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[Notes 6]
On the first-order conditions for the full commitment solution

Q Could you explain more precisely how to derive the three first-order conditions at the bottom on p. 14?

A As stated, on p 14, when finding the full commitment solution, one is searching for the optimal paths of the output gap and inflation. For that purpose, one imagines that the central planner can control these at will, and by implication can control their expectations also. Note that we know from the time-inconsistency literature that this requires commitment abilities, which a policymaker may not have. Nevertheless, the full commitment solution is a useful benchmark to examine, so as to explore the best possible policy actions.

Let us repeat the Lagrangian for convenience:

$$\mathcal{L} = -\frac{1}{2}\mathrm{E}_t\left\{\sum_{i=0}^{\infty}\beta^i\left[\alpha x_{t+i}^2+\pi_{t+i}^2\right.\right. \quad (4.17')$$
$$\left.\left.+2\phi_{t+i}\left(\pi_{t+i}-\beta\pi_{t+1+i}-\lambda x_{t+i}-u_{t+i}\right)\right]\right\}$$

where $2\phi_{t+i}$ is the multiplier on the Phillips curve (we do not need the IS-curve as the nominal interest rate can be adjusted freely). We then find the optimal paths as solutions to

$$\frac{\partial \mathcal{L}}{\partial x_{t+i}} = 0, \qquad i \geq 0, \quad (*)$$
$$\frac{\partial \mathcal{L}}{\partial \pi_{t+i}} = 0, \qquad i \geq 0. \quad (**)$$

Let's start by finding (*). We get

$$\frac{\partial \mathcal{L}}{\partial x_t} = 0 \quad \Leftrightarrow \quad \alpha x_t - \lambda\phi_t = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_{t+1}} = 0 \quad \Leftrightarrow \quad \beta\alpha x_{t+1} - \beta\lambda\phi_{t+1} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_{t+2}} = 0 \quad \Leftrightarrow \quad \beta^2\alpha x_{t+2} - \beta^2\lambda\phi_{t+2} = 0,$$
$$\vdots$$

implying that we can simply write

$$\alpha x_{t+i} - \lambda \phi_{t+i} = 0,$$

which is the first of the three conditions stated on page 14.

Then, let's find (**). We get

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \pi_t} &= 0 \quad \Leftrightarrow \quad \pi_t + \phi_t = 0, \\
\frac{\partial \mathcal{L}}{\partial \pi_{t+1}} &= 0 \quad \Leftrightarrow \quad \beta \pi_{t+1} + \beta \phi_{t+1} - \beta^0 \beta \phi_t = 0, \\
\frac{\partial \mathcal{L}}{\partial \pi_{t+2}} &= 0 \quad \Leftrightarrow \quad \beta^2 \pi_{t+2} + \beta^2 \phi_{t+2} - \beta \beta \phi_{t+1} = 0, \\
\frac{\partial \mathcal{L}}{\partial \pi_{t+3}} &= 0 \quad \Leftrightarrow \quad \beta^3 \pi_{t+3} + \beta^3 \phi_{t+3} - \beta^2 \beta \phi_{t+2} = 0, \\
&\vdots
\end{aligned}$$

implying that we can write

$$\begin{aligned}
\pi_{t+i} + \phi_{t+i} - \phi_{t+i-1} &= 0, \quad i > 0, \\
\pi_t + \phi_t &= 0,
\end{aligned}$$

which are the last two of the three conditions stated on page 14. As mentioned in Footnote 6, the difference in first-order conditions for inflation in period t and in periods $t+i$, $i > 0$, shows mathematically the time-inconsistency of the optimal plan. When period $t+1$ arrives, it would be optimal "forget about period t," set up a new Lagrangian, and aim at:

$$\begin{aligned}
\pi_{t+1+i} + \phi_{t+1+i} - \phi_{t+i} &= 0, \quad i > 0, \\
\pi_{t+1} + \phi_{t+1} &= 0.
\end{aligned}$$

Hence, the full commitment plan at time t is optimal at time t, but not at time $t+1$; it is a time-inconsistent plan.