1. Cash-in-Advance models
   
a. Basic model under certainty
b. Extended model in stochastic case

Literature: Walsh (2010, Chapter 3, pp. 91-115) (NB: Material on shopping-time models "only" recommended)
Cash-in-Advance Models

Basic model and optimal choices under certainty

- Takes the transactions purpose of money literally:
  - Having cash, is \textit{by assumption needed} to purchase some (or all) goods
  - A “Cash-in-Advance” constraint is introduced

- Certainty case
  - Shocks involve further complications

- Utility (endogenous leisure/labor dropped for starters)

\[
\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1. \tag{3.15}
\]

- Budget constraint:

\[
\omega_t \equiv f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{m_{t-1} + (1 + i_{t-1}) b_{t-1}}{1 + \pi_t} \\
= c_t + k_t + m_t + b_t, \tag{3.18}
\]

(where \(b_t\) is real bond holdings per capita).
• Cash-in-advance (CIA) constraint on consumption goods:

\[ c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t \]  \hspace{1cm} (3.16)

• Note, as opportunity cost of holding money is \( i_t \), the CIA constraint *always holds with equality for* \( i_t > 0 \) *in an environment of certainty.* Why would one hold more money than needed?

  Not necessarily the case in stochastic case— one could “end up with too much cash” (when surprised by a negative income shock)

• Alternative formulation of the CIA constraint:

\[ c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t - b_t \]  \hspace{1cm} (3.20)

  – This will always hold with equality when \( i_t > 0 \), also in a stochastic setting

• Under (3.20) the goods market “opens” after the financial market; under (3.16) the goods market “opens” before
• Optimization: \( \omega_t \) and \( m_{t-1} \) are state variables so we have the value function:

\[
V(\omega_t, m_{t-1}) = \max \{ u(c_t) + \beta V(\omega_{t+1}, m_t) \}
\]

Maximization is over \( c, m, b, k \) and subject to budget constraint and CIA constraint.

• From \( \omega_t = c_t + k_t + m_t + b_t \), one can eliminate \( b_t \) from budget constraint:

\[
\omega_{t+1} = f(k_t) + \pi_{t+1} + (1 - \delta) k_t + \frac{m_t}{1 + \pi_{t+1}} + (1 + r_t) (\omega_t - c_t - k_t - m_t)
\]

with \( 1 + r_t \equiv (1 + i_t) / (1 + \pi_{t+1}) \) (Note: Walsh does not make this substitution)
• Let $\mu_t$ denote the Lagrange multiplier associated with the CIA constraint
  
  – Kuhn-Tucker conditions for optimum include $\mu_t \geq 0$: If $\mu_t > 0$ the constraint binds with equality; if $\mu_t = 0$ the constraint does not bind
  
  – Kuhn-Tucker conditions for optimum include the “complementary slackness” condition:
    
    $$
    \mu_t \left( c_t - \frac{m_{t-1}}{1 + \pi_t} - \tau_t \right) = 0
    $$

  
• The optimization problem is stated as
    
    $$
    V (\omega_t, m_{t-1}) = \max \left\{ u(c_t) + \beta V (\omega_{t+1}, m_t) - \mu_t \left( c_t - \frac{m_{t-1}}{1 + \pi_t} - \tau_t \right) \right\},
    $$
    
    where maximization is over $c$, $k$, and $m$. (And $\omega_{t+1}$ follows from the budget constraint.)

• First-order condition with respect to $c_t$:
    
    $$
    u_c (c_t) = \beta (1 + r_t) V_\omega (\omega_{t+1}, m_t) + \mu_t
    $$
    
    Marginal utility of consumption equals the marginal losses, which are the discounted marginal value of next-period wealth plus the “price” of holding cash as measured by $\mu_t$ (cost of liquidity services provided by money)
  
    – Marginal cost of consumption is higher when the CIA constraint binds, $\mu_t > 0$
• First-order condition with respect to $k_t$:

$$
\beta V_\omega (\omega_{t+1}, m_t) [f_k (k_t) + 1 - \delta] = \beta (1 + r_t) V_\omega (\omega_{t+1}, m_t)
$$

Marginal gain in terms of more next-period wealth equals the marginal loss in terms of less next-period wealth due to lower bond holdings

- Implies $r_t = f_k (k_t) - \delta$ as $k$ and $b$ are perfect substitutes

• First-order condition with respect to $m_t$:

$$
\beta \frac{1}{1 + \pi_{t+1}} V_\omega (\omega_{t+1}, m_t) + \beta V_m (\omega_{t+1}, m_t) = \beta (1 + r_t) V_\omega (\omega_{t+1}, m_t)
$$

