m\delta - \rho), \tag{27}$$

where $z \equiv qY/K$ (the modified output–capital ratio) and $m \equiv pq$ with p denoting the shadow price of the capital good in terms of the consumption good. Thus, z is a modified output–capital ratio and m is the shadow price of newly produced investment

[20] Differences of minor importance from our perspective include, first, that Boucekkine et al. (2003) let the embodied learning effect come from accumulated (net) investment *per capita* (presumably to avoid any kind of scale effect), second, that they combine this effect with a disembodied learning effect.

goods in terms of the consumption good. Let $v \equiv qcL/K$ (the modified consumption–capital ratio), so that, by (24), the growth rate of K is $g_K = z - v - \delta$. Further, let $h \equiv \gamma Y/q^{1/\beta}$, so that, by (26), the growth rate of q is $g_q = (1 - v/z)h$; that is, $1 - v/z$ is the saving rate, which we will denote s, and h is the highest possible growth rate of the quality of newly produced investment goods. Then, combining the first-order conditions and the dynamic constraints (24) and (26) yields the dynamic system:

$$\dot{m} = \left[\frac{1-m}{m}(\delta m - \alpha z) + \left(1 - \frac{v}{z}\right)h\right]m, \tag{28}$$

$$\dot{v} = \left[\frac{1}{\theta}(\alpha z - \delta m - \rho) - (z - v - \delta - n) + \left(1 - \frac{v}{z}\right)h\right]v, \tag{29}$$

$$\dot{z} = \left[-(1-\alpha)(z - v - \delta - n) + \left(1 - \frac{v}{z}\right)h\right]z, \tag{30}$$

$$\dot{h} = \left[\alpha(z - v - \delta - n) + n - \frac{1}{\beta}\left(1 - \frac{v}{z}\right)h\right]h. \tag{31}$$

Consider a steady state, (m^*, v^*, z^*, h^*), of this system. In steady state, if $n > 0$, the economy follows a balanced growth path (BGP for short) with constant growth rates of K, q, Y, and c. Indeed, from (30) and (31) we find the growth rate of K to be

$$g_K^* = z^* - v^* - \delta = \frac{(1-\alpha)(1+\beta)}{1-\alpha(1+\beta)}n > n \quad \text{iff } n > 0. \tag{32}$$

The inequality is due to the parameter condition

$$\alpha < 1/(1+\beta) \tag{33}$$

which is equivalent to $\beta < (1-\alpha)/\alpha$, the condition assumed in (25). Then, from (30),

$$g_q^* = s^* h^* = \left(1 - \frac{v^*}{z^*}\right)h^* = \frac{(1-\alpha)\beta}{1-\alpha(1+\beta)}n = \frac{\beta}{1+\beta}g_K^*. \tag{34}$$

In view of constancy of $h \equiv \gamma Y/q^{1/\beta}$,

$$g_Y^* = \frac{1}{\beta}g_q^* = \frac{1}{1+\beta}g_K^*. \tag{35}$$

That is, owing to the embodiment of technical progress Y does not grow as fast as K. This is in line with the empirical evidence mentioned above. Inserting (27), (32), and (34) into (29) we find

$$g_c^* = \frac{1}{\theta}(\alpha z^* - m^*\delta - \rho) = \frac{\alpha\beta}{1-\alpha(1+\beta)}n > 0 \quad \text{iff } n > 0. \tag{36}$$

This result is, of course, also obtained if we use constancy of v^*/z^* to conclude that $g_c^* = g_Y^* - n$. To ensure boundedness of the discounted utility integral we impose the parameter restriction

$$(1-\theta)\frac{\alpha\beta}{1-\alpha(1+\beta)}n < \rho - n, \tag{37}$$

which is equivalent to $(1-\theta)g_c^* < \rho - n$.

With these findings we get from (28)

$$m^* = \frac{\alpha(\theta g_c^* + \rho)}{(1-\alpha+\alpha\theta)g_c^* + \alpha\rho} = \frac{\theta\alpha\beta n + [1-\alpha(1+\beta)]\,\rho}{(1-\alpha+\alpha\theta)n + [1-\alpha(1+\beta)]\,\rho} \leq 1, \tag{38}$$

if $n \geq 0$, respectively. The parameter restriction (37) implies $m^* > \alpha$. Next, from (36),

$$z^* = \frac{\theta\beta}{1-\alpha(1+\beta)}n + \frac{\rho + \delta m^*}{\alpha} > 0, \tag{39}$$

so that, from (32),

$$v^* = \frac{\theta\beta - (1-\alpha)(1+\beta)}{1-\alpha(1+\beta)}n + \frac{\rho+\delta m^*}{\alpha} - \delta, \tag{40}$$

and

$$s^* \equiv 1 - \frac{v^*}{z^*} = \alpha\frac{(1-\alpha)(1+\beta)n + [1-\alpha(1+\beta)]\,\delta}{\theta\alpha\beta n + [1-\alpha(1+\beta)]\,(\rho+\delta m^*)} \in (0, 1). \tag{41}$$

That $s^* > 0$ is immediate from the formula. And $s^* < 1$ is implied by $v^* < z^*$, which immediately follows by comparing (40) and (39). Finally, we have from (34)

$$h^* = \frac{g_q^*}{s^*} = \frac{(1-\alpha)\beta n}{[1-\alpha(1+\beta)]\,s^*} \geq 0 \quad \text{for } n \geq 0, \tag{42}$$

respectively.

