2. Money's role in flexible-price general equilibrium models*

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Abstract

Notes for the course "Monetary Economics: Macro Aspects," Spring 2006. The relevant literature behind these notes is:

Walsh (2003, Chapter 2);
Walsh (2003, Chapter 3, pp. 95-120; pp. 126-131);
Walsh (2003, Chapter 4, pp. 135-164; 172-187; 192-195).

1 The Tobin effect and money-in-the-utility function models

1.1 Introductory remarks

- The standard model for (exogenous) economic growth is the simple Solow model featuring a fixed savings rate and a law of motion for physical capital accumulation (gross investment)
- When this model is extended with optimizing savings behavior by representative households, we get the Ramsey model of neoclassical growth
- Both models have in their standard formulation no role for money and, thus, monetary policy
- Purpose of the following models/analyses is to introduce a role for money in these type of models. Even though the way this is done is simple in words, it takes some computations to back out the implications of different monetary policies
- Money is introduced in various ways: These are often modelling **short-cuts,** which hopefully captures realistic features of e.g. money demand, while retaining tractability
  - So, short-cuts are helpful for understanding simple features, and the more robust results are to particular short cuts, of course, the better
- The so-called **Tobin model** extends the Solow growth model by *postulating* a demand for money (the short cut)
  - Highlights the implications of inflation for the choice between investment in physical and financial assets
- The so-called **Money-in-the-Utility function models** extend the Ramsey model by *postulating* that money gives utility (the short cut)
  - They highlight — when compared with the Tobin model — the importance of microfoundations and optimizing private-sector behavior (absent in the Tobin model). Microfoundations ensure that potential behavioral changes are accounted for, when one examines the effects of a policy change. I.e., the model is not vulnerable to the Lucas critique.

1.2 The Tobin model

- Like in the standard Solow model, output is produced by a CRS production function:

$$Y_t = F\left(K_{t-1}, N_t\right),$$

which in "intensive" form is:

$$y_t = f\left(k_{t-1}\right), \quad y_t \equiv Y_t/N_t, \quad k_{t-1} \equiv K_{t-1}/N_{t-1}$$

(population grows at a constant rate $N_t/N_{t-1} = 1{+}n$) where $f\left(k_{t-1}\right) \equiv F\left(k_{t-1}/\left(1+n\right), 1\right)$. We have that f is a strictly concave function with $f\left(0\right) = 0$, and where the Inada conditions $f\left(0\right) = 0$, $f\left(\infty\right) = \infty$, $\lim_{k\to 0} f' = \infty$ and $\lim_{k\to\infty} f' = 0$ are satisfied.

- Households invest in capital or money, M_t (of which one unit buys $1/P_t$ goods). Real per capita wealth, a_t, is the sum of real per capita physical capital and real money per capita:

$$a_t \equiv k_t + m_t, \qquad m_t \equiv \left(M_t/P_t\right)/N_t$$

- The government makes real lump-sum transfers (or withdrawals) to households in the form of money supply changes:

$$\tau_t = \frac{\Delta M_t}{P_t N_t}$$

- The real economy-wide household budget constraint is (for simplification there is no depreciation of capital)

$$Y_t + \tau_t N_t + \frac{M_{t-1}}{P_t} = C_t + \Delta K_t + \frac{M_t}{P_t},$$

or

$$Y_t + \tau_t N_t + \frac{M_{t-1}}{\left(1+\pi_t\right) P_{t-1}} = C_t + \Delta K_t + \frac{M_t}{P_t}.$$

Note how inflation $\pi_t \equiv \left(P_t/P_{t-1}\right) - 1$ *erodes* initial resources available for consumption, investment in capital and future real money holdings

- The budget constraint is rewritten such that it depicts resources available for consumption, for investment in capital and *accumulation* of real money holdings (subtract M_{t-1}/P_{t-1} from both sides of the above expression):

$$\begin{aligned} Y_t + \tau_t N_t + \frac{M_{t-1}}{\left(1+\pi_t\right) P_{t-1}} - \frac{M_{t-1}}{P_{t-1}} &= C_t + \Delta K_t + \Delta \frac{M_t}{P_t} \\ Y_t + \tau_t N_t - \frac{\pi_t M_{t-1}}{\left(1+\pi_t\right) P_{t-1}} &= C_t + \Delta K_t + \Delta \frac{M_t}{P_t} \end{aligned}$$

Note that inflation *erodes* available resources

- As in Solow model, households are assumed to accumulate assets according to a simple time-invariant decision rule. It is assumed that the use a fraction $0 < s < 1$ of available resources to save in available assets (physical capital and real money), while the remaining fraction, $1 - s$, is consumed. Formally we thus have:

$$\Delta K_t + \Delta \frac{M_t}{P_t} = s \left(Y_t + \tau_t N_t - \frac{\pi_t M_{t-1}}{(1+\pi_t) P_{t-1}} \right)$$

- In per capita terms this becomes

$$\Delta k_t + \Delta \left(\frac{M_t}{P_t} \right) \frac{1}{N_t} = s \left(y_t + \tau_t - \frac{\pi_t}{(1+\pi_t)(1+n)} m_{t-1} \right) - \frac{n}{1+n} k_{t-1},$$

and expressed exclusively as a physical capital accumulation expression, we get:

$$\begin{aligned} \Delta k_t &= s \left(y_t + \tau_t - \frac{\pi_t}{(1+\pi_t)(1+n)} m_{t-1} \right) - \frac{n}{1+n} k_{t-1} \\ &\quad - \frac{\theta_t - \pi_t}{(1+\pi_t)(1+n)} m_{t-1} \end{aligned}$$

where $\theta_t \equiv \Delta M_t / M_{t-1}$ is the growth rate of the nominal money supply.

  - From this equation we see that per capita physical capital changes with *total* savings (first term on the right hand side), net of "depreciation" due to population growth (second term on the right hand side), and net of changes in real per capital money (third term on the right hand side).

- Use the definition of transfers

$$\tau_t \equiv \frac{\Delta M_t}{P_t N_t} = \frac{\theta_t}{(1+\pi_t)(1+n)} m_{t-1},$$

and we can rewrite the capital accumulation equation as

$$\Delta k_t = sf(k_{t-1}) - (1-s) \frac{\theta_t - \pi_t}{(1+\pi_t)(1+n)} m_{t-1} - \frac{n}{1+n} k_{t-1}$$

- The second term of the right-hand side summarizes how the introduction of money affects capital accumulation (if absent, the expression would the standard discrete time expression from the Solow model). The term

$$\frac{\theta_t - \pi_t}{(1+\pi_t)(1+n)} m_{t-1}$$

is the same as changes in real money per capita. I.e.,

$$\Delta \left(\frac{M_t}{P_t} \right) / N_t = \frac{\theta_t - \pi_t}{(1+\pi_t)(1+n)} m_{t-1}$$

To see this note that

$$\begin{aligned}
\triangle\left(\frac{M_t}{P_t}\right)\frac{1}{N_t} &\equiv \left(\frac{M_t}{P_t}-\frac{M_{t-1}}{P_{t-1}}\right)\frac{1}{N_t}, \\
&= \left(\frac{M_t P_{t-1}}{P_t M_{t-1}}-1\right)\frac{M_{t-1}}{P_{t-1}}\frac{1}{N_t}, \\
&= \left(\frac{M_t P_{t-1}}{P_t M_{t-1}}-1\right)\frac{M_{t-1}N_{t-1}}{P_{t-1}N_{t-1}}\frac{1}{N_t}, \\
&= \left(\frac{M_t P_{t-1}}{P_t M_{t-1}}-1\right)\frac{N_{t-1}}{N_t}m_{t-1}, \\
&= \left(\frac{M_t P_{t-1}}{P_t M_{t-1}}-1\right)\frac{1}{1+n}m_{t-1}, \\
&= \left(\frac{1+\theta_t}{1+\pi_t}-1\right)\frac{1}{1+n}m_{t-1}, \\
&= \frac{\theta_t-\pi_t}{(1+\pi_t)(1+n)}m_{t-1}.
\end{aligned}$$

An increase in real money per capita have two effects on capital accumulation, i.e., on Δk_t:

a) It crowds out Δk_t *completely* for *given total* savings

b) It increases savings by a fraction $s < 1$

- The net effect, $s-1$, is *negative*: Increases in real money per capita divert resources away from physical capital accumulation
- Now consider a steady state where real per capita variables are constant, i.e., $\Delta m_t = \Delta k_t = 0$
  - $\Delta m_t = 0$ implies $\Delta m_t/m_{t-1} = 0$ and therefore

$$\begin{aligned}
(1+\theta) &= (1+\pi)(1+n) \\
\pi &\approx \theta - n
\end{aligned}$$

  - $\Delta k_t = 0$ implies

$$\begin{aligned}
sf(k^{ss}) &= (1-s)\frac{\theta-\pi}{(1+\pi)(1+n)}m^{ss}+\frac{n}{1+n}k^{ss} \\
sf(k^{ss}) &= (1-s)\frac{1+\theta-(1+\pi)}{(1+\theta)}m^{ss}+\frac{n}{1+n}k^{ss} \\
sf(k^{ss}) &= (1-s)\frac{n}{(1+n)}m^{ss}+\frac{n}{1+n}k^{ss},
\end{aligned}$$

condensed as

$$\begin{aligned}
sf(k^{ss}) &= \left[(1-s)\phi^{ss}+1\right]\overline{n}k^{ss} \\
\phi^{ss} &\equiv \frac{m^{ss}}{k^{ss}}, \qquad \overline{n} \equiv \frac{n}{1+n}
\end{aligned}$$

- Note that ϕ^{ss} will affect the steady-state value of the per capita level of physical capital and thus per-capita output
- In a standard Solow diagram, ϕ^{ss} is an additional part in investment needed to maintain a constant value of the per capita capital stock
- Hence, *higher* ϕ^{ss} *decreases* k^{ss}.
- What determines the steady-state real money to physical capital-ratio?
  - This is found by an assumption: It is assumed that it is determined by capital and money's relative real yields
    * Physical capital yields $f_k(k)$ (the marginal product of capital, i.e., the real interest rate, r)
    * Real money "yields" $-\pi/(1+\pi)$ (as seen before; inflation erodes the value)
  - It is therefore assumed that $\phi \equiv \phi(r,\pi) = \phi(f_k(k),\pi)$ with both partial derivatives negative.
- Steady-state relationship then becomes

$$sf(k^{ss}) = [(1-s)\,\phi(f_k(k^{ss}),\pi^{ss}) + 1]\,\overline{n}k^{ss}$$

- Total differentiation yields

$$dk^{ss} = \frac{(1-s)\,\phi_\pi \overline{n}k^{ss}}{sf_k - \overline{n}\,[1 + (1-s)\,\phi + (1-s)\,\phi_r f_{kk}]}d\pi \qquad ((*))$$

- One then can show that[1]

$$\frac{dk^{ss}}{d\pi} > 0$$

- This is the (Mundell-) Tobin effect: Higher inflation decreases m^{ss}/k^{ss}, since it implies a substitution away from real money towards physical capital
- Steady-state result: **Higher money growth** and the associated higher inflation causes **higher output**

[1] The numerator of (*) is negative as $\phi_\pi < 0$ by assumption. The denominator is signed as follows. The term sf_k is the slope of the total steady-state savings schedule as in the standard Solow model. The term $\overline{n}\,[1 + (1-s)\,\phi + (1-s)\,\phi_r f_{kk}]$ is the slope of the total depreciation schedule. In a steady state, total savings equals total depreciation. Since the former schedule is increasing and concave with infinite slope at $k = 0$ (by the Inada condition), and the latter is linear with positive slope (for given m), a stable steady state is one where sf_k is lower than $\overline{n}\,[1 + (1-s)\,\phi + (1-s)\,\phi_r f_{kk}]$; hence the denominator of (*) is negative. It thus follows that $dk^{ss}/d\pi > 0$.

1.2.1 Discussion

- Neutrality of money holds in model (different *levels* of M_t has no real effects — only effects on P_t)
- One-for-one long run relationship between steady state nominal money growth and inflation as in data
- The Tobin effect posits a positive long-run relationship between money growth and output
  - This means that superneutrality of money **fails** (a model features superneutrality of money, if the growth rate of money has no effects of real variables)
  - It is a particular relationship not strongly featured in data
  - Hence, *if* the Tobin effect is important in real life, other implications of money growth not modelled here must neutralize it
- However, a relevant question is: Why is money in the model? It only has negative effects — at least if output is a relevant proxy for well being! So one may wonder whether the short cut to modelling money is a good one?
- All behavior is postulated (not only the constant savings rate as in the Solow model, but the mere existence of money is just postulated). So is it reasonable to evaluate the effects of different monetary policies? It begs, at a minimum, the question of what optimizing agents would do?

1.3 Money in the utility function: The basic model

- This model seeks to answer some of the questions that the Tobin model leaves open. The model amends the standard Ramsey model (optimizing households) with money
  - I.e., an extension of the Tobin model in terms of private sector behavior, just like the Ramsey model extends the Solow model with endogenous, optimal, savings behavior.
- The short-cut here for introducing a role for money is that real money *provides utility* per se
  - One interpretation (that will be developed formally later) is that money in the hands of households facilitates their transactions on the market by, e.g., reducing "shopping time" (money as such is an otherwise useless commodity). Another phrasing: money provides "liquidity services," which the households value (without money, households would technically have to resort to barter, and that is easy to imagine as time consuming, if feasible at all).

- It is therefore assumed that the per-period utility function of infinitely lived, representative households is

$$U_t = u\left(c_t, m_t\right)$$

with u being increasing and concave in both arguments. (It is, of course, *real* money that enters in u, i.e., M_t's value relative to what it can buy at price P_t.)

- Often, to ensure existence of an equilibrium where money is held (a "monetary equilibrium"), it is assumed that for some $\overline{m}$, $u_m\left(c_t, \overline{m}\right) = 0$ and $u_m\left(c_t, m_t\right) < 0$ for $m_t > \overline{m}$

- The aim of the representative household is to maximize:

$$W = \sum_{t=0}^{\infty} \beta^t u\left(c_t, m_t\right), \qquad 0 < \beta < 1. \tag{2.1}$$

- The relevant constraint for households is similar to the one presented by the Tobin model. Here it is assumed that physical capital depreciates at rate $0 < \delta < 1$, so the real economy-wide budget constraint is (ignoring, for simplicity, financial asset holdings B_t used in Walsh, 2003)

$$Y_t + \tau_t N_t + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t}, \tag{2.2'}$$

The left-hand side is available resources at t (income, transfers, and real wealth carried over from last period) used for consumption, physical capital and real money

- In per-capita version, and assuming no population growth as in Walsh (2003) (i.e., set $n = 0$), we thus get

$$y_t + \tau_t + (1-\delta) k_{t-1} + \frac{1}{1+\pi_t} m_{t-1} = c_t + k_t + m_t$$

- Assuming again a standard neoclassical production function $y_t = f\left(k_{t-1}\right)$, we get

$$\omega_t \equiv f\left(k_{t-1}\right) + \tau_t + (1-\delta) k_{t-1} + \frac{1}{1+\pi_t} m_{t-1} = c_t + k_t + m_t \tag{2.4'}$$

- Hence, ω_t denotes total available resources, treated *as given* at t by the households (it contains terms which are all either given by the past or by the government).

  - The variable ω_t can then be considered as the relevant *state variable* when choosing the optimal paths of c, k, and m at date t

- The representative household's optimization problem is solved by *dynamic programming* (could be done with Lagrangian methods as well)

- This method involves use of the *value function* — the value of W,
  - given optimal behavior of the household in period t and onwards
  - given the current state, ω_t
- The value function, denoted by V, therefore satisfies

$$\begin{aligned} V(\omega_t) &= \max\left\{u(c_t, m_t) + \beta u(c_{t+1}, m_{t+1}) + \beta^2 u(c_{t+2}, m_{t+2}) + \beta^3 u(c_{t+3}, m_{t+3}) + ...\right\}, \\ &= \max\left\{u(c_t, m_t) + \beta\left[u(c_{t+1}, m_{t+1}) + \beta u(c_{t+2}, m_{t+2}) + \beta^2 u(c_{t+3}, m_{t+3}) + ...\right]\right\}, \end{aligned}$$

  which can be written compactly as

$$V(\omega_t) = \max\{u(c_t, m_t) + \beta V(\omega_{t+1})\}.$$

  Maximization is over c_t, m_t, k_t subject to the budget constraint and the definition of ω_{t+1} (available resources one period ahead)[2]

- To make it simple, one substitutes out ω_{t+1} by (2.4') and then substitutes out k_t by using that $k_t = \omega_t - c_t - m_t$.
- One is then left with an unconstrained maximization problem over c_t and m_t (as we will see later, one can not always turn the maximization problem into an unconstrained one):

$$\max\left\{\begin{array}{c} u(c_t, m_t) \\ +\beta V\left(\underbrace{f(k_t) + \tau_t + (1-\delta)k_t + \frac{1}{1+\pi_{t+1}}m_t}_{\omega_{t+1}}\right) \end{array}\right\}$$

  and thus

$$\max\left\{\begin{array}{c} u(c_t, m_t) \\ +\beta V\left(\begin{array}{c} f(\omega_t - c_t - m_t) + \tau_t \\ +(1-\delta)(\omega_t - c_t - m_t) + \frac{1}{1+\pi_{t+1}}m_t \end{array}\right) \end{array}\right\}$$

[2]Note that one can loosely interpret the value function as the dynamic equivalent of the indirect utility function, which you encounter in basic micro theory. The value function gives the optimal life-time utility (discounted sum of all current and future per-period utilities) of the households, conditional on the state, i.e., the variables outside the households' control at t. So it is like the indirect utility function v of a standard static consumer optimization problem:

$$\begin{aligned} v(p, y) &= \max\ u(c) \\ s.t.\quad pc &= y \end{aligned}$$

where price p and income y are taken as given when choosing consumption c.

