1. Shopping-time models

2. Cash-in-Advance Models (certainty)

   Literature: Walsh (2003, Chapter 3, pp. 95-111)

3. Plan for next lecture
Shopping-time Models

• In MIU models, money provides utility directly in order to secure a demand for money
  
  – Motivation: Approximation to utility from saved time on transactions; “liquidity services”

• Shopping-time models formalize this
  
  – Helps put restrictions on, e.g., the signs of $u_{cm}$ and $u_{lm}$ under the MIU approach
    (these signs determined how labor supply and output would react to changes in money growth)

A simple model to prove the point

• Utility function (increasing and concave in both arguments):

$$v(c_t, l_t)$$

Leisure:

$$l = 1 - n - n^s$$

$n$ is fraction of time spent on work

$n^s$ is fraction of time spent on shopping (buying consumption goods)
• Assumption: **Transaction services**, \( \psi \), are needed for consumption purchases

\[ \psi = c \]

Assumption: Transaction services are facilitated by real money and are higher the more one shops:

\[ \psi = c = \psi (m, n^s) \quad (3.1) \]

\[ \psi_m \geq 0, \quad \psi_{n^s} \geq 0 \]

This is restated in terms of shopping time:

\[ n^s = g (c, m) \]

\[ g_c > 0, \quad g_m < 0 \]

• Per-period utility function can be rewritten as:

\[ u (c_t, m_t, l_t) \equiv v [c_t, 1 - n_t - g (c_t, m_t)] \]

While not an explicit argument in \( v \), money enters implicitly — and positively — in \( u \) through the negative impact on shopping time.
Optimal behavior

- Budget constraint as in MIU model with endogenous labor:

  \[ f(k_{t-1}, n_t) + \tau_t + (1 - \delta) k_{t-1} + \frac{m_{t-1}}{1 + \pi_t} = c_t + k_t + m_t \]

  (again, bonds are ignored)

Optimization is characterized by the value function:

\[ V(a_t, k_{t-1}) = \max \{ v[c_t, 1 - n_t - g(c_t, m_t)] + \beta V(a_{t+1}, k_t) \} \]

with

\[ a_t \equiv \tau_t + \frac{m_{t-1}}{1 + \pi_t} \]

Optimization is over \( c, n, m, k, \) and \( a \) subject to the budget constraint and the definition of \( a_t \).

- The budget constraint and definition of \( a_t \) are used to substitute out \( a_{t+1} \) and \( k_t \), to get an unconstrained maximization problem
To assess how money affects the consumption-leisure choice, consider the first-order conditions w.r.t. $c_t$ and $n_t$

First-order condition with respect to $c_t$:

$$v_c (c_t, 1 - n_t - g (c_t, m_t)) = v_l (c_t, 1 - n_t - g (c_t, m_t)) g_c (c_t, m_t) + \beta V_k (a_{t+1}, k_t)$$

Marginal utility of consumption is equated to the marginal losses: lost leisure due to more time spent on shopping and the marginal value of lower next-period capital

First-order condition with respect to $n_t$:

$$v_l (c_t, 1 - n_t - g (c_t, m_t)) = f_n (k_t, n_t) \beta V_k (a_{t+1}, k_t)$$

Marginal loss of labor is equated to the marginal gain which is the addition to next-period capital (determined by the MPL = real wage) times capital’s marginal value

Using these first-order conditions provides the condition for the consumption-leisure choice:

$$\frac{u_l}{u_c} = \frac{v_l}{v_c - v_l g_c} = f_n (k_t, n_t).$$

The marginal rate of substitution between leisure and consumption equals the real wage

This holds in every period, and therefore also in the steady state. So, how is it affected by, e.g., a raise in $m_t$?

- This will be indicative for “how” superneutrality fails in the long run
• The marginal utility of leisure is affected as

\[ u_{lm} = v_{lm} = -v_{ll}g_m < 0 \]

More money reduces marginal utility of leisure; it increases leisure for given consumption (by reducing shopping time).