Marginal gains in terms of more next-period wealth and money per se (for transactions), equals marginal loss in terms of less next-period wealth due to lower bond holdings
• Relationships between partial derivatives of the value function from the envelope theorem:

\[ V_\omega (\omega_t, m_{t-1}) = \beta (1 + r_t) V_\omega (\omega_{t+1}, m_t) \]  
\((3.27)+(3.29)\)

- In optimum, equality between the period \( t \) marginal value of wealth and the discounted next-period marginal value of wealth (times the gross real interest rate)

\[ V_m (\omega_t, m_{t-1}) = \mu_t \frac{1}{1 + \pi_t} \]  
(3.28)

- Marginal value of money carried into period \( t \) equals their marginal cost in terms of the “price” of holding cash as measured by \( \mu_t / (1 + \pi_t) \)

- Note: Marginal value of money is zero if \( \mu_t = 0 \); i.e., if the CIA constraint does not bind
What is the nominal interest rate, and does the CIA constraint bind?

- Let $\lambda_t \equiv V_\omega(\omega_t, m_{t-1})$ define the marginal value of wealth
  (= the Lagrange multiplier on the budget constraint in Walsh)
- From $V_\omega(\omega_t, m_{t-1}) = \beta (1 + r_t) V_\omega(\omega_{t+1}, m_t)$ one can write
  \[
  \lambda_t = \beta (1 + r_t) \lambda_{t+1}
  \]  
  (3.29)
- From f.o.c. w.r.t. $m_t$:
  \[
  \beta \frac{1}{1 + \pi_{t+1}} V_\omega(\omega_{t+1}, m_t) + \beta V_m(\omega_{t+1}, m_t) = \beta (1 + r_t) V_\omega(\omega_{t+1}, m_t),
  \]
  using (3.28),
  one can write
  \[
  \beta \frac{1}{1 + \pi_{t+1}} \lambda_{t+1} + \beta \mu_{t+1} \frac{1}{1 + \pi_{t+1}} = \beta (1 + r_t) \lambda_{t+1}.
  \]
  Hence,
  \[
  \frac{1}{1 + \pi_{t+1}} (\lambda_{t+1} + \mu_{t+1}) = \frac{1 + i_t}{1 + \pi_{t+1}} \lambda_{t+1}
  \]
  \[
  i_t = \frac{\mu_{t+1}}{\lambda_{t+1}}
  \turns\ (3.32)
- The nominal interest rate is positive when money has liquidity services, and then the CIA constraint binds, $\mu_{t+1} > 0$
Now note that the first-order condition for consumption can be rewritten as

\[ u_c(c_t) = \lambda_t + \mu_t \]
\[ = \lambda_t \left( 1 + \frac{\mu_t}{\lambda_t} \right) \]

With the expression for the nominal interest rate:

\[ u_c(c_t) = \lambda_t (1 + i_{t-1}) \]

A positive interest rate raises the marginal cost of consumption above the marginal value of wealth.

The “price” of consumption goods in terms of output has increased by a positive \( i_{t-1} \) due to the need for holding cash (foregoing interest income) to purchase goods.

The nominal interest rate is equivalent to a “consumption tax.” However, it is a non-distorting tax in the long run as it:

a) Does not affect long-run capital accumulation
b) Does not distort any intratemporal trade-offs
Steady-state properties: Superneutrality or not?

- From the steady-state condition $R^{ss} = 1/\beta$ and the capital accumulation condition one gets:
  \[
  f_k(k^{ss}) + 1 - \delta = 1/\beta
  \]
  Hence, long-run capital and output per capita are neutral w.r.t. monetary factors.

Steady-state consumption follows from the national account as (note, $b^{ss} = 0$: in a closed economy bonds are in zero net supply/demand in equilibrium)

\[
c^{ss} = f(k^{ss}) - \delta k^{ss}
\]

I.e., long-run superneutrality holds.