- The first-order condition concerning the choice of c_t is found as

$$u_c(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial c_t} = 0,$$

$$u_c(c_t, m_t) = \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta]. \tag{2.6'}$$

In optimum, the marginal utility of period t consumption equals its marginal loss (in terms of the discounted marginal value of future capital)

- The first-order condition concerning choice of m_t follows as

$$u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial m_t} = 0,$$

$$\begin{aligned} & u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \frac{1}{1+\pi_{t+1}} \\ = \; & \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta]. \end{aligned} \tag{2.8'}$$

In optimum, the marginal gain of period t real money (in terms of direct utility plus discounted marginal value of more future real money) equals marginal loss (in terms of the marginal value of less future capital)

- Furthermore, transversality conditions must be satisfied:

$$\begin{aligned} \lim_{t\to\infty} \beta^t u_c(c_t, m_t) k_t &= 0 \\ \lim_{t\to\infty} \beta^t u_c(c_t, m_t) m_t &= 0 \end{aligned}$$

(otherwise over-accumulation of k and m is taking place — lifetime utility could be improved through higher consumption by accumulating less).

- We have now optimality conditions expressed by the derivative of the value function; a function we do not know. What can we do about that? It turns out that by use of the so-called envelope theorem, on can eliminate the derivative of the value function V.

- First note that optimal period t consumption and money holding choices will be (unknown) functions of the current state ω_t

- Define these as $c_t \equiv c(\omega_t)$ and $m_t \equiv m(\omega_t)$, respectively. It could appear as we are walking on thin ice now: we are trying to eliminate the derivative of an unknown function by introducing more unknown functions. But the ice is actually thickening as will be clear in a moment.

- With the definition of the functions c and m, the value function is *by definition* characterized as

$$V(\omega_t) = u(c(\omega_t), m(\omega_t)) + \beta V(\omega_{t+1}). \tag{*}$$

Since (*) holds for *any* value of ω_t, it of course follows that the derivatives of the left-hand and right-hand sides of (*) with respect to ω_t also are equal. I.e.,

$$\begin{aligned} V_\omega(\omega_t) &= u_c(c(\omega_t), m(\omega_t)) c'(\omega_t) + u_m(c(\omega_t), m(\omega_t)) m'(\omega_t) \\ &\quad + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial \omega_t}. \end{aligned} \tag{**}$$

holds. Now, find $\partial\omega_{t+1}/\partial\omega_t$ when $c_t = c(\omega_t)$ and $m_t = m(\omega_t)$ applies. Remember that

$$\begin{aligned} \omega_{t+1} &= f(k_t) + \tau_t + (1-\delta) k_t + \frac{1}{1+\pi_{t+1}} m_t, \\ &= f(\omega_t - c_t - m_t) + \tau_t \\ &\quad + (1-\delta)(\omega_t - c_t - m_t) + \frac{1}{1+\pi_{t+1}} m_t. \end{aligned}$$

One therefore gets

$$\begin{aligned} \frac{\partial \omega_{t+1}}{\partial \omega_t} &= [f_k(k_t) + 1 - \delta](1 - c'(\omega_t) - m'(\omega_t)) \\ &\quad + \frac{1}{1+\pi_{t+1}} m'(\omega_t). \end{aligned}$$

Combining this with (**) yields:

$$\begin{aligned} V_\omega(\omega_t) &= u_c(c(\omega_t), m(\omega_t)) c'(\omega_t) + u_m(c(\omega_t), m(\omega_t)) m'(\omega_t) \\ &\quad + \beta V_\omega(\omega_{t+1}) \left\{ \begin{array}{c} [f_k(k_t) + 1 - \delta](1 - c'(\omega_t) - m'(\omega_t)) \\ + \frac{1}{1+\pi_{t+1}} m'(\omega_t) \end{array} \right\}. \end{aligned}$$

- Now collect the $c'(\omega_t)$ and $m'(\omega_t)$ terms:

$$\begin{aligned} V_\omega(\omega_t) &= [u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta)] c'(\omega_t) \\ &\quad + \left[\begin{array}{c} u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1}) \frac{1}{1+\pi_{t+1}} \\ -\beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] \end{array} \right] m'(\omega_t) \\ &\quad + \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] \end{aligned}$$

- Then note the beautiful insight:

$$\begin{aligned} V_\omega(\omega_t) &= \left[\underbrace{u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta)}_{=\ 0 \text{ by (2.6')}} \right] c'(\omega_t) \\ &\quad + \left[\underbrace{\begin{array}{c} u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1}) \frac{1}{1+\pi_{t+1}} \\ -\beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] \end{array}}_{=\ 0 \text{ by (2.8')}} \right] m'(\omega_t) \\ &\quad + \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] \end{aligned}$$

- I.e., the terms in front of $c'(\omega_t)$ and $m'(\omega_t)$, respectively, *are zero* by the first-order conditions!
  - This makes sense intuitively, as these terms indeed capture the marginal values of c and m. In an optimum these must, of course, be zero. Otherwise, they were not optimal.
- Therefore, (**) reduces immediately to

$$V_\omega(\omega_t) = \beta V_\omega(\omega_{t+1})\left[f_k(k_t) + 1 - \delta\right]$$

- Then use the first-order condition for consumption choice,

$$u_c(c_t, m_t) - \beta V_\omega(\omega_{t+1})\left[f_k(k_t) + 1 - \delta\right] = 0,$$

to obtain Walsh's expression (which technically **is** the envelope theorem):

$$V_\omega(\omega_t) = u_c(c_t, m_t). \tag{2.10}$$

  - Marginal utility of consumption equals marginal value of wealth; a familiar restatement of optimal intratemporal consumption choice.
- The first-order conditions can then be rewritten as

$$u_c(c_t, m_t) = \beta u_c(c_{t+1}, m_{t+1})\left[f_k(k_t) + 1 - \delta\right];$$

this is just a discrete-time, money version of the standard "Keynes-Ramsey rule" for optimal allocation of consumption over time, and

$$u_m(c_t, m_t) + \beta\frac{u_c(c_{t+1}, m_{t+1})}{1 + \pi_{t+1}} = u_c(c_t, m_t);$$

which states that the marginal gain of m_t equals the marginal loss in terms of lower capital in period $t+1$ — equal to the marginal utility of c_t by the "Keynes-Ramsey rule"). These conditions, together with the budget constraint fully characterize the optimal paths of c, k, and m

- We will, for now, concentrate on the long-run properties of the model; i.e., a steady state with $\Delta c_t = \Delta k_t = \Delta m_t = 0$. First, from the "Keynes-Ramsey rule" it follows that in steady state

$$1 = \beta\left[f_k(k^{ss}) + 1 - \delta\right],$$

or,

$$f_k(k^{ss}) + 1 - \delta = \frac{1}{\beta} \tag{2.18'}$$

This — independently of any monetary factors — defines the steady-state capital per capita (and thus output per capita). Hence, it is equivalent to the conventional determination of the steady-state capital stock in the Ramsey model. This contrasts strongly with the Tobin model

- The difference between the two models' results on money and capital is due to the fact that the MIU model envisions optimal, endogenous savings behavior.

  - If, e.g., $k_t < k^{ss}$ the current marginal product of capital is relatively high (as $f_{kk} < 0$). This means, from the Keynes-Ramsey rule, that it is optimal to postpone consumption to later. Hence, capital is accumulated until $f_k(k^{ss}) + 1 - \delta = 1/\beta$ holds again
  - If one imagined that a Tobin effect was present; one would be contradicting optimal behavior:
    * Assume that higher inflation increases capital two a new higher steady state
    * Then the marginal product of capital is lower, and optimizing households would want to consume now rather than later. That is, they would *endogenously* save less and capital would decumulate until k^{ss} is reached again, such that $f_k(k^{ss}) + 1 - \delta = 1/\beta$ holds again.
    * Inflation has no steady-state effect on capital. Possible in Tobin model, where individuals are modelled as having an *exogenous* savings rate, and thus do not reduce their propensity to save

- Now, what about consumption in steady state? The budget constraint in steady state is given as

$$f(k^{ss}) + \tau^{ss} + (1-\delta)k^{ss} + \frac{1}{1+\pi}m^{ss} = c^{ss} + k^{ss} + m^{ss}.$$

  Transfers are determined as

$$\begin{aligned} \tau_t &= (M_t - M_{t-1})/P_t, \\ &= \theta_t M_{t-1}/P_t, \\ &= \frac{\theta_t}{1+\pi_t} m_{t-1}, \end{aligned}$$

  so in steady state:

$$\tau^{ss} = \frac{\theta}{1+\pi} m^{ss}$$

- Since a constant m implies $\pi = \theta$, as in the Tobin model with no population growth, one gets

$$f(k^{ss}) - \delta k^{ss} = c^{ss}.$$

  This is simply the economy's resource constraint (national account): Output less gross investment equals consumption. The implication is that c^{ss} is determined exclusively by k^{ss}; and thus also independent of money growth

- This simple MIU model therefore exhibits *superneutrality of money.*

- This, however, does not imply that money growth and inflation do not affect anything. Inflation affect the opportunity cost of holding real money, and thus the steady-state value of real money. To see this, note that relative demand for consumption versus real money is given by [use first-order condition for money holdings and divide through by $u_c(c_t, m_t)$]

$$\begin{aligned}\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{u_c(c_t, m_t)}{u_c(c_t, m_t)} - \frac{1}{1+\pi_{t+1}} \frac{\beta u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \\ &= 1 - \frac{1}{1+\pi_{t+1}} \frac{1}{(f_k(k_t) + 1 - \delta)} \\ &= 1 - \frac{1}{(1+\pi_{t+1})} \frac{1}{(1+r_t)}\end{aligned}$$

  with $r_t \equiv f_k(k_t) - \delta$ being the real interest rate. Note that the real interest rate is the nominal rate net of expected inflation:

$$1 + r_t = (1 + i_t) / (1 + \pi_{t+1}), \qquad (r_t \approx i_t - \pi_{t+1})$$

  Hence,

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1+i_t} \equiv I_t \tag{2.12}$$

  So, as nominal interest rate is determined by the Fisher relationship, $i_t \approx r_t + \pi_{t+1}$, higher inflation leads to a higher nominal interest rate, and for given c_t, m_t is likely to fall (as $u_{mm} < 0$). This is consistent with empirical evidence on money demand relationships (and (2.12) can be seen as one microfoundation for the usual money demand function postulated in *ad hoc* IS/LM models).[3]

- A comparison of the Tobin model and the MIU model outlined here shows that both have one for one relationships between inflation and money growth. The main difference is that the Tobin model does not have long-run superneutrality. The reason for the difference is the postulated and policy-invariant private-sector behavior in the Tobin model. This difference highlights importance of microfoundations in order to avoid the Lucas critique

- Still, the MIU approach *is* a short-cut

1.3.1 Welfare Costs of Inflation

- In the Tobin model money and inflation were just put into the model by assumption, and one could not meaningfully talk about the welfare consequences of different

[3] Will a unique steady-state value for m exist? One must solve $u_m(c^{ss}, m^{ss}) = I^{ss} u_c(c^{ss}, m^{ss})$. Not necessarily unique. What about stability properties? For separable utility, $u(c_t, m_t) = v(c_t) + \gamma\phi(m_t)$, the resulting difference equation (from the first-order condition) will imply a saddle-point stable $m^{ss} > 0$ (m' in Walsh, 2003, Figure 2.1, page 56). The problem is, however, that one cannot necessarily rule out the paths with falling m below steady-state ("speculative" hyperinflations), leading to $m^{ss} = 0$. Typically, however, such "bubble paths" are just assumed away for convenience.

monetary regimes. But in the MIU model, one has a welfare metric, namely the utility of the representative household. Above, we saw that inflation affects real money holdings by affecting nominal interest rates and thus the opportunity cost of holding money:

$$\frac{u_m(c^{ss}, m^{ss})}{u_c(c^{ss}, m^{ss})} = \frac{i^{ss}}{1+i^{ss}} = I \qquad \text{(money demand)}$$

$$i^{ss} = r^{ss} + \pi^{ss} \qquad \text{(Fisher relationship)}$$

- Can one use this to can something about the welfare costs (or gains?) of inflation? Is there an optimal rate of inflation in the model?

On the optimal rate of inflation

- The Bailey/Friedman intuition:
  - Private marginal cost of holding money is increasing in the nominal interest rate
  - Social marginal cost of creating money is essentially zero (in the big picture the production of small pieces of paper is not a big post on the national account)

  $\Longrightarrow$ A social optimal allocation is characterized by equality of private and social marginal costs. Equating private and social marginal cost thus requires *a zero nominal interest rate*
  - Using the Fisher relationship it follows that $\pi^{ss} = -r^{ss} < 0$ is optimal
  - I.e., the optimal rate of change in prices involves *deflation* equal to the real interest rate. This result is in the monetary economics literature now referred to as "The Friedman Rule"

- In the MIU model, this intuitive result is formally confirmed when searching for the utility maximizing nominal money growth rate subject to resource constraint of economy. The first-order condition guiding the optimal choice of steady-state nominal money growth rate is:

$$u_c(f(k^{ss}) - \delta k^{ss}, m^{ss}) \frac{\partial (f(k^{ss}) - \delta k^{ss})}{\partial m^{ss}} \frac{\partial m^{ss}}{\partial \theta^{ss}} + u_m(c^{ss}, m^{ss}) \frac{\partial m^{ss}}{\partial \theta^{ss}} = 0$$

or,

$$u_m(c^{ss}, m^{ss}) = 0$$

- From the (money demand) equation, this indeed implies $i^{ss} = 0$, and the condition $u_m(c^{ss}, m^{ss}) = 0$ determines what Friedman called the "optimal quantity of money." Note that with some finite $\overline{m}$ defined as $u_m(c^{ss}, \overline{m}) = 0$, and $u_m(c^{ss}, m^*) < 0$ for $m^* > \overline{m}$, this $\overline{m}$ is the optimal quantity of money.

What are then the welfare costs of inflation?

- It could be computed as the area under the money demand curve at a given positive nominal interest rate (Figure 2.2 in Walsh, 2003)
  - This is the "consumer surplus" lost by a positive nominal interest rate, and thus price changes higher than what is dictated by the Friedman rule. Some estimates indicate that an inflation rate at 10% corresponds to a loss equivalent to 1-3% of GDP per year
- It could also, as Lucas suggests, be computed as the percentage increase in steady-state consumption needed to compensate for a suboptimal low real money stock caused by too high $i/(1+i) \equiv \Upsilon$.
  - If one normalizes consumption to $c^{ss} = 1$, this implies that the cost of inflation, denoted by $w(\Upsilon)$, is implicitly given as

    $$u(1 + w(\Upsilon), m(\Upsilon)) = u(1, m^*)$$

    where m^* is the optimal quantity of real money balances, and $m(\Upsilon)$ is the money demand function, $m' < 0$.
  - With a specific form of the utility function, and using numbers from estimated money demand functions, Lucas finds that a 10% nominal interest rate is equivalent of around 1.3% lost steady-state consumption. Is that a large number or not? This is subjective, but if it is lost on a yearly basis, the present value terms is quite large.[4]

1.4 Money in the utility function: An extended model

- Is superneutrality of money a robust feature of the MIU model? In the simple model above, superneutrality arose because endogenous savings behavior uniquely defined the steady-state capital stock. Capital is accumulated or decumulated until its net marginal product (real interest rate) equals households' subjective real interest rate:

  $$f_k(k^{ss}) + 1 - \delta = \frac{1}{\beta}$$

  Therefore, long-run superneutrality can will fail in such a model framework if the marginal product of capital is affected by monetary policy and inflation. This can

[4] Lucas has elsewhere argued that the costs of business cycle fluctuations corresponds to (at most) around a tenth of a percentage of steady-state consumption. This is considered as a very small number, and has provoked a large body of literature seeking to overturn this finding, as many economists find that business cycle fluctuations are very costly. The point of mentioning this is that viewed in light of that debate, the 1.3 percentage consumption loss of having a 10% nominal interest rate is rather large.

only be possible if the production function contains another endogenous input factor, which is affected by inflation

- "Candidates"? The obvious one: *Money* in the production function: $y_t = f(k_{t-1}, m_t)$, $f_m > 0$
  - E.g., if $f_{km} > 0$ (more money makes capital more productive) higher inflation leads to lower real money balances and lower steady-state capital

- Although some literature has formulated models with real money as a productive input (the motivation for such an assumption is that it provides liquidity services to firms, making other inputs more productive), a more natural possibility is *endogenous* labor input in the production function. Such endogeneity arises in a MIU model if it is amended by a labor supply choice by households. This is achieved by having *leisure* entering in utility function:

$$u_t = u(c_t, m_t, l_t)$$

- Assuming that l is the fraction of time spent on leisure, the production function is then given by

$$y_t = f(k_{t-1}, 1 - l_t),$$

  or,

$$y_t = f(k_{t-1}, n_t)$$

  with $n_t = 1 - l_t$ being fraction of time spent on work.[5]

- Households now face **an additional decision**: How much time should be devoted to work; how much to leisure? The relevant optimality condition can readily be stated:

$$u_l(c_t, m_t, l_t) = u_c(c_t, m_t, l_t) f_n(k_{t-1}, 1 - l_t) \tag{2.34'}$$

  Marginal gain of leisure is equated to the marginal cost, which is the utility loss from lower consumption times the marginal product of labor (the real wage).

