0. End last lectures’ slides on superneutrality and what is affected by money growth in MIU model

1. Money in the utility function (continued)
   a. Welfare costs of inflation
   b. Potential non-superneutrality of money
   c. Dynamics and calibration

Literature: Walsh (2010, Chapter 2, pp. 52–86, so check the Appendix as well; i.e., get a grip on the linearization technique)
Welfare Costs of Inflation

- Inflation affects real money holdings by affecting the nominal interest rate (the opportunity cost of holding money):

\[
\frac{u_m(c^{ss}, m^{ss})}{u_c(c^{ss}, m^{ss})} = 1 - \frac{1}{(1 + \pi^{ss})(1 + r^{ss})} = \frac{i^{ss}}{1 + i^{ss}} \quad (\equiv I)
\]

(money demand)

Using

\[1 + i^{ss} = (1 + r^{ss})(1 + \pi^{ss})\]

(Fisher relationship)

Within the MIU model framework with household utility as welfare measure, what are the welfare effects of inflation?

- Is there an optimal rate of inflation?

On the optimal long-run inflation rate

- Bailey/Friedman intuition:
  - Private marginal cost of holding money is increasing in the nominal interest rate
  - Social marginal cost of creating money is essentially zero
  - Equating private and social marginal cost requires a zero nominal interest rate
  - By the Fisher relationship it follows that \( \pi^{ss} \approx -r^{ss} < 0 \) is optimal
    - I.e., the optimal rate of change in prices involves deflation equal to the real interest rate—“The Friedman Rule”
• This is formally confirmed in the model when finding the utility maximizing nominal money growth rate subject to resource constraint of economy

Solve:

\[
\begin{align*}
\max_{\theta^{ss}} & \quad u(c^{ss}, m^{ss}) \\
\max_{\theta^{ss}} & \quad u(f(k^{ss}) - \delta k^{ss}, m^{ss})
\end{align*}
\]

First-order condition:

\[
uc(f(k^{ss}) - \delta k^{ss}, m^{ss}) \frac{\partial (f(k^{ss}) - \delta k^{ss})}{\partial m^{ss}} \frac{\partial m^{ss}}{\partial \theta^{ss}} + um(c^{ss}, m^{ss}) \frac{\partial m^{ss}}{\partial \theta^{ss}} = 0.
\]

So,

\[
u_m(c^{ss}, m^{ss}) = 0
\]

• With

\[
\frac{u_m(c^{ss}, m^{ss})}{uc(c^{ss}, m^{ss})} = \frac{i^{ss}}{1 + i^{ss}},
\]

this implies \(i^{ss} = 0\), and the condition determines what Friedman called the “optimal quantity of money”

- Note that with some finite \(\bar{m}\) defined as \(u_m(c^{ss}, \bar{m}) = 0\), and \(u_m(c^{ss}, m^{ss}) < 0\) for \(m^{ss} > \bar{m}\), this \(\bar{m}\) is the optimal quantity of money
Potential non-superneutrality of money

- Is superneutrality of money a robust feature of the MIU model?

In the model of previous lecture, endogenous savings behavior uniquely defines steady-state capital

- Capital is accumulated or decumulated until its net marginal product (real interest rate) equals households’ subjective real interest rate:

\[ f_k(k^{ss}) + 1 - \delta = \frac{1}{\beta} \]

- Hence, long-run superneutrality can only fail if the marginal product of capital is affected by inflation
• Natural extension is endogenous labor input in production

Arises in MIU model when amended by a labor supply choice by households

This is achieved by having leisure enter in utility function:

\[ u_t = u(c_t, m_t, l_t). \]

The production function is

\[ y_t = f(k_{t-1}, n_t), \]
\[ n_t \equiv 1 - l_t \]

(\(n_t\) is not population growth!)

• Households now face an additional decision: How much time should be devoted to work; how much to leisure?