In a BGP the shadow price p ($\equiv m/q$) of the capital good in terms of the consumption good is falling since m is constant while q is rising. Indeed,

$$g_p^* = -g_q^* = -\frac{(1-\alpha)\beta}{1-\alpha(1+\beta)}n = -\frac{\beta}{1+\beta}g_K^*. \tag{43}$$

Thus, at the same time as Y/K is falling, the *value* capital–output ratio $Y/(pK)$ stays constant in a BGP. If r denotes the social planner's marginal net rate of return in terms of the consumption good, we have $r = [\partial Y/\partial K - (p\delta - \dot{p})]/p$. Since $p \equiv m/q$ and $z \equiv qY/K$, we have $(\partial Y/\partial K)/p = \alpha Y/(pK) = \alpha z^*/m^*$. Along the BGP, therefore,

$$r^* = \alpha \frac{z^*}{m^*} - (\delta - g_p^*) = \frac{\theta\alpha\beta}{1-\alpha(1+\beta)} n + \rho = \theta g_c^* + \rho, \tag{44}$$

as expected. Since the investment good and the consumption good are produced by the same technology, we can alternatively calculate r as the marginal net rate of return to investment: $r = (\partial Y/\partial K - p\delta)\,\partial \dot{K}/\partial I = (\alpha Y/K - p\delta)q$. In the BGP we then get $r^* = \alpha z^* - m^*\delta$, which according to (36) amounts to the same as (44).

We have hereby shown that if the learning parameter satisfies (33), a steady state of the dynamic system is feasible and features exponential semi-endogenous growth if $n > 0$.[21] On the other hand, violation of (33) combined with a positive n implies a growth potential so enormous that a steady state of the system is infeasible and growth tends to be explosive. But what if $n = 0$?

5.2 The case with zero population growth

With $n = 0$ the formulas above are still valid. As a result, the growth rates g_K^*, g_q^*, g_c^*, and g_p^* are all zero, whereas $m^* = 1$, $z^* = (\rho + \delta)/\alpha$, $v^* = (\rho + \delta)/\alpha - \delta$, $s^* = \alpha\delta/(\rho+\delta) = \delta/z^*$, and $h^* = 0$. By definition we have $h \equiv \gamma Y/q^{1/\beta} > 0$ for all t. So the vanishing value of h^* tells us that the economic system can never attain the steady state. We will now show, however, that the system converges towards this steady state, which is, therefore, an asymptotic steady state.

When $n = 0$ and $\alpha < 1/(1 + \beta)$, we have from purely technological reasons that $\lim_{t\to\infty} h = 0$ (for details, see Appendix C). This implies that for $t \to \infty$ the dynamics of m, v, and z approach the simpler form

$$\begin{aligned}
\dot{m} &= (1-m)(\delta m - \alpha z), \\
\dot{v} &= \left[\frac{1}{\theta}(\alpha z - \delta m - \rho) - (z - v - \delta)\right] v, \\
\dot{z} &= -(1-\alpha)(z - v - \delta)z.
\end{aligned}$$

The associated Jacobian is

$$J = \begin{bmatrix} \rho & 0 & 0 \\ -\frac{\delta}{\theta}v^* & v^* & (\frac{\alpha}{\theta} - 1)v^* \\ 0 & (1-\alpha)z^* & -(1-\alpha)z^* \end{bmatrix}.$$

[21] The standard transversality conditions are satisfied at least if $\theta \geq 1$ (see Appendix C). Owing to non-concavity of the maximized Hamiltonian, however, we have not been able to establish *sufficient* conditions for optimality.

This is block-triangular and so the eigenvalues are ρ and those of the lower right 2×2 sub-matrix of J. Note that this sub-matrix is identical to the Jacobian in the learning-by-doing example of Sect. 4. Accordingly, its eigenvalues are of opposite sign. Since m and v are jump variables and z is pre-determined, it follows that the asymptotic steady state is locally saddle-point stable.[22]

For $t \to \infty$ we, therefore, have $s \equiv 1 - v/z \to 1 - v^*/z^* \equiv s^*$ and $K \to L(q/z^*)^{1/(1-\alpha)}$ (from the definition of z). So, from (26) and (23) it follows that ultimately

$$\dot{q} = \gamma q^{\frac{\beta-1}{\beta}} s^* K^\alpha L^{1-\alpha} = \gamma L s^* z^{*\frac{-\alpha}{1-\alpha}} q^{1-\frac{1-\alpha(1+\beta)}{(1-\alpha)\beta}} \equiv C q^{1-\xi}, \tag{45}$$

where C and ξ are implicitly defined constants. This Bernoulli equation has the solution

$$q(t) = (q_0^\xi + \xi C t)^{\frac{1}{\xi}} = q_0(1+\mu t)^{\frac{1}{\xi}}, \quad \text{where } \mu \equiv \xi L \gamma \delta \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} q_0^{-\xi},$$

using the solutions for s^* and z^* above. This shows that in the long run the productivity of newly produced investment goods features regular growth with damping coefficient $\xi = [1-\alpha(1+\beta)] \,/\, [(1-\alpha)\beta] > 0$ and growth momentum μ (which, as expected, is seen to incorporate a scale effect reflecting the non-rival character of learning). The corresponding long-run path for capital is

$$K(t) = L\left(\frac{q}{z^*}\right)^{\frac{1}{1-\alpha}} = L\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} q_0^{\frac{1}{1-\alpha}} (1+\mu t)^{\frac{1}{(1-\alpha)\xi}}$$

and for output

$$Y(t) = K(t)^\alpha L^{1-\alpha} = L\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} q_0^{\frac{\alpha}{1-\alpha}} (1+\mu t)^{\frac{\alpha}{(1-\alpha)\xi}}.$$