- In steady state one have then the following three relationships (the last two is the same as in the simple MIU model, just with the new production function)::

$$u_l(c^{ss}, m^{ss}, l^{ss}) = u_c(c^{ss}, m^{ss}, l^{ss}) f_n(k^{ss}, 1 - l^{ss}) \qquad (l \text{ versus } c \text{ choice})$$

$$f_k(k^{ss}, 1 - l^{ss}) + 1 - \delta = \frac{1}{\beta} \qquad \text{(constant capital)}$$

$$c^{ss} = f(k^{ss}, 1 - l^{ss}) - \delta k^{ss} \qquad \text{(national account)}$$

[5] Do not confuse this lowercase n with population growth from the Tobin model.

- Then note the following. If $u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{ss}, m^{ss}, l^{ss})$ is independent of m^{ss}, these equations determine k^{ss}, l^{ss} and c^{ss} independent of monetary factors. Long-run superneutrality holds!
  - This will, for example, be the case if utility is *separable* in money; e.g. $u = v(c, l) g(m)$ (u_l and u_c are affected by m in the same way)
  - Also, of course, it will be the case if u_l and u_c are not affected by m at all
- But if $u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{ss}, m^{ss}, l^{ss})$ *depends* on m^{ss}, long-run superneutrality will not hold. Different inflation rates will imply different nominal interest rates and thus different money demands. If this affect the consumption-leisure decision, different monetary policies will imply different labor supplies and different steady-state outputs.
- But why would u_l or u_c depend on m? And if they did, in which direction would they be affected? There is no immediate obvious answer to these questions.[6]
- We therefore examine a specific functional form of utility function, which allows for an appropriate balance between tractability and flexibility in the sense that money may affect the consumption-leisure choice in :

$$u(c_t, m_t, l_t) = \frac{\left(ac_t^{1-b} + (1-a) m_t^{1-b}\right)^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi \frac{l_t^{1-\eta}}{1-\eta},$$

$0 < a < 1$, $b > 0$, $\eta > 0$, $\Phi > 0$, $\Psi > 0$ $(b, \eta, \Phi \neq 1)$

  - Φ is coefficient of constant relative risk aversion (CRRA preferences)
  - b is inverse nominal interest elasticity of money demand — see Walsh (2003) equation (2.25), p. 57
- What is the marginal rate of substitution between leisure and consumption, $u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{s}$ with this specification? It is

$$\frac{u_l}{u_c} = \frac{\Psi l_t^{-\eta}}{a\left(ac_t^{1-b} + (1-a) m_t^{1-b}\right)^{\frac{b-\Phi}{1-b}} c_t^{-b}}$$

The marginal utility of leisure is not affected by money, so the marginal rate of substitution is affected (potentially) through money's role for u_c.

  - In the knife-edge case of $\Phi = b$, it follows that $u_{cm} = 0$ and superneutrality holds in the short *and* the long run

[6] Note that if u_l and u_c are independent of m, then superneutrality holds in the short run as well. Dynamics "collapse" into a real Ramsey-style model as tthe "Keynes-Ramsey rule" depicting the evolution of marginal utility of consumption will no longer be affected by money.

- But if $\Phi < b$, then $u_{cm} > 0$; i.e., more real balances increase the marginal utility of consumption. This relationship between Φ and b is the empirically relevant case as the coefficient of relative risk aversion is typically estimated to be around 1-5, and since the interest sensitivity of money demand is typically found to be of magnitude 0.1 or less. Hence, b is larger than 10.
  - Higher expected inflation will increase the nominal interest rate, and reduce real money balances and decrease marginal utility of consumption
  - Households thus substitute towards leisure, and labor supply decreases. So superneutrality fails in the short and long run, and in a direction opposite that of the Tobin model. Higher inflation reduces labor supply and output.
- If, on the other hand, $\Phi > b$ (which is empirically a less plausible condition), then $u_{cm} < 0$ and superneutrality fails "in the opposite direction"

- It is important here to stress that it is *anticipated* changes in inflation that causes real effects. An unanticipated, temporary, change in π_t has *no* effects, as it does not affect the nominal interest rate that ultimately determines money demand, and then marginal utility of consumption. Only when $\mathrm{E}_t\pi_{t+1}$ is affected, is the nominal interest rate affected through the Fisher relationship, $i_t = \mathrm{E}_t\pi_{t+1} + r_t$

1.4.1 Dynamics of the model and calibration

- Given that superneutrality is likely to fail in the short run due to endogenous labor choice, a relevant issue is whether the MIU model has short-run properties, which match the data. Only then is the model appropriate for analysis of monetary policymaking at business cycle frequencies. I.e., how are monetary shocks transmitted to the real economy, and how will monetary policy be able to play a stabilizing role?

- For this purpose a stochastic version of the model is formulated. The model is extended to include exogenous shocks bringing the economy away from steady state. Focus will be on *technology shocks* and *shocks to the growth rate of nominal money supply*

Model and private sector optimization. General case (a la Appendix in Walsh, 2003, Chapter 2)

- The production function is amended to

$$y_t = f\left(k_{t-1}, 1 - l_t, z_t\right)$$

where z_t is a technology shock. It is assumed that

$$z_t = \rho z_{t-1} + e_t, \qquad |\rho| < 1,$$

with e_t being a mean-zero, white-noise shock.

- Nominal money growth is assumed to be given by

$$\theta_t = \theta^{ss} + u_t$$

where u_t is a shock to the growth rate. It is assumed that

$$u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t, \qquad 0 \leq \gamma < 1, \quad \phi \lesseqgtr 0$$

with φ_t being a mean-zero, white-noise shock.

  – Note that there may or may not be serial correlation in the shocks to nominal money growth (depending upon whether γ is zero or not).
  – Note also that money growth may or may not respond toward past period's technology shock. One may say that money is *procyclical* when $\phi > 0$ as money growth is then increased when the economy has experienced an expansion of production possibilities, $z_{t-1} > 0$ (which is expected to continue into period t for $\rho > 0$. In the mirror case with $\phi < 0$ we label money as being *countercyclical.*

- The per-period utility function and the budget constraint are, respectively,

$$u(c_t, m_t, l_t)$$

and

$$y_t + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1} = c_t + k_t + m_t.$$

(ignoring financial assets b_t as previously). As in basic MIU model without endogenous labor, households maximize discounted lifetime utility subject to the budget constraint, and again, dynamic programming method is applied.

- Note now, however, that since l_t is a choice variable, it is inappropriate to treat available resources (the left-hand-side of the budget constraint) as the state variable at period t. It is not something that can be taken as given, since income, y_t, will depend on how much the household chooses to work. Instead, state variables will therefore be k_{t-1} and $a_t \equiv \tau_t + m_{t-1}/(1 + \pi_t)$. These are variable outside the household's control.

- The optimization is then characterized by the value function

$$V(a_t, k_{t-1}) = \max \mathrm{E}_t \{u(c_t, m_t, l_t) + \beta V(a_{t+1}, k_t)\}$$

where the maximization is over c, m, k, and l subject to the budget constraint and the definition of the state variable a_t. E_t is the rational expectations operator.

- One substitutes the budget constraint and definition of a_t so as to eliminate k_t and a_{t+1} and so as to obtain an unconstrained maximization problem. The first-order condition with respect to c_t is

$$u_c(c_t, m_t, l_t) = \mathrm{E}_t \beta V_k(a_{t+1}, k_t) \tag{2.51'}$$

(as $\partial a_{t+1}/\partial c_t = 0$ by the definition of a). It has the usual interpretation: The marginal gain of consumption (marginal utility of consumption) must equal the expected marginal loss in terms of lower capital in the next period. The first-order condition with respect to m_t is

$$u_m(c_t, m_t, l_t) + \beta \mathrm{E}_t V_a(a_{t+1}, k_t) \frac{1}{1+\pi_{t+1}} = \beta \mathrm{E}_t V_k(a_{t+1}, k_t). \tag{2.53'}$$

It also has the usual interpretation: The marginal gain in terms of current marginal utility of money and expected discounted marginal value of next-period monetary wealth, must equal the expected marginal loss in terms of lower capital in the next period. Finally, the first-order condition with respect to l_t is given by

$$u_l(c_t, m_t, l_t) = \mathrm{E}_t \beta V_k(a_{t+1}, k_t) f_n(k_{t-1}, 1-l_t, z_t) \tag{2.54'}$$

The marginal gain of leisure is equated to the marginal cost, which is the expected marginal value loss from less next-period capital, times the marginal product of labor (the real wage)

- To eliminate the partial derivatives of the value function, we use that we know that optimum will be characterized by optimal values of c_t, m_t, and l_t as functions of the state variables. Call these functions

$$c_t = c(a_t, k_{t-1}), \quad m_t = m(a_t, k_{t-1}), \quad l_t = l(a_t, k_{t-1}).$$

The value function is thus by definition given as

$$V(a_t, k_{t-1}) = u(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) + \beta \mathrm{E}_t V(a_{t+1}, k_t)$$

This holds for all a_t, k_{t-1} so we have

$$\begin{aligned} V_a(a_t, k_{t-1}) &= u_c(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) c_a(a_t, k_{t-1}) \\ &\quad + u_m(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) m_a(a_t, k_{t-1}) \\ &\quad + u_l(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) l_a(a_t, k_{t-1}) \\ &\quad + \beta \mathrm{E}_t V_a(a_{t+1}, k_t) \frac{\partial a_{t+1}}{\partial a_t} + \beta \mathrm{E}_t V_k(a_{t+1}, k_t) \frac{\partial k_t}{\partial a_t} \\ &= \beta \mathrm{E}_t V_k(a_{t+1}, k_t) \end{aligned}$$

where last equality follows from the fact that when $c_t = c(a_t, k_{t-1})$, $m_t = m(a_t, k_{t-1})$, $l_t = l(a_t, k_{t-1})$, it follows that

$$\begin{aligned} \frac{\partial a_{t+1}}{\partial a_t} &= \frac{1}{1+\pi_{t+1}} m_a(a_t, k_{t-1}), \\ \frac{\partial k_t}{\partial a_t} &= 1 - c_a(a_t, k_{t-1}) - m_a(a_t, k_{t-1}) - f_n(k_{t-1}, 1-l_t, z_t) l_a(a_t, k_{t-1}), \end{aligned}$$

such that all the terms in front of $c_a(a_t, k_{t-1})$, $m_a(a_t, k_{t-1})$ and $l_a(a_t, k_{t-1})$ cancel out. I.e., at an optimum, the marginal value of changing c, m, or l must be zero. Likewise we get

$$V_k(a_t, k_{t-1}) = \beta \mathrm{E}_t V_k(a_{t+1}, k_t)\left[f_k(k_{t-1}, 1 - l_t, z_t) + 1 - \delta\right]$$

- So, by the first-order condition guiding c:

$$u_c(c_t, m_t, l_t) = V_a(a_t, k_{t-1})$$

- We also get the corresponding relationship for guiding money choice similar to simple MIU model:

$$\begin{aligned} u_m(c_t, m_t, l_t) + \beta \mathrm{E}_t u_c(c_{t+1}, m_{t+1}, l_{t+1}) \frac{1}{1+\pi_{t+1}} &= \beta \mathrm{E}_t V_k(a_{t+1}, k_t) \\ &= u_c(c_t, m_t, l_t) \qquad (*) \end{aligned}$$

- Also, the condition guiding consumption can be modified by use of $V_k(a_t, k_{t-1}) = \beta \mathrm{E}_t V_k(a_{t+1}, k_t)\left[f_k(k_{t-1}, n_t, z_t) + 1 - \delta\right]$:

$$\begin{aligned} u_c(c_t, m_t, l_t) &= \mathrm{E}_t \beta V_k(a_{t+1}, k_t) \\ &= \beta \mathrm{E}_t \beta \mathrm{E}_{t+1} V_k(a_{t+2}, k_{t+1})\left[f_k(k_t, 1 - l_{t+1}, z_{t+1}) + 1 - \delta\right] \end{aligned}$$

and using that $V_a(a_t, k_{t-1}) = \beta \mathrm{E}_t V_k(a_{t+1}, k_t)$, gives $V_a(a_{t+1}, k_t) = \beta \mathrm{E}_{t+1} V_k(a_{t+2}, k_{t+2})$ and thus

$$\begin{aligned} u_c(c_t, m_t, l_t) &= \beta \mathrm{E}_t \mathrm{E}_{t+1} V_a(a_{t+1}, k_t)\left[f_k(k_t, 1 - l_{t+1}, z_{t+1}) + 1 - \delta\right] \\ &= \beta \mathrm{E}_t R_t u_c(c_{t+1}, m_{t+1}, l_{t+1}) \qquad (**) \end{aligned}$$

with $R_t \equiv f_k(k_t, 1 - l_{t+1}, z_{t+1}) + 1 - \delta$. I.e., (**) is the "modified Keynes-Ramsey rule"

- Finally, we get the condition for the choice of l_t, which becomes

$$u_l(c_t, m_t, l_t) = u_c(c_t, m_t, l_t) f_n(k_{t-1}, 1 - l_t, z_t); \qquad (***)$$

just like the "postulated" optimality condition in the steady-state analysis above. Hence, the equations (*), (**) and (***), together with the budget constraint, provide solutions for the paths of c, m, l, and k.

Particular functional forms of utility and production functions

- The model is by numerical methods under assumptions about particular functional forms for utility and production function. The per-period utility function is (as above):

$$u\left(c_t, m_t, l_t\right) = \frac{\left(a c_t^{1-b} + (1-a)\, m_t^{1-b}\right)^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi \frac{l_t^{1-\eta}}{1-\eta}.$$

  The production function is assumed to be Cobb-Douglas:

$$y_t = k_{t-1}^{\alpha} n_t^{1-\alpha} \mathrm{e}^{z_t}, \qquad 0 < \alpha < 1.$$

- Consider briefly the steady-state solution: Note that the real interest rate,

$$R_t = f_k\left(k_t, 1 - l_{t+1}, z_{t+1}\right) + 1 - \delta$$

  with the Cobb-Douglas production function becomes

$$\begin{aligned} R_t &= \alpha \mathrm{E}_t k_t^{\alpha-1} \left(1 - l_{t+1}\right)^{1-\alpha} \mathrm{e}^{z_{t+1}} + 1 - \delta \\ &= \alpha \frac{\mathrm{E}_t y_{t+1}}{k_t} + 1 - \delta. \end{aligned}$$

  By the steady-state condition $R^{ss} = 1/\beta$, this "only" determines the *ratio* y^{ss}/k^{ss}. This *ratio* will be independent of monetary factors, but the *levels* y^{ss}, k^{ss}, c^{ss} may not, if superneutrality fails so that l^{ss} will be affected

Dynamic effects of money and technology shocks

- To assess the quantitative effects of money and technology shocks, the model is *calibrated* and *simulated*. In the calibration, one assigns empirically plausible values to the parameters of the model (parameters in the utility function, production function, shock processes). The numerical simulation one usually does involves first a linearization of the model's dynamic equations (such that everything is expressed as percentage deviations from steady state). This resulting linear system of equations is then solved by numerical methods (various simulation programs are available on the internet). From the solutions, one create artificial time series data for various key variables (output, consumption, labor, inflation, etc.).