• The marginal utility of consumption is affected as

\[ u_{cm} = -v_{cl}g_m - v_{ll}g_cg_m + v_{lg}g_{cm} \leq 0 \quad (3.2') \]

This is ambiguous, but:

– Effect 1) is positive if \( v_{cl} > 0 \): More money frees up leisure, increasing the marginal utility of consumption “directly”

– Effect 2) is positive: More money frees up leisure, which decreases marginal utility of leisure =⇒ less utility loss from transaction costs

– Effect 3) is positive if \( g_{cm} < 0 \): More money reduces the marginal transaction costs (in terms of lost leisure)
• So,
  - unless $v_{cl} << 0$; i.e., consumption and leisure are strong substitutes
  - and/or unless $g_{cm} >> 0$; i.e., more money increases marginal transactions costs
  - . . . $u_{cm} > 0$ and higher $m$ will lead to more work

• So,
  - As $u_{lm} < 0$ and $u_{cm}$ is probably positive, $v_l/v_c$ increases when $m$ falls
    * Lower real money balances reduces employment and output as in benchmark MIU model calibration
    * (Lower real money balances follow from higher nominal money growth, inflation and nominal interest rate.)

• A shopping-time model is therefore equivalent to a MIU approach, but . . .
  - . . . formalizes the idea of utility from liquidity services of money
  - . . . one knows better how and why superneutrality fails
Welfare implications of inflation and nominal interest rates? Just as in MIU model; first-order condition governing optimal money holdings:

\[-v_t g_m + \beta \frac{V_a (a_{t+1}, k_t)}{1 + \pi_{t+1}} = \beta V_k (a_{t+1}, k_t)\]

Marginal gains of money in terms of more current leisure and next-period money wealth are equated to the marginal cost in terms of lower next-period capital.

Using the value function relationships (by the envelope theorem)

\[V_k (a_t, k_{t-1}) = \beta V_k (a_{t+1}, k_t) [f_k (k_{t-1}, n_t) + 1 - \delta] = \beta R_{t-1} V_k (a_{t+1}, k_t) \tag{**}\]

and

\[V_a (a_t, k_{t-1}) = \beta V_k (a_{t+1}, k_t) \tag{**}\]

to get

\[-v_t g_m + \beta \frac{V_a (a_{t+1}, k_t)}{1 + \pi_{t+1}} = V_a (a_t, k_{t-1})\]

Hence,

\[-v_t g_m = V_a (a_t, k_{t-1}) \left[ 1 - \beta \frac{V_a (a_{t+1}, k_t)}{(1 + \pi_{t+1}) V_a (a_t, k_{t-1})} \right]\]
• But

\[
\frac{V_a(a_{t+1}, k_t)}{V_a(a_t, k_{t-1})} = \frac{V_k(a_{t+2}, k_{t+1})}{V_k(a_{t+1}, k_t)} \quad \text{(using (**))}
\]

\[
= \frac{1}{\beta R_t} \quad \text{(using (*))}
\]

So,

\[
-vigm = V_a(a_t, k_{t-1}) \left[1 - \frac{1}{(1 + \pi_{t+1}) R_t}\right]
\]

The Fisher relationship, \(1 + i_t = R_t (1 + \pi_{t+1})\), implies

\[
-vigm = V_a(a_t, k_{t-1}) \frac{i_t}{1 + i_t}
\]

• As in MIU model, it is optimal to have \(i_t = 0\) so the private marginal product of real money balances equals zero; namely at \(g_m = 0\).

  – Friedman rule is optimal
  – Money balances are then at a level high enough to minimize shopping time
Cash-in-Advance Models

Basic model and optimal choices under certainty

- Takes the transactions purpose of money literally:
  - Having cash, is *by assumption needed* to purchase some (or all) goods
  - A “Cash-in-Advance” *constraint* is introduced

- Certainty case (“simple”)
  - Uncertainty involves further complications: One may suddenly hold too little or too much cash (former case leads to suboptimal low consumption; latter case leads to suboptimal low savings)

- Utility (endogenous leisure dropped for starters)
  \[
  \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1. \tag{3.12}
  \]

- Budget constraint:
  \[
  \omega_t \equiv f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{m_{t-1} + (1 + i_{t-1}) b_{t-1}}{1 + \pi_t} \\
  = c_t + k_t + m_t + b_t \tag{3.15'}
  \]

\((b_t \text{ is real bond holdings per capita})\).
• Cash-in-advance (CIA) constraint on consumption goods:

\[ c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t \]  

(3.13)

• Note, as opportunity cost of holding money is \( i_t \), the CIA constraint always holds with equality for \( i_t > 0 \). Why would one hold more money than needed?

(Not necessarily the case with uncertainty—one could “end up with too much cash.”)

• Optimization is characterized by (\( \omega_t \) and \( m_{t-1} \) are state variables):

\[ V (\omega_t, m_{t-1}) = \max \{ u(c_t) + \beta V (\omega_{t+1}, m_t) \} \]

Maximization is over \( c, m, b, k \) and subject to budget constraint and CIA constraint.