- Nominal money growth affects inflation and inflation affects the nominal interest rate (through the Fisher relationship):
  \[
  \pi^{ss} = \theta^{ss}, \quad i^{ss} \approx r^{ss} + \pi^{ss}
  \]

- Analogy with MIU model concerning relative marginal values of real money balances (in terms of liquidity services) and consumption:
  \[
  \frac{\mu}{u_c} = \frac{\mu}{\lambda (1 + i)} = \frac{i}{1 + i}
  \]

- Difference with MIU approach: No steady-state welfare costs of inflation; only $c^{ss}$ matters for utility, and $c^{ss}$ is independent of inflation and $i$. 
Extensions yielding non-superneutrality and a well-defined optimal inflation rate

- Introduction of consumption-leisure trade-off:
  - Leisure can be “purchased” without money, so the CIA constraint “taxes” consumption relative to leisure (distorts the trade-off)
  - Households choose more leisure relative to consumption; output is lower

- “Cash goods” and “credit goods”
  - Subset of consumption goods can be bought on credit; i.e., the CIA constraint does not apply
  - The CIA constraint “taxes” cash goods, but not credit goods (distorts relative demand)

- CIA restriction on investment in physical capital
  - Then, accumulation of capital becomes “taxed,” and steady state capital will be lower (investment decision is distorted)

- All cases strongly qualify the “any inflation rate goes” result of the simple CIA model: It will be optimal to have \( i^{ss} = 0 \), i.e., to eliminate any distortion arising from the CIA constraint
  \( \Rightarrow \) Implement the Friedman rule!
Cash-in-Advance Models: Stochastic case

- Simple CIA model under certainty exhibited long-run superneutrality

- As discussed, it is possible that this is not the case, and issues are therefore:
  - What is the potential nature of non-superneutralities in the short and the long run?
  - What are the quantitative implications of inflation and the CIA constraint?
  - Will dynamics match data in a stochastic version?
  - Can monetary policy play a stabilizing role?

- Issues addressed in stochastic CIA model (solved, calibrated, and simulated—just as the stochastic MIU model)
  Exogenous shocks bringing the economy off steady state are technology shocks and nominal money growth shocks
  The main channel causing non-superneutrality here is (as in the MIU approach) endogenous labor supply
Model and private sector optimization

- Production is given by a Cobb-Douglas function
  \[ y_t = f (k_{t-1}, n_t, z_t) = e^{z_t k_{t-1}^\alpha n_t^{1-\alpha}}, \quad 0 \leq \alpha \leq 1, \]
  \( z_t \) is a technology shock. Model now features endogenous labor

- Assumption about the technology shock as in stochastic MIU model:
  \[ z_t = \rho z_{t-1} + e_t, \quad |\rho| < 1, \]
  with \( e_t \) being a mean-zero, white-noise shock

- Nominal money growth rate:
  \[ \theta_t = \theta^{ss} + u_t \]
  where \( u_t \) is a shock to the growth rate
  As in stochastic MIU model the shock process is
  \[ u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t, \quad 0 \leq \gamma < 1, \quad \phi \geq 0 \]
  with \( \varphi_t \) being a mean-zero, white-noise shock
  - As in MIU model, there may or may not be serial correlation in shocks to money growth
  - As in MIU model, money growth may or may not respond to past technology shocks
    * Pro-cyclical (\( \phi > 0 \)) or countercyclical (\( \phi < 0 \))
• Per-period utility function:

\[ u(c_t, 1 - n_t) = \frac{(c_t)^{1-\Phi}}{1 - \Phi} + \Psi \frac{(1 - n_t)^{1-\eta}}{1 - \eta}, \tag{3.34'} \]

\( \eta, \Phi, \Psi > 0 \) with \( \Phi, \eta > 0 \) being coefficients of relative risk aversion.

Compared with simple CIA model under certainty, leisure provides utility, and a consumption-leisure decision will potentially be affected by the CIA constraint.

The CIA constraint is (on consumption goods):

\[ c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t - b_t \tag{3.35} \]

I.e., the version where financial markets open before goods markets, so real bond selling \( (b_t < 0) \) gives liquidity for purchases.

The budget constraint is:

\[ e^{zt} k_{t-1}^\alpha n_t^{1-\alpha} + (1 - \delta) k_{t-1} + \frac{m_{t-1}}{1 + \pi_t} + \tau_t + i_t b_t = c_t + k_t + m_t \tag{3.36'} \]

• Optimization is characterized by \( (k_{t-1}, b_{t-1} \text{ and } m_{t-1}) \) are state variables):

\[ V(k_{t-1}, b_{t-1}, m_{t-1}) = \max \left\{ \frac{(c_t)^{1-\Phi}}{1 - \Phi} + \Psi \frac{(1 - n_t)^{1-\eta}}{1 - \eta} + \beta E_t V(k_t, b_t, m_t) \right\} \]

Maximization is over \( c, m, n, k, \) and \( b \) subject to the CIA constraint and the budget constraint.
• Trick: Eliminate $k_t$ a by the budget constraint, and one “only” maximizes over $c$, $m$, $n$ and $b$ subject to the CIA constraint