- By these artificial data one evaluates the properties of the model in terms of:
  - Standard deviations of various relevant variables, and their standard deviation relative to output
  - Correlation coefficients of various variables with output
  - Impulse response patterns of variables when shocks bring key variables off the steady state and back

- **Main results from the simulation exercise in Walsh (2003)**
  - The calibration is characterized by $b > \Phi$, implying $u_{cm} > 0$
  - Steady-state non-superneutrality is thus of the form of: Higher θ implies lower money holdings and thus lower marginal utility of consumption, less labor supply, and lower output
  - If money shocks, φ_t-shocks, shall pay a role in the short run, persistence in money growth is necessary, i.e., $\gamma > 0$ is needed. If not, a shock will *not* affect expected next-period inflation, and thus — through the Fisher equation — the period t nominal interest rate will not be affected. Real money holdings are then unaltered and the consumption-leisure trade-off unaltered. With persistence in the money supply process (which is empirically plausible), a φ_t-shock affects next period's money growth and inflation, and thus the period t nominal interest rate.
  - The effects of money shocks on labor and output are, not surprisingly then, stronger the more persistence in money growth (the higher is γ), but the effects of realistically sized money shocks are quantitatively **very** small when one looks at output.
  - If technology shocks are met with procyclical money, output is more stable (expansive money growth dampens labor supply which dampens the output effect of a positive technology shock). The magnitude of this monetary stabilization when gauging the reduction in the standard deviation of output, however, is modest
  - The main effects of money shocks are on inflation and nominal interest rates
  - Positive money shocks lead to *higher* nominal interest rates. This contrasts starkly with the usual IS/LM story where a *liquidity effect* is present: nominal rates fall to increase money demand and secure equilibrium on the money market. The reason for this difference is that prices are flexible in the MIU model contrary to the sticky-price IS/LM model. In the MIU model prices adjust instantaneously so as to *reduce* real money supply, matching the fall in demand resulting from higher nominal interest rates.

1.5 Summary on MIU models

- The MIU framework provides a setting in which the welfare costs of inflation can be assessed, and where the optimal inflation rate can be determined. This, in turn, is equivalent of determining the "optimal quantity of money" (some aspects of inflation for public finance and taxation, however, are neglected; more on this later).

- The stochastic, dynamic model without the superneutrality property can be used to assess the importance of monetary shocks for economic fluctuations
- In the calibrated, MIU model with endogenous labor, money matters for business cycle fluctuations, but **not very much**
- This is one indication that flexible-price models may be ill-suited for analysis of monetary phenomena in the short run. Its short run behavior is very much akin to its long run behavior because of the flexible price assumption.

2 Shopping-time Models

- In MIU models, money provides utility directly in order to secure a demand for money. One justification for putting money directly into the utility function was that it was a way of depicting utility from saved time on transactions; "liquidity services"
- Shopping-time models formalize this argument, and helps putting restrictions on, e.g., the signs of u_{cm} and u_{lm} under the MIU approach. Remember that these signs determined how labor supply and output would react to changes in monetary regimes.

2.1 A simple model to prove the point

- Per-period utility function (increasing and concave in both arguments):

$$v(c_t, l_t)$$

- Leisure is now defined as:

$$l = 1 - n - n^s$$

where n is fraction of time spent on work, and n^s is fraction of time spent on shopping (buying consumption goods)[7]

- Crucial assumption: **Transaction services**, ψ, are needed for consumption purchases. For simplicity it is just assumed that there is a one-for-one relationship between transaction services and consumption:

$$\psi = c.$$

It is then assumed that transaction services are facilitated by real money, and are higher the more one shops:

$$\begin{aligned} \psi &= c = \psi(m, n^s) \qquad (3.1) \\ \psi_m &\geq 0, \qquad \psi_{n^s} \geq 0 \end{aligned}$$

[7] To some gender of the population it may appear heretical that time spent on shopping thus is a direct utility loss. For the opposite gender, however, it may seem as a fantastic assumption.

Thus, for given money holdings, more time spend on shopping provides more transaction services. For given shopping time, more money holdings provide more transaction services. This is restated in terms of shopping time:

$$\begin{aligned} n^s &= g(c,m) \\ g_c &> 0, \qquad g_m < 0 \end{aligned}$$

More consumption requires more shopping time, but for given consumption more money reduces shopping time.

- Per-period utility function can then be rewritten as:

$$u(c_t, m_t, l_t) \equiv v[c_t, 1 - n_t - g(c_t, m_t)]$$

While not an explicit argument in basic utility function v, money now enters *implicitly* — and positively — in u through its negative impact on shopping time. So we have provided a more rigorous argument for the inclusion of money in the utility function.

Optimal behavior

- The budget constraint as in the MIU model with endogenous labor:

$$f(k_{t-1}, n_t) + \tau_t + (1-\delta) k_{t-1} + \frac{m_{t-1}}{1+\pi_t} = c_t + k_t + m_t$$

(again, bonds are ignored). Optimization is characterized by the value function

$$V(a_t, k_{t-1}) = \max\{v[c_t, 1 - n_t - g(c_t, m_t)] + \beta V(a_{t+1}, k_t)\}$$

with

$$a_t \equiv \tau_t + \frac{m_{t-1}}{1+\pi_t}$$

Optimization is over c, n, m, k, and a subject to the budget constraint and the definition of a_t. The budget constraint and definition of a_t are used to substitute out a_{t+1} and k_t, to get an unconstrained maximization problem

- To asses how money affects the consumption-leisure choice, consider the first-order conditions w.r.t. c_t and n_t. The first-order condition with respect to c_t is

$$\begin{aligned} v_c(c_t, 1 - n_t - g(c_t, m_t)) &= v_l(c_t, 1 - n_t - g(c_t, m_t))\, g_c(c_t, m_t) \\ &\quad + \beta V_k(a_{t+1}, k_t) \end{aligned}$$

The marginal utility of consumption is equated to the marginal losses, which consist of lost leisure due to more time spent on shopping and the marginal value of lower next-period capital. The first-order condition with respect to n_t is

$$v_l\left(c_t, 1-n_t-g\left(c_t, m_t\right)\right)=\beta V_k\left(a_{t+1}, k_t\right) f_n\left(k_t, 1-n_t\right)$$

The marginal loss of labor is equated to the marginal gain which is the addition to next-period capital (of a magnitude determined by the marginal product of labor, i.e., the real wage)

- Using these first-order conditions provides the condition for the consumption-leisure choice:
$$\frac{u_l}{u_c}=\frac{v_l}{v_c-v_l g_c}=f_n\left(k_t, 1-n_t\right).$$
The marginal rate of substitution between leisure and consumption equals the real wage

- This expression holds in each and every period, and therefore also in the steady state. So, how is it affected by, e.g., a raise in m_t? This will be indicative for in which manner superneutrality fails in the long run.

- The marginal utility of leisure is affected according to

$$u_{lm}=v_{lm}=-v_{ll} g_m<0$$

More money reduces marginal utility of leisure; it increases leisure for given consumption (by reducing shopping time). This will tend to increase labor and output

- The marginal utility of consumption is affected according to

$$u_{cm}=\underbrace{-\overset{?}{v_{cl}}\overset{-}{g_m}}_{1)}\underbrace{+\overset{-}{v_{ll}}\overset{+}{g_c}\overset{-}{g_m}}_{2)}\underbrace{-\overset{+}{v_l}\overset{?}{g_{cm}}}_{3)} \lesseqgtr 0 \tag{3.2'}$$

The sign is thus ambiguous, and can, as indicated, be seen to be determined by three effects:

  - Effect 1): More money increases u_{cm} if $v_{cl}>0$: More money frees up leisure, increasing the marginal utility of consumption "directly"
  - Effect 2): More money increases u_{cm} unambiguously: More money frees up leisure, which decreases marginal utility of leisure, which implies less utility loss from transaction costs
  - Effect 3); More money increases u_{cm} if $g_{cm}<0$: More money reduces the marginal transaction costs (in terms of lost leisure)

- So,

– unless $v_{lc} << 0$; i.e., consumption and leisure are **strong substitutes**
– and/or unless $g_{cm} >> 0$; i.e., more money **increases** marginal transactions costs
– then higher m will lead to more work. This means that higher money growth and inflation reduces employment and output as in benchmark MIU model calibration

- A shopping-time model is therefore equivalent to a MIU approach, but it provides a formalization of the idea of utility from liquidity services of money, and one knows better how and why superneutrality fails

- Once again, one can assess the welfare implications of inflation and nominal interest rates. Just as in MIU model, we get a first-order condition governing optimal money holdings:
$$-v_l g_m + \beta \frac{V_a(a_{t+1}, k_t)}{1+\pi_{t+1}} = \beta V_k(a_{t+1}, k_t)$$
The marginal gains of money in terms of more current leisure and next-period money wealth are equated to the marginal cost in terms of lower next-period capital. Then use the value function relationships. (These follow by use of the the envelope theorem; see Appendix A for a derivation of (**) below — (*) follows by same exercise.)
$$\begin{aligned} V_k(a_t, k_{t-1}) &= \beta V_k(a_{t+1}, k_t)[f_k(k_{t-1}, n_t) + 1 - \delta] \\ &= \beta R_{t-1} V_k(a_{t+1}, k_t) \qquad (*) \end{aligned}$$
and
$$V_a(a_t, k_{t-1}) = \beta V_k(a_{t+1}, k_t) \qquad (**)$$
to get
$$-v_l g_m + \beta \frac{V_a(a_{t+1}, k_t)}{1+\pi_{t+1}} = V_a(a_t, k_{t-1})$$
Hence,
$$-v_l g_m = V_a(a_t, k_{t-1})\left[1 - \beta \frac{V_a(a_{t+1}, k_t)}{(1+\pi_{t+1}) V_a(a_t, k_{t-1})}\right]$$
But
$$\begin{aligned} \frac{V_a(a_{t+1}, k_t)}{V_a(a_t, k_{t-1})} &= \frac{V_k(a_{t+2}, k_{t+1})}{V_k(a_{t+1}, k_t)} \qquad \text{(using (**))} \\ &= \frac{1}{\beta R_t} \qquad \text{(using (*))} \end{aligned}$$
So, eventually
$$-v_l g_m = V_a(a_t, k_{t-1})\left[1 - \frac{1}{(1+\pi_{t+1}) R_t}\right].$$
As the Fisher relationship implies $1 + i_t = R_t(1+\pi_{t+1})$, it follows that
$$-v_l g_m = V_a(a_t, k_{t-1}) \frac{i_t}{1+i_t}.$$

- As in MIU model, it is optimal to have $i_t = 0$ so the private marginal product of real money balances **equals zero**; namely at $g_m = 0$. In other words, the Friedman rule is optimal in this model because money balances are then at a level high enough to minimize shopping time

3 Cash-in-Advance Models

3.1 Basic model under certainty

- This approach introduces money into a general equilibrium, flexible price setting by takes the transactions purpose of money literally. Having cash, is *by assumption needed* to purchase some (or all) goods. Formally, a "Cash-in-Advance" constraint is introduced. It is, in other words, posited that one has to hold money to purchase goods.

- The case of certainty case is relatively simple to analyze. Introducing uncertainty involves further complications: One may suddenly hold too little or too much cash (former case leads to suboptimal low consumption; latter case leads to suboptimal low savings). We take the case of certainty first.

- Utility of the representative household is given by (endogenous leisure is dropped for simplicity)

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1. \tag{3.12}$$

- The budget constraint is given by

$$\begin{aligned} \omega_t &\equiv f(k_{t-1}) + \tau_t + (1-\delta) k_{t-1} + \frac{m_{t-1} + (1+i_{t-1}) b_{t-1}}{1+\pi_t} \\ &= c_t + k_t + m_t + b_t \end{aligned} \tag{3.15'}$$

(b_t is real bond holdings per capita; and is not ignored here, as it is necessary to operate with a nominal interest rate explicitly from the beginning, as will be clear in a moment). The cash-in-advance (CIA) constraint on consumption goods takes the following form:

$$c_t \leq \frac{m_{t-1}}{1+\pi_t} + \tau_t \tag{3.13}$$

- Note, as opportunity cost of holding money is i_t, the CIA constraint will *always hold with equality for* $i_t > 0$. An optimizing agent would never want to hold more money than is necessary for desired consumption (this is not necessarily the case with uncertainty — one could risk "ending up with too little cash")

- Optimization is characterized by (ω_t and m_{t-1} are state variables):

$$V(\omega_t, m_{t-1}) = \max \{u(c_t) + \beta V(\omega_{t+1}, m_t)\},$$

where maximization is over c, m, b, k and subject to budget constraint and CIA constraint.

- From $\omega_t = c_t + k_t + m_t + b_t$, one can eliminate b_t from budget constraint:

$$\omega_{t+1} = f(k_t) + \tau_{t+1} + (1-\delta) k_t + \frac{m_t}{1+\pi_{t+1}} + R_t (\omega_t - c_t - k_t - m_t)$$

  with $R_t \equiv (1+i_t) / (1+\pi_{t+1})$ being the real interest rate. Note that Walsh (2003) does not make this substitution.

- Let μ_t denote the Lagrange multiplier associated with the CIA constraint (so, if $\mu_t > 0$ the constraint binds with equality; if $\mu_t = 0$ the constraint does not bind with equality).

- The optimization problem is thus characterized by

$$V(\omega_t, m_{t-1}) = \max \left\{ u(c_t) + \beta V(\omega_{t+1}, m_t) - \mu_t \left(c_t - \frac{m_{t-1}}{1+\pi_t} - \tau_t \right) \right\},$$

  where the maximization is over c, k, and m. One may ask why the Lagrange multiplier μ_t is suddenly introduced in the maximization problem? Note that the optimization problem involves choosing c, k, m, and b, subject to the budget constraint and the CIA constraint. One should in principle have Lagrange multipliers on both the budget constraint and the CIA constraint, but as seen above, one can eliminate bond holdings from the problem by the definition of ω and then simplify the optimization problem to a choice over c, k, and m, subject to the CIA constraint. Had there been no such constraint (as in the MIU model), the problem would now be an unconstrained maximization problem, as one effectively has substituted the budget constraint into the utility function.[8] Now, however, there *is* such a constraint, and one cannot avoid dealing with it. This is done by restating the value function in terms of the constraint and Lagrange multiplier.

- The first-order condition with respect to c_t is

$$u_c(c_t) = \beta R_t V_\omega(\omega_{t+1}, m_t) + \mu_t$$

[8] Think of a simple static problem:

$$\max \; U(c_1, c_2) \text{ s.t. } c_1 + c_2 = y.$$

where maximization is over c_1 and c_2, and y is exogenous. This problem one could solve by setting up the Lagrangian

$$\mathcal{L} = U(c_1, c_2) - \lambda (c_1 + c_2 - y),$$

with λ being the Lagrange multipier on the constraint. From the two first-order conditions and the constraint one could then solve for the three unknowns: c_1, c_2 and λ.

However, it is much simpler to eliminate, say c_2, by use of the budget constraint to get the *unconstrained* maximization problem

$$\max \; U(c_1, y - c_1)$$

From the single first-order condition and the constraint, one then recovers the two unknowns, c_1 and c_2. Try both approaches, and verify that they give the same result.

The marginal utility of consumption equals the marginal losses, which are the discounted marginal value of next-period wealth **plus** the "price" of holding cash as measured by μ_t (cost of liquidity services provided by money). Note that the marginal cost of consumption is higher when the CIA constraint binds**,** $\mu_t > 0$. The first-order condition with respect to k_t is

$$\beta V_\omega(\omega_{t+1}, m_t)[f_k(k_t) + 1 - \delta] = \beta R_t V_\omega(\omega_{t+1}, m_t).$$

Marginal gain in terms of more next-period wealth equals the marginal loss in terms of less next-period wealth due to lower bond holdings (note that this implies that $R_t = f_k(k_t) + 1 - \delta$; i.e., as there is perfect capital markets the return on bonds and physical capital are equalized). The first-order condition with respect to m_t is

$$\beta \frac{1}{1+\pi_{t+1}} V_\omega(\omega_{t+1}, m_t) + \beta V_m(\omega_{t+1}, m_t) = \beta R_t V_\omega(\omega_{t+1}, m_t).$$

The marginal gains in terms of more next-period wealth and money per se (for transactions), equals marginal loss in terms of less next-period wealth due to lower bond holdings

- Relationships between partial derivatives of the value function from the envelope theorem:

  $$V_\omega(\omega_t, m_{t-1}) = \beta R_t V_\omega(\omega_{t+1}, m_t)$$

  In optimum, equality between the period t marginal value of wealth and the discounted next-period marginal value of wealth (times the gross real interest rate). Also,

  $$V_m(\omega_t, m_{t-1}) = \mu_t \frac{1}{1+\pi_t}$$

  The marginal value of money carried into period t equals their marginal cost in terms of the "price" of holding cash as measured by $\mu_t/(1+\pi_t)$. Note that the marginal value of money is zero if $\mu_t = 0$; i.e., if the CIA constraint does not bind (one has already sufficient money for transactions).