The damping coefficient for Y is, thus, $(1-\alpha)\xi/\alpha = [1-\alpha(1+\beta)]\,/(\alpha\beta)$, so that more-than-arithmetic growth arises if $\frac{1}{2}(1-\alpha)/\alpha < \beta < (1-\alpha)/\alpha$ and less-than-arithmetic growth if β is beneath the lower end of this interval. The same is then true for the marginal product of labor, $w(t) = (1-\alpha)Y(t)/L$, and for per capita consumption, $c(t) = (1-s(t))Y(t)/L$, which tends to $(1-\delta/z^*)Y(t)/L$. For the capital–output ratio we ultimately have $K(t)/Y(t) = q(t)/z^*$, which implies more-than-arithmetic growth if $\beta > 1-\alpha$ and less-than-arithmetic growth if $\beta < 1-\alpha$.

A new interesting facet compared with the learning-by-doing example of Sect. 4 is that the shadow price, p, of capital goods remains falling, although at a decreasing rate. This follows from the fact that the shadow price, $m \equiv pq$, of newly produced investment goods in terms of the consumption good tends to a constant at the same

[22] The unique converging path unconditionally satisfies the standard transversality conditions (see Appendix C).

time as q is growing, although at a decreasing rate. Finally, the *value* output–capital ratio $Y/(pK)$ tends to the constant $(qY/K)m = z^*m^* = z^* = (\rho + \delta)/\alpha$ and the marginal net rate of return to investment tends to $r^* = \alpha Y/(pK) - \delta = \rho$.

These results hold when, in addition to $n = 0$, we have $\alpha < 1/(1 + \beta)$. In the limiting case, $\alpha = 1/(1 + \beta)$, the growth formulas above no longer hold and instead exponential growth arises. Indeed, the system (28), (29), (30), and (31) is still valid and so is (45) in a steady state of the system. But now $\xi = 0$. We, therefore, have in a steady state that $\dot{q} = Cq$, which has the solution $q(t) = q_0 e^{Ct}$, where $C \equiv \gamma L s^* z^{*\frac{-\alpha}{1-\alpha}} > 0$. By constancy of h in the steady state, $g_Y = \alpha g_K = g_q/\beta = C/\beta = \alpha C/(1 - \alpha)$ so that also Y and K grow exponentially. This is the fully endogenous growth case of the model. If instead $\alpha > 1/(1 + \beta)$ we get $\xi < 0$ in (45), implying explosive growth, a not plausible scenario.

We conclude this section with a remark on why, when exponential growth cannot be sustained in a model, sometimes quasi-arithmetic growth results and sometimes complete stagnation. In the present context, where we focus on learning, it is the *source* of learning that matters. Suppose that, contrary to our assumption above, learning is associated with *net* investment, as in Boucekkine et al. (2003). If with respect to the value of the learning parameter we rule out both the knife-edge case leading to exponential growth and the explosive case, then $n = 0$ will lead to complete stagnation. Even if there is an incentive to maintain the capital stock, this requires no net investment and so learning tends to stop. When learning is associated with *gross* investment, however, maintaining the capital stock implies sustained learning. In turn, this induces more investment than needed to replace wear and tear and so capital accumulates, although at a declining rate. Even if there are diminishing marginal returns to capital, this is countervailed by the rising productivity of investment goods due to learning. Similarly, in the learning-by-doing example of Sect. 4, where learning is simply associated with working, learning occurs even if the capital stock is just maintained. Therefore, instead of mere stagnation we get quasi-arithmetic growth.

6 Conclusion

The search for exponential growth paths can be justified by analytical simplicity and the approximate constancy of the long-run growth rate for more than a century in, for example, the US. Yet, this paper argues that growth theory needs a more general notion of regularity than that of exponential growth. We suggest that paths along which the rate of decline of the growth rate is proportional to the growth rate itself deserve attention; this criterion defines our concept of *regular growth*. Exponential growth is the limiting case where the factor of proportionality, the "damping coefficient", is zero. When the damping coefficient is positive, there is less-than-exponential growth, yet this growth exhibits a certain regularity and is sustained in the sense that $Y/L \to \infty$ for $t \to \infty$. We believe that such a broader perspective on growth will prove particularly useful for discussions of the prospects of economic growth in the future, where population growth (and thereby the expansion of the ultimate source of new ideas) is likely to come to an end.

