Technical notes to Walsh (2003), Section 5.3.2

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Abstract

This note presents some derivations of expressions in Walsh (2003), Section 5.3.1.

1 Derivation of the expression for P_t following equation (5.22) on p. 218

Profits are zero if revenues equals total costs, i.e., if

$$P_t Y_t = \int_0^1 P_t(i) Y_t(i) \, \mathrm{d}i.$$

Then replace $Y_t(i)$ with the demand function, (5.22) to get

$$P_t Y_t = \int_0^1 P_t(i) \left[\frac{P_t}{P_t(i)} \right]^{\frac{1}{1-q}} Y_t \mathrm{d}i.$$

We immediately see that Y_t drops out such that

$$P_t = \int_0^1 P_t(i) \left[\frac{P_t}{P_t(i)} \right]^{\frac{1}{1-q}} \mathrm{d}i.$$

The condition is then rewritten as

$$P_t^{1-\frac{1}{1-q}} = \int_0^1 P_t(i)^{1-\frac{1}{1-q}} \, \mathrm{d}i,$$

$$P_t^{\frac{q}{q-1}} = \int_0^1 P_t(i)^{\frac{q}{q-1}} \, \mathrm{d}i,$$

and finally

$$P_t = \left[\int_0^1 P_t(i)^{\frac{q}{q-1}} \, \mathrm{d}i \right]^{\frac{q-1}{q}},$$

which is the expression on p. 218.

2 Derivation of expression (5.30)

The expression is an approximation of (5.29) where discounting is ignored (i.e., $R_t = 1$). One therefore has to take a first-order Taylor approximation of the expression

$$\overline{P}_t = \frac{\mathrm{E}_t \left[P_t^{\theta} V_t Y_t + P_{t+1}^{\theta} V_{t+1} Y_{t+1} \right]}{q \mathrm{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]},$$

around the steady state (where variables are constants; i.e., the price level is constant and thus there is assumed no inflation). The left-hand side needs no approximation. Now, let an undated variable indicate its steady-state value. A first-order Taylor expansion of the right-hand side then yields:

$$\begin{aligned}
& \frac{\mathrm{E}_t \left[P_t^{\theta} V_t Y_t + P_{t+1}^{\theta} V_{t+1} Y_{t+1} \right]}{q \mathrm{E}_t \left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1} \right]} \\
\approx \; & \frac{2P^{\theta} V Y}{q 2 P^{\frac{1}{1-q}} Y} + \frac{\theta P^{\theta - 1} V Y \left(2 q P^{\frac{1}{1-q}} Y \right) - q \frac{1}{1-q} P^{\frac{1}{1-q} - 1} Y \left(2 P^{\theta} V Y \right)}{\left(q 2 P^{\frac{1}{1-q}} Y \right)^2} \left(\overline{P}_t - P \right) \\
& + \frac{\theta P^{\theta - 1} V Y \left(2 q P^{\frac{1}{1-q}} Y \right) - q \frac{1}{1-q} P^{\frac{1}{1-q} - 1} Y \left(2 P^{\theta} V Y \right)}{\left(q 2 P^{\frac{1}{1-q}} Y \right)^2} \mathrm{E}_t \left(\overline{P}_{t+1} - P \right) \\
& + \frac{P^{\theta} Y}{q \left(2 P^{\frac{1}{1-q}} Y \right)} \left(V_t - V \right) + \frac{P^{\theta} Y}{q \left(2 P^{\frac{1}{1-q}} Y \right)} \mathrm{E}_t \left(V_{t+1} - V \right) \\
& + \frac{P^{\theta} V \left(2 q P^{\frac{1}{1-q}} Y \right) - q P^{\frac{1}{1-q}} \left(2 P^{\theta} V Y \right)}{q \left(2 P^{\frac{1}{1-q}} Y \right)^2} \left(Y_t - Y \right) \\
& + \frac{P^{\theta} V \left(2 q P^{\frac{1}{1-q}} Y \right) - q P^{\frac{1}{1-q}} \left(2 P^{\theta} V Y \right)}{q \left(2 P^{\frac{1}{1-q}} Y \right)^2} \mathrm{E}_t \left(Y_{t+1} - Y \right).
\end{aligned}$$