- What is the nominal interest rate, and **does** the CIA constraint bind? First let $\lambda_t \equiv V_\omega(\omega_t, m_{t-1})$ define the marginal value of wealth (= the Lagrange multiplier on the budget constraint in Walsh, 2003). Then from $V_\omega(\omega_t, m_{t-1}) = \beta R_t V_\omega(\omega_{t+1}, m_t)$ one can write

  $$\lambda_t = \beta R_t \lambda_{t+1}$$

  From

  $$\beta \frac{1}{1+\pi_{t+1}} V_\omega(\omega_{t+1}, m_t) + \beta V_m(\omega_{t+1}, m_t) = \beta R_t V_\omega(\omega_{t+1}, m_t)$$

  one can write

  $$\beta \frac{1}{1+\pi_{t+1}} \lambda_{t+1} + \beta \mu_{t+1} \frac{1}{1+\pi_{t+1}} = \beta R_t \lambda_{t+1}$$

Hence, it follows that

$$\frac{1}{1+\pi_{t+1}}\left(\lambda_{t+1}+\mu_{t+1}\right)=\frac{1+i_t}{1+\pi_{t+1}}\lambda_{t+1}$$

$$\Longrightarrow i_t=\frac{\mu_{t+1}}{\lambda_{t+1}} \tag{3.29}$$

The nominal interest rate is positive, *only* when the CIA constraint binds, $\mu_{t+1}>0$, i.e., when there is a cost of "liquidity services" provided by real money holdings

- Note how λ_t/P_t, the utility-value of *nominal* money, equals the present value of marginal values of money, cf. (3.28) in Walsh (2003). Money is "priced" like a conventional asset; as the present value of all future "returns" in terms of liquidity services. If $V_m=0$ the "price" is zero.

- Now note that the first-order condition for consumption can be rewritten as

$$\begin{aligned} u_c\left(c_t\right) &= \lambda_t+\mu_t \\ &= \lambda_t\left(1+\frac{\mu_t}{\lambda_t}\right). \end{aligned}$$

With the expression for the nominal interest rate on finds

$$u_c\left(c_t\right)=\lambda_t\left(1+i_{t-1}\right)$$

Hence, a positive interest rate increases the marginal cost of consumption *above* the marginal value of wealth. The "price" of consumption goods in terms of output has increased by a positive i_{t-1} due to the need for holding cash (foregoing interest income) to purchase goods the nominal interest rate is equivalent to a "consumption tax." However, it is a **non-distorting tax in the long run** as it:

a) Does not affect long-run capital accumulation

b) Does not distort any intratemporal trade-offs

- Note with $\mu_t>0$ — a binding CIA constraint — the model features a strong version of the quantity theory of money:

$$c_t=m_t=\frac{M_t}{P_t}$$

Constant consumption-based velocity — unrealistic and due to certainty and the simplicity of the model

Steady-state properties: Superneutrality or not?

- From the steady-state condition $R^{ss} = 1/\beta$ and the capital accumulation condition one gets the familiar condition:

$$f_k(k^{ss}) + 1 - \delta = 1/\beta$$

Hence, long-run capital and output per capita are **invariant** to monetary factors. Steady-state consumption follows from the national account as

$$c^{ss} = f(k^{ss}) - \delta k^{ss}$$

I.e., **long-run superneutrality holds**

- Nominal money growth affects inflation and inflation affects the nominal interest rate (through the Fisher relationship):

$$\begin{aligned} \pi_t &= \theta^{ss} \\ i^{ss} &\approx R^{ss} + \pi^{ss} \end{aligned}$$

- Analogy with MIU model concerning relative marginal values of real money balances (in terms of liquidity services) and consumption:

$$\frac{\mu}{u_c} = \frac{\mu}{\lambda(1+i)} = \frac{i}{1+i}$$

- Marked difference with MIU approach (and shopping time approach): There are no steady-state welfare costs of inflation (or, deflation); only c^{ss} matters for utility, and c^{ss} is independent of inflation and the nominal interest rate. There is thus no optimal long run rate of inflation. Utility is invariant to various monetary policies in the long run.

3.1.1 Extensions yielding non-superneutrality and a well-defined optimal inflation rate

- Again, a natural extension involves introducing a consumption-leisure trade-off:
  - Note that leisure is a "commodity" that can be "purchased" without money, so the CIA constraint "taxes" consumption **relative** to leisure (hence, it **distorts** the trade-off)
  - Households choose more leisure relative to consumption when the CIA constrain binds; output will be lower the higher in money growth, inflation and the nominal interest rate

- Coexistence of "Cash goods" and "credit goods":
  - A subset of consumption goods can be bought on credit; i.e., the CIA constraint does not apply

- The CIA constraint then "taxes" cash goods, but not credit goods (hence, it **distorts** relative demand)
- This can account for a time-varying velocity as $m = c^{\text{cashgoods}}$, such that m/c^{allgoods} varies with expected inflation and the nominal interest rate (higher i is likely to reduce $m/c^{\text{allgoods}} \approx$ higher velocity; this positive relationship between nominal interest rate an velocity is consistent with data)

- CIA restriction on investment in physical capital
  - In that case, accumulation of capital becomes "taxed," and steady state capital will be lower (i.e., the investment decision is **distorted**)
- All cases strongly qualifies the "any inflation rate goes" result of simple CIA model above. Now it will be optimal to have $i^{ss} = 0$, i.e., to eliminate any distortion arising from the CIA constraint. In other words, in all these cases implementation of the Friedman rule is optimal

3.2 Basic model in the stochastic case

- The simple CIA model under certainty exhibited long-run superneutrality
- It is possible that this is not the case, and issues are therefore:
  - What is the potential nature of non-superneutralities in the short and the long run?
  - What are the quantitative implications of inflation and the CIA constraint?
  - Will dynamics match the data in a stochastic version?
  - Can monetary policy play a stabilizing role?
- These issues are addressed in a stochastic CIA model (solved, calibrated, and simulated — just as the stochastic MIU model)
- Exogenous shocks bringing the economy off steady state are — as in the stochastic MIU model — technology shocks and nominal money growth shocks
- The channel causing non-superneutrality here is (as in the MIU approach) an endogenous labor supply decision

Model and private sector optimization

- Production is given by a Cobb-Douglas function

$$y_t = f(k_{t-1}, n_t, z_t) = \mathrm{e}^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}, \qquad 0 < \alpha < 1, \tag{3.33'}$$

where z_t is a technology shock given by (as in stochastic MIU model)

$$z_t = \rho z_{t-1} + e_t, \qquad |\rho| < 1,$$

with e_t being a mean-zero, white-noise shock. Nominal money growth rate is also modelled as in the MIU model:

$$\theta_t = \theta^{ss} + u_t$$

where

$$u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t, \qquad 0 \leq \gamma < 1, \quad \phi \lesseqgtr 0$$

with φ_t being a mean-zero, white-noise shock. As in MIU model there may or may not be serial correlation in the shocks to nominal money growth, and as in MIU model, money growth may or may not respond toward past technology shocks, and may be either pro-cyclical ($\phi > 0$) or countercyclical ($\phi < 0$).

- The per-period utility function is given by (again like that in stochastic MIU model, just without money and $a = 1$):

$$u(c_t, 1 - n_t) = \frac{(c_t)^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-n_t)^{1-\eta}}{1-\eta},$$

η, Φ, $\Psi > 0$ (Φ, η are coefficients of relative risk aversion). Compared to the simple CIA model under certainty, leisure provides utility, and a consumption-leisure decision will potentially be affected by the CIA constraint (as already hinted in the discussion of potential sources of non-superneutrality of money).

- The CIA constraint is (on consumption goods):

$$c_t \leq \frac{m_{t-1}}{1+\pi_t} + \tau_t \equiv a_t, \tag{3.50'}$$

while the budget constraint is (ignoring nominal debt, b, for simplicity):

$$\mathrm{e}^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha} + (1-\delta) k_{t-1} + a_t = c_t + k_t + m_t \tag{3.51'}$$

- Optimization is characterized by (k_{t-1} and a_t are state variables):

$$V(k_{t-1}, a_t) = \max \left\{ \frac{(c_t)^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-n_t)^{1-\eta}}{1-\eta} + \beta \mathrm{E}_t V(k_t, a_{t+1}) \right\}.$$

Maximization is over c, m, n, k, and a subject to the CIA constraint, budget constraint and the definition of a. The by now usual trick is used: Eliminate k_t and a_{t+1}

by the budget constraint and definition of a, and one then "only" maximizes over c, m and n subject to the CIA constraint. Let μ_t denote the Lagrange multiplier associated with the CIA constraint. The first-order condition with respect to c_t is

$$c_t^{-\Phi} = \beta \mathrm{E}_t V_k(k_t, a_{t+1}) + \mu_t \tag{3.52'}$$

Marginal utility of consumption equals the marginal losses, which are the expected, discounted marginal value of next-period capital **plus** the "price" of holding cash as measured by μ_t (cost of liquidity services provided by money when nominal interest rate is positive). As in simple CIA model: the marginal cost of consumption is **higher** when the CIA constraint binds. The first-order condition with respect to m_t is

$$\beta \mathrm{E}_t \left[\frac{V_a(k_t, a_{t+1})}{1+\pi_{t+1}} \right] = \beta \mathrm{E}_t V_k(k_t, a_{t+1}) \tag{3.54'}$$

The expected marginal value in terms of more next-period money wealth, equals the expected marginal value in terms of lower capital holdings. Finally, the first-order condition with respect to n_t is

$$\Psi(1-n_t)^{-\eta} = (1-\alpha)\beta \mathrm{E}_t V_k(k_t, a_{t+1}) \mathrm{e}^{z_t} k_{t-1}^{\alpha} n_t^{-\alpha}. \tag{3.55'}$$

The marginal loss in terms of less leisure equals the expected value of higher future capital (which is higher the higher is the marginal product of labor). The relationships between the partial derivatives of the value function follows from the envelope theorem, and are, respectively,

$$V_k(k_{t-1}, a_t) = \beta \mathrm{E}_t V_k(k_t, a_{t+1}) \left[\alpha \mathrm{e}^{z_t} k_{t-1}^{\alpha-1} n_t^{1-\alpha} + 1 - \delta \right] \tag{3.57'}$$

(the marginal value of current capital equals the expected marginal value of future capital "corrected for" the net marginal product of current capital; Keynes-Ramsey rule "in disguise") and

$$V_a(k_{t-1}, a_t) = \mu_t + \beta \mathrm{E}_t V_k(k_t, a_{t+1}) \tag{3.56'}$$

(the marginal value of real balances per se equals the marginal costs in terms of the "price" of the CIA constraint and the expected value of lower capital).

Steady state, and the form of non-superneutrality

- Let $\lambda_t \equiv \beta \mathrm{E}_t V_k(k_t, a_{t+1})$ be discounted, expected the marginal value of capital. We then get (from the first-order condition with respect to c)

$$c_t^{-\Phi} = \lambda_t + \mu_t \tag{3.59}$$

and (from the first-order condition with respect to m)

$$\beta \mathrm{E}_t \left[\frac{\lambda_{t+1} + \mu_{t+1}}{1+\pi_{t+1}} \right] = \lambda_t. \tag{3.60}$$

Remember the marginal value of money, $V_a(k_t, a_{t+1}) / (1+\pi_{t+1})$, indeed are μ and λ [from (3.56') and the definition of λ]. We also get (from relationship between the derivatives of the value function):

$$\lambda_t = \beta \mathrm{E}_t R_t \lambda_{t+1} \tag{3.62}$$

where $R_t = \alpha e^{z_{t+1}} k_t^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta = \alpha (y_{t+1}/k_t) + 1 - \delta$. Finally, we get (from the first-order condition with respect to n):

$$\Psi (1-n_t)^{-\eta} = \lambda_t (1-\alpha) \left(\frac{y_t}{n_t} \right)$$

- In steady state we have $\beta R^{ss} = 1$. This determines y^{ss}/k^{ss} **independently of monetary factors.** From the resource constraint, $y^{ss} = c^{ss} + \delta k^{ss}$ one identifies $(c^{ss}/k^{ss}) = (y^{ss}/k^{ss}) - \delta$. From the production function, one gets $(n^{ss}/k^{ss}) = (y^{ss}/k^{ss})^{1/(1-\alpha)}$. What then determines n^{ss}? Essentially, the consumption-leisure decision:

$$\frac{(c^{ss})^{-\Phi}}{\Psi (1-n^{ss})^{-\eta}} = \frac{\lambda^{ss} + \mu^{ss}}{\lambda^{ss} (1-\alpha) (y^{ss}/n^{ss})}$$

  A higher μ^{ss} tends to make consumption more costly relative to leisure (the nominal interest rate is positive implying positive costs of holding money for transactions); hence, less labor is supplied. The CIA constraint corresponds to a distorting consumption tax. More specifically, in steady state

$$\begin{aligned} \beta \left[\frac{\lambda^{ss} + \mu^{ss}}{1+\pi^{ss}} \right] &= \lambda^{ss} \\ \beta \left[\frac{1 + \mu^{ss}/\lambda^{ss}}{1+\pi^{ss}} \right] &= 1 \\ \frac{1 + \mu^{ss}/\lambda^{ss}}{1+\pi^{ss}} &= \frac{1}{\beta} = R^{ss} \end{aligned}$$

- So (as in simple CIA model)

$$i^{ss} = \mu^{ss}/\lambda^{ss}$$

  and consumption leisure choice becomes

$$\frac{(c^{ss})^{-\Phi}}{\Psi (1-n^{ss})^{-\eta}} = \frac{1+i^{ss}}{(1-\alpha)(y^{ss}/n^{ss})}$$

  I steady-state, higher money growth and inflation will **raise the nominal interest rate** and induce a **substitution away from the cash good** (consumption) towards the "non-cash" good, leisure:

$$\frac{\partial n^{ss}}{\partial \theta^{ss}} < 0$$

- Note in contrast with the MIU model with leisure, the non-neutrality is non-ambiguous and thus independent of Φ. In the MIU model with leisure, a higher nominal interest rate reduced m, and depending upon $u_{cm} \gtrless 0$ it reduced or increased n. In the CIA model, the effect of money growth is more "direct": Consumption is being **taxed** by a positive nominal interest rate, while leisure is not.

3.2.1 Dynamics

- Method used is as in stochastic MIU model: Calibration: Assign plausible values the parameters of the model. Values chosen to conform with basics of MIU model. Simulation: perform a log-linearization of the model's dynamic equations (everything is expressed as percentage deviations from steady state). Then solve this system by numerical methods (various simulation programs are available on the internet), and create artificial time series data from the system. From the artificial data one evaluates the statistical properties of the model

Main results from the simulations presented by Walsh (2003)

- As in MIU model, if money shocks, φ_t-shocks, shall pay a role, persistence in money growth is necessary ($\gamma > 0$). Otherwise, the shock will *not* affect expected next-period inflation, and thus — through the Fisher relationship — period t nominal interest rate. In effect, the "consumption tax" does not vary! Hence, only "anticipated money" matters

- The effects of money shocks on labor and output are stronger the more persistence in money growth, and the effects are generally stronger than in MIU model. The reason is mainly that the effects of variations in the nominal interest rate are having a **direct** effect on the consumption-leisure choice. By contrast, in MIU model the effect were generated **indirectly** through money demand and the utility function properties (i.e., the cross-derivative u_{cm})

- If technology shocks are met with procyclical money, output is more stable (as in MIU model with $\Phi < b$). The magnitude of this stabilization gain is small (but stronger than in MIU model). The reason for the more stable output is that when a positive technology shock is met by an increase in money growth, the nominal interest rate increases and discourages labor supply, thereby dampening the increase in output.

- No "liquidity effect of monetary shocks" (as in IS/LM models): Positive money shock *increases* nominal interest rates. The difference between the CIA model and the IS/LM model stems from the fact that the former has flexible prices while the latter has fixed prices. Again, the model's short-run behavior is much like its long run behavior (which is not too surprising when prices are flexible).