The main advantages of the generalized regularity concept are as follows: (1) The concept allows researchers to reduce the number of problematic parameter restrictions, which underlie both standard neoclassical and endogenous growth models. (2) Since the resulting dynamic process has one more degree of freedom compared to exponential growth, it is at least as plausible in empirical terms. (3) The concept covers a continuum of sustained growth processes which fill the whole range between exponential growth and complete stagnation, a range which may deserve more attention in view of the likely future demographic development in the world. (4) As our analyses of zero population growth in the Jones (1995) model, a learning-by-doing model, and an embodied technical change model show, falling growth rates need not mean that economic development grinds to a halt. (5) Finally, at least for these three examples, we have demonstrated not only the presence of the generalized regularity pattern, but also the asymptotic stability of this pattern.

The examples considered are based on a representative agent framework. Our conjecture is that with heterogeneous agents the generalized notion of regular growth could be of use as well. Likewise, an elaboration of the embodied technical change approach of Sect. 5 might be of empirical interest. For example, Solow (1996) indicates that vintage effects tend to be more visible against a background of less-than exponential growth. As Solow has also suggested,[23] there is an array of "behavioral" assumptions waiting for application within growth theory, in particular growth theory without the straightjacket of exponential growth.

Appendix A: The canonical Ramsey example

This appendix derives the results reported for Case 4 in Sect. 3. The Hamiltonian for the optimal control problem is:

$$H(K, A, c, u, \lambda_1, \lambda_2, t) = \frac{c^{1-\theta} - 1}{1-\theta} L + \lambda_1 (Y - cL) + \lambda_2 \gamma A^{\varphi} (1-u) L,$$

where $Y = A^{\sigma} K^{\alpha} (uL)^{1-\alpha}$ and λ_1 and λ_2 are the co-state variables associated with physical capital and knowledge, respectively. Applying the catching-up optimality criterion, necessary first-order conditions (see Seierstad and Sydsaeter 1987, pp. 232–234) for an interior solution are:

$$\frac{\partial H}{\partial c} = c^{-\theta} L - \lambda_1 L = 0, \tag{46}$$

$$\frac{\partial H}{\partial u} = \lambda_1 (1-\alpha) \frac{Y}{u} - \lambda_2 \gamma A^{\varphi} L = 0, \tag{47}$$

$$\frac{\partial H}{\partial K} = \lambda_1 \alpha \frac{Y}{K} = -\dot{\lambda}_1, \tag{48}$$

$$\frac{\partial H}{\partial A} = \lambda_1 \sigma \frac{Y}{A} + \lambda_2 \varphi \gamma A^{\varphi-1} (1-u) L = -\dot{\lambda}_2. \tag{49}$$

[23] Private communication.

Combining (46) and (48) gives the Keynes–Ramsey rule

$$g_c = \frac{1}{\theta}\alpha A^{\sigma} K^{\alpha-1}(uL)^{1-\alpha}. \tag{50}$$

Given the definition $p = \lambda_2/\lambda_1$, (47), (48), and (49) yield

$$g_p = \alpha A^{\sigma} K^{\alpha-1}(uL)^{1-\alpha} - \frac{\sigma\gamma A^{\varphi-1}uL}{1-\alpha} - \varphi\gamma A^{\varphi-1}(1-u)L. \tag{51}$$

Let $x \equiv pA/K$. Log-differentiating x w.r.t. time and using (47), (6), (5), and (4) gives (7). Log-differentiating (47) w.r.t. time, using (51), (5), (4) and (6), gives (8). Finally, log-differentiating $1 - s \equiv cL/Y$, using (50), (4), (6) and (5), gives (9).

In the text we defined $\mu \equiv (1-\varphi)\gamma(1-u^*)LA_0{}^{\varphi-1}$.

Lemma 1 *In a steady state of the system* (7), (8)*, and* (9)

$$\lambda_1(t)K(t) = \lambda_1(0)K_0(1+\mu t)^{\omega}, \quad \text{and} \tag{52}$$

$$\lambda_2(t)A(t) = \lambda_2(0)A_0(1+\mu t)^{\omega}, \tag{53}$$

where

$$\omega \equiv \frac{\sigma + 1 - \varphi - \theta\left[\sigma + \alpha(1-\varphi)\right]}{(1-\alpha)(1-\varphi)}.$$

Proof As shown in the text, in a steady state of the system we have $Y(t)/K(t) = (Y(0)/K_0)(1+\mu t)^{-1}$ so that

$$\int_0^t \frac{Y(\tau)}{K(\tau)}\mathrm{d}\tau = \frac{Y(0)}{K_0}\mu^{-1}\ln(1+\mu t) = \frac{\theta\left[\sigma+\alpha(1-\varphi)\right]}{\alpha(1-\alpha)(1-\varphi)}\ln(1+\mu t),$$

where the latter equality follows from (13), (11), (10), and the definition of μ. Therefore, by (48) and (13),

$$\begin{aligned}\lambda_1(t)K(t) &= \lambda_1(0)\mathrm{e}^{-\alpha\int_0^t \frac{Y(\tau)}{K(\tau)}\mathrm{d}\tau} K_0(1+\mu t)^{\frac{\sigma+1-\varphi}{(1-\alpha)(1-\varphi)}} \\ &= \lambda_1(0)K_0(1+\mu t)^{\frac{\sigma+1-\varphi}{(1-\alpha)(1-\varphi)} - \frac{\theta[\sigma+\alpha(1-\varphi)]}{(1-\alpha)(1-\varphi)}},\end{aligned}$$

which proves (52).