The relevant optimality condition is:

\[ u_l(c_t, m_t, l_t) = u_c(c_t, m_t, l_t) f_n(k_{t-1}, n_t) \quad (2.43') \]

Marginal gain of leisure is equated to the marginal cost, which is the utility loss from lower consumption times the marginal product of labor (the real wage)
• In steady state, three relationships will apply:

\[ u_l(c^{ss}, m^{ss}, l^{ss}) = u_c(c^{ss}, m^{ss}, l^{ss}) f_n(k^{ss}, 1 - l^{ss}) \]  

(\textit{l versus c choice})

\[ f_k(k^{ss}, 1 - l^{ss}) + 1 - \delta = \frac{1}{\beta} \]  

(constant capital)

\[ c^{ss} = f(k^{ss}, 1 - l^{ss}) - \delta k^{ss} \]  

(national account)

• If \( u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{ss}, m^{ss}, l^{ss}) \) is independent of \( m^{ss} \), these equations determine \( k^{ss}, l^{ss} \) and \( c^{ss} \). Long-run superneutrality holds!

  – This will be the case if utility is \textit{separable} in money; e.g. \( u = v(c, l) g(m) \) (\( u_l \) and \( u_c \) are affected by \( m \) in the same way)

  – Also, of course, it will be the case if \( u_l \) and \( u_c \) are not affected by \( m \) at all

• If \( u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{ss}, m^{ss}, l^{ss}) \) \textit{depends} on \( m^{ss} \), long-run superneutrality will not hold
• Specific functional form of utility function (covering all possibilities)

\[ u(c_t, m_t, 1 - n_t) = \frac{(ac_t^{1-b} + (1-a)m_t^{1-b})^{\frac{1}{1-\Phi}}}{1-\Phi} + \frac{\Psi(1-n_t)^{1-\eta}}{1-\eta}, \quad ((2.64)) \]

0 < a < 1, \ b > 0, \ \eta > 0, \ \Phi > 0, \ \Psi > 0 \quad (b, \eta, \Phi \neq 1)

– \Phi \text{ is coefficient of relative risk aversion}

– \( b \text{ is inverse nominal interest rate elasticity of money demand—cf. (2.32), p. 49} \)

• What is \( u_l(c^{ss}, m^{ss}, l^{ss}) / u_c(c^{ss}, m^{ss}, l^{ss}) \) with this specification?

\[ \frac{u_l}{u_c} = \frac{\Psi l_t^{-\eta}}{a \left( ac_t^{1-b} + (1-a)m_t^{1-b} \right)^{\frac{b-\Phi}{1-b}} c_t^{-b}} \]

– Hence, if \( \Phi = b, \ u_{cm} = 0, \) and superneutrality holds in the short and the long run

– If \( \Phi < b \) (empirically plausible), then \( u_{cm} > 0. \)

* Higher expected inflation will reduce real money balances and decrease marginal utility of consumption

* Households substitute towards leisure, and labor supply decreases

* Superneutrality fails in the short and long run

– If \( \Phi > b \) (empirically less plausible), then \( u_{cm} < 0 \) and superneutrality fails “in the opposite direction”
Dynamics and calibration

• Given that superneutrality may fail in the short run due to endogenous labor choice, a relevant issue is whether the MIU model has short-run properties which match the data.

  I.e., how is monetary shocks transmitted to the real economy, and will monetary policy be able to play a stabilizing role?

  ⇒ a stochastic version of the model is formulated

  Exogenous shocks bringing the economy away from steady state will be technology shocks and shocks to the growth rate of nominal money supply.

Model and private sector optimization. General case

• Production function is amended to

\[ y_t = f(k_{t-1}, 1 - l_t, z_t) \]

  where \( z_t \) is a technology shock:

\[ z_t = \rho_z z_{t-1} + e_t, \quad |\rho_z| < 1, \]

  with \( e_t \) being a mean-zero, white-noise shock.
Nominal money growth follows

\[ \theta_t = \theta^{ss} + u_t \]

where \( u_t \) is a shock to the growth rate

Assumption:

\[ u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t, \quad 0 \leq \rho_u < 1, \quad \phi \geq 0 \]

with \( \varphi_t \) being a mean-zero, white-noise shock.

- Note that there may or may not be serial correlation in the shocks to nominal money growth
- Note that money growth may or may not respond toward past technology shocks, and may be either procyclical \((\phi > 0)\) or countercyclical \((\phi < 0)\).