• From \( \omega_t = c_t + k_t + m_t + b_t \), one can eliminate \( b_t \) from budget constraint:

\[ \omega_{t+1} = f(k_t) + \tau_{t+1} + (1 - \delta) k_t + \frac{m_t}{1 + \pi_{t+1}} + R_t (\omega_t - c_t - k_t - m_t) \]

with \( R_t \equiv \frac{1 + i_t}{1 + \pi_{t+1}} \) (Note: Walsh does not make this substitution)
• Let \( \mu_t \) denote the Lagrange multiplier associated with the CIA constraint

  – Kuhn-Tucker conditions for optimum include \( \mu_t \geq 0 \): If \( \mu_t > 0 \) the constraint binds with equality; if \( \mu_t = 0 \) the constraint does not bind

  – Kuhn-Tucker conditions for optimum include the “complementary slackness” condition:

\[
\mu_t \left( c_t - \frac{m_{t-1}}{1 + \pi_t} - \tau_t \right) = 0
\]

• The optimization problem is stated as

\[
V(\omega_t, m_{t-1}) = \max \left\{ u(c_t) + \beta V(\omega_{t+1}, m_t) - \mu_t \left( c_t - \frac{m_{t-1}}{1 + \pi_t} - \tau_t \right) \right\},
\]

where maximization is over \( c, k, \) and \( m \). (And \( \omega_{t+1} \) follows from the budget constraint.)

• First-order condition with respect to \( c_t \):

\[
u_c(c_t) = \beta R_t V_\omega(\omega_{t+1}, m_t) + \mu_t
\]

Marginal utility of consumption equals the marginal losses, which are the discounted marginal value of next-period wealth plus the “price” of holding cash as measured by \( \mu_t \) (cost of liquidity services provided by money)

  – NB: Marginal cost of consumption higher when the CIA constraint binds, \( \mu_t > 0 \)
First-order condition with respect to $k_t$:

$$
\beta V_\omega (\omega_{t+1}, m_t) [f_k (k_t) + 1 - \delta] = \beta R_t V_\omega (\omega_{t+1}, m_t)
$$

Marginal gain in terms of more next-period wealth equals the marginal loss in terms of less next-period wealth due to lower bond holdings

- Implies familiar $R_t = f_k (k_t) + 1 - \delta$ as $k$ and $b$ are perfect substitutes

First-order condition with respect to $m_t$:

$$
\beta \frac{1}{1 + \pi_{t+1}} V_\omega (\omega_{t+1}, m_t) + \beta V_m (\omega_{t+1}, m_t) = \beta R_t V_\omega (\omega_{t+1}, m_t)
$$

Marginal gains in terms of more next-period wealth and money per se (for transactions), equals marginal loss in terms of less next-period wealth due to lower bond holdings
• Relationships between partial derivatives of the value function from the envelope theorem:

\[ V_\omega (\omega_t, m_{t-1}) = \beta R_t V_\omega (\omega_{t+1}, m_t) \]

- In optimum, equality between the period \( t \) marginal value of wealth and the discounted next-period marginal value of wealth (times the gross real interest rate)

\[ V_m (\omega_t, m_{t-1}) = \mu_t \frac{1}{1 + \pi_t} \]

- Marginal value of money carried into period \( t \) equals their marginal cost in terms of the “price” of holding cash as measured by \( \mu_t / (1 + \pi_t) \)

- Note: Marginal value of money is zero if \( \mu_t = 0 \); i.e., if the CIA constraint does not bind
• What is the nominal interest rate, and **does** the CIA constraint bind?