From (49) and $p \equiv \lambda_2/\lambda_1$ follows that in steady state

$$\frac{\dot{\lambda}_2}{\lambda_2} = -\frac{\sigma Y}{pA} - \varphi\gamma A^{\varphi-1}(1-u^*)L = -\gamma\left(\frac{\sigma u^*}{1-\alpha} + \varphi(1-u^*)\right)LA_0^{\varphi-1}(1+\mu t)^{-1},$$

where the latter equality follows from (4), (12), and (11). Hence,

$$\int_0^t \frac{\dot{\lambda}_2(\tau)}{\lambda_2(\tau)} d\tau = -\frac{\varphi - \sigma - \alpha + \theta\,[\sigma + \alpha(1-\varphi)]}{(1-\alpha)(1-\varphi)} \ln(1+\mu t),$$

by (10) and the definition of μ. Therefore,

$$\begin{aligned}\lambda_2(t)A(t) &= \lambda_2(0)e^{\int_0^t \frac{\dot{\lambda}_2(\tau)}{\lambda_2(\tau)} d\tau} A_0(1+\mu t)^{\frac{1}{1-\varphi}} \\ &= \lambda_2(0)A_0(1+\mu t)^{\frac{1}{1-\varphi} - \frac{\varphi-\sigma-\alpha+\theta[\sigma+\alpha(1-\varphi)]}{(1-\alpha)(1-\varphi)}},\end{aligned}$$

which proves (53). □

We have $\omega < 0$ if and only if

$$\theta > (\sigma + 1 - \varphi)/\,[\sigma + \alpha(1-\varphi)]\,. \tag{54}$$

Hence, by Lemma 1 follows that the "standard" transversality conditions, $\lim_{t\to\infty} \lambda_1(t)K(t) = 0$ and $\lim_{t\to\infty} \lambda_2(t)A(t) = 0$, hold along the unique regular growth path if and only if (54) is satisfied. This condition is a little stronger than $\theta > 1$. Our conjecture is that these transversality conditions together with the first-order conditions are necessary and sufficient for an optimal solution. This guessed necessity and sufficiency is based on the saddle-point stability of the steady state (see below). Yet, we have so far no proof. The maximized Hamiltonian is not jointly concave in (K, A) unless $\sigma = \varphi(1-\alpha)$. Thus, the Arrow sufficiency theorem does not apply; hence, neither does the Mangasarian sufficiency theorem (see Seierstad and Sydsaeter 1987). So, we only have a conjecture. (This is of course not a satisfactory situation, but we might add that this situation is quite common in the semi-endogenous growth literature, although authors are often silent about the issue.)

As to the stability question it is convenient to transform the dynamic system. We do that in two steps. First, let $z \equiv xu$ and $q \equiv (1-s)xu$. Then the system (7), (8), and (9) becomes

$$\begin{aligned}\dot{z} &= \gamma L A^{\varphi-1}\left(1-\varphi+\frac{\sigma}{\alpha}-z-(1-\varphi)u\right)z, \\ \dot{u} &= \gamma L A^{\varphi-1}\left(\frac{\sigma}{\alpha}+\frac{\sigma}{1-\alpha}u-\frac{q}{1-\alpha}\right)u, \\ \dot{q} &= \gamma L A^{\varphi-1}\left(1-\varphi+\frac{\alpha-\theta}{(1-\alpha)\theta}z-(1-\varphi)u+\frac{1}{1-\alpha}q\right)q.\end{aligned}$$

The steady state of this system is $(z^*, u^*, q^*) = (x^*u^*, u^*, (1-s^*)x^*u^*)$. Second, this system can be converted into an autonomous system in "transformed time" $\tau = \ln A(t) \equiv f(t)$. With $u(t) < 1$, $f'(t) = \gamma A(t)^{\varphi-1}(1-u(t))L > 0$ and we have $t = f^{-1}(\tau)$. Thus, considering $\tilde{z}(\tau) \equiv z(f^{-1}(\tau))$, $\tilde{u}(\tau) \equiv u(f^{-1}(\tau))$ and $\tilde{q}(\tau) \equiv q(f^{-1}(\tau))$, the above system is converted into

$$\frac{d\tilde{z}}{d\tau} = \left(1 - \varphi + \frac{\sigma}{\alpha} - \tilde{z} - (1-\varphi)\tilde{u}\right)\frac{\tilde{z}}{1-\tilde{u}},$$
$$\frac{d\tilde{u}}{d\tau} = \left(\frac{\sigma}{\alpha} + \frac{\sigma}{1-\alpha}\tilde{u} - \frac{\tilde{q}}{1-\alpha}\right)\frac{\tilde{u}}{1-\tilde{u}},$$
$$\frac{d\tilde{q}}{d\tau} = \left(1 - \varphi + \frac{\alpha-\theta}{(1-\alpha)\theta}\tilde{z} - (1-\varphi)\tilde{u} + \frac{1}{1-\alpha}\tilde{q}\right)\frac{\tilde{q}}{1-\tilde{u}}.$$