Per-period utility function and budget constraint are

\[ u(c_t, m_t, l_t) \]

and

\[ c_t + k_t + m_t = y_t + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1}. \quad (2.45' \text{ and } 2.46') \]

(ignoring financial assets \( b_t \) as in last lecture . . .)

As in MIU model without endogenous labor, households maximizes discounted lifetime utility subject to the budget constraint

Again, dynamic programming is used
• **Note**, however, that since $l_t$ is a choice variable, it is inappropriate to treat available resources as the state variable at period $t$. Instead, state variables will therefore be $k_{t-1}$ and

$$a_t \equiv \tau_t + m_{t-1}/(1 + \pi_t)$$

• The optimization problem is then characterized by the value function

$$V (a_t, k_{t-1}) = \max \mathbb{E}_t \{u(c_t, m_t, l_t) + \beta V (a_{t+1}, k_t)\}$$

where the maximization is over $c, m, k, l$ subject to the budget constraint and the definition of the state variable $a_t$. $\mathbb{E}_t$ is the rational expectations operator.

• One substitutes the constraint and definition so as to eliminate $k_t$ and $a_{t+1}$ and get an unconstrained maximization problem (over $c, m, l$)
• First-order condition with respect to $c_t$:

$$u_c(c_t, m_t, l_t) = \mathbb{E}_t \beta V_k(a_{t+1}, k_t)$$

(2.47')

(as $\partial a_{t+1}/\partial c_t = 0$ by the definition of $a$). Marginal gain of consumption must equal the expected marginal loss in terms of lower capital in next period

• First-order condition with respect to $m_t$:

$$u_m(c_t, m_t, l_t) + \beta\mathbb{E}_t V_a(a_{t+1}, k_t) \frac{1}{1 + \pi_{t+1}} = \beta\mathbb{E}_t V_k(a_{t+1}, k_t)$$

(2.50')

Marginal gain in terms of current utility and expected next period monetary wealth must equal the expected marginal loss in terms of lower capital in next period

• First-order condition with respect to $l_t$:

$$u_l(c_t, m_t, l_t) = \mathbb{E}_t \beta V_k(a_{t+1}, k_t) f_n(k_{t-1}, 1 - l_t, z_t)$$

(2.48')

Marginal gain of leisure is equated to the marginal cost, which is the value loss from less next-period capital, times the marginal product of labor (the real wage)
Mathematical digression “not for lecturing”, but for reading: Elimination of the value function

• We know that optimum will be characterized by optimal values of \( c_t, m_t, \) and \( l_t \) as functions of the state variables. Call these functions

\[
c_t = c(a_t, k_{t-1}), \quad m_t = m(a_t, k_{t-1}), \quad l_t = l(a_t, k_{t-1}).
\]

• The value function is thus by definition given as

\[
V(a_t, k_{t-1}) = u(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) + \beta E_t V(a_{t+1}, k_t)
\]

This holds for all \( a_t, k_{t-1} \) so we have

\[
V_a(a_t, k_{t-1}) = u_c(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) c_a(a_t, k_{t-1})
\]

\[
+ u_m(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) m_a(a_t, k_{t-1})
\]

\[
+ u_l(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) l_a(a_t, k_{t-1})
\]

\[
+ \beta E_t V_a(a_{t+1}, k_t) \frac{\partial a_{t+1}}{\partial a_t} + \beta E_t V_k(a_{t+1}, k_t) \frac{\partial k_t}{\partial a_t}.
\]

We have that

\[
\frac{\partial a_{t+1}}{\partial a_t} = \frac{1}{1 + \pi_{t+1}} m_a(a_t, k_{t-1}),
\]

\[
\frac{\partial k_t}{\partial a_t} = 1 - c_a(a_t, k_{t-1}) - m_a(a_t, k_{t-1}) - f_n(k_{t-1}, 1 - l_t, z_t) l_a(a_t, k_{t-1}),
\]
- Use this in the expression for $V_a(a_t, k_{t-1})$:

$$V_a(a_t, k_{t-1}) = u_c(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) c_a(a_t, k_{t-1})$$

$$+ u_m(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) m_a(a_t, k_{t-1})$$