  - Let $\lambda_t \equiv V_\omega(\omega_t, m_{t-1})$ define the marginal value of wealth
    (= the Lagrange multiplier on the budget constraint in Walsh)
  - From $V_\omega(\omega_t, m_{t-1}) = \beta R_t V_\omega(\omega_{t+1}, m_t)$ one can write
    \[
    \lambda_t = \beta R_t \lambda_{t+1}
    \]
  - From
    \[
    \beta \frac{1}{1 + \pi_{t+1}} V_\omega(\omega_{t+1}, m_t) + \beta V_m(\omega_{t+1}, m_t) = \beta R_t V_\omega(\omega_{t+1}, m_t)
    \]
    one can write
    \[
    \beta \frac{1}{1 + \pi_{t+1}} \lambda_{t+1} + \beta \mu_{t+1} \frac{1}{1 + \pi_{t+1}} = \beta R_t \lambda_{t+1}
    \]
    Hence,
    \[
    \frac{1}{1 + \pi_{t+1}} (\lambda_{t+1} + \mu_{t+1}) = \frac{1 + i_t}{1 + \pi_{t+1}} \lambda_{t+1}
    \]
    \[
    i_t = \frac{\mu_{t+1}}{\lambda_{t+1}} \tag{3.29}
    \]
  - The nominal interest rate is positive, *only* when the CIA constraint binds, $\mu_{t+1} > 0$, i.e., when there *is* a cost of “liquidity services” provided by real money holdings
Now note that the first-order condition for consumption can be rewritten as

\[ u_c(c_t) = \lambda_t + \mu_t \]
\[ = \lambda_t \left( 1 + \frac{\mu_t}{\lambda_t} \right) \]

With the expression for the nominal interest rate:

\[ u_c(c_t) = \lambda_t (1 + i_{t-1}) \]

A positive interest rate raises the marginal cost of consumption above the marginal value of wealth. The “price” of consumption goods in terms of output has increased by a positive \( i_{t-1} \) due to the need for holding cash (foregoing interest income) to purchase goods.

The nominal interest rate is equivalent to a “consumption tax.” However, it is a non-distorting tax in the long run as it:

a) Does not affect long-run capital accumulation
b) Does not distort any intratemporal trade-offs
Steady-state properties: Superneutrality or not?

- From the steady-state condition $R^{ss} = 1/\beta$ and the capital accumulation condition one gets:
  $$f_k(k^{ss}) + 1 - \delta = 1/\beta$$

  Hence, long-run capital and output per capita are neutral w.r.t. monetary factors.

Steady-state consumption follows from the national account as (note, $b^{ss} = 0$)

  $$c^{ss} = f(k^{ss}) - \delta k^{ss}$$

  I.e., long-run superneutrality holds.

- Nominal money growth affects inflation and inflation affects the nominal interest rate (through the Fisher relationship):
  $$\pi_t = \theta^{ss}$$
  $$i^{ss} \approx R^{ss} + \pi^{ss}$$

- Analogy with MIU model concerning relative marginal values of real money balances (in terms of liquidity services) and consumption:
  $$\frac{\mu}{u_c} = \frac{\mu}{\lambda (1 + i)} = \frac{i}{1 + i}$$

- Difference with MIU approach, no steady-state welfare costs of inflation; only $c^{ss}$ matters for utility, and $c^{ss}$ is independent of inflation and $i$.  

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Extensions yielding non-superneutrality and a well-defined optimal inflation rate

- Introduction of consumption-leisure trade-off:
  - Leisure can be “purchased” without money, so the CIA constraint “taxes” consumption relative to leisure (distorts the trade-off)
  - Households choose more leisure relative to consumption; output is lower

- “Cash goods” and “credit goods”
  - Subset of consumption goods can be bought on credit; i.e., the CIA constraint does not apply
  - The CIA constraint “taxes” cash goods, but not credit goods (distorts relative demand)

- CIA restriction on investment in physical capital
  - Then, accumulation of capital becomes “taxed,” and steady state capital will be lower (investment decision is distorted)

- All cases strongly qualify the “any inflation rate goes” result of the simple CIA model: It will be optimal to have $i^{ss} = 0$, i.e., to eliminate any distortion arising from the CIA constraint
  => Implement the Friedman rule!
Summary

- MIU models can be rationalized by shopping-time models
- Most properties are the same, but shopping-time models help restricting the cross-derivatives of the utility function in MIU models

- Cash-in-advance constraint is a direct portrait of money’s role as a means of transactions
- Simple model exhibits superneutrality, but no well-defined optimal inflation rate (a positive nominal interest rate is a non-distorting “consumption tax”)
- Amending the simple model will cause non-superneutrality and a well-defined optimal inflation rate. The Friedman rule again!
- Business cycle properties of CIA models? Next time; with model including endogenous labour supply
Plan for next lectures

(Remember next week (number 9) is teaching free.)

Wednesday, March 10

1. Cash-in-Advance models (stochastic)

   Literature: Walsh (Chapter 3, pp.126-131). Material on Real Resource Costs is supplementary “only.”

Monday, March 15

Exercises:

“QUESTION 2” from June 15 exam, 2006 (CIA constraint on investment purchases)