The Jacobian of this system, evaluated in steady state, is

$$J = \begin{bmatrix} -z^* & -(1-\varphi)z^* & 0 \\ 0 & \frac{\sigma}{1-\alpha}u^* & -\frac{1}{1-\alpha}u^* \\ \frac{\alpha-\theta}{(1-\alpha)\theta}q^* & -(1-\varphi)q^* & \frac{1}{1-\alpha}q^* \end{bmatrix} \cdot \frac{1}{1-u^*}.$$

The determinant is

$$\det J = -\frac{\sigma\theta + (\theta-1)(1-\varphi)\alpha}{(1-\alpha)^2\theta} z^* u^* q^* < 0,$$

in view of $\theta > 1$. The trace is

$$\mathrm{tr} J = \frac{(\alpha - s^*)x^* + \sigma}{1-\alpha}\frac{u^*}{1-u^*} = \frac{[\sigma + \alpha(1-\varphi)](2\theta-1) - \sigma - 1 + \varphi}{(1-\alpha)(\theta-1)[\sigma+\alpha(1-\varphi)]}\frac{\sigma u^*}{1-u^*} > 0,$$

in view of the transversality condition (54). Thus, J has one negative eigenvalue, η_1, and two eigenvalues with positive real part. All three variables, $\tilde{z}$, $\tilde{u}$ and $\tilde{q}$, are jump variables, but $\tilde{z}$ and $\tilde{u}$ are linked through

$$\tilde{z} = \frac{1-\alpha}{\gamma L^\alpha} A^{\sigma+1-\varphi}\left(\frac{\tilde{u}}{K}\right)^{1-\alpha} \equiv h(\tilde{u}, A, K). \tag{55}$$

In order to check existence and uniqueness of a convergent solution, let $\mathbf{x} = (x_1, x_2, x_3) \equiv (\tilde{z}, \tilde{u}, \tilde{q})$ and $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \equiv (z^*, u^*, q^*)$. Then, in a small neighborhood of $\bar{\mathbf{x}}$ any convergent solution is of the form $\mathbf{x}(\tau) = Cv e^{\eta_1 \tau} + \bar{\mathbf{x}}$, where C is a constant, depending on initial A and K, and $v = (v_1, v_2, v_3)$ is an eigenvector associated with η_1 so that

$$(-z^* - \eta_1)v_1 - (1-\varphi)z^* v_2 = 0, \tag{56a}$$

$$0 + \left(\frac{\sigma}{1-\alpha}u^* - \eta_1\right)v_2 - \frac{1}{1-\alpha}u^* v_3 = 0, \tag{56b}$$

$$\frac{\alpha-\theta}{(1-\alpha)\theta}q^* v_1 - (1-\varphi)q^* v_2 + \left(\frac{1}{1-\alpha}q^* - \eta_1\right)v_3 = 0. \tag{56c}$$

We see that $v_i \neq 0, i = 1, 2, 3$. Initial transformed time is $\tau_0 = \ln A_0$ and we have $\mathbf{x}(\tau_0) = (h(u(0), A_0, K_0), u(0), q(0))$ for $A(0) = A_0$, and $K(0) = K_0$ (both pre-determined), where we have used (55) for $t = 0$. Hence, coordinate-wise,

$$x_1(\tau_0) = C v_1 \mathrm{e}^{\eta_1 \tau_0} + z^* = h(u(0), A_0, K_0), \tag{57}$$

$$x_2(\tau_0) = C v_2 \mathrm{e}^{\eta_1 \tau_0} + u^* = u(0), \tag{58}$$

$$x_3(\tau_0) = C v_3 \mathrm{e}^{\eta_1 \tau_0} + q^* = q(0). \tag{59}$$

This system has a unique solution in $(C, u(0), q(0))$; indeed, substituting (58) and (59) into (57), setting $v_1 = 1$ and using $z^* = x^* u^*$, gives

$$\frac{1}{v_2} u(0) + u^*(x^* - \frac{1}{v_2}) = h(u(0), A_0, K_0). \tag{60}$$

It follows from Lemma 2 that, given $\theta > 1$, (60) has a unique solution in $u(0)$. With the pre-determined initial value of the ratio, $A^{\sigma+1-\varphi}/K^{1-\alpha}$, in a small neighborhood of its steady state value (which is $\gamma L^\alpha x^* u^{*\alpha}/(1-\alpha)$), the solution for $u(0)$ is close to u^*; hence it belongs to the open interval $(0, 1)$.

Lemma 2 *Assume* $\theta > 1$. *Then* $1/v_2 > x^*$.