$$+ u_l(c(a_t, k_{t-1}), m(a_t, k_{t-1}), l(a_t, k_{t-1})) l_a(a_t, k_{t-1})$$

$$+ \beta E_t V_a(a_{t+1}, k_t) \left[ \frac{1}{1 + \pi_{t+1}} m_a(a_t, k_{t-1}) \right]$$

$$+ \beta E_t V_k(a_{t+1}, k_t) \left[ 1 - c_a(a_t, k_{t-1}) - m_a(a_t, k_{t-1}) - f_n(k_{t-1}, 1 - l_t, z_t) l_a(a_t, k_{t-1}) \right].$$

- By the Envelope theorem, all the terms in front of $c_a(a_t, k_{t-1})$, $m_a(a_t, k_{t-1})$ and $l_a(a_t, k_{t-1})$ are zero. I.e., at an optimum, the marginal value of changing $c$, $m$, or $l$ must be zero.

- Therefore:

$$V_a(a_t, k_{t-1}) = \beta E_t V_k(a_{t+1}, k_t)$$
• Likewise, we get
\[ V_k (a_t, k_{t-1}) = \beta E_t V_k (a_{t+1}, k_t) [f_k (k_{t-1}, 1 - l_t, z_t) + 1 - \delta] \]

• So, by the first-order condition guiding \( c \):

\[ u_c (c_t, m_t, l_t) = V_a (a_t, k_{t-1}) \]

• We then get the corresponding relationship for guiding money choice similar to simple MIU model:

\[ u_m (c_t, m_t, l_t) + \beta E_t u_c (c_{t+1}, m_{t+1}, l_{t+1}) \frac{1}{1 + \pi_{t+1}} = \beta E_t V_k (a_{t+1}, k_t) \]
\[ = u_c (c_t, m_t, l_t) \] (\(*\))
• Also, the condition guiding consumption can be modified by use of

\[ V_k(a_t, k_{t-1}) = \beta E_t V_k(a_{t+1}, k_t) \left[ f_k(k_{t-1}, n_t, z_t) + 1 - \delta \right] \]

\[ u_c(c_t, m_t, l_t) = E_t \beta V_k(a_{t+1}, k_t) = \beta E_t \beta E_{t+1} V_k(a_{t+2}, k_{t+1}) \left[ f_k(k_t, 1 - l_{t+1}, z_{t+1}) + 1 - \delta \right] \]

and using that \( V_a(a_t, k_{t-1}) = \beta E_t V_k(a_{t+1}, k_t) \), gives \( V_a(a_{t+1}, k_t) = \beta E_{t+1} V_k(a_{t+2}, k_{t+2}) \) and thus

\[ u_c(c_t, m_t, l_t) = \beta E_t \beta E_{t+1} V_a(a_{t+1}, k_t) \left[ f_k(k_t, 1 - l_{t+1}, z_{t+1}) + 1 - \delta \right] \]

\[ = \beta E_t (1 + r_t) u_c(c_{t+1}, m_{t+1}, l_{t+1}) \quad (***) \]

with \( r_t \equiv f_k(k_t, 1 - l_{t+1}, z_{t+1}) - \delta \). I.e., (***) is the “money-modified Keynes-Ramsey rule”

• Finally, we get the condition for the choice of \( l_t \), which becomes

\[ u_l(c_t, m_t, l_t) = u_c(c_t, m_t, l_t) f_n(k_{t-1}, 1 - l_t, z_t) \quad (***) \]

• Hence, the equations (*), (**) and (***) , together with the budget constraint, provide solutions for the paths of \( c, m, l, \) and \( k \).