3.2.2 Real resource costs of transactions

- In shopping-time models, transactions took *time* (which is a valuable commodity). In CIA models, transactions *must* be carried out using money. Another possibility is that transactions involve direct resource costs. I.e., *transaction costs*. E.g., more volume of goods traded in the market, more resources are being "wasted." The general idea is then that presence of money can reduce transaction costs

- Thus money does not provide utility directly or indirectly, but frees up resources spent on transactions. Transaction costs are specified as:

$$\Upsilon(c,m) \quad \Upsilon_c > 0, \quad \Upsilon_m < 0$$

So, the higher is consumption, the higher are transaction costs (which are reduced by money). This function will then show up in the budget constraint:

$$\begin{aligned} & f(k_{t-1}) + (1-\delta)k_{t-1} + \tau_t + \frac{m_{t-1}}{1+\pi_t} + \frac{(1+i_{t-1})b_{t-1}}{1+\pi_t} \\ = \; & c_t + m_t + b_t + k_t + \Upsilon(c_t, m_t) \end{aligned}$$

- Robert Feenstra (1986) demonstrated equivalence of the MIU approach and the transaction cost approach (analogy to showing that the shopping-time approach also implies money-in-the-utility implicitly). Indeed, with plausible restrictions on Υ, a definition of a money-in-the utility function $W(x,m)$, (with x to be specified), and introduction of a standard consumption-based utility function $U(c)$, then it turns out that the following two problems have the **same solution** (ignore capital):

$$\max \quad U(c) \quad \text{s.t.} \quad y = c + \Upsilon(c,m) + b + m \tag{3.37'}$$

–

$$\max \quad V(x,m) = U(W(x,m)) \quad \text{s.t.} \quad y = x + b + m \tag{3.38'}$$

where $x \equiv c + \Upsilon(c,m)$

- I.e., the transactions cost problem has solution c^*, b^*, m^* and money-in-the utility function has solution $x^* = c^* + \Upsilon(c^*, m^*)$, b^*, m^*. Hence, transactions cost idea is another candidate behind the money in the utility approach (note however, for equivalence to hold, one has consumption **plus** transactions cost as an argument in the MIU model)

3.3 Summary on CIA models

- MIU-models, shopping time models, CIA models and other models of money, are . . . . just models. Models, nevertheless, are useful, consistent abstractions to use for thinking about economics

- The micro-founded flex-price models analyzed so far are:

  - Suitable for long run-analyses of links between money and inflation, and potential real allocation
  - Suitable for thinking about why money exists and what is the value of money (direct utility, liquidity service, saved leisure....)
  - Suitable for thinking about the optimal rate of inflation (robustness of Friedman rule)
  - Less suitable for analyzing the short run implications of monetary shocks as the models, by nature, exhibits monetary neutrality (although not necessarily superneutrality)
  - To remedy the short-run failure of such models, one *must* introduce incomplete nominal adjustment (more on this later in the course)

4 Monetary policy and public finance

- Inflation or the nominal interest rate have been viewed as a **tax** on household's resources in the previous models. In particular through the erosion of real money balances. However, the flip-side of the coin have been ignored: The proceedings from the tax have not been modelled; i.e., the government budget has been (almost) neglected. This is an omission, as one ignores important questions like:

  - Can inflation be effectively be used as a means of financing public expenditures, deficits and debt?
  - Will monetary and fiscal policy interact in ways that qualify some of the conclusions reached so far in the flex-price general equilibrium models?
  - Will debt and deficit policy have implications for monetary policy and thus, e.g., inflation?
  - If inflation is used as a tax, what will its optimal value be? (Will the Friedman rule be robust to public finance considerations?)

4.1 Public budget accounting, inflation and debt

- The core element linking monetary and fiscal policy is the public budget constraint. First consider the fiscal branch of government (the Treasury). It faces the following budget identity in nominal terms

$$G_t + i_{t-1}B^T_{t-1} = T_t + \left(B^T_t - B^T_{t-1}\right) + RCB_t \tag{4.1}$$

Superscript T denotes total public debt; RCB_t is receipts from the central bank. Then consider the central bank's budget identity

$$\left(B^M_t - B^M_{t-1}\right) + RCB_t = i_{t-1}B^M_{t-1} + (H_t - H_{t-1}) \tag{4.2}$$

Superscript M denotes public debt held by the central bank; H_t is "high-powered money" — the monetary base; the central bank's own liabilities. Now, let $B_t = B^T_t - B^M_t$ be public debt held by the private sector. Then the two budget identities are combined to the **consolidated budget identity**

$$G_t + i_{t-1}B_{t-1} = T_t + (B_t - B_{t-1}) + (H_t - H_{t-1}). \tag{4.3}$$

- Express things in real terms and in relation to total nominal income $P_t y_t$, and ignore population and output growth:

$$g_t + \overline{r}_{t-1}b_{t-1} = t_t + (b_t - b_{t-1}) + \frac{H_t - H_{t-1}}{P_t y_t} \tag{4.4'}$$

(lower-case letters are variables relative to total nominal income). The term

$$\overline{r}_{t-1} = \frac{1 + i_{t-1}}{1 + \pi_t} - 1$$

is the **ex post** real interest rate. Define

$$r_{t-1} = \frac{1 + i_{t-1}}{1 + \pi^e_t} - 1$$

as the **ex ante** real interest rate, and the budget identity becomes

$$g_t + r_{t-1}b_{t-1} = t_t + (b_t - b_{t-1}) + \frac{(1 + r_{t-1})\left(\pi_t - \pi^e_t\right)}{1 + \pi_t}b_{t-1} + \frac{H_t - H_{t-1}}{P_t y_t} \tag{4.5'}$$

- Note the revenue caused by $\pi_t > \pi^e_t$ as higher-than-expected inflation **erodes** the real servicing costs of debt; but inflation in itself, when anticipated, does not reduce the real debt burden. The last term of (4.5') is **seigniorage**, the real income generated by "issuing non-interest bearing debt": money. Seigniorage relative to total nominal income is

$$s_t \equiv h_t - \frac{1}{1 + \pi_t}h_{t-1} = h_t - h_{t-1} + \frac{\pi_t}{1 + \pi_t}h_{t-1}. \tag{4.6'}$$

Importantly, even in steady state with $h_t - h_{t-1} = 0$, one has

$$s = \frac{\pi}{1 + \pi}h,$$

which is **positive** for **positive inflation.** Alternative intuition: As h is a real liability for the government, which **does not** pay interest, inflation reduces the real return, and thereby reduces "real interest" payments on money

- What if inflation is zero? Will there then be no revenue from seigniorage? Yes, by issuing non-interest bearing debt (=money) instead of interest bearing debt (bonds), **the government saves interest payments.** To see this, rewrite budget identity, defining $d_t = b_t + h_t$ as **total liabilities**:

$$\begin{aligned} & g_t + r_{t-1}\left(d_{t-1} - h_{t-1}\right) \\ = \; & t_t + \left(d_t - d_{t-1}\right) + \frac{\left(1 + r_{t-1}\right)\left(\pi_t - \pi_t^e\right)}{1 + \pi_t}\left(d_{t-1} - h_{t-1}\right) + \frac{\pi_t}{1 + \pi_t} h_{t-1} \end{aligned}$$

and thus

$$\begin{aligned} g_t + r_{t-1} d_{t-1} &= t_t + \left(d_t - d_{t-1}\right) + \frac{\left(1 + r_{t-1}\right)\left(\pi_t - \pi_t^e\right)}{1 + \pi_t} d_{t-1} \\ &+ \left[\frac{\pi_t - \left(1 + r_{t-1}\right)\left(\pi_t - \pi_t^e\right)}{1 + \pi_t} + r_{t-1}\right] h_{t-1} \end{aligned}$$

$$g_t + r_{t-1} d_{t-1} = t_t + \left(d_t - d_{t-1}\right) + \frac{\left(1 + r_{t-1}\right)\left(\pi_t - \pi_t^e\right)}{1 + \pi_t} d_{t-1} + \frac{i_{t-1}}{1 + \pi_t} h_{t-1}$$

So, when constraint is formulated in terms of total liabilities, the steady-state seigniorage is

$$\overline{s} = \frac{i}{1 + \pi} h \tag{4.9}$$

So, a Friedman rule eliminates seigniorage, and requires offsetting fiscal policy changes to maintain budget identity (e.g., raising conventional taxes). In general, irrespective of definition of seigniorage, a change in monetary financing requires offsetting changes in either taxes, spending or debt

- In which manner monetary changes affect fiscal policy depends on the definition of fiscal policy

  - If it is in terms of fixed spending and interest rate bearing debt, changes in s and the offsetting changes in taxes are monetary policy
  - If it is in terms of fixed spending and total liabilities, changes in $\overline{s}$ and offsetting taxes and composition of liabilities are monetary policy

- So, which definition is relevant, depends on how fiscal policy is conducted

- Note an implication of a simple version of budget identity (ignoring unanticipated inflation)

$$g_t + r b_{t-1} = t_t + b_t - b_{t-1} + s_t,$$

can be solved forward to obtain the "solvency requirement":

$$\left(1 + r\right) b_{t-1} + \sum_{i=0}^{\infty} \frac{g_{t+i}}{\left(1 + r\right)^i} = \sum_{i=0}^{\infty} \frac{t_{t+i} + s_{t+i}}{\left(1 + r\right)^i}$$

which holds when $\lim_{i\to\infty}(1+r)^{-i} b_{t+i} = 0$ (no "Ponzi games"). So, if a government has initial debt, it must at some point run surpluses, $t_{t+i} + s_{t+i} > g_{t+i}$, generated through either taxes or seigniorage. This raises the issue of whether government debt or deficits will ultimately create seigniorage and thus inflation

- This will generally depend on the **fiscal-monetary regime**
  - If fiscal policy is "dominant" (or "active"), monetary policy must be passive, and secure solvency
  - If monetary policy is "dominant", fiscal policy must secure solvency
- Hence, in regimes of fiscal dominance, it may be the case that debt and deficits will be inflationary
- Also, it may be the case that **monetary contractions** (e.g., aimed at reducing inflation) will reduce seigniorage revenues, increasing debt, which ultimately requires increased seigniorage, and thus, **inflation in the future** (Sargent and Wallace's "Unpleasant Monetarist Arithmetic")
- This emphasizes that treating money as independent of fiscal policy, could be **very misleading**, as monetary policy changes could very well be the result of changes in fiscal policy (and the effects on the macroeconomy could be very different depending upon the source of the change in monetary policy).

4.2 A simple model to prove the point

- Assume government spending is zero. The budget constraint then becomes

$$(1+r_{t-1})\,b_{t-1} = t_t + b_t + s_t \tag{4.15}$$

Let the present value of taxes cover **a fraction** of existing government liabilities $(1+r_{t-1})\,b_{t-1}$:

$$T_t \equiv \sum_{s=t}^{\infty}\left(\frac{1}{1+r_s}\right)^{s-t} t_s = \psi\,(1+r_{t-1})\,b_{t-1}, \qquad 0<\psi\leq 1$$

For $\psi = 1$, any debt is fully **backed** by taxes (this is sometimes referred to as a Ricardian fiscal policy — or, non-dominance in fiscal policy). For $\psi < 1$ only a fraction is backed, and to secure solvency some seigniorage is required. I.e., some fiscal dominance is present. One can write

$$T_t = t_t + \frac{1}{1+r_t} T_{t+1}$$

(forward it successively, and one gets the present value expression). Hence,

$$T_t = t_t + \frac{1}{1+r_t}\psi\left(1+r_t\right)b_t = t_t + \psi b_t$$

From the assumption about T_t one gets

$$\psi\left(1+r_{t-1}\right)b_{t-1} = t_t + \psi b_t$$

Note that with $\psi = 1$ one indeed gets the government budget constraint with $s_t = 0$. So, for $\psi < 1$, some seigniorage is required

- Now consider the households' budget constraint (y_t is an endowment)

$$y_t + \left(1+r_{t-1}\right)b_{t-1} + \frac{m_{t-1}}{1+\pi_t} - t_t = c_t + m_t + b_t \tag{4.16}$$

Inserting the expression for t_t from the government budget constraint one gets

$$y_t + \left(1-\psi\right)\left(1+r_{t-1}\right)b_{t-1} + \frac{m_{t-1}}{1+\pi_t} = c_t + m_t + \left(1-\psi\right)b_t$$

For $\psi = 1$, one sees that debt disappears. In equilibrium it will play no role for determination of the price level; only the money stock matters. For $\psi < 1$, debt will matter for price level determination. To exemplify the role of debt for prices and inflation, consider a MIU preference specification for households, where the per-period utility function is given by

$$u\left(c_t, m_t\right) = \ln c_t + \delta \ln m_t, \qquad \delta > 0,$$

The households then maximize life-time discounted utility subject to the budget constraint. The maximization is characterized by

$$V\left(b_{t-1}, m_{t-1}\right) = \max\left\{\ln c_t + \delta \ln m_t + \beta V\left(b_t, m_t\right)\right\}$$

where maximization is over c, m, and b and subject to the budget constraint. As usual, use the budget constraint to eliminate b_t to obtain an unconstrained maximization problem over c and m. The first-order condition w.r.t. c is

$$\frac{1}{c_t} = \frac{\beta}{1-\psi}V_b\left(b_t, m_t\right), \tag{i}$$

and the first-order condition w.r.t. m is

$$\frac{\delta}{m_t} + \beta V_m\left(b_t, m_t\right) = \frac{\beta}{1-\psi}V_b\left(b_t, m_t\right). \tag{ii}$$

Relationships between partial derivatives of the value function by the envelope theorem can be found as

$$V_b\left(b_{t-1}, m_{t-1}\right) = \beta\left(1+r_{t-1}\right)V_b\left(b_t, m_t\right), \tag{iii}$$

$$V_m\left(b_{t-1}, m_{t-1}\right) = \frac{\beta}{\left(1+\pi_t\right)\left(1-\psi\right)}V_b\left(b_t, m_t\right). \tag{iv}$$

Forward (iii) and multiply by $\beta/(1-\psi)$ on both sides to get:

$$\frac{\beta}{1-\psi}V_b(b_t, m_t) = \frac{\beta^2}{1-\psi}(1+r_t)V_b(b_{t+1}, m_{t+1}).$$

Then use (i) to obtain the consumption Euler equation (the Keynes-Ramsey rule):

$$\begin{aligned} \frac{1}{c_t} &= \beta(1+r_t)\frac{1}{c_{t+1}} \\ c_{t+1} &= \beta(1+r_t)c_t \end{aligned} \tag{v}$$

Then use (iv) on (ii) to get

$$\begin{aligned} \frac{\delta}{m_t} + \frac{\beta^2}{(1+\pi_{t+1})(1-\psi)}V_b(b_{t+1}, m_{t+1}) &= \frac{\beta}{1-\psi}V_b(b_t, m_t) \\ \frac{\delta}{m_t} + \frac{\beta}{(1+\pi_{t+1})}\frac{1}{c_{t+1}} &= \frac{1}{c_t} \\ \frac{\delta}{m_t} + \frac{1}{(1+\pi_{t+1})(1+r_t)}\frac{1}{c_t} &= \frac{1}{c_t} \end{aligned}$$

where the last two equations follows from applying (iv) and (i), and finally (v). From this, the money demand relationship follows as:

$$\begin{aligned} m_t &= \delta\left[1 - \frac{1}{(1+\pi_{t+1})(1+r_t)}\right]^{-1} c_t \\ m_t &= \delta\left[1 - \frac{1}{(1+i_t)}\right]^{-1} c_t \end{aligned}$$

and thus

$$m_t = \delta\left(\frac{1+i_t}{i_t}\right)c_t \tag{vi}$$

- Note that this follows immediately from the general characterization of optimal money demand from the MIU approach in Walsh (2003, Chapter 2):

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1+i_t},$$

which indeed with the particular utility function used here gives

$$\frac{\delta/m_t}{1/c_t} = \frac{i_t}{1+i_t},$$

and thus the money demand function (vi). Now define

$$w_t \equiv m_t + (1-\psi)b_t$$

as the "total" real wealth net of taxes: Real money balances plus non-tax-backed government debt. The budget constraint then becomes

$$y_t + (1+r_{t-1})w_{t-1} - \frac{i_{t-1}}{1+\pi_t}m_{t-1} = c_t + w_t.$$

Note the term

$$\frac{i_{t-1}}{1+\pi_t} m_{t-1}$$

which is the deduction from total net wealth arising from the fact that real money pays no interest. Remark that this term **corresponds exactly** to the seigniorage definition $\overline{s}$, (4.9), which is the relevant definition when a fixed fiscal policy is one with a fixed path for **total** liabilities: money and debt

- Use now the lagged money demand function and the consumption Euler equation to get

$$\begin{aligned} y_t + (1+r_{t-1}) w_{t-1} - \frac{i_{t-1}}{1+\pi_t} \delta \frac{1+i_{t-1}}{i_{t-1}} c_{t-1} &= c_t + w_t \\ y_t + (1+r_{t-1}) w_{t-1} - \delta (1+r_{t-1}) c_{t-1} &= c_t + w_t \\ y_t + (1+r_{t-1}) w_{t-1} - \frac{\delta}{\beta} c_t &= c_t + w_t \end{aligned}$$

Use that $y_t = c_t$ in equilibrium and examine the steady state:

$$w^{ss} = \frac{\delta}{\beta r^{ss}} y^{ss} = \frac{M^{ss} + (1-\psi) B^{ss}}{P^{ss}}$$

The price level is thus determined as

$$P^{ss} = \frac{\beta r^{ss}}{\delta y^{ss}} \left[M^{ss} + (1-\psi) B^{ss} \right]$$

Hence, government debt matters for the price level when $\psi < 1$. Only when all government debt is backed by taxes, $\psi = 1$, will there be the usual proportionality between nominal money and prices. Otherwise, higher B^{ss} will, for $\psi < 1$, require more reliance on seigniorage, leading to an increase in the price level.