Proof From (56a),

$$v_2 = \frac{-z^* - \eta_1}{(1-\varphi) z^*}. \tag{61}$$

Substituting $v_1 = 1$ together with (56b) into (56c) gives

$$\frac{\alpha - \theta}{(1-\alpha)\theta} q^* - (1-\varphi) q^* v_2 + \left(\frac{1}{1-\alpha} q^* - \eta_1\right)\left(\sigma - \frac{(1-\alpha)\eta_1}{u^*}\right) v_2 \equiv Q(v_2, \eta_1) = 0.$$

Replacing η_1 and v_2 in (61) by η and $w(\eta)$, respectively, we see that $P(\eta) \equiv Q(w(\eta), \eta)$ is the characteristic polynomial of degree 3 corresponding to J. Now,

$$P(-z^*) = \frac{\alpha - \theta}{(1-\alpha)\theta} q^* < 0,$$

as $\theta > 1$. Consider $\eta_0 \equiv -(1-\varphi) z^*/x^* - z^* < -z^*$. Clearly, $w(\eta_0) = 1/x^*$. If $P(\eta_0) > 0$, then the unique negative eigenvalue η_1 satisfies $\eta_0 < \eta_1 < -z^*$, implying that $v_2 \equiv w(\eta_1) < 1/x^*$, in view of $w'(\eta) < 0$; hence $1/v_2 > x^*$. It remains to prove that $P(\eta_0) > 0$. We have

$$P(\eta_0) = \frac{\alpha-\theta}{(1-\alpha)\theta}q^* - (1-\varphi)q^* w(\eta_0) + \left(\frac{1}{1-\alpha}q^* - \eta_1\right)\left(\sigma - \frac{(1-\alpha)\eta_1}{u^*}\right) w(\eta_0)$$

$$= \frac{\alpha-\theta}{(1-\alpha)\theta}q^* - \frac{(1-\varphi)q^*}{x^*} + \left(\frac{1}{1-\alpha}q^* - \eta_0\right)\left(\sigma - \frac{(1-\alpha)\eta_0}{u^*}\right)\frac{1}{x^*}$$

$$= \frac{\alpha(1-\theta)\left[(1-\alpha)(1-\varphi)+\sigma\right](1-s^*)x^* u^*}{(1-\alpha)\theta\sigma}$$

$$+ \frac{\left[(1-\alpha)(1-\varphi)+\sigma\right](1-s^*) + (1-\alpha)\sigma}{1-\alpha}u^* + \frac{1-\varphi}{x^*}\sigma u^* + \frac{1-\alpha}{u^* x^*}\eta_0^2$$

$$= \frac{\theta-1}{\theta}\left[\sigma + \alpha(1-\varphi)\right] + \frac{1-\alpha}{u^* x^*}\eta_0^2 > 0,$$

where the third equality is based on reordering and the definition of q^*, whereas the last equality is based on the formulas for x^*, u^*, and s^* in (10); finally, the inequality is due to $\theta > 1$. □

Appendix B: The learning-by-doing example

By (19), the transversality condition (21) can be written

$$\lim_{t\to\infty} c(t)^{-\theta} \mathrm{e}^{-\rho t} K(t) = 0,$$

which is obviously satisfied along the asymptotic regular growth path, since $\rho > 0$, and c and K feature *less* than exponential growth. In the text we claimed that the first-order conditions together with the transversality condition are sufficient for an optimal solution. Indeed, this follows from the Mangasarian sufficiency theorem, since H is jointly concave in (K, c) and the state and co-state variables are non-negative for all $t \geq 0$, cf. Seierstad and Sydsaeter (1987, pp. 234–235). Uniqueness of the solution follows because H is *strictly* concave in (K, c) for all $t \geq 0$.

Appendix C: The investment-specific learning example

The current-value Hamiltonian for the optimal control problem is:

$$H(K, q, c, \lambda_1, \lambda_2, t) = \frac{c^{1-\theta}-1}{1-\theta}L + \lambda_1\left[q(Y - cL) - \delta K\right] + \lambda_2 \gamma q^{\frac{\beta-1}{\beta}}(Y - cL),$$

where $Y = K^{\alpha}L^{1-\alpha}$ and λ_1 and λ_2 are the co-state variables associated with physical capital and the quality of newly produced investment goods, respectively. An interior solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial c} = c^{-\theta}L - \lambda_1 qL - \lambda_2 \gamma q^{\frac{\beta-1}{\beta}} L = 0, \tag{62}$$

$$\frac{\partial H}{\partial K} = \lambda_1 (q\alpha \frac{Y}{K} - \delta) + \lambda_2 \gamma q^{\frac{\beta-1}{\beta}} \alpha \frac{Y}{K} = \rho\lambda_1 - \dot{\lambda}_1, \tag{63}$$

$$\frac{\partial H}{\partial q} = \lambda_1 (Y - cL) + \lambda_2 \gamma \frac{\beta-1}{\beta} q^{\frac{-1}{\beta}} (Y - cL) = \rho\lambda_2 - \dot{\lambda}_2. \tag{64}$$

The first-order conditions imply

Lemma 3 $\frac{\mathrm{d}}{\mathrm{d}t}(c^{-\theta}) = c^{-\theta}(\rho - \alpha q \frac{Y}{K}) + \lambda_1 q\delta.$