End of mathematical digression “not for lecturing”
Particular functional forms of utility and production functions

- Model is solved by numerical methods under assumptions about particular functional forms for utility and production function

Utility (as before):

$$u(c_t, m_t, 1 - n_t) = \frac{(ac_t^{1-b} + (1 - a)m_t^{1-b})^{\frac{1-\Phi}{1-b}}}{1 - \Phi} + \frac{\psi(1 - n_t)^{1-\eta}}{1 - \eta}$$

Production function, Cobb-Douglas:

$$y_t = k_{t-1}^\alpha n_t^{1-\alpha} e^{zt}, \quad 0 < \alpha < 1$$

(2.63)

- Steady-state solution: Note that the real interest rate,

$$r_t = f_k(k_t, n_{t+1}, z_{t+1}) = \delta$$

becomes

$$r_t = \alpha E_t k_t^{\alpha-1} (n_{t+1})^{1-\alpha} e^{zt+1} - \delta$$

$$= \frac{E_t y_{t+1}}{k_t} - \delta.$$  

By the steady-state condition $1 + r^{ss} = 1/\beta$, this only determines the ratio $y^{ss}/k^{ss}$ [holds for any CRS production function; cf. (2.61)]

The ratio is independent of monetary factors, but the levels $y^{ss}$, $k^{ss}$, $c^{ss}$ are not when, e.g., $n^{ss}$ is affected by $m$. Then, superneutrality fails.
Dynamic effects of money and technology shocks

• To assess the quantitative effects of money and technology shocks, the model is calibrated and simulated

Calibration: Assign empirically plausible values to the parameters of the model and check that steady-state ratios are empirically plausible (like $c^{ss}/y^{ss}$ etc.)

Simulation:

– Perform a log linearization of the model’s dynamic equations (everything is expressed as percentage deviations from steady state); see Appendix 2.7.2, p. 85 in Walsh (2010)
– Solve the model by numerical methods (various simulation programs are available on the internet; I use “Dynare” – an intuitive “add on” to MatLab);
– Create artificial time series data from the system

• From the artificial data, one evaluates the properties of the model relative to data in terms of:

  – Standard deviations of various relevant variables, and their s.d. relative to output
  – Correlation coefficients of various variables with output
  – Impulse response patterns of variables when shocks hit
• Main results (when $\Phi < b$; implying $u_{cm} > 0$)

  – Steady-state non-superneutrality is of the form: Higher $\theta \Rightarrow$ lower output
  – If money shocks, $\varphi_t$—shocks, shall pay a role, $\rho_u > 0$ is needed. Then, the shock will affect expected next-period inflation, and thus—through the Fisher equation—period-$t$ nominal interest rate.

    * Unanticipated changes in money growth have no effects: They do not affect $i_t$

  – The effects of money shocks on labor and output are stronger the more persistence in money growth, but the effects are quantitatively very small: 0.9% increase in money growth (at quarterly rate), lowers output by around 0.02% in baseline

  – If technology shocks are met with procyclical money growth, output is more stable. The magnitude, however, is very modest

  – Main effects of money shocks are on inflation and nominal interest rates

  – Positive money shocks lead to higher nominal interest rates. In contrast with usual IS/LM story (where a liquidity effect is present: nominal rates fall to increase money demand). Reason is flexible prices in the MIU model (contrary to the sticky price IS/LM model).

    * Prices adjust instantaneously so as to reduce real money supply, matching the fall in demand resulting from higher nominal interest rates.

    * Wildly at odds with empirical evidence on short run correlations ($corr(M, i) < 0$)
Summary

- The MIU framework provides a setting in which the welfare costs of inflation can be assessed, and where the optimal long-run inflation rate can be determined.
- This, in turn, is equivalent of determining the “optimal quantity of money.”

- The stochastic, dynamic model without the superneutrality property can be used to assess the importance of monetary shocks for economic fluctuations.
- In the calibrated, MIU model with endogenous labor, money matters for business cycle fluctuations, but not very much. Moreover, the missing liquidity effect is a major weakness.
- This is an indication that flex-price models may be ill-suited for analysis of monetary phenomena in the short run, i.e., at business cycle frequency.
Plan for next lectures

Monday, February 23

1. A Cash-in-Advance Model

Literature: Walsh (2010, Chapter 3, pp. 91–115) (NB: Material on shopping-time models “only” recommended)

Thursday, February 26

1. Public budget accounting, inflation and debt
2. Equilibrium seigniorage

Literature: Walsh (Chapter 4, pp. 135–162)

Monday, March 2:

1. Optimal taxation and seigniorage
2. Robustness of the Friedman rule?

Literature: Walsh (Chapter 4, pp. 170–184; pp. 188–191)