Notes to Svensson's (1997) Appendix B

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Abstract

Some notes explaining how to solve the Svensson (1997) model.

Some may wonder about the relationship between minimization with respect to the nominal interest rate and then minimization of V with respect to output. Moreover, on page 1141 after equation (B5) the Envelope theorem is applied. The following hopefully clear up what is happening

In this part of the appendix, Svensson considers a variant of the model from the main body of the paper, where the nominal interest rate in period t affects output immediately in period t (as opposed to the one-year control lag for output in the main text). He retains a one year transmission lag between output and inflation such that the Phillips curve is

$$\pi_{t+1} = \pi_t + \alpha_1 y_t + \epsilon_{t+1}. \tag{B.2}$$

Hence there is only a one-period control lag for inflation. In any case, as with the optimization exercises in Clarida et al. (1999, *Journal of Economic Literature*), it is not necessary to formulate the problem as one of choosing the nominal interest rate subject to the aggregate demand and supply curves. As the nominal interest rate affects period t output, but not inflation directly, one can safely take period t output to be the control variable. Subsequently, one can infer which nominal interest rate is consistent with the found optimal output value.[1]

Generally, think of the problem of maximizing

$$u(x, y)$$

[1] Similarly, when he solves the model with a two-year control lag for inflation (on page 1142), he considers $y_{t+1|t}$ as the instrument, as that is the variable that the period t nominal interest rate can affect.

with respect to z subject to

$$\begin{aligned} x &= f(z, y), \\ y &= g(x). \end{aligned}$$

Using z as the instrument, one would get the first-order condition

$$u_x(x, y) f_z(z, y) + u_y(x, y) g'(x) f_z(z, y) = 0,$$

or,

$$u_x(x, y) + u_y(x, y) g'(x) = 0$$

which together with $y = g(x)$ identify y and x. The control is then found by $x = f(z, y)$. One could, however just as we had solved the problem by treating x as the instrument. The first-order condition would be:

$$u_x(x, y) + u_y(x, y) g'(x) = 0,$$

and the same results obtain.

The use of the envelope theorem is made in order to identify the unknown parameter k in the conjectured "value function" (note that it represents *losses*). From (B.3) one immediately gets

$$V_\pi(\pi_t) = k(\pi_t - \pi^*). \tag{*}$$

Strictly speaking, this is not really an application of the envelope theorem, as it is simply by definition the first-order derivative of the conjectured value function with respect to the state variable π_t. We have from (B.1) that in an equilibrium the value function is by definition given by

$$V(\pi_t) = \frac{1}{2}\left[(\pi_t - \pi^*)^2 + \lambda y_t^2\right] + \delta \mathrm{E}_t V(\pi_{t+1}),$$

which by use of the Phillips curve becomes

$$V(\pi_t) = \frac{1}{2}\left[(\pi_t - \pi^*)^2 + \lambda y_t^2\right] + \delta \mathrm{E}_t V(\pi_t + \alpha_1 y_t + \epsilon_{t+1}),$$

We know that in a solution, variables will be functions of the states; i.e. $y_t = y(\pi_t)$. Using this, and differentiating the value function yields

$$V_\pi(\pi_t) = (\pi_t - \pi^*) + \lambda y_t y'(\pi_t) + \delta \mathrm{E}_t V_\pi(\pi_{t+1})\left[1 + \alpha_1 y'(\pi_t)\right].$$

The envelope theorem gives us that the parts in front of $y'(\pi_t)$ are zero; indeed:

$$\lambda y_t + \delta \mathrm{E}_t V_\pi(\pi_{t+1}) \alpha_1 = 0$$

is the first-order condition for optimal y_t. Clearly, in an optimum, a change in y_t as a consequence of a marginal change in the state variable, should leave the value function unchanged (otherwise $y(\pi_t)$ was not the optimal solution for output). We therefore get that

$$\begin{aligned} V_\pi(\pi_t) &= (\pi_t - \pi^*) + \delta \mathrm{E}_t V_\pi(\pi_{t+1}) \\ &= (\pi_t - \pi^*) + \delta k \left(\pi_{t+1|t} - \pi^*\right). \end{aligned} \tag{**}$$

Combining (*) and (**) gives

$$k(\pi_t - \pi^*) = (\pi_t - \pi^*) + \delta k \left(\pi_{t+1|t} - \pi^*\right). \tag{***}$$

Now, from the first-order condition

$$\pi_{t+1|t} - \pi^* = -\frac{\lambda}{\delta \alpha_1 k} y_t,$$

and the Phillips curve in expectation

$$\pi_{t+1|t} = \pi_t + \alpha_1 y_t,$$

one finds

$$\pi_{t+1|t} - \pi^* = -\frac{\lambda}{\delta \alpha_1^2 k}\left(\pi_{t+1|t} - \pi_t\right)$$

$$\pi_{t+1|t}\frac{\delta \alpha_1^2 k + \lambda}{\delta \alpha_1^2 k} - \pi^* = \frac{\lambda}{\delta \alpha_1^2 k}\pi_t$$

$$\pi_{t+1|t}\frac{\delta \alpha_1^2 k + \lambda}{\delta \alpha_1^2 k} - \pi^*\frac{\delta \alpha_1^2 k + \lambda}{\delta \alpha_1^2 k} = \frac{\lambda}{\delta \alpha_1^2 k}(\pi_t - \pi^*)$$

and thus

$$\pi_{t+1|t} - \pi^* = \frac{\lambda}{\delta \alpha_1^2 k + \lambda}(\pi_t - \pi^*)$$

This is inserted into (***):

$$\begin{aligned} k(\pi_t - \pi^*) &= (\pi_t - \pi^*) + \delta k \frac{\lambda}{\delta \alpha_1^2 k + \lambda}(\pi_t - \pi^*) \\ &= \left(1 + \frac{\delta k \lambda}{\delta \alpha_1^2 k + \lambda}\right)(\pi_t - \pi^*). \end{aligned}$$

This must hold for all possible states of the world, so k is the solution to

$$k = 1 + \frac{\delta k \lambda}{\delta \alpha_1^2 k + \lambda}.$$