4.3 Equilibrium seigniorage

- Before considering how inflation is *optimally* used as a tax, and before (re)considering the optimality of the Friedman rule, we assess:
  - What can seigniorage achieve in terms of financing given deficits? Anything? Or are there limits?
  - What are the inflationary implications of relying on seigniorage? Can hyperinflations result from seigniorage collection?

Are there limits to collection of seigniorage?

- In one word: Yes! In more words:

- On the one hand, higher inflation and nominal interest rates, increase seigniorage for given real money balances
- On the other hand, real money balances will fall as inflation and nominal interest rates increase (a money demand response)
- Hence, as the inflation tax goes up, the **tax base** is going down; if the latter effect is strong total inflation revenues may eventually fall

• This is shown formally in a MIU model. Here, it suffices to find the money demand function from the usual first-order condition:

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1+i_t} \equiv I_t$$

Specific form of per-period utility function is

$$u(c_t, m_t) = \ln c_t + m_t (B - D \ln m_t), \quad B > D > 0$$

The resulting money demand relation is then

$$\frac{B - D \ln m_t - D}{1/c_t} = I_t$$

leading to

$$m_t = A\mathrm{e}^{-I_t/Dc_t}, \quad A \equiv \mathrm{e}^{(B-D)/D} \tag{4.24'}$$

A standard specification of money demand, which is often replaced by

$$m_t = K\mathrm{e}^{-\alpha\pi_t^e}, \quad \alpha > 0,$$

in studies of hyperinflations (where the underlying assumption is that the real interest rate and consumption/output fluctuates relatively little), or for steady-state/long-run analyses, where by the Fisher relationship inflation and nominal interest rate moves one-for-one

• Steady state seigniorage is (from (4.9))

$$\begin{aligned}\bar{s} &= \frac{i}{1+\pi} m \\ &= (1+r) \frac{i}{1+i} m \\ &= (1+r) I\, m\end{aligned}$$

Using the money demand function one gets

$$\bar{s} = (1+r) I\, A \exp\left[-\frac{I}{Dc}\right]$$

For I close to zero, seigniorage is close to zero; this will thus be a characteristic of relatively **low inflation.** For I very high, seigniorage is also close to zero; this will thus be a characteristic of relatively **high inflation**. Hence, with inflation increasing from a low level, seigniorage is increasing, but eventually the falling money demand reduces seigniorage. A maximal amount of seigniorage thus exists. In other words an inflation rate π^* exists for which seigniorage is at a maximum.

  - For $\pi > \pi^*$ equilibrium seigniorage is decreasing in π
  - For $\pi < \pi^*$ equilibrium seigniorage is increasing in π
  - A seigniorage "Laffer curve" is faced by the government (cross-country studies for developing countries — which are countries that relies on seigniorage to a much larger extent than industrialized countries — confirm the presence of such a "Laffer curve" relationship).

Inflationary implications of relying on seigniorage

- Remark that the Laffer curve property implies that **two** steady-state inflation rates can finance the same deficit; a **high** and a **low** inflation rate. Also, there are limits as to how much one can finance by seigniorage. Both aspects can potentially lead to hyperinflation (often defined as monthly inflation rates of 50% or more).
- What are the stability properties of the two steady states, both accomplishing the same financing target?
- Assume sluggish adjustment of money demand due to, e.g., adjustment costs or "slow" reactions in inflation expectations (as in Walsh, 2003).
- Consider the high-inflation steady-state
  - If inflation temporarily goes up, seigniorage goes up as money demand reacts little
  - Households start gradually to reduce real money balances; this causes seigniorage to fall
  - If inflation remains permanently at the higher level, seigniorage will eventually become lower than target
  - Hence inflation must increase further to maintain required seigniorage revenues
  - ...and the process continues with ever increasing inflation
- Consider the low-inflation steady-state

  - If inflation temporarily goes up, seigniorage goes up as money demand reacts little
  - Households start gradually to reduce real money balances; this causes seigniorage to fall
  - If inflation remains permanently at the higher level, seigniorage will eventually become higher than target
  - Hence, inflation must decrease to maintain required seigniorage revenues
  - ...and the process continues back to the low-inflation steady-state

- Hence, low-inflation steady state is stable, while high-inflation steady state is unstable

- Therefore, if some shock brings inflation above the high-inflation steady state, the result will be hyperinflation. Also, **if financing requirement suddenly increases** above what is feasible to finance by seigniorage, the government may engage in futile financing attempts by printing money at a faster and faster rate, thereby driving money balances down, leading to attempts to raise revenue by increasing money growth even further, etc. In both cases, hyperinflations can only be stopped by fiscal reform (e.g., contract monetary policy and raise conventional taxes, a policy package that may be painful). Note that in both "stories" of hyperinflations, inflation was caused by money growth; i.e., inflation was based on fundamentals

- As an aside, note that hyperinflations can be **non-fundamental**; i.e., occur in isolation of money growth. These are labelled **speculative** hyperinflations (or "bubble paths"). As an example, let money demand be (now lower case variables are denoting logarithms):
$$m_t - p_t = -\alpha \left(\mathrm{E}_t p_{t+1} - p_t \right), \quad \alpha > 0,$$
This is rearranged as an expression for the (log of) price level:
$$p_t = \frac{1}{1+\alpha} m_t + \frac{\alpha}{1+\alpha} \mathrm{E}_t p_{t+1}$$
This is a first-order expectational difference equation in p_t (as it depends on its own expected future value). Money is for simplicity assumed to be constant, $m_t = \theta_0$.The parameter θ_0 is the model's **fundamental**, and a solution of p_t depending only on the fundamental and parameters is a fundamental solution. To find this solution, use the **method of undetermined coefficients:**

  - Conjecture a form of the solution by undetermined coefficients

  - Forward the solution, take expectations and insert into the expectational difference equation in p_t
  - Identify the coefficients

- Here: Conjecture the solution $p_t = X\theta_0$, where X is the undetermined coefficient. Forward the conjecture and take expectations: $\mathrm{E}_t p_{t+1} = X\theta_0$. Insert into difference equation:

$$\begin{aligned} p_t &= \frac{1}{1+\alpha}\theta_0 + \frac{\alpha}{1+\alpha}X\theta_0 \\ &= \frac{1+\alpha X}{1+\alpha}\theta_0 \end{aligned}$$

The undetermined coefficient is now identified as it must be the case that

$$X = \frac{1+\alpha X}{1+\alpha}$$

and thus $X = 1$ Hence, $p_t = \theta_0$ is the fundamental solution.

- However, it is easy to see that infinitely many solutions of the form

$$p_t = \theta_0 + bub_t, \quad bub_t \lesseqgtr 0$$

exists when

$$\mathrm{E}_t bub_{t+1} = \frac{1+\alpha}{\alpha} bub_t.$$

To see this, note that it *is* consistent with the difference equation:

$$\begin{aligned} \theta_0 + bub_t &= \frac{1}{1+\alpha}\theta_0 + \frac{\alpha}{1+\alpha}\mathrm{E}_t\left[\theta_0 + bub_{t+1}\right] \\ \theta_0 + bub_t &= \frac{1}{1+\alpha}\theta_0 + \frac{\alpha}{1+\alpha}\left[\theta_0 + \frac{1+\alpha}{\alpha}bub_t\right] \\ \theta_0 &= \theta_0 \end{aligned}$$

- Hence, if $bub_t > 0$, we have ever rising prices, even though the money supply is constant. Prices are rising just because they are (rationally!) expected to do so. It resembles the idea behind asset price bubbles, which can arise due to self-fulfilling expectations. If all traders expect asset prices to go up, all buy the assets and drive up the price, thereby confirming the expected.

4.3.1 Summary

- Monetary and fiscal policy are linked through the public budget constraint
- Ignoring this may be valid if governments have access to lump-sum taxation and follow policies that fully back interest-bearing debt with taxes

- Otherwise, important channels from monetary policy to fiscal policy and vice versa may be overlooked, as the financing properties of inflation is ignored

- Also, it is important to stress that an observed change in monetary policy may or may not be due to fiscal considerations, and therefore have different implications for the real economy depending on the source of the change

- While a potential financing tool, one must be aware of the dangers of hyperinflation associated with reliance on seigniorage as a means of financing public expenditures

4.4 Optimal taxation and seigniorage

- *If* inflation is used as a tax What should its optimal value be? Is the Friedman rule robust, even under public finance considerations? Are there (other) circumstances under which inflation can be harmful when taxation is considered?

- Basic idea goes back to Edmund Phelps (1972): If inflation is a tax, then it should be used along "conventional" taxes in the design of an optimal tax structure. Optimal tax policies trade off the distortionary effects of various taxes (on savings, labour supply, etc.). So, seigniorage should be part of that "optimal tax structure problem"! And this may call for the optimality of positive inflation to achieve seigniorage (so as to reduce the distortions from other taxes)

- This idea is examined in a simple partial equilibrium model due to N. Gregory Mankiw (1987). Assume an exogenous stream of expenditures. Government issues debt, collects taxes and seigniorage (the real interest rate is assumed constant). The government flow budget constraint is given by:

$$g + Rb_{t-1} = \tau_t + s_t + b_t, \tag{4.41'}$$

  and the government's intertemporal budget constraint (as $\lim_{i\to\infty} R^{-i} b_{t+i} = 0$ is imposed: "No Ponzi-Games") is:

$$Rb_{t-1} + \frac{R}{R-1} g = \mathrm{E}_t \left[\sum_{i=0}^{\infty} R^{-i} \left(\tau_{t+i} + s_{t+i} \right) \right] \tag{4.43'}$$

  Debt liability plus interest and present value of current and future spending equals (expected) present value of current and future taxes and seigniorage

- The government's choice: Set taxes and seigniorage (commit to a whole path for them), given the path of g and initial debt, and the intertemporal budget constraint. The government's objective is described by an *ad hoc* specification of preferences: The government is assumed to **minimize** present value of the **distortions** arising from taxes and seigniorage. These distortions are assumed to be stochastic and quadratic in taxes and seigniorage:

- Per-period loss from tax distortions:

$$\frac{1}{2}(\tau_t + \phi_t)^2, \quad \phi_t \text{ is capturing stochastic shifts in tax distortions}$$

- Per-period loss from seigniorage distortions:

$$\frac{1}{2}(s_t + \varepsilon_t)^2, \quad \varepsilon_t \text{ is capturing stochastic shifts in seign. distortions}$$

Hence, the objective is to minimize

$$\frac{1}{2}\sum_{i=0}^{\infty} R^{-i}\mathrm{E}_t\left[\left(\tau_{t+i} + \phi_{t+i}\right)^2 + \left(s_{t+i} + \varepsilon_{t+i}\right)^2\right]$$

subject to the intertemporal budget constraint. Let λ be the Lagrange multiplier on the constraint. First order conditions then become:

$$\begin{aligned}\mathrm{E}_t\left(\tau_{t+i} + \phi_{t+i}\right) &= \lambda, \quad i \geq 0, \\ \mathrm{E}_t\left(s_{t+i} + \varepsilon_{t+i}\right) &= \lambda, \quad i \geq 0,\end{aligned}$$

- We then have **intra**temporal optimality conditions: $\mathrm{E}_t\left(\tau_{t+i} + \phi_{t+i}\right) = \mathrm{E}_t\left(s_{t+i} + \varepsilon_{t+i}\right) = \lambda$. I.e., marginal losses of distortions are *equalized* within each period. The implication is that if financing needs go up (e.g., g or b_{t-1} go up causing λ to go up), taxes and seigniorage move in *similar direction*

- We have **inter**temporal optimality conditions: $\mathrm{E}_t\left(\tau_{t+i} + \phi_{t+i}\right) = \mathrm{E}_t\left(\tau_{t+i+1} + \phi_{t+i+1}\right)$ and $\mathrm{E}_t\left(s_{t+i} + \varepsilon_{t+i}\right) = \mathrm{E}_t\left(s_{t+i+1} + \varepsilon_{t+i+1}\right)$. Marginal losses of each instrument are equalized across periods. This is known as "tax smoothing." If $\mathrm{E}_t\varepsilon_{t+1} = \varepsilon_t$, future seigniorage is unpredictable; i.e., it follows a "random walk."

- Empirical evidence of Mankiw model mixed....

  - For some countries, the positive relationship between taxes and seigniorage are present, for some not (in particular industrialized countries)
  - For some countries the (near) random walk behavior of inflation is observed in data; however, this could have other explanations
  - For U.S., seigniorage are linked to deficit rather than taxes

- Model is based on *ad hoc* government loss function; no explicit formulation of money demand, and thereby how distortions arise. A fully formulated model may provide more "restrictions" on optimal inflation; e.g., through its interaction with consumption changes

- Optimal seigniorage with temporary changes in financing needs? In the model, taxes and seigniorage are linked to permanent changes in expenditures. If model is extended to include temporary variation in expenditures, it will be optimal that these are met with changes in deficits (as these have no distortionary costs)

- But why would there be costs of temporary, unanticipated, seigniorage??
  - In MIU and CIA models, all distortions from inflation came from *anticipated* inflation (as this affected nominal interest rates)
  - Unanticipated inflation will just have income effects (through the budget); hence, non-distortionary
- Indeed, it can be shown in micro-founded model that optimal behavior has seigniorage responding temporarily to temporary changes in financing needs (in contrast with Mankiw's model). This is in accordance with the data from the US (and shows importance of micro-founded models)

4.4.1 Robustness of the Friedman rule?

- Will public finance considerations render the Friedman rule invalid, as positive inflation necessarily will involve a positive nominal interest rate? The quick answer is "**maybe**," but surprisingly the Friedman rule may still be optimal in some circumstances. To analyze the issue, one must move beyond the *ad hoc* model of optimal seigniorage with postulated distortions (in that type of model, positive seigniorage is optimal by definition!) Here the focus is on the optimal inflation tax in a CIA and a MIU model

The Friedman rule in a CIA model

- Two consumption goods. A "cash" good (i.e., subject to a CIA constraint) and a "credit" good. Taxes on labor income and consumption goods (commodity taxes). For simplicity, no capital and a linear production technology is assumed. Per-period utility function is

$$U\left(c_{1,t}, c_{2,t}, l_t\right)$$

where $c_{1,t}$ is the cash good and $c_{2,t}$ is the credit good; l_t is leisure. The budget constraint in nominal terms is

$$\begin{aligned} & \left(1+\tau^c\right) Q_t \left(c_{1,t}+c_{2,t}\right) + M_t + B_t \\ = & \left(1-\tau^h\right) Q_t \left(1-l_t\right) + \left(1+i_{t-1}\right) B_{t-1} + M_{t-1} \end{aligned}$$

τ^c is the commodity tax (*uniform* = identical on both goods); τ^h is the wage tax and Q_t is producer price. Let $P_t = \left(1+\tau^c\right) Q_t$ be consumer prices, and let $w_t = \left(M_t + B_t\right)/P_t = m_t + b_t$ be total real wealth. The budget constraint then becomes:

$$\begin{aligned} c_{1,t} + c_{2,t} + w_t &= (1-\tau)\left(1-l_t\right) + \left(1+r_{t-1}\right) w_{t-1} - \frac{i_{t-1}}{1+\pi_t} m_{t-1} \\ (1-\tau) &\equiv \frac{\left(1-\tau^h\right)}{\left(1+\tau^c\right)} \end{aligned}$$

Note last term is seigniorage definition $\overline{s}$, relevant with this wealth definition

- CIA constraint is $c_{1,t} \leq (1+\pi_t)^{-1} m_{t-1}$. Optimality condition guiding relative demand between the two consumption goods:
$$\frac{U_1(c_{1,t}, c_{2,t}, l_t)}{U_2(c_{1,t}, c_{2,t}, l_t)} = 1 + i_t$$
As in CIA model of Walsh (2003, Chapter 3), the CIA constraint is a tax on the "cash" good relative to the credit good. What is the optimal tax structure?