• Let $\mu_t$ denote the Lagrange multiplier associated with the CIA constraint

• First-order condition with respect to $c_t$:

$$c_t^{-\Phi} = \beta E_t V_k (k_t, b_t, m_t) + \mu_t$$

Marginal utility of consumption equals the marginal losses, which are the expected, discounted marginal value of next-period capital plus the “price” of holding cash as measured by $\mu_t$

– As in simple CIA model: Marginal cost of consumption is higher when the CIA constraint binds

• First-order condition with respect to $m_t$:

$$\beta E_t V_m (k_t, b_t, m_t) = \beta E_t V_k (k_t, b_t, m_t)$$

Expected marginal gain of real money balances equals the expected marginal loss in terms of lower capital holdings
• First-order condition with respect to $n_t$:

$$\Psi (1 - n_t)^{-\eta} = \beta E_t V_k(k_t, b_t, m_t) (1 - \alpha) e^{z_t} k_{t-1}^\alpha n_t^{-\alpha}$$

Marginal loss in terms of less leisure equals the expected value of higher future capital (which is higher the higher is the marginal product of labor)

• First-order condition with respect to $b_t$:

$$\beta E_t V_b(k_t, b_t, m_t) + \beta E_t V_k(k_t, b_t, m_t) i_t = \mu_t$$

Expected marginal gain of bonds [(per se and through higher capital (though nominal interest rate payments)] equals marginal costs in terms of lower liquidity services
• Relationships between partial derivatives of the value function from the envelope theorem:

\[ V_k(k_{t-1}, b_{t-1}, m_{t-1}) = \beta E_t V_k(k_t, b_t, m_t) \left[ \alpha e z l k_{t-1}^{\alpha-1} n_{t-1}^{1-\alpha} + 1 - \delta \right] \]  

The marginal value of current capital equals the expected marginal value of future capital “corrected for” the net marginal product of current capital (Keynes-Ramsey rule “in disguise”)

\[ V_b(k_{t-1}, b_{t-1}, m_{t-1}) = 0 \]  

The marginal value of bonds is zero; by assumption “bygones are bygones” (the model is set up such that a purchased bond is sold with interest within the period)

\[ V_m(k_{t-1}, b_{t-1}, m_{t-1}) = \frac{\mu_t}{1 + \pi_t} + \beta E_t V_k(k_t, b_t, m_t) \frac{1}{1 + \pi_t} \]  

The marginal value of real balances equals the marginal costs in terms of the “price” of the CIA constraint and the expected value of lower capital
Steady state, and the form of non-superneutrality

- Let \( \lambda_t \equiv \beta E_t V_k (k_t, b_t, m_t) \) be the discounted, expected the marginal value of capital
  
  - We get (f.o.c. for \( c \))
    \[
    c_t^{-\Phi} = \lambda_t + \mu_t
    \] (3.37)
  
  - We get [f.o.c. for \( m \) and (***)]
    \[
    \beta E_t \left[ \frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}} \right] = \lambda_t
    \] (3.41)
    
    (remember the marginal value of money, \( V_m (k_t, b_t, m_t) \) indeed are \( (\mu_{t+1} + \lambda_{t+1}) / (1 + \pi_{t+1}) \).)
  
  - We also get [from (*)]:
    \[
    \lambda_t = \beta E_t (1 + r_t) \lambda_{t+1}
    \] (3.39)
    
    where \( r_t = \alpha e^{\pi_{t+1} k_t^{\alpha-1} n_{t+1}^{1-\alpha}} - \delta = \alpha (y_{t+1} / k_t) - \delta \)
  
  - Finally, we get (from f.o.c. for \( n \)):
    \[
    \Psi (1 - n_t)^{-\eta} = \lambda_t (1 - \alpha) \left( \frac{y_t}{n_t} \right)
    \] (3.38)
• In steady state we have $\beta (1 + r^{ss}) = 1$. This determines $y^{ss}/k^{ss}$ independently of monetary factors

From resource constraint, $y^{ss} = c^{ss} + \delta k^{ss}$ one identifies $(c^{ss}/k^{ss}) = (y^{ss}/k^{ss}) - \delta$

From production function, one gets $(n^{ss}/k^{ss}) = (y^{ss}/k^{ss})^{1/(1-\alpha)}$

• What then determines $n^{ss}$?