This expression is reduced to

$$\begin{aligned}
& \frac{\mathrm{E}_t\left[P_t^{\theta} V_t Y_t + P_{t+1}^{\theta} V_{t+1} Y_{t+1}\right]}{\mathrm{E}_t\left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1}\right]} \\
\approx\; & \frac{1}{q} P^{\theta-\frac{1}{1-q}} V + 2qV \frac{\theta P^{\theta-1}\left(P^{\frac{1}{1-q}}\right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1}\left(P^{\theta}\right)}{\left(q2P^{\frac{1}{1-q}}\right)^2}\left(\overline{P}_t - P\right) \\
& +2qV \frac{\theta P^{\theta-1}\left(P^{\frac{1}{1-q}}\right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1}\left(P^{\theta}\right)}{\left(q2P^{\frac{1}{1-q}}\right)^2} \mathrm{E}_t\left(\overline{P}_{t+1} - P\right) \\
& +\frac{1}{2q} P^{\theta-\frac{1}{1-q}}\left(V_t - V\right) + \frac{1}{2q} P^{\theta-\frac{1}{1-q}} \mathrm{E}_t\left(V_{t+1} - V\right)
\end{aligned}$$

Now use that $\theta = (2-q)/(1-q)$ in order to get

$$\begin{aligned}
& \frac{\mathrm{E}_t\left[P_t^{\theta} V_t Y_t + P_{t+1}^{\theta} V_{t+1} Y_{t+1}\right]}{\mathrm{E}_t\left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1}\right]} \\
\approx\; & \frac{1}{q} P^{\theta-\frac{1}{1-q}} V + 2qV \frac{\frac{2-q}{1-q} P^{\frac{1}{1-q}}\left(P^{\frac{1}{1-q}}\right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} P^{\frac{2-q}{1-q}}}{\left(q2P^{\frac{1}{1-q}}\right)^2}\left(\overline{P}_t - P\right) \\
& +2qV \frac{\frac{2-q}{1-q} P^{\frac{1}{1-q}}\left(P^{\frac{1}{1-q}}\right) - \frac{1}{1-q} P^{\frac{1}{1-q}-1} P^{\frac{2-q}{1-q}}}{\left(q2P^{\frac{1}{1-q}}\right)^2} \mathrm{E}_t\left(\overline{P}_{t+1} - P\right) \\
& +\frac{1}{2q} P^{\theta-\frac{1}{1-q}}\left(V_t - V\right) + \frac{1}{2q} P^{\theta-\frac{1}{1-q}} \mathrm{E}_t\left(V_{t+1} - V\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\mathrm{E}_t\left[P_t^{\theta} V_t Y_t + P_{t+1}^{\theta} V_{t+1} Y_{t+1}\right]}{\mathrm{E}_t\left[P_t^{\frac{1}{1-q}} Y_t + P_{t+1}^{\frac{1}{1-q}} Y_{t+1}\right]} \approx\; & \frac{1}{q} P^{\theta-\frac{1}{1-q}} V + \frac{1}{2q} V\left(\overline{P}_t - P\right) \\
& +\frac{1}{2q} V \mathrm{E}_t\left(\overline{P}_{t+1} - P\right) \\
& +\frac{1}{2q} P^{\theta-\frac{1}{1-q}}\left(V_t - V\right) + \frac{1}{2q} P^{\theta-\frac{1}{1-q}} \mathrm{E}_t\left(V_{t+1} - V\right),
\end{aligned}$$

We thus have the approximated relationship

$$\begin{aligned}
\overline{P}_t \;=\; & \frac{1}{q} P^{\theta-\frac{1}{1-q}} V + \frac{1}{2q} V\left(\overline{P}_t - P\right) + \frac{1}{2q} V \mathrm{E}_t\left(\overline{P}_{t+1} - P\right) \\
& +\frac{1}{2q} P^{\theta-\frac{1}{1-q}}\left(V_t - V\right) + \frac{1}{2q} P^{\theta-\frac{1}{1-q}} \mathrm{E}_t\left(V_{t+1} - V\right),
\end{aligned}$$

or as $\overline{P} = \frac{1}{q}P^{\theta - \frac{1}{1-q}}V = \frac{1}{q}PV$:

$$\begin{aligned}\overline{P}_t - \overline{P} &= \frac{1}{2q}V\left(\overline{P}_t - P\right) + \frac{1}{2q}V\mathrm{E}_t\left(\overline{P}_{t+1} - P\right) \\ &\quad + \frac{1}{2q}P\left(V_t - V\right) + \frac{1}{2q}P\mathrm{E}_t\left(V_{t+1} - V\right),\end{aligned}$$

Divide the left-hand side by $\overline{P}$ and the right-hand side by $\frac{1}{q}PV$, and the expression becomes

$$\begin{aligned}\frac{\overline{P}_t - \overline{P}}{\overline{P}} &= \frac{1}{2}\frac{\overline{P}_t - P}{P} + \frac{1}{2}\mathrm{E}_t\left(\frac{\overline{P}_{t+1} - P}{P}\right) \\ &\quad + \frac{1}{2}\frac{V_t - V}{V} + \frac{1}{2}\mathrm{E}_t\left(\frac{V_{t+1} - V}{V}\right).\end{aligned}$$

Letting lower-case letters denote percentage deviations from steady state one recovers

$$\overline{p}_t = \frac{1}{2}\left(p_t + \mathrm{E}_t p_{t+1}\right) + \frac{1}{2}\left(v_t + \mathrm{E}_t v_{t+1}\right),$$

which is equation (5.30).

3 Derivation of the last part of expression (5.31)

The second line in (5.31) states

$$\overline{p}_t = \frac{1}{2}\overline{p}_{t-1} + \frac{1}{2}\mathrm{E}_t\overline{p}_{t+1} + \gamma\left[m_t - p_t + \mathrm{E}_t m_{t+1} - \mathrm{E}_t p_{t+1}\right].$$

One then uses that the aggregate price level is an average of prices set in the current and last period; i.e.,

$$p_t = \frac{1}{2}\left(\overline{p}_{t-1} + \overline{p}_t\right).$$

This is inserted into the equation above:

$$\begin{aligned}\overline{p}_t &= \frac{1}{2}\overline{p}_{t-1} + \frac{1}{2}\mathrm{E}_t\overline{p}_{t+1} \\ &\quad + \gamma\left[m_t - \frac{1}{2}\left(\overline{p}_{t-1} + \overline{p}_t\right) + \mathrm{E}_t m_{t+1} - \mathrm{E}_t\frac{1}{2}\left(\overline{p}_t + \overline{p}_{t+1}\right)\right]\end{aligned}$$

Solving for $\overline{p}_t$ yields:

$$\begin{aligned}\overline{p}_t\left(1 + \frac{\gamma}{2} + \frac{\gamma}{2}\right) &= \frac{1}{2}\overline{p}_{t-1} + \frac{1}{2}\mathrm{E}_t\overline{p}_{t+1} \\ &\quad + \gamma\left[m_t - \frac{1}{2}\overline{p}_{t-1} + \mathrm{E}_t m_{t+1} - \mathrm{E}_t\frac{1}{2}\overline{p}_{t+1}\right]\end{aligned}$$

This immediately yields

$$\overline{p}_t = \frac{1}{2}\frac{1-\gamma}{1+\gamma}\left[\overline{p}_{t-1} + \mathrm{E}_t\overline{p}_{t+1}\right] + \frac{\gamma}{1+\gamma}\left[m_t + \mathrm{E}_t m_{t+1}\right],$$

which is the last line of equation (5.31).

4 Application of the method of undetermined coefficients in derivation of (5.32)

One makes the conjecture that the solution of (5.31) takes the form

$$\overline{p}_t = a\overline{p}_{t-1} + bm_t.$$

Forward this one period and take period t expectations:

$$\mathrm{E}_t\overline{p}_{t+1} = a\overline{p}_t + bm_t,$$

where it has been used that money supply follows a random walk ($\mathrm{E}_t m_{t+1} = m_t$). Insert this expression in (5.31) (and use again that $\mathrm{E}_t m_{t+1} = m_t$):

$$\overline{p}_t = \frac{1}{2}\frac{1-\gamma}{1+\gamma}\left[\overline{p}_{t-1} + a\overline{p}_t + bm_t\right] + \frac{\gamma}{1+\gamma}2m_t,$$

and solve for $\overline{p}_t$:

$$\overline{p}_t\left[1 - \frac{a}{2}\frac{1-\gamma}{1+\gamma}\right] = \frac{1}{2}\frac{1-\gamma}{1+\gamma}\left[\overline{p}_{t-1} + bm_t\right] + \frac{\gamma}{1+\gamma}2m_t,$$

$$\overline{p}_t = \frac{1-\gamma}{2(1+\gamma) - a(1-\gamma)}\overline{p}_{t-1} + \frac{(1-\gamma)b + 4\gamma}{2(1+\gamma) - a(1-\gamma)}m_t.$$

This verifies the form of the conjecture, and shows that the undetermined coefficients must satisfy

$$\begin{aligned} a &= \frac{1-\gamma}{2(1+\gamma) - a(1-\gamma)}, \\ b &= \frac{(1-\gamma)b + 4\gamma}{2(1+\gamma) - a(1-\gamma)}. \end{aligned}$$

The first determines a from

$$-a^2(1-\gamma) + 2(1+\gamma)a - (1-\gamma) = 0,$$

or from

$$a^2 - \frac{2(1+\gamma)}{(1-\gamma)}a + 1 = 0,$$

as stated by Walsh. The coefficient b can then be determined subsequently. Indeed, it is easy to verify that $b = 1 - a$.