Proof Let

$$u \equiv c^{-\theta} - \lambda_1 q = \lambda_2 \gamma q^{\frac{\beta-1}{\beta}} = \lambda_2 \frac{\dot{q}}{I}, \tag{65}$$

by (62) and (26), respectively. Then, using (64) and $I \equiv Y - cL$,

$$g_u = g_{\lambda_2} + \frac{\beta-1}{\beta} g_q = \rho - \left(\frac{\lambda_1}{\lambda_2} + \gamma \frac{\beta-1}{\beta} q^{\frac{-1}{\beta}} \right) I + \frac{\beta-1}{\beta} \gamma q^{\frac{-1}{\beta}} I = \rho - \frac{\lambda_1}{\lambda_2} I, \tag{66}$$

so that

$$\dot{u} = \rho u - \frac{\lambda_1}{\lambda_2} I u = \rho u - \lambda_1 \dot{q}, \tag{67}$$

by (65). Rewriting (65) as $c^{-\theta} = \lambda_1 q + u$, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(c^{-\theta}) &= \lambda_1 \dot{q} + \dot{\lambda}_1 q + \dot{u} = \rho u + \dot{\lambda}_1 q = \rho c^{-\theta} - (\rho\lambda_1 - \dot{\lambda}_1) q \text{ (from (67) and (65))} \\ &= \rho c^{-\theta} - \left[(\lambda_1 q + \lambda_2 \gamma q^{\frac{\beta-1}{\beta}}) \alpha \frac{Y}{K} - \lambda_1 \delta \right] q = \rho c^{-\theta} - c^{-\theta} \alpha q \frac{Y}{K} + \lambda_1 q \delta,\end{aligned}$$

where the two latter equalities come from (63) and (62), respectively. □

From Lemma 3 follows

$$g_c = -\frac{1}{\theta} \frac{\frac{d}{\mathrm{d}t}(c^{-\theta})}{c^{-\theta}} = \frac{1}{\theta} (\alpha z - m\delta - \rho),$$

using that $z \equiv qY/K$ and

$$m \equiv pq \equiv (\lambda_1 / c^{-\theta}) q = \frac{\lambda_1}{\lambda_1 q + \lambda_2 \gamma q^{\frac{\beta-1}{\beta}}} q, \tag{68}$$

by (62). This proves (27).

The conjectured necessary and sufficient transversality conditions are $\lim_{t\to\infty} \lambda_1(t)\mathrm{e}^{-\rho t}K(t) = 0$ and $\lim_{t\to\infty} \lambda_2(t)\mathrm{e}^{-\rho t}q(t) = 0$. We now check whether these conditions hold in the steady state. First, note that (63) and (65) give

$$\begin{aligned} g_{\lambda_1} &= \rho + \delta - \frac{c^{-\theta}}{\lambda_1}\alpha\frac{Y}{K} = \rho + \delta - \alpha\frac{Y}{pK} = \rho + \frac{1}{m}(m\delta - \alpha z) \\ &= \rho - \frac{1}{m^*}(\theta g_c^* + \rho) = \frac{(1-\alpha+\alpha\theta)g_c^*}{\alpha} \end{aligned}$$

in steady state, by (38). Further, we have in steady state $g_K^* = g_c^*/\alpha + n$. Hence, $g_{\lambda_1}^* + g_K^* - \rho = (1-\theta)g_c^* + n - \rho < 0$, by the parameter restriction (37). Thus the first transversality condition holds for all $\theta > 0$.

From (66)

$$\begin{aligned} g_{\lambda_2} + g_q - \rho &= \rho - \frac{\lambda_1}{\lambda_2}I - \frac{\beta-1}{\beta}g_q + g_q - \rho = -\frac{m}{1-m}\gamma q^{\frac{-1}{\beta}}I + \frac{1}{\beta}g_q \text{ (by (68))} \\ &= -\frac{m}{1-m}g_q + \frac{1}{\beta}g_q = -(\theta g_c^* + \rho) + \frac{1-\alpha}{\alpha\beta}g_c^* = \frac{1-\alpha(1+\theta\beta)}{1-\alpha(1+\beta)}n - \rho, \end{aligned}$$

in steady state, by (38), (34), and (36). It follows that $\theta \geq 1$ is sufficient for the second transversality condition to hold. If $n = 0$, no particular condition on θ is needed to ensure this transversality condition.

It remains to show:

Lemma 4 *If* $n = 0$ *and* $\alpha < 1/(1+\beta)$, *then for purely technological reasons* $\lim_{t\to\infty} h = 0$.

Proof Let $n = 0$ and $\alpha < 1/(1+\beta)$. We have $h \equiv Y/q^{1/\beta} = K^\alpha L^{1-\alpha}/q^{1/\beta}$, where L is constant and q is always non-decreasing, by (26). There are two cases to consider. *Case 1*: $q \nrightarrow \infty$ for $t \to \infty$. Then, by (25), for $t \to \infty$, $I \to 0$, hence $\dot{K} \to -\delta K$, and so $K \to 0$, whereby $h \to 0$. *Case 2*: $q \to \infty$ for $t \to \infty$. If $K \nrightarrow \infty$ for $t \to \infty$, we are finished. Suppose $K \to \infty$ for $t \to \infty$. Then, for $t \to \infty$ we must have $g_K = sz - \delta \geq 0$ so that $z \nrightarrow 0$, in view of $\delta > 0$. In addition, defining $x \equiv zh^\beta$, we get $x = K^{\alpha(1+\beta)-1}L^{(1-\alpha)(1+\beta)} \to 0$ for $t \to \infty$. It follows that $h \equiv (x/z)^{1/\beta} \to 0$ for $t \to \infty$, since $\alpha < 1/(1+\beta)$. □

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