Essentially, the consumption-leisure decision:

\[
\frac{u_c}{u_{1-n}} \equiv \frac{(c^{ss})^{-\Phi}}{\Psi (1 - n^{ss})^{-\eta}} = \frac{\lambda^{ss} + \mu^{ss}}{\lambda^{ss} (1 - \alpha) (y^{ss}/n^{ss})} = \frac{\lambda^{ss} + \mu^{ss}}{\lambda^{ss}} \frac{1}{f_n}
\]

• A higher $\mu^{ss} > 0$ makes consumption more costly relative to leisure; hence, less labor is supplied. The CIA constraint implies a distorting consumption tax
• How are $\mu$ and the nominal interest rate related?

• From the first-order condition for $b$ we have

$$\beta V^s_s + \lambda^s s_i^s = \mu^s s$$

Since by (**) $V^s_s = 0$, we get (as in simple CIA model):

$$i^s s = \frac{\mu^s s}{\lambda^s s}$$

and consumption-leisure choice is given by

$$\frac{(c^s s)^{-\Phi}}{\Psi (1 - n^s s)^{-\eta}} = \frac{(1 + \mu^s s / \lambda^s s) \frac{1}{f_n}}{1 + \frac{i^s s}{\frac{1}{(1 - \alpha) \left(\frac{y^s s}{n^s s}\right)}}}$$

• I steady-state, higher money growth and inflation raise the nominal interest rate, and induce a substitution away from the cash good (consumption) towards the non-cash good (leisure):

$$\frac{\partial n^s s}{\partial \theta^s s} < 0$$

• We also see that $i^s s = 0$ provides the non-distorted consumption-leisure choice; i.e., optimality of the Friedman rule
• Note in contrast with the MIU model with leisure, the non-neutrality is non-ambiguous and thus independent of $\Phi$

(In MIU model with leisure, a higher nominal interest rate reduced $m$, and depending upon $u_{cm} \geq 0$ it reduced or increased $n$)

• In CIA model, the effect of money growth is “direct”: Consumption is being taxed by a positive nominal interest rate, while leisure is not

Dynamics

• Method as in stochastic MIU model:

  Calibration: Assign plausible values the parameters of the model. Values chosen to conform with basics of MIU model

  Simulation:

  – Perform a log-linearization of the model’s dynamic equations (everything is expressed as percentage deviations from steady state)
  – Solve this system by numerical methods
    (various simulation programs are available on the internet)
  – Create artificial time series data from the system

From the artificial data one evaluates the statistical properties of the model
Main results

– As in MIU model, if money shocks, $\varphi_t$—shocks, shall pay a role, persistence in money growth is necessary ($\gamma > 0$). Only then, will the shock affect expected next-period inflation, and thus — through the Fisher relationship — period $t$ nominal interest rate.

  * Hence, only “anticipated money” matters

– The effects of money shocks on labor and output are stronger the more persistent is money growth, and the effects are stronger than in MIU model

  * Reason: Effects of variations in the nominal interest rate are having a direct effect on the consumption-leisure choice; in MIU model the effect were indirect through money demand and $u_{cm}$

– If technology shocks are met with procyclical money, output is more stable (as in MIU model with $\Phi > b$). The magnitude is small (but stronger than in MIU model)

  * Reason: When a positive technology shock is met by an increase in money growth, the nominal interest rate increases and discourages labor supply

– No liquidity effect of monetary shocks: Positive money shock increases nominal interest rate (Figure 3.1).

  * In strong opposition to dynamics of business cycles, where $cov(M, i) < 0$
Summary

- MIU-models, shopping time models, CIA models and other models of money, are . . . . just models
- Models, nevertheless, are useful, consistent abstractions to use for thinking about economics

- The micro-founded flex-price models analyzed so far are:
  — Suitable for long run-analyses of links between money and inflation, and potential real allocation
  — Suitable for thinking about why money exists and what is the value of money (direct utility, liquidity service, saved leisure,...)
  — Suitable for thinking about the long-run optimal rate of inflation
  — Less suitable for analyzing the short run implications of monetary shocks as the models, by nature, exhibits monetary neutrality (although not necessarily superneutrality)

  — To remedy the short-run failure of such models, one must introduce incomplete nominal adjustment
Plan for next lectures

Thursday, February 28
Exercises:
“QUESTION 2” from June 15 exam, 2006 (CIA constraint on investment purchases)

Tuesday, March 5, Lectures:

   Money in the short run: Incomplete nominal adjustment (I)
   Flexible prices and imperfect information; the Lucas model

Literature: Walsh (Chapter 5, pp. 195–203 plus relevant appendix)