- One uses a famous result from public finance literature (Atkinson and Stiglitz, 1972): **Uniform commodity taxes are optimal when preferences are homothetic and weakly separable in leisure**
  - Homothetic utility: Any monotonic function of a function that is homogeneous of degree one in its arguments
  - Weakly separability implies that ratios of marginal utilities of everything but leisure, are independent of leisure
  - Example: $U(c_{1,t}, c_{2,t}, l_t) \equiv V[\varphi(c_{1,t}, c_{2,t}), l_t]$ where $\varphi(c_{1,t}, c_{2,t})$ is homogeneous of degree one in $c_{1,t}, c_{2,t}$

- Why does result hold? And why is it interesting here?
  - **Why does it hold:**
    * When utility is homogeneous of degree one, marginal utility is homogeneous of degree zero. Hence
    $$\frac{\varphi_{c_1}(c_{1,t}, c_{2,t})}{\varphi_{c_2}(c_{1,t}, c_{2,t})} = \frac{\varphi_{c_1}(c_{1,t}/c_{2,t}, 1)}{\varphi_{c_2}(c_{1,t}/c_{2,t}, 1)} = f\left(\frac{c_{1,t}}{c_{2,t}}\right)$$
    * The marginal rate of substitution between the goods determines the *ratio* of the goods, and this will be *independent of* leisure, and thus *income*!
    * Optimal to have the marginal rate of substitution equal to the relative price of goods = 1; hence producer prices should not be distorted by different commodity taxes — irrespective of the distortions taxes may have on income
  - **Why is this interesting here?**
    * Irrespective of the labour supply distortions or other income distortions caused by the tax system, optimality requires:
    $$\begin{aligned}\frac{U_1(c_{1,t}, c_{2,t}, l_t)}{U_2(c_{1,t}, c_{2,t}, l_t)} &= \frac{V_1[\varphi(c_{1,t}, c_{2,t}), l_t]\,\varphi_{c_1}(c_{1,t}, c_{2,t})}{V_1[\varphi(c_{1,t}, c_{2,t}), l_t]\,\varphi_{c_2}(c_{1,t}, c_{2,t})} \\ &= \frac{\varphi_{c_1}(c_{1,t}, c_{2,t})}{\varphi_{c_2}(c_{1,t}, c_{2,t})} = 1\end{aligned}$$

* We had before
$$\frac{U_1(c_{1,t}, c_{2,t}, l_t)}{U_2(c_{1,t}, c_{2,t}, l_t)} = 1 + i_t,$$
and therefore $i_t = 0$ is the optimal monetary policy. The Friedman rule is optimal again!
* No seigniorage will be optimal in the optimal tax mix under the assumptions about the utility function
* It only distorts the demand away from cash goods

- Note: If there were no credit good, the result is even more general:
  - A positive nominal interest rate distorts the consumption-leisure trade-off
  - ...but government already has a (conventional) labor tax
  - ...so no need to use seigniorage as an "additional labor tax"

The Friedman rule in a MIU model

- Same procedure, and usual first-order condition guiding money versus consumption:
$$\frac{U_m(c_t, m_t, l_t)}{U_c(c_t, m_t, l_t)} = \frac{i_t}{1 + i_t}$$
So if utility is homothetic in c and m and weakly separable in l, then we have again that the social optimum is one where the marginal rate of substitution between money and consumption should equal the relative social price, **which is zero.** Hence, $i_t = 0$ is optimal again under these restrictions on the utility function. The Friedman rule applies again.

- So, both in MIU and CIA models, one can restore the Friedman rule, albeit under some restrictions on the utility function (note, however, that the specific utility functions typically used in quantitative models indeed satisfy these restrictions)

4.4.2 Inflation and an unindexed tax system: An example of negative consequences of inflation

- Usually, taxes are levied on nominal interest income on bonds and nominal capital gains on capital. Inflation can then seriously **distort** the savings decision. In absence of taxes, budget constraint in nominal terms is
$$\begin{aligned} & P_t f(k_{t-1}) + (1 + i_{t-1}) B_{t-1} + P_t T_t + P_t (1 - \delta) k_{t-1} \\ = \; & P_t c_t + P_t k_t + B_t \end{aligned}$$

Nominal resources equal nominal outlays. Reformulate the left-hand side to be nominal current income *flow*:

$$\begin{aligned} & P_t f(k_{t-1}) + i_{t-1}B_{t-1} + P_t T_t + (P_t - P_{t-1})(1-\delta) k_{t-1} \\ = \; & P_t c_t + P_t k_t - P_{t-1}(1-\delta) k_{t-1} + B_t - B_{t-1} \end{aligned}$$

Assume this nominal income is taxed by τ. The constraint then becomes

$$\begin{aligned} & (1-\tau)\left[P_t f(k_{t-1}) + i_{t-1}B_{t-1} + P_t T_t + (P_t - P_{t-1})(1-\delta) k_{t-1}\right] \\ = \; & P_t c_t + P_t k_t - P_{t-1}(1-\delta) k_{t-1} + B_t - B_{t-1} \end{aligned}$$

(note mistake in Walsh p. 192: he has $P_t(1-\delta)k_{t-1}$ on right-hand side instead of $P_{t-1}(1-\delta)k_{t-1}$). In real terms (deflate by P_t):

$$\begin{aligned} & (1-\tau)\left[f(k_{t-1}) + \frac{i_{t-1}}{1+\pi_t} b_{t-1} + T_t + \frac{\pi_t}{1+\pi_t}(1-\delta) k_{t-1}\right] \\ = \; & c_t + k_t - \frac{1}{1+\pi_t}(1-\delta) k_{t-1} + b_t - \frac{1}{1+\pi_t} b_{t-1} \end{aligned}$$

Isolate the part involving capital gains (on physical capital):

$$\begin{aligned} & (1-\tau)\left[f(k_{t-1}) + \frac{i_{t-1}}{1+\pi_t} b_{t-1} + T_t\right] - \tau \frac{\pi_t}{1+\pi_t}(1-\delta) k_{t-1} \\ = \; & c_t + k_t - (1-\delta) k_{t-1} + b_t - \frac{1}{1+\pi_t} b_{t-1} \end{aligned}$$

With $\tau > 0$, inflation involves a *reduction* in real income available for consumption, physical investment and bond accumulation. Household optimization can be characterized by

$$V(k_{t-1}, b_{t-1}) = \max\{u(c_t) + \beta V(k_t, b_t)\}$$

The first-order condition for k_{t-1} (using envelope theorem):

$$V_k(k_{t-1}, b_{t-1}) = \beta V_k(k_t, b_t)\left[(1-\tau) f_k(k_t) + \left(1 - \tau \frac{\pi_{t+1}}{1+\pi_{t+1}}\right)(1-\delta)\right]$$

Marginal value of k_{t-1} equals the discounted future marginal value of k_t, taking into account $\partial k_t / \partial k_{t-1}$. Note: higher π_{t+1} reduces the after-tax return on k_{t-1}

- In steady state:

$$(1-\tau) f_k(k^{ss}) + \left(1 - \tau \frac{\pi^{ss}}{1+\pi^{ss}}\right)(1-\delta) = \frac{1}{\beta}$$

$$\tau > 0 \Longrightarrow \frac{\partial k^{ss}}{\partial \pi^{ss}} < 0$$

- The reduction in real after-tax return on capital accumulation results in lower steady-state capital stock. This is an example of distortion of a tax on nominal return distorting the after tax real return. Generally, the real after-tax return is, when nominal return is taxed,

$$
\begin{aligned}
r_a &= (1-\tau)\, i - \pi \\
&= (1-\tau)\, r - \pi + (1-\tau)\, \pi \\
&= (1-\tau)\, r - \tau\pi
\end{aligned}
$$

  Hence, with $\tau > 0$, inflation reduces the real after-tax return. Result: Higher pre-tax return, r, in equilibrium $\iff$ lower k. With indexed, or with real taxation, $r^a = (1-\tau)\, r$, inflation has no independent role for the after-tax return

- The distortionary effects of inflation in an unindexed tax system can be **severe** (cf. Martin Feldstein's, 1996, computations of the welfare gains of reducing inflation from 2% to zero ≈ 1 % permanent raise in GDP). A strong argument against using inflation as a source of revenue

4.4.3 Summary

- Optimal seigniorage may be a tax that helps equating marginal costs of all tax distortions within and across periods (Mankiw)
- In fully specified, micro-founded models with distortionary taxes, however, the Friedman rule may still hold (under some restrictions on utility functions)
- Inflation may have negative effects in a non-indexed tax system

- So, although monetary and fiscal policy **is** linked through the public budget constraint, the normative case for using inflation as a tax is not clear

Appendix

A Finding a value function relationship in the shopping-time model

Remember that the maximization problem as expressed by the value function is

$$V\left(a_t, k_{t-1}\right) = \max\left\{v\left[c_t, 1 - n_t - g\left(c_t, m_t\right)\right] + \beta V\left(a_{t+1}, k_t\right)\right\},$$

where maximization is over c_t, m_t and n_t, subject to the budget constraint

$$f\left(k_{t-1}, n_t\right) + (1-\delta) k_{t-1} + a_t = c_t + k_t + m_t,$$

and the definition

$$a_{t+1} = \tau_{t+1} + \frac{m_t}{1+\pi_{t+1}}.$$

One then substitutes out k_t and a_{t+1} to obtain an unconstrained maximization problem. The first-order conditions w.r.t. c_t, m_t and n_t are as stated given by, respectively

$$\begin{aligned} v_c\left(c_t, 1 - n_t - g\left(c_t, m_t\right)\right) &= v_l\left(c_t, 1 - n_t - g\left(c_t, m_t\right)\right) g_c\left(c_t, m_t\right) \\ &\quad + \beta V_k\left(a_{t+1}, k_t\right), \end{aligned}$$

$$-v_l g_m + \beta \frac{V_a\left(a_{t+1}, k_t\right)}{1+\pi_{t+1}} = \beta V_k\left(a_{t+1}, k_t\right)$$

$$v_l\left(c_t, 1 - n_t - g\left(c_t, m_t\right)\right) = \beta V_k\left(a_{t+1}, k_t\right) f_n\left(k_t, 1 - n_t\right)$$

To derive (**) in the main text one makes use of the fact that the optimal period t choices of consumption, real balances and labour supply are all functions of the state variables a_t and k_{t-1}. Denote these functions by

$$c_t = c\left(a_t, k_{t-1}\right), \qquad m_t = m\left(a_t, k_{t-1}\right), \qquad n_t = n\left(a_t, k_{t-1}\right).$$

The value function is then by definition given by

$$\begin{aligned} V\left(a_t, k_{t-1}\right) &= v\left[c\left(a_t, k_{t-1}\right), 1 - n\left(a_t, k_{t-1}\right) - g\left(c\left(a_t, k_{t-1}\right), m\left(a_t, k_{t-1}\right)\right)\right] \\ &\quad + \beta V\left(a_{t+1}, k_t\right) \end{aligned}$$

with (from the budget constraint and the definition of a_{t+1}):

$$k_t = f\left(k_{t-1}, n\left(a_t, k_{t-1}\right)\right) + (1-\delta) k_{t-1} + a_t - c\left(a_t, k_{t-1}\right) - m\left(a_t, k_{t-1}\right)$$

$$a_{t+1} = \tau_{t+1} + \frac{m\left(a_t, k_{t-1}\right)}{1+\pi_{t+1}}.$$

As this value function definition holds for all a_t and k_{t-1} we can differentiate on both sides with respect to these variables. Consider differentiation with respect to a_t:

$$\begin{aligned} V_a(a_t, k_{t-1}) &= v_c c_a(a_t, k_{t-1}) - v_l n_a(a_t, k_{t-1}) \\ &\quad - v_l g_c c_a(a_t, k_{t-1}) - v_l g_m m_a(a_t, k_{t-1}) + \beta V_a(a_{t+1}, k_t) \frac{m_a(a_t, k_{t-1})}{1+\pi_{t+1}} \\ &\quad + \beta V_k(a_{t+1}, k_t) \left[f_n n_a(a_t, k_{t-1}) + 1 - c_a(a_t, k_{t-1}) - m_a(a_t, k_{t-1}) \right]. \end{aligned} \tag{1}$$

Upon closer examination, we see that all the expressions multiplied on $c_a(a_t, k_{t-1})$, $n_a(a_t, k_{t-1})$ and $m_a(a_t, k_{t-1})$, respectively, sums to zero. This reflects the envelope theorem. When, say, consumption is chosen optimally (i.e., satisfies the first-order condition for consumption choice) a marginal change in state variables cannot affect the value function through changes in consumption. If it could, consumption would not have been chosen optimally. To see it formally, note that the coefficients to $c_a(a_t, k_{t-1})$ are

$$v_c - v_l g_c - \beta V_k(a_{t+1}, k_t),$$

which is zero by the first-order condition for optimal consumption choice. The same can be seen about the coefficients to $n_a(a_t, k_{t-1})$ and $m_a(a_t, k_{t-1})$. One can then reduce (1) to the much simpler expression

$$V_a(a_t, k_{t-1}) = \beta V_k(a_{t+1}, k_t),$$

which is indeed equation (**). (Equation (*) follows by the same principles.)

B Key concepts you should know

B.1 The Tobin model

- The simple budget accounting of the Tobin-Solow model
- How inflation erodes available resources
- Real money's effect on capital accumulation
- Steady-state inflation rate
- Impact of real money to capital ratio on steady-state capital
- Real money's postulated relationship with inflation
- The Tobin effect: Inflation's impact on steady state capital labor ratio
- Monetary neutrality versus superneutrality

B.2 Money in the utility function

- Dynamic programming method of deriving optimal household behavior
- Interpretation of the first-order conditions for optimal behavior
- Elimination of the partial derivatives of the value function
- Steady state properties of the simple MIU model
- Contrast with Tobin model
- Superneutrality or not?
- Real money holdings and the nominal interest rate
- The Fisher relationship
- The optimal rate of inflation
- The Friedman rule
- Welfare costs of inflation
- What can generate non-superneutrality in a MIU model?
- MIU model with endogenous labor; the consumption-leisure trade off
- Properties of utility function determining superneutrality or not
- Constant relative risk aversion utility function
- Distinction between effects of anticipated and unanticipated inflation
- Stochastic version of MIU model
- Requirement on money supply process to obtain non-superneutrality
- Role of money shocks for the real economy

B.3 Shopping-time models

- The relationship between shopping time, consumption and real money holdings
- The utility function as an indirect function of real money balances
- The consumption leisure trade-off and the impact of an increase in real money
- Optimal money demand and the optimal quantity of money in the dynamic shopping-time model

B.4 Cash-in-advance models

- The Cash-in-Advance constraint
- The Lagrange multiplier on the CIA constraint
- The optimal consumption choice with a binding CIA constraint
- The relationship between the nominal interest rate and the Lagrange multiplier on the CIA constraint
- Consumption and the nominal interest rate; the nominal interest rate as a "consumption tax"
- The long-run non-distortionary implications of the "consumption tax"
- Superneutrality in model with CIA constraint only on consumption goods, and utility only depending on consumption
- Causes of non-superneutrality: A consumption-leisure trade-off; cash and credit goods; CIA restrictions on investment on physical capital
- The optimal rate of inflation in cases of non-superneutrality
- The stochastic CIA model with endogenous labor supply
- The channel of monetary shocks through the consumption-leisure trade-off
- Qualitative effect of money shocks independent of value of constant rate of relative risk aversion
- Why only anticipated money matters
- The stabilizing effect of a procyclical monetary policy (against supply shocks)

Money and real costs of transactions

- Money, consumption and transactions costs
- Feenstra's equivalence result (just knowing it exists is sufficient)

B.5 Public budget accounting, inflation and debt

- The consolidated public budget constraint
- Seigniorage
- Real money balances as a real liability for the government paying no interest => inflation reduces the real return
- Seigniorage possible even with zero inflation when nominal interest rates are positive (by changing the composition of total liabilities from bonds towards money)
- Monetary policy's influence on fiscal policy and vice versa
- The "No-Ponzi-Game" condition
- Debt and deficits' influence on inflation
  - Dependence of the fiscal-monetary regime
  - Fiscal versus monetary "dominance"
  - With fiscal "dominance," debt may be inflationary
- Debt and price level determination when only a fraction of debt is backed by taxes

Equilibrium seigniorage

- The limits to collection of seigniorage
- The seigniorage "Laffer curve"
- Seigniorage and hyperinflations
- Hyperinflations as non-fundamental based

Optimal taxation and seigniorage

- Seigniorage as part of the optimal tax structure problem
- Positive correlation between taxes and seigniorage in Mankiw model
- Tax smoothing
- Taxes and seigniorage when financing needs are permanent versus when they are temporary

Robustness of the Friedman rule?

- The Friedman rule in a CIA model with a cash and a credit good
- The optimality of the Friedman rule even with distortionary taxes:
  - With homothetic utility, and weak separability in leisure, uniform commodity taxes are optimal
  - A positive interest rate distorts demand away from cash good
- The Friedman rule in a MIU model
  - With homothetic utility in consumption and money, and weak separability in leisure, the Friedman rule is optimal

Inflation and an unindexed tax system

- Undesirable effects of inflation through the tax system
- Inflation reduces after-tax return on capital
- Lower savings and lower steady-state physical